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Nonlocal constants of motion in Lagrangian Dynamics of any order

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ABSTRACT

We describe a recipe to generate "nonlocal" constants of motion for ODE Lagrangian systems. As a sample application, we recall a nonlocal constant of motion for dissipative mechanical systems, from which we can deduce global existence and estimates of solutions under fairly general assumptions. Then we review a generalization to Euler–Lagrange ODEs of order higher than two, leading to first integrals for the Pais–Uhlenbeck oscillator and other systems. Future developments may include adaptations of the theory to Euler–Lagrange PDEs.

1. Introduction

We are interested in constants of motion for the *Euler–Lagrange equation*

$$\frac{d}{dt}\partial_{\dot q}L\big(t,q(t),\dot q(t)\big)-\partial_q L\big(t,q(t),\dot q(t)\big)=0, \tag{1.1}$$

where $L(t,q,\dot q)$ is a smooth scalar valued Lagrangian function, $t\in\mathbb{R}$, $q,\dot q\in\mathbb{R}^n$. In the paper[1] the first and the last author revisited Noether's Theorem, which links first integrals with symmetries of the Lagrangian L. Leaving aside asynchronous perturbations and boundary terms and other issues, here we single out the following simple result. (Notation: the central dot is the scalar product in $\mathbb{R}^n$).

Theorem 1. *Let $q(t)$ be a solution to the Euler–Lagrange equation. Suppose we have a smooth family $q_\lambda(t)$, $\lambda\in\mathbb{R}$, of perturbed motions, so that $q_0(t)\equiv q(t)$. Then the following function of t is constant:*

$$\partial_{\dot q}L\big(t,q(t),\dot q(t)\big)\cdot\partial_\lambda q_\lambda(t)\Big|_{\lambda=0}-\int_{t_0}^{t}\frac{\partial}{\partial\lambda}L\big(s,q_\lambda(s),\dot q_\lambda(s)\big)\Big|_{\lambda=0}ds. \tag{1.2}$$

The proof is straightforward: we just take the derivative of the function in (1.2), use the Euler–Lagrange equation and reverse the order of a double derivative.

We call (1.2) the *constant of motion associated to the family* $q_\lambda(t)$. For a random family, we may expect the constant of motion to be trivial or inconsequential. In general it is *nonlocal*, which means that its value at a time t depends not only on the current state $(t,q(t),\dot q(t))$ at time t, but also on the whole history between t_0 and t.

In the original spirit of Noether's theorem we can concentrate the attention to families $q_\lambda(t)$ which make the integrand in (1.2) vanish whenever L enjoys an invariance property. For instance, for the Lagrangian of a particle in the plane under a central force field

$$L(t,q,\dot q):=\frac{1}{2}m\|\dot q\|^2-U\big(t,\|q\|\big),\quad q=(q_1,q_2)\in\mathbb{R}^2, \tag{1.3}$$

the rotation family

$$q_\lambda(t):=\begin{pmatrix}\cos\lambda & -\sin\lambda\\ \sin\lambda & \cos\lambda\end{pmatrix}\begin{pmatrix}q_1(t)\\ q_2(t)\end{pmatrix}, \tag{1.4}$$

exploits the invariance of L under rotations when plugged into (1.2), and leads to the conservation of the angular momentum $\det(q(t),\dot q(t))$, which is definitely a *local* first integral.

In a series of papers we have used Theorem 1 to find numerous *nonlocal* constants of motion which are useful. Here is a partial list.

- Consider a particle moving in a time-independent potential field $U(q)$, $q\in\mathbb{R}^n$, and *viscous*, i.e., linear, fluid resistance. We see at once that the solutions to the Euler–Lagrange equation exist globally in the future. What about the existence in the past? For a quite natural choice of the perturbed motions $q_\lambda(t)$, after

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an integration by parts the integrand in (1.2) becomes negative, provided we assume $U \geq 0$. Under these conditions the first term in (1.2) is an increasing function of time, which permits to prove the *global existence in the past*. The paper[2] also obtains other estimates for the solutions for this system and for some Lane–Emden equation for which global existence is proved too. We have picked this example for a detailed exposition in Section 2 below, as an illustration of one way of using our nonlocal constants of motion.

- General time-independent homogeneous potentials of degree −2, for instance Calogero's potential for the one dimensional n-body problem with inversely quadratic pair potentials, taken from Ref. 1, for which it is possible to write an explicit formula for $\|q(t)\|$ as a function of time, even though we may ignore how the orbit is shaped.
- The Maxwell–Bloch equations for laser dynamics in the conservative case taken from Ref. 3, where (1.2) by derivation permits to separate the equations (in a nonstandard way) into a system exhibiting "fish dynamics" and a system with central force.

Theorem 1 can be generalized to *nonvariational* Lagrange equations, as we do in Ref. 4, among other results on Killing-type equations, and applications for:

- A particle under a time-independent potential field $U(q)$, $q \in \mathbb{R}^n$, and *hydraulic*, i.e., quadratic, fluid resistance, a result taken from Gorni–Zampieri,[5] for which the result in dynamics is: whenever $0 \leq U \leq U_{\text{sup}} < +\infty$, every solution for which the initial kinetic energy is strictly greater than U_{sup} explodes in the past in finite time.
- The dissipative Maxwell–Bloch equations for laser dynamics, taken from Gorni–Residori–Zampieri,[6] for which a quite natural family $q_\lambda(t)$ yields a constant of motion (1.2) which, for a special choice of the parameters, turns out to be a genuine first integral $N(t, q, \dot{q})$ since the integral term vanishes. The first integral permits some kind of separation of variables.

The paper[5] presents the result on hydraulic fluid resistance and gives a survey on all other applications we mentioned till now. The novelty in the present survey is the introduction of Scomparin's generalization of Theorem 1 to higher-order Lagrangian systems, which recently appeared in Ref. 7.

For $N = 1, 2, \ldots$ consider the *higher-order Euler–Lagrange equation*

$$\sum_{k=0}^{N}(-1)^k \frac{d^k}{dt^k}\,\partial_{q^{(k)}} L(t, q, \ldots, q^{(N)}) = 0, \tag{1.5}$$

where the Nth-order Lagrangian L is a smooth function with $t \in \mathbb{R}$, and $q, \ldots, q^{(N)} \in \mathbb{R}^n$. We use $q^{(k)} \equiv d^k q/dt^k$.

Within the higher-order framework, a *first integral* is a smooth function

$$K(t, q, q^{(1)}, q^{(2)}, \ldots), \tag{1.6}$$

that keeps constant along each solution of Eq. (1.5). The celebrated *Noether's Theorem* establishes a relation between invariance properties of a Lagrangian and its first integrals.[8,9]

Scomparin's paper[7] generalizes Theorem 1 to Nth-order Lagrangians:

Theorem 2. *Suppose that the function $t \mapsto q(t)$ satisfies the Euler–Lagrange equation for smooth $L(t, q, \ldots, q^{(N)})$, and that $q_\lambda(t)$, $\lambda \in \mathbb{R}$, is a smooth family of motions, for which $q_0(t) \equiv q(t)$. Then the following function does not depend on t:*

$$\sum_{j=1}^{N}\sum_{k=0}^{j-1}(-1)^k \frac{d^k}{dt^k}\,\partial_{q^{(j)}} L(t, q, \ldots, q^{(N)}) \cdot \partial_\lambda q_\lambda^{(j-k-1)}\Big|_{\lambda=0}$$
$$-\int_{t_0}^{t} \frac{\partial}{\partial\lambda} L(s, q_\lambda, \ldots, q_\lambda^{(N)})\Big|_{\lambda=0} ds. \tag{1.7}$$

A basic higher-order mechanical system is the *Pais–Uhlenbeck oscillator*,[10] whose Lagrangian function is

$$L^{\text{PU}} = \frac{1}{2}q^{(2)2} - \frac{1}{2}(w_1^2 + w_2^2)\,q^{(1)2} + \frac{1}{2}w_1^2 w_2^2 q^2, \tag{1.8}$$

with $w_1, w_2 > 0$. Its Euler–Lagrange equation is

$$q^{(4)} + (w_1^2 + w_2^2)q^{(2)} + w_1^2 w_2^2\, q = 0. \tag{1.9}$$

With higher-order Lagrangians it is possible to build models for modified gravity theories,[11] quantum-loop cosmologies,[12] and string theories.[13] Approaching higher-order mechanics from a new nonlocal point of view provides new perspectives to identify novel first integrals without necessarily requiring invariance proprieties on the already difficult to investigate structure of higher-order Lagrangians. Hopefully Theorem 2 will provide a valuable tool to give a novel insight into stability proprieties of higher-order models and boundedness of related solutions as is done for second order equations in Kaparulin[14,15] and in the papers mentioned above.

Section 3 below summarizes the main results in Scomparin's paper.[7]

2. An application to dissipative dynamics

The results in this section are taken from Gorni–Zampieri.[2]

Let us consider the first integral of *energy* which is generally derived in Noether's framework by the use of asynchronous perturbations but can also be treated by means of Theorem 1. For a time independent Lagrangian function $L(t, q, \dot{q}) = \mathcal{L}(q, \dot{q})$, and the *time-translation* family $q_\lambda(t) = q(t + \lambda)$ we have

$$\partial_\lambda L\big(t, q_\lambda(t), \dot{q}_\lambda(t)\big)\Big|_{\lambda=0} = \partial_q \mathcal{L} \cdot \dot{q}(t) + \partial_{\dot{q}} \mathcal{L} \cdot \ddot{q}(t) = \frac{d}{dt}\mathcal{L}(q(t), \dot{q}(t)). \tag{2.1}$$

Thus the constant of motion (1.2) is

$$\partial_{\dot{q}} \mathcal{L} \cdot \dot{q}(t) - \int_{t_0}^{t} \frac{d}{ds}\mathcal{L}(q(s), \dot{q}(s))\, ds$$
$$= \partial_{\dot{q}} \mathcal{L}\big(q(t), \dot{q}(t)\big) \cdot \dot{q}(t) - \mathcal{L}\big(q(t), \dot{q}(t)\big) + \mathcal{L}\big(q(t_0), \dot{q}(t_0)\big), \tag{2.2}$$

which differs from the *energy*

$$E(q, \dot{q}) = \partial_{\dot{q}} \mathcal{L}(q, \dot{q}) \cdot \dot{q} - \mathcal{L}(q, \dot{q}), \tag{2.3}$$

by the constant $\mathcal{L}(q(t_0), \dot{q}(t_0))$.

Now, we turn to dissipation. Consider $k > 0$, a smooth potential function $U : \mathbb{R}^n \to \mathbb{R}$ defined on the whole space. The equation of motion of a particle of mass $m > 0$ under this potential and viscous dissipation is

$$m\ddot{q} = -k\dot{q} - \nabla U(q), \quad q \in \mathbb{R}^n. \tag{2.4}$$

The energy first integral (2.3) for $k = 0$ is

$$E(q, \dot{q}) = \frac{1}{2}m\|\dot{q}\|^2 + U(q). \tag{2.5}$$

For $k > 0$ this function decreases along solutions

$$\dot{E} = m\dot{q} \cdot \frac{1}{m}\big(-k\dot{q} - \nabla U(q)\big) + \nabla U(q) \cdot \dot{q} = -k\|\dot{q}\|^2 \leq 0. \tag{2.6}$$

In the sequel we assume that *the potential is bounded from below*, say $U \geq 0$. Then, for any solution $q(t)$ the velocity $\dot{q}(t)$ is bounded in the future:

$$\frac{1}{2}m\|\dot{q}(t)\|^2 \leq \frac{1}{2}m\|\dot{q}(t)\|^2 + U\big(q(t)\big)$$
$$\leq \frac{1}{2}m\|\dot{q}(t_0)\|^2 + U\big(q(t_0)\big), \quad t \geq t_0, \tag{2.7}$$

so $q(t)$ is bounded in every bounded interval of time and we deduce that all solutions are *global in the future. What can be said* about the past?

Notice that for $k = 0$, with no dissipation, we have global existence since the above argument holds in the past too.

Our (2.4) can be seen as the Euler–Lagrange equation for the Lagrangian

$$L = e^{kt/m}\left(\frac{1}{2}m\|\dot{q}\|^2 - U(q)\right). \tag{2.8}$$

It is quite natural to consider the family $q_\lambda(t) := q(t + \lambda\, e^{at})$ with $a \in \mathbb{R}$ new parameter and $q(t)$ solution. Indeed for $a = 0$ the family reduces to

the time-shift used for energy conservation as $k=0$ and the exponential function is easily inspired by the one in (2.8).

Then

$$\frac{\partial}{\partial\lambda}L\big(t,q_\lambda(t),\dot q_\lambda(t)\big)\Big|_{\lambda=0}=\frac{d}{dt}\left[-2e^{\left(a+\frac{k}{m}\right)t}U\big(q(t)\big)\right]$$
$$+\,e^{\left(a+\frac{k}{m}\right)t}\left[\Big(a-\tfrac{k}{m}\Big)m\|\dot q(t)\|^2+2\Big(a+\frac{k}{m}\Big)U\big(q(t)\big)\right], \tag{2.9}$$

where we eliminated $\ddot q(t)$ using the differential equation. For $a=k/m$ it simplifies and we have the constant of motion

$$t\mapsto \partial_{\dot q}L\big(t,q(t),\dot q(t)\big)\cdot\partial_\lambda q_\lambda(t)\Big|_{\lambda=0}- \tag{2.10}$$
$$-\int_{t_0}^{t}\frac{\partial}{\partial\lambda}L\big(s,q_\lambda(s),\dot q_\lambda(s)\big)\Big|_{\lambda=0}ds$$
$$=e^{2kt/m}\Big[m\|\dot q(t)\|^2+2U\big(q(t)\big)\Big]+4\,\frac{k}{m}\int_t^{t_0}e^{2ks/m}U\big(q(s)\big)\,ds. \tag{2.11}$$

Since we assumed $U\ge 0$, the last integral decreases for $t\le t_0$ and the function

$$t\mapsto e^{2kt/m}\Big[m\,\|\dot q(t)\|^2+2U(q(t))\Big], \tag{2.12}$$

increases with t for all $t\le t_0$.

Finally, we have the estimate for $t\le t_0$:

$$m\,\|\dot q(t)\|^2\le e^{2k(t_0-t)/m}\Big[m\,\big\|\dot q(t_0)\big\|^2+2U\big(q(t_0)\big)\Big]. \tag{2.13}$$

Since in any time interval of the form $(t_1,t_0]$ the velocity $\dot q(t)$ is bounded, so is $q(t)$, and we have global existence of solutions. Summing up:

Theorem 3. *If $k>0$ and U is a smooth potential on $\mathbb{R}^n$ which is bounded from below, then all solutions of the dissipative equation $\ddot q=-k\dot q-\nabla U(q)$ are defined for all $t\in\mathbb{R}$.*

3. First integrals for higher-order Lagrangians

Generally, we cannot expect that Theorem 2 yields true first integrals for a random choice of the family $q_\lambda(t)$. However, few and precious Lagrangians make our machinery work. The simplest example is for autonomous Lagrangians:

Theorem 4. *Let $t\mapsto q(t)$ be a solution to the Euler–Lagrange equation for a time-independent smooth $\mathcal{L}(q,\dots,q^{(N)})$. Then the following function is a first integral:*

$$K_1(q,\dots,q^{(2N-1)})=$$
$$=\sum_{i=1}^{N}\sum_{k=0}^{i-1}(-1)^k\frac{d^k}{dt^k}\partial_{q^{(i)}}\mathcal{L}(q,\dots,q^{(N)})\cdot q^{(i-k)}-\mathcal{L}(q,\dots,q^{(N)}). \tag{3.1}$$

It is important to notice that Theorem 4 recovers the Noetherian result of Ref. 16 for $N=2$ Lagrangians.

Since the Pais–Uhlenbeck Lagrangian L^{PU} of formula (1.8) is time-independent, we deduce the following first integral:

$$2K_1^{\mathrm{PU}}=q^{(2)2}-(w_1^2+w_2^2)q^{(1)2}-2q^{(3)}q^{(1)}-w_1^2w_2^2q^2. \tag{3.2}$$

In Ref. 1 the authors proved energy conservation for the canonical harmonic oscillator starting from nonlocal space-changes. Consequently, using Theorem 2, we deduce that first integrals are easy to be found if $\partial_{q^{(j)}}L\propto d^j\partial_qL/dt^j$ for all $j=1,\dots,N$:

Theorem 5. *Consider a smooth Lagrangian $L(t,q,\dots,q^{(N)})$ for which there exists a set of constant parameters $\rho_1\dots\rho_N\in\mathbb{R}$ such that*

$$\partial_{q^{(i)}}L=\rho_i\frac{d^i}{dt^i}\partial_qL\qquad\text{for all motions and } i\in\{1,\dots,N\}. \tag{3.3}$$

Take a solution $t\mapsto q(t)$ to the Euler–Lagrange equation, and define $F^{(0)}=\sum_{j=1}^{N}(-1)^{j+1}d^{j-1}\partial_{q^{(j)}}L/dt^{j-1}$. Then $F^{(\ell)}=d^{\ell-1}\partial_qL/dt^{\ell-1}$ with $\ell\in\{1,\dots,2N\}$, and the following function is a first integral:

$$K_2(t,q,\dots,q^{(3N-1)})=$$
$$=\sum_{i=1}^{N}\rho_i\left[\sum_{k=0}^{i-1}(-1)^kF^{(i+k+1)}\cdot F^{(i-k-1)}-\frac{1}{2}\|F^{(i)}\|^2\right]-\frac{1}{2}\|F^{(0)}\|^2. \tag{3.4}$$

Notice that L^{PU} satisfies Theorem 5 with $\rho_1=-(w_1^2+w_2^2)w_1^{-2}w_2^{-2}$ and $\rho_2=w_1^{-2}w_2^{-2}$. Hence, $F^{(0)}=-(w_1^2+w_2^2)q^{(1)}-q^{(3)}$ and $F^{(\ell)}=w_1^2w_2^2q^{(\ell-1)}$ ($\ell=0,1,2$) give

$$2K_2^{\mathrm{PU}}=\big(w_1^4+w_1^2w_2^2+w_2^4\big)q^{(1)2}+q^{(3)2}+2w_1^2w_2^2\,qq^{(2)}+$$
$$+\,(w_1^2+w_2^2)\big(2q^{(3)}q^{(1)}+w_1^2w_2^2q^2\big). \tag{3.5}$$

If combined, K_1^{PU} and K_2^{PU} recover the two first integrals recently proposed by Ref. 15.

Another interesting situation generating first integrals arises when the Lagrangian, evaluated on a perturbed motion $q_\lambda(t)$, has constant derivative at $\lambda=0$.

Theorem 6. *Consider a smooth Lagrangian, and suppose that for a given smooth family of perturbed motions $q_\lambda(t)$ there exists a constant $\mu\in\mathbb{R}$ such that $\partial_\lambda L(t,q_\lambda,\dots,q_\lambda^{(N)})\big|_{\lambda=0}=\mu$. Then, the following function is a first integral:*

$$K_3(t,q,\dots,q^{(2N-1)})=$$
$$=\sum_{i=1}^{N}\sum_{k=0}^{i-1}(-1)^k\frac{d^k}{dt^k}\partial_{q^{(i)}}L(t,q,\dots,q^{(N)})\cdot\partial_\lambda q_\lambda^{(i-k-1)}\Big|_{\lambda=0}-\mu t\,. \tag{3.6}$$

Let $q=(q_1,q_2)\in\mathbb{R}^2$ and consider the rotation family $q_\lambda(t)$ in (1.4). When evaluated on $q_\lambda(t)$, L^{PU} does not depend on λ, hence Theorem 6 gives the new angular-momentum-like first integral

$$K_3^{\mathrm{PU}}=(w_1^2+w_2^2)\det\big(q^{(1)},q\big)+\det\big(q^{(1)},q^{(2)}\big)+\det\big(q^{(3)},q\big). \tag{3.7}$$

In Scomparin,[7] the machinery is also applied to analyze a higher-order generalization of the Pais–Uhlenbeck oscillator,[17,18] and a simple Degenerate Higher-Order model of Scalar-Tensor (DHOST) theory that, in these last years, inspired many modified gravity theories.[11] Again, the full consistency of the machinery is confirmed.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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