A new criterion for $\mathcal{M}, \mathcal{N}$-adhesivity, with an application to hierarchical graphs*

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Abstract. *Adhesive categories* provide an abstract framework for the algebraic approach to rewriting theory, where many general results can be recast and uniformly proved. However, checking that a model satisfies the adhesivity properties is sometimes far from immediate. In this paper we present a new criterion giving a sufficient condition for $\mathcal{M}, \mathcal{N}$*-adhesivity*, a generalisation of the original notion of adhesivity. We apply it to several existing categories, and in particular to *hierarchical graphs*, a formalism that is notoriously difficult to fit in the mould of algebraic approaches to rewriting and for which various alternative definitions float around.

1 Introduction

The introduction of *adhesive categories* marked a watershed moment for the algebraic approaches to the rewriting of graph-like structures [16,9]. Until then, key results of the approaches on e.g. parallelism and confluence had to be proven over and over again for each different formalism at hand, despite the obvious similarity of the procedure. Differently from previous solutions to such problems, as the one witnessed by the *butterfly lemma* for graph rewriting [8, Lemma 3.9.1], the introduction of adhesive categories provided such a disparate set of formalisms with a common abstract framework where many of these general results could be recast and uniformly proved once and for all.

Despite the elegance and effectiveness of the framework, proving that a given category satisfies the conditions for being adhesive can be a daunting task. For this reason, we look for simpler general criteria implying adhesivity for a class of categories. Similar criteria have been already provided for the core framework of adhesive categories; e.g., every elementary topos is adhesive [17], and a category is (quasi)adhesive if and only if can be suitably embedded in a topos [15,12]. This covers many useful categories such as sets, graphs, etc.; on the other hand, there are many categories of interest which are not (quasi)adhesive, such as directed graphs, posets, and many of their subcategories. In these cases we can try to prove the more general $\mathcal{M}, \mathcal{N}$-adhesivity for suitable $\mathcal{M}, \mathcal{N}$; however, so far this

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has been achieved only by means of *ad hoc* arguments. To this end, one of the main contributions of this paper is a new criterion for $\mathcal{M}, \mathcal{N}$-adhesivity, based on the verification of some properties of functors connecting the category of interest to a family of suitable adhesive categories. This criterion allows us to prove in a uniform and systematic way some previous results about the adhesivity of categories built by products, exponents, and comma construction.

Moreover, it is well-known that categorical properties are often *prescriptive*, indicating abstractly the presence of some good behaviour of the modelled system. Adhesivity is one such property, as it is highly sought after when it comes to rewriting theories. Thus, our criterion for proving $\mathcal{M}, \mathcal{N}$-adhesivity can be seen also as a "litmus test" for the given category. This is useful in situations that are not completely settled, and for which different settings have been proposed. An important example is that of *hierarchical graphs*, for which we roughly can find two alternative proposals: on the one hand, algebraic formalisms where the edges have some algebraic structures, so that the nesting is a side effect of the term construction; on the other hand, combinatorial approaches where the topology of a standard graph is enriched by some partial order, either on the nodes or on the edges, where the order relation indicates the presence of nesting. By applying our criterion, we can show that the latter approach yields indeed an $\mathcal{M}, \mathcal{N}$-adhesive category, confirming and overcoming the limitations of some previous approaches to hierarchical graphs [21,23,24], which we briefly recall next.

The more straightforward proposal is by Palacz [24], using a poset of edges instead of just a set; however, the class of rules has to be restricted in order to apply the approach, which in any case predates the introduction of adhesive categories. Our work allows to rephrase in terms of adhesive properties and generalise Palacz's proposal, dropping his constraint on rules. Another attempt are Mylonakis and Orejas' *graphs with layers* [21], for which $\mathcal{M}$-adhesivity is proved for a class of monomorphisms in the category of symbolic graphs; however, nodes between edges at different layers cannot be shared. Padberg [23] goes for a coalgebraic presentation via a peculiar "superpower set" functor; this gives immediately $\mathcal{M}$-adhesivity provided that this superpower set functor is well-behaved with respect to limits. However this approach is rather *ad hoc*, not modular and not very natural for actual modelling.

Summarising, the main contributions of this work are: (a) a new general criterion for assessing $\mathcal{M}, \mathcal{N}$-adhesivity; (b) new proofs of $\mathcal{M}, \mathcal{N}$-adhesivity for some relevant categories, systematising previous known proofs; (c) the first proof that a category of hierarchical graph is $\mathcal{M}, \mathcal{N}$-adhesive.

Synopsis. After having recalled some basic notions, in Section 2 we introduce the new criterion for $\mathcal{M}, \mathcal{N}$-adhesivity; using it, we show $\mathcal{M}, \mathcal{N}$-adhesivity of several constructions, such as products and comma categories. In Section 3 we apply this theory to various example categories, such as directed (acyclic) graphs, trees and term graphs. We show also the adhesivity of several categories obtained by combining adhesive ones, and in particular of the elusive category of hierarchical graphs. Conclusions and directions for future work are in Section 4. An extended version of this paper is available at [6].

2 $\mathcal{M},\mathcal{N}$-adhesivity via creation of (co)limits

In this section we recall some definitions and results about $\mathcal{M},\mathcal{N}$-adhesive categories and provide a new criterion to prove this property.

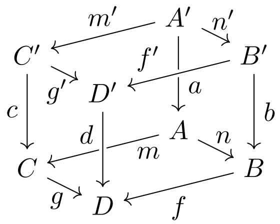
2.1 $\mathcal{M},\mathcal{N}$-adhesive categories

Intuitively, an adhesive category is one in which pushouts of monomorphisms exist and "behave more or less as they do in the category of sets" [16]. Formally, we require pushouts of monomorphisms to be Van Kampen colimits.

Definition 2.1. *A* Van Kampen square *in a category* $\mathbf{A}$ *is a pushout square*

$$\begin{array}{ccc} A & \xrightarrow{n} & B \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow[g]{} & D \end{array}$$

such that for any cube as follows, where the back faces are pullbacks,



the top face is a pushout if and only if the front faces are pullbacks.

Pushout squares which enjoy the "if" of this condition are called stable.

Given a category $\mathbf{A}$ we will denote by $\mathsf{Mor}(\mathbf{A}), \mathsf{Mono}(\mathbf{A}), \mathsf{Reg}(\mathbf{A})$ respectively the classes of morphisms, monomorphisms and regular monomorphisms of $\mathbf{A}$.

Definition 2.2. *Let* $\mathbf{A}$ *be a category and* $\mathcal{A} \subseteq \mathsf{Mor}(\mathbf{A})$. *Then we say that* $\mathcal{A}$ *is*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle n} \\ C & \xrightarrow[g]{} & D \end{array}$$

- stable under pushouts *if for every pushout square as aside, if* $m \in \mathcal{A}$ *then* $n \in \mathcal{A}$*;*
  stable under pullbacks *if for every pullback square as aside, if* $n \in \mathcal{A}$ *then* $m \in \mathcal{A}$*;*
- closed under composition *if* $g, f \in \mathcal{A}$ *implies* $g \circ f \in \mathcal{A}$ *whenever* g *and* f *are composable;*
- closed under $\mathcal{B}$-decomposition *(where* $\mathcal{B}$ *is another subclass of* $\mathsf{Mor}(\mathbf{A})$*) if* $g \circ f \in \mathcal{A}$ *and* $g \in \mathcal{B}$ *implies* $f \in \mathcal{A}$*;*
- closed under decomposition *if it is closed under* $\mathcal{A}$*-decomposition.*

Remark 2.1. Clearly, "decomposition" corresponds to "left cancellation", but we prefer to stick to the name commonly used in literature (see e.g. [14]).

We are now ready to give the definition of $\mathcal{M},\mathcal{N}$-adhesive category [14,25].

Definition 2.3. *Let* $\mathbf{A}$ *be a category and* $\mathcal{M} \subseteq \mathsf{Mono}(\mathbf{A})$, $\mathcal{N} \subseteq \mathsf{Mor}(\mathbf{A})$ *where*

(i) $\mathcal{M}$ *and* $\mathcal{N}$ *contain all isomorphisms and are closed under composition and decomposition;*
(ii) $\mathcal{N}$ *is closed under* $\mathcal{M}$*-decomposition;*
(iii) $\mathcal{M}$ *and* $\mathcal{N}$ *are stable under pullbacks and pushouts.*

Then we say that $\mathbf{A}$ *is* $\mathcal{M},\mathcal{N}$-adhesive *if*

(a) every cospan $C \xrightarrow{g} D \xleftarrow{m} B$ *with* $m \in \mathcal{M}$ *can be completed to a pullback (such pullbacks will be called* $\mathcal{M}$-pullbacks*);*
(b) every span $C \xleftarrow{m} A \xrightarrow{n} B$ *with* $m \in \mathcal{M}$ *and* $n \in \mathcal{N}$ *can be completed to a pushout; such pushouts will be called* $\mathcal{M},\mathcal{N}$-pushouts*;*
(c) $\mathcal{M},\mathcal{N}$*-pushouts are Van Kampen squares.*

Remark 2.2. $\mathcal{M}$*-adhesivity* as defined in [2] coincides with $\mathcal{M}, \mathsf{Mor}(\mathbf{A})$-adhesivity, while *adhesivity* and *quasiadhesivity* [16,12] coincide with $\mathsf{Mono}(\mathbf{A})$-adhesivity and $\mathsf{Reg}(\mathbf{A})$-adhesivity, respectively. Notice that, in the $\mathcal{M}$-adhesive case, stability under pushouts of $\mathcal{M}$ derives from properties (a)–(c) of Definition 2.3, while closure under decomposition follows from stability under pullbacks in any category, so there is no need to prove it independently.

Other authors have introduced weaker notions of $\mathcal{M}$-adhesivity; see, e.g., [9,11,28], where our $\mathcal{M}$-adhesive categories are called *adhesive HLR categories.*

In general, proving that a given category is $\mathcal{M},\mathcal{N}$-adhesive by verifying the conditions of Definition 2.3 may be long and tedious; hence, we seek criteria which are sufficient for adhesivity, and simpler to prove. A prominent example is the following result due to Lack and Sobociński.

Theorem 2.1 ([17], Thm. 26**).** *Any elementary topos is an adhesive category.*

In particular the category **Set** of sets and any presheaf category are adhesive. However, there are many important categories for (graph) rewriting which are not toposes, hence the need for more general criteria.

2.2 A new criterion for $\mathcal{M},\mathcal{N}$-adhesivity

In this section we present our main result, i.e., that $\mathcal{M},\mathcal{N}$-adhesivity is guaranteed by the existence of a family of functors with sufficiently nice properties. We will adapt some definitions from [1].

Definition 2.4. *Let* $I : \mathbf{I} \to \mathbf{C}$ *be a diagram and* J *a set. We say that a family* $F = \{F_j\}_{j\in J}$ *of functors* $F_j : \mathbf{C} \to \mathbf{D}_j$

1. jointly preserves (co)limits *of* I *if given a (co)limiting (co)cone* $(L, l_i)_{i\in\mathbf{I}}$ *for* I*, every* $(F_j(L), F_j(l_i))_{i\in\mathbf{I}}$ *is (co)limiting for* $F_j \circ I$*;*
2. jointly reflects (co)limits *of* I *if a (co)cone* $(L, l_i)_{i\in\mathbf{I}}$ *is (co)limiting for* I *whenever* $(F_j(L), F_j(l_i))_{i\in\mathbf{I}}$ *is (co)limiting for* $F_j \circ I$ *for every* $j \in J$*;*

3. jointly lifts (co)limits *of I if given a (co)limiting (co)cone $(L_j, l_{j,i})_{i\in\mathbf{I}}$ for every $F_j \circ I$, there exists a (co)limiting (co)cone $(L, l_i)_{i\in\mathbf{I}}$ for I such that $(F_j(L), F_j(l_i))_{i\in\mathbf{I}} = (L_j, l_{j,i})_{i\in\mathbf{I}}$ for every $j \in J$;*
4. jointly creates (co)limits *of I if $F_j \circ I$ has a (co)limit for every $j \in J$, I has a (co)limit and F jointly preserves and reflects it.*

Remark 2.3. Joint preservation, reflection, lifting or creation of (co)limits of $F = \{F_j : \mathbf{A} \to \mathbf{B}_j\}_{j\in J}$ is equivalent to the usual preservation, reflection, lifting or creation of (co)limits for the functor $\mathbf{A} \to \prod_{j\in J} \mathbf{B}_j$ induced by F. Notice that our notion of creation follows [22], which is more lax than, e.g., [19, Def. V.1].

Theorem 2.2. *Let $\mathbf{A}$ be a category, $\mathcal{M} \subset \mathsf{Mono}(\mathbf{A})$, $\mathcal{N} \subset \mathsf{Mor}(\mathbf{A})$ satisfying conditions (i)–(iii) of Definition 2.3, and F a non empty family of functors $F_j : \mathbf{A} \to \mathbf{B}_j$ such that $\mathbf{B}_j$ is $\mathcal{M}_j,\mathcal{N}_j$-adhesive.*

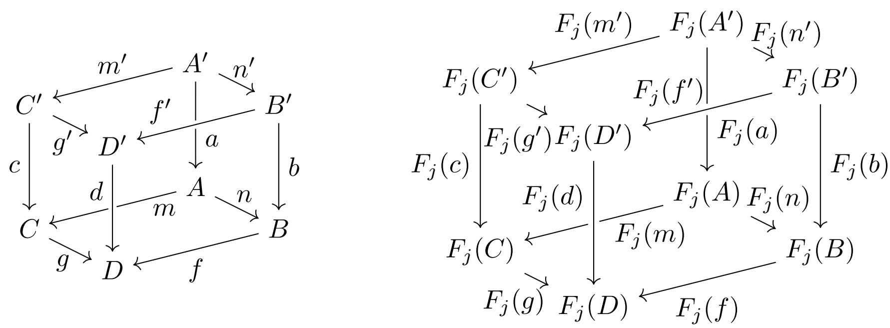
1. *If every F_j preserves pullbacks, $F_j(\mathcal{M}) \subset \mathcal{M}_j$ and $F_j(\mathcal{N}) \subset \mathcal{N}_j$ for every $j \in J$, F jointly preserves $\mathcal{M},\mathcal{N}$-pushouts, and jointly reflects pushout squares*

$$\begin{array}{ccc} F_j(A) & \xrightarrow{F_j(f)} & F_j(B) \\ {\scriptstyle F_j(m)}\downarrow & & \downarrow{\scriptstyle F_j(n)} \\ F_j(C) & \xrightarrow{F_j(g)} & F_j(D) \end{array}$$

   with $m, n \in \mathcal{M}$ and $f \in \mathcal{N}$, then $\mathcal{M},\mathcal{N}$-pushouts in $\mathbf{A}$ are stable.
   Moreover if in addition F jointly reflects $\mathcal{M}$-pullbacks and $\mathcal{N}$-pullbacks then $\mathcal{M},\mathcal{N}$-pushouts are Van Kampen squares.
2. *If F satisfies the assumptions of the previous points and jointly creates both $\mathcal{M}$-pullbacks and $\mathcal{N}$-pullbacks, then $\mathbf{A}$ is $\mathcal{M},\mathcal{N}$-adhesive.*
3. *If F jointly creates all pushouts and all pullbacks, then $\mathbf{A}$ is $\mathcal{M}_F,\mathcal{N}_F$-adhesive, where*

$$\mathcal{M}_F := \{m \in \mathsf{Mor}(\mathbf{A}) \mid F_j(m) \in \mathcal{M}_j \text{ for every } j \in J\}$$
$$\mathcal{N}_F := \{n \in \mathsf{Mor}(\mathbf{A}) \mid F_j(n) \in \mathcal{N}_j \text{ for every } j \in J\}$$

Proof. (1.) Take a cube in which the bottom face is an $\mathcal{M},\mathcal{N}$-pushout and all the vertical faces are pullbacks (below, left). Applying any $F_j \in F$ we get another cube in $\mathbf{B}_j$ (below, right) in which the bottom face is an $\mathcal{M}_j,\mathcal{N}_j$-pushout (because $F_j(m) \in \mathcal{M}_j$ and $F_j(n) \in \mathcal{N}_j$) and the vertical faces are pullbacks, thus the top face of the second cube is a pushout for every $j \in J$



Now $m', f' \in \mathcal{M}$ and $n' \in \mathcal{N}$ since they are the pullbacks of m, f and n and thus we can conclude.

Suppose now that F jointly reflects $\mathcal{M}$-pullbacks and $\mathcal{N}$-pullbacks, we have to show that the front faces of the first cube above are pullbacks if the top one is a pushout. In the second cube, the bottom and top face are $\mathcal{M}_j, \mathcal{N}_j$-pushouts and the back faces are pullbacks, then the front faces are pullbacks too by $\mathcal{M}_j, \mathcal{N}_j$-adhesivity. Now, notice that $f \in \mathcal{M}$ and $g \in \mathcal{N}$ (since $\mathcal{M}$ and $\mathcal{N}$ are closed under pushouts) and thus we can conclude since F jointly reflects pullbacks along arrows in $\mathcal{M}$ or in $\mathcal{N}$.

(2.) Let us show properties (a), (b), (c) defining $\mathcal{M}, \mathcal{N}$-adhesivity.

(a) Given a cospan $C \xrightarrow{g} D \xleftarrow{m} B$ in $\mathbf{A}$ with $m \in \mathcal{M}$ we can apply $F_j \in F$ to it and get $F_j(C) \xrightarrow{F_j(g)} F_j(D) \xleftarrow{F_j(m)} F_j(B)$ which is a cospan in $\mathbf{B}_j$ with $F_j(g) \in \mathcal{M}_j$, thus, by hypothesis it has a limiting cone $(P_j, p_{F_j(B)}, p_{F_j(C)})$ in $\mathbf{B}_j$. Since F jointly creates $\mathcal{M}$-pullbacks there exists a limiting cone (P, p_B, p_C) for the cospan $C \xrightarrow{g} D \xleftarrow{m} B$.

(b) Analogously: for every span $C \xleftarrow{m} A \xrightarrow{n} B$ in $\mathbf{A}$ with $m \in \mathcal{M}$ and $n \in \mathcal{N}$, we have $F_j(C) \xleftarrow{F_j(m)} F_j(A) \xrightarrow{F_j(n)} F_j(B)$ in each $\mathbf{B}_j$ with $F_j(m) \in \mathcal{M}_j$ and $F_j(n) \in \mathcal{N}_j$ and thus there exists a colimiting cocone $(Q_j, q_{F_j(B)}, q_{F_j(C)})$ in $\mathbf{B}_j$. Now we can conclude because F jointly creates $\mathcal{M}, \mathcal{N}$-pushouts.

(c) This follows at once by the second half of the previous point.

(3.) By the previous point it is enough to show that $\mathcal{M}_F$ and $\mathcal{N}_F$ satisfy conditions (i)–(iii) of Definition 2.3.

(i) If $f \in \mathsf{Mor}(\mathbf{A})$ is an isomorphism then so is $F_j(f)$ for every $F_j \in F$. Thus $F_j(f)$ belongs to $\mathcal{M}_j$ and $\mathcal{N}_j$ for every $j \in J$, implying f is in $\mathcal{M}_F$ and in $\mathcal{N}_F$. The parts regarding composition and decomposition follow immediately by functoriality of each $F_j \in F$.

(ii) Suppose that $g \circ f \in \mathcal{N}_F$, with $g \in \mathcal{M}_F$ then for every $j \in F$ $F_j(g \circ f) = F_j(g) \circ F_j(f) \in \mathcal{N}_j$ and $F_j(g) \in \mathcal{M}_j$, thus $F_j(f) \in \mathcal{N}_j$ and so $f \in \mathcal{N}_F$.

(iii) Take a square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle n} \\ C & \xrightarrow[g]{} & D \end{array}$$

and suppose that it is a pullback with $n \in \mathcal{M}_F$ ($\mathcal{N}_F$), then applying any $F_j \in F$ we get that $F_j(m)$ is the pullback of $F_j(n)$ along $F_j(g)$, since $F_j(n)$ is in $\mathcal{M}_j$ (in $\mathcal{N}_j$), which implies that $F_j(m) \in \mathcal{M}_j$ ($\mathcal{N}_j$). This is true for every $j \in J$, from which the thesis follows. Stability under pushouts is proved applying the same argument to m. □

Applying the previous theorem to the families given by, respectively, projections, evaluations and the inclusion we get immediately the following three corollaries (cfr. also [9, Thm. 4.15]).

Corollary 2.1. *Let* $\{\mathbf{A}\}_{i\in I}$ *be a family of categories such that each* $\mathbf{A}_i$ *is* $\mathcal{M}_i,\mathcal{N}_i$*-adhesive. Then the product category* $\prod_{i\in I}\mathbf{A}_i$ *is* $\prod_{i\in I}\mathcal{M}_i,\prod_{i\in I}\mathcal{N}_i$*-adhesive, where*

$$\prod_{i\in I}\mathcal{M}_i := \{(m_i)_{i\in I}\in \mathsf{Mor}(\prod_{i\in I}\mathbf{A}_i) \mid m_i\in\mathcal{M}_i \textit{ for every } i\in I\}$$
$$\prod_{i\in I}\mathcal{N}_i := \{(n_i)_{i\in I}\in \mathsf{Mor}(\prod_{i\in I}\mathbf{A}_i) \mid n_i\in\mathcal{N}_i \textit{ for every } i\in I\}$$

Corollary 2.2. *Let* $\mathbf{A}$ *be an* $\mathcal{M},\mathcal{N}$*-adhesive category. Then for every other category* $\mathbf{C}$*, the category of functors* $\mathbf{A}^{\mathbf{C}}$ *is* $\mathcal{M}^{\mathbf{C}},\mathcal{N}^{\mathbf{C}}$*-adhesive, where*

$$\mathcal{M}^{\mathbf{C}} := \{\eta\in\mathsf{Mor}(\mathbf{A}^{\mathbf{C}}) \mid \eta_C\in\mathcal{M} \textit{ for every object } C \textit{ of } \mathbf{C}\}$$
$$\mathcal{N}^{\mathbf{C}} := \{\eta\in\mathsf{Mor}(\mathbf{A}^{\mathbf{C}}) \mid \eta_C\in\mathcal{N} \textit{ for every object } C \textit{ of } \mathbf{C}\}$$

Corollary 2.3. *Let* $\mathbf{A}$ *be a full subcategory of an* $\mathcal{M},\mathcal{N}$*-adhesive category* $\mathbf{B}$ *and* $\mathcal{M}'\subset\mathsf{Mono}(\mathbf{A})$*,* $\mathcal{N}'\subset\mathsf{Mor}(\mathbf{A})$ *satisfying the first three conditions of Definition 2.3 such that* $\mathcal{M}'\subset\mathcal{M}$*,* $\mathcal{N}'\subset\mathcal{N}$ *and* $\mathbf{A}$ *is closed in* $\mathbf{B}$ *under pullbacks and* $\mathcal{M}',\mathcal{N}'$*-pushouts. Then* $\mathbf{A}$ *is* $\mathcal{M}',\mathcal{N}'$*-adhesive.*

2.3 Comma categories

In this section we show how to apply Theorem 2.2 to the comma construction [19] in order to guarantee some adhesivity properties under suitable hypotheses.

Definition 2.5. *For any two functors* $L:\mathbf{A}\to\mathbf{C}$*,* $R:\mathbf{B}\to\mathbf{C}$*, the* comma category $L{\downarrow}R$ *is the category in which*

- *objects are triples* (A,B,f) *with* $A\in\mathbf{A}$*,* $B\in\mathbf{B}$*, and* $f:L(A)\to R(B)$*;*
- *a morphism* $(A,B,f)\to(A',B',g)$ *is a pair* (h,k) *with* $h:A\to A'$*,* $k:B\to B'$ *such that the following diagram commutes*

$$\begin{array}{ccc} L(A) & \xrightarrow{L(h)} & L(A') \\ f\downarrow & & \downarrow g \\ R(C) & \xrightarrow[R(k)]{} & R(C') \end{array}$$

We have two obvious forgetful functors

$$\begin{array}{ccc} U_L : L{\downarrow}R \to \mathbf{A} & & \\ (A,B,f) & \longmapsto & A \\ (h,k)\downarrow & & \downarrow h \\ (A',B',g) & \longmapsto & A' \end{array} \qquad \begin{array}{ccc} U_R : L{\downarrow}R \to \mathbf{B} & & \\ (A,B,f) & \longmapsto & B \\ (h,k)\downarrow & & \downarrow k \\ (A',B',g) & \longmapsto & B' \end{array}$$

Example 2.1. **Graph** is equivalent to the comma category made from the identity functor on **Set** and the product functor sending X to $X\times X$.

We have a classic result relating limits and colimits in the comma category with those preserved by L or R.

Lemma 2.1. *Let $I : \mathbf{I} \to L{\downarrow}R$ be a diagram such that L preserves the colimit (if it exists) of $U_L \circ I$. Then the family $\{U_L, U_R\}$ jointly creates colimits of I.*

Corollary 2.4. *The family $\{U_L, U_R\}$ jointly creates limits along every diagram $I : \mathbf{I} \to L{\downarrow}R$ such that R preserves the limit of $U_R \circ I$.*

Proof. Apply the previous lemma to $R^{op}{\downarrow}L^{op}$ which is equivalent to $(L{\downarrow}R)^{op}$.

We are now able to deduce the following result from Theorem 2.2.

Theorem 2.3. *Let $\mathbf{A}$ and $\mathbf{B}$ be respectively $\mathcal{M},\mathcal{N}$-adhesive and $\mathcal{M}',\mathcal{N}'$-adhesive categories, $L : \mathbf{A} \to \mathbf{C}$ a functor that preserves $\mathcal{M},\mathcal{N}$-pushouts, and $R : \mathbf{B} \to \mathbf{C}$ a pullback preserving one. Then $L{\downarrow}R$ is $\mathcal{M}{\downarrow}\mathcal{M}',\mathcal{N}{\downarrow}\mathcal{N}'$-adhesive, where*

$$\mathcal{M}{\downarrow}\mathcal{M}' := \{(h,k) \in \mathsf{Mor}(L{\downarrow}R) \mid h \in \mathcal{M}, k \in \mathcal{M}'\}$$
$$\mathcal{N}{\downarrow}\mathcal{N} := \{(h,k) \in \mathsf{Mor}(L{\downarrow}R) \mid h \in \mathcal{N}, k \in \mathcal{N}'\}.$$

3 Some paradigmatic examples

In this section we apply the results provided in Section 2, to some important categories, such as directed (acyclic) graphs, hierarchical (hyper)graphs, directed (acyclic) hypergraphs, and term graphs. These examples have been chosen for their importance in graph rewriting, and because we can recover their $\mathcal{M},\mathcal{N}$-adhesivity in a uniform and systematic way. In fact, in the case of hierarchical (hyper)graphs we give the first proof of $\mathcal{M},\mathcal{N}$-adhesivity, to our knowledge.

3.1 Directed (acyclic) graphs

Among visual formalisms, directed (also known as "simple") graphs represent one of the most-used paradigms, since they adhere to the classical view of graphs as relations included in the cartesian product of vertices. It is also well-known that directed graphs are not quasiadhesive [15], not even in their acyclic variant. In this section we are going to exploit Corollary 2.3 to show that these categories of (acyclic) graphs have nevertheless adhesivity properties.

Definition 3.1. *A* directed multigraph *is a 4-tuple (E, V, s, t) where E and V are sets, called the set of* edges *and* nodes *respectively, and $s, t : E \to V$ are functions, called* source *and* target*. An edge e is* between *v and w if $s(e) = v$ and $t(e) = w$, $E(v, w)$ is the set of edges between v and w. A morphism $(E, V, s, t) \to (F, W, s', t')$ is a pair (f, g) of functions $f : E \to F$, $g : V \to W$ such that the following diagrams commute*

$$\begin{array}{ccc} E & \xrightarrow{s} & V \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ F & \xrightarrow[s']{} & W \end{array} \qquad \begin{array}{ccc} E & \xrightarrow{t} & V \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ F & \xrightarrow[t']{} & W \end{array}$$

We will denote by **Graph** *the category so defined. A* directed graph *is a directed multigraph in which there is at most one edge between two nodes,* **DGraph** *is the full subcategory of* **Graph** *given by directed graphs.*

A path $[e_i]_{i=1}^n$ *in a directed multigraph is a finite list of edges such that* $t(e_i) = s(e_{i+1})$ *for all* $1 \leq i \leq n-1$*. A path is called a* cycle *if* $s(e_1) = t(e_n)$*. A* directed acyclic graph *is a directed graph without cycles, directed acyclic graphs form a full subcategory* **DAG** *of* **DGraph** *and* **Graph***.*

Remark 3.1. **Graph** is equivalent to the category of presheaves on $\bullet \rightrightarrows \bullet$, the category with just two objects and only two parallel arrows between them (besides the identities), thus it is a topos and as such adhesive. Notice that this also implies that limits and colimits are computed component-wise and that an arrow in **Graph** is mono if and only if both its underlying functions are injective.

Remark 3.2. Notice that if $(f, g) : (E, V, s, t) \to (F, W, s', t')$ is an arrow in **DGraph** with f injective, then g is injective too.

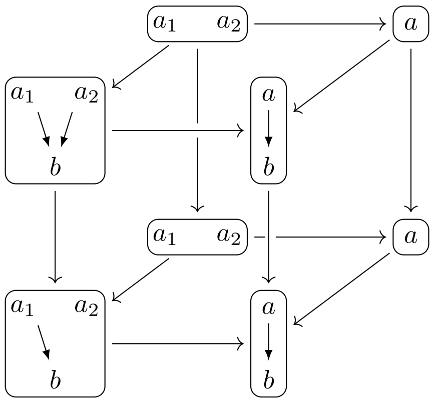
We will state now two categorical properties of **DGraph** that will be useful in the following.

Proposition 3.1. *The following properties hold*

1. *the inclusion functor* I : **DGraph** $\to$ **Graph** *has a left adjoint* L : **Graph** $\to$ **DGraph** *which sends a graph* (V, E, s, t) *to the graph on the same vertices but in which edges with the same source and target are identified;*
2. *an arrow* $(f, g) : (E, V, s, t) \to (F, W, s', t')$ *of* **DGraph** *is a regular monomorphism if and only if* f *is injective and* $E(v_1, v_2)$ *is non empty whenever* $F(f(v_1), f(v_2)) \neq \emptyset$*.*

Remark 3.3. Notice that, since L does not modify the vertices part of a graph, Remark 3.2 implies that L preserves monomorphisms.

Example 3.1. In [15] it is shown that **DGraph** is not quasiadhesive. Take the cube

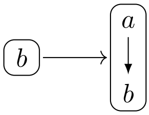


By the results of Proposition 3.1 the top and bottom faces are pushouts along regular monos and the back faces are pullbacks, but the front one is not, contradicting the Van Kampen property. The same example shows that even **DAG** is not quasiadhesive.

Definition 3.2. *A monomorphism* $(f, g) : (E, V, s, t) \to (F, W, s', t')$ *in* **Graph** *is said to be* downward closed *if, for all* $e \in F$, $e \in f(E)$ *whenever* $t'(e) \in g(V)$ *(in particular this implies that* $s'(e) \in g(V)$ *too). We denote by* $\mathsf{dclosed}$, $\mathsf{dclosed_d}$ *and* $\mathsf{dclosed_{da}}$ *the classes of downward closed morphisms in* **Graph**, **DGraph** *and* **DAG** *respectively.*

Remark 3.4. The functor L of Proposition 3.1 sends downward closed morphisms to downward closed morphisms.

Remark 3.5. By Proposition 3.1 it is clear that any downward closed morphism is regular. The vice-versa does not hold: a counterexample is given by



Lemma 3.1. **DGraph** *and* **DAG** *are closed in* **Graph** *under pullbacks. Moreover,* **DGraph** *is closed under* $\mathsf{Reg}(\mathbf{DGraph}), \mathsf{Mono}(\mathbf{DGraph})$*-pushouts and* **DAG** *under* $\mathsf{dclosed_{da}}, \mathsf{Mono}(\mathbf{DAG})$*-pushouts.*

Theorem 3.1. *The category* **DGraph** *is* $\mathsf{Reg}(\mathbf{DGraph}), \mathsf{Mono}(\mathbf{DGraph})$*- and* $\mathsf{Mono}(\mathbf{DGraph}), \mathsf{Reg}(\mathbf{DGraph})$*-adhesive, while* **DAG** *is* $\mathsf{dclosed_{da}}, \mathsf{Mono}(\mathbf{DAG})$*-adhesive.*

3.2 Tree Orders

In this section we present *trees* as partial orders and show that the resulting category is actually a topos of presheaves, hence adhesive. This fact will be exploited in Section 3.3 to construct a category of hierarchical graphs, where the hierarchy between edges is modelled by trees.

Definition 3.3. *A* tree order *is a partial order* $(E, \leq)$ *such that for every* $e \in E$, ${\downarrow}e$ *is a finite set totally ordered by the restriction of* $\leq$*. Since* ${\downarrow}e$ *is a finite chain we can define the* immediate predecessor function

$$i_E : E \to E \sqcup \{*\} \qquad e \mapsto \begin{cases} \max({\downarrow}e \smallsetminus \{e\}) & {\downarrow}e \neq \{e\} \\ * & {\downarrow}e = \{e\} \end{cases}$$

Let i^0_E *be the inclusion* $E \to E \sqcup \{*\}$*; then, for any* $k \in \mathbb{N}_+$*, the* k^{th} predecessor function $i^k_E : E \to E \sqcup \{*\}$ *is defined by induction as follows:*

$$e \mapsto \begin{cases} i_E(i^{k-1}_E(e)) & i^{k-1}_E(e) \in E \\ * & i^{k-1}_E(e) = * \end{cases}$$

Let $f : (E, \leq) \to (F, \leq)$ be a monotone map and $f_* : E \sqcup \{*\} \to F \sqcup \{*\}$ be its extension sending $*$ to $*$. We say that f is strict *if the following diagram commutes*

$$\begin{array}{ccc} E & \xrightarrow{i_E} & E \sqcup \{*\} \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f_*} \\ F & \xrightarrow[i_F]{} & F \sqcup \{*\} \end{array}$$

We define the category **Tree** *as the subcategory of* **Poset** *given by tree orders and strict morphisms.*

Example 3.2. A strict morphisms is simply a monotone function that preserves immediate predecessors (and thus every predecessor). For instance the function $\{0\} \to \{0, 1\}$ sending 0 to 1 and where we endow the codomain with the order $0 \leq 1$, is not a strict morphism.

Remark 3.6. Clearly $i^1_E = i_E$ and it holds that $i^k_E(e) = *$ if and only if $|{\downarrow}e| \leq k$. In this case an easy induction shows that $\left|{\downarrow} i^k_E(e)\right| = |{\downarrow}e| - k$.

Remark 3.7. We have an obvious forgetful functor

$$\begin{array}{ccc} & |-| : \mathbf{Tree} \to \mathbf{Set} & \\ (E, \leq) & \longmapsto & E \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ (F, \leq) & \longmapsto & F \end{array}$$

Remark 3.8. Let $(E, \leq)$ be an object of **Tree** and ω the first infinite ordinal, then we can define its *associated presheaf* $\widehat{E} : \omega^{op} \to \mathbf{Set}$ sending n to the set

$$\{e \in E \mid |{\downarrow}e \smallsetminus \{e\}| = n\}$$

If $n \leq m$ in ω, we can define a function

$$\iota^E_{n,m} : \widehat{E}(m) \to \widehat{E}(n) \qquad e \mapsto i^{m-n}_E(e)$$

which is well defined since $|{\downarrow}e| > m - n$ so

$$\left|{\downarrow} i^{m-n}_E(e)\right| = |{\downarrow}e| - m + n = m + 1 - m + n = n + 1$$

Notice that if $m = n$, $i^{m-n}_E(e)$ is the identity, while for any $k \leq n \leq m$ we have

$$\iota^E_{k,n}(\iota^E_{n,m}(e)) = i^{n-k}_E(i^{m-n}_E(e)) = i^{n-k+m-n}_E(e) = i^{m-k}_E(e) = \iota^E_{m-k}(e)$$

so $\widehat{E}$ is really a presheaf on ω.

Theorem 3.2. *There exists an equivalence of categories $\widehat{(-)} : \mathbf{Tree} \to \mathbf{Set}^{\omega^{op}}$ sending $(E, \leq)$ to $\widehat{E}$.*

Corollary 3.1. **Tree** *is adhesive and the forgetful functor $|-| : \mathbf{Tree} \to \mathbf{Set}$ preserves all colimits.*

3.3 Various kinds of hierarchical graphs

In this section we construct several categories of hierarchical graphs combining sufficiently adhesive categories of preorders or graphs (modelling the hierarchy between the edges) and the wanted structure on the nodes. For each of them we can readily prove suitable adhesivity properties, leveraging the modularity provided by Theorem 2.2. Besides hypergraphs and interfaces, this methodology can be applied to other settings such as Petri nets (see [10]).

Hierarchical graphs We can use trees to produce a category of hierarchical graphs [24], which, in addition, can be equipped with an interface, modelled by a function into the set of nodes.

Definition 3.4. *The category* **HIGraph** *of* hierarchical graphs with interface *has as objects* 6*-tuples* $((E,\leq),V,X,f,s,t)$ *where* $(E,\leq)$ *is a tree order,* f *is a function* $X \to V$ *and* s,t *are functions* $E \to V$*, and as arrows triples* $(h,k,l) : ((E,\leq),V,X,f,s,t) \to ((F,\leq),W,Y,g,s',t')$ *with* $h : (E,\leq) \to (F,\leq)$ *in* **Tree***,* $k : V \to W$ *and* $l : X \to Y$ *in* **Set** *such that the following squares commute*

$$\begin{array}{ccc} E \xrightarrow{s} V & E \xrightarrow{t} V & X \xrightarrow{f} V \\ h\downarrow \quad \downarrow k & h\downarrow \quad \downarrow k & l\downarrow \quad \downarrow k \\ F \xrightarrow{s'} W & F \xrightarrow{t'} W & Y \xrightarrow{g} W \end{array}$$

We can realise **HIGRaph** as a comma category: as L we take the functor $|-| : \mathbf{Tree} \to \mathbf{Set}$ of Remark 3.7, while as R we take the composition of $\mathsf{cod} : \mathbf{Set}^2 \to \mathbf{Set}$, sending an arrow to its codomain, with the functor $\mathbf{Set} \to \mathbf{Set}$ that sends a set X to $X \times X$. Notice that cod preserves limits since it coincides with the forgetful functor $\mathsf{id}_{\mathbf{Set}} {\downarrow} \mathsf{id}_{\mathbf{Set}}$, so we can apply Theorem 2.3 to get the following.

Theorem 3.3. **HIGraph** *is an adhesive category.*

The next step is to move to hypergraphs, using the Kleene star $(-)^\star : \mathbf{Set} \to \mathbf{Set}$ (the monoid monad) instead of the product functor. This step is not trivial: it relies on the fact that the monoid monad preserves all connected limits (such monads are called *cartesian*), which in turn rests upon the fact that the theory of monoids is a *strongly regular theory* (see [5, Sec. 3] and [18, Ch.4] for details).

Hierarchical hypergraphs A variation on the previous example is obtained by allowing an edge to be mapped to an arbitrary subset of nodes. In this way, we obtain a category of hypergraphs whose edges form a tree order, corresponding to Milner's (pure) bigraphs [20], with possibly infinite edges[3].

Definition 3.5. *The category* **HHGraph** *of* hierarchical hypergraphs with interface *has as objects* 5*-tuples* $((E,\leq),V,X,f,e)$ *where* $(E,\leq)$ *is a tree order and* $f : X \to V$*,* $e : E \to V^\star$ *two functions; arrows are triples* $(h,k,l) : ((E,\leq),V,X,f,e) \to ((F,\leq),W,Y,g,e')$ *with* $h : (E,\leq) \to (F,\leq)$ *in* **Tree***,* $k : V \to W$ *and* $l : X \to Y$ *in* **Set** *such that the following squares commute*

[3] In bigraph terminology, "controls" and "edges" correspond to our edges and nodes.

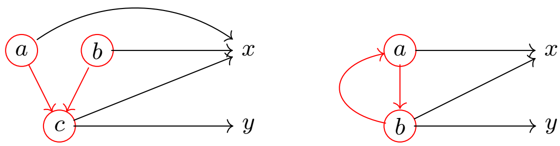


Fig. 1. A **DAG**-hypergraph (left) and a **DGraph**-hypergraph corresponding to the CCS process $P = a(x).b(xy).P$ (right). Relation between edges is depicted in red.

$$\begin{array}{ccc} E & \xrightarrow{e} & V^* \\ h\downarrow & & \downarrow k^* \\ F & \xrightarrow[e']{} & W^* \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{f} & V \\ l\downarrow & & \downarrow k \\ Y & \xrightarrow[g]{} & W \end{array}$$

Even in this case **HHGraph** is a comma category: on the left side we take $|-|$ as before, on the right side we take the composition of cod with the Kleene star, so even in this case we can deduce adhesivity.

Theorem 3.4. **HHGRaph** *is adhesive.*

DGraph *and* **DAG**-*hypergraphs* We can consider more general relations between edges, besides tree orders. An interesting case is when edges form a directed acyclic graph, yielding the category of **DAG**-*hypergraphs*; this corresponds to (possibly infinite) *bigraphs with sharing*, where an edge can have more than one parent, as in [27] (see also Fig. 1, left). Even more generally, we can consider any relation between edges, i.e., the edges form a generic directed graph possibly with cycles, yielding the category of **DGraph**-*hypergraphs*. These can be seen as "recursive bigraphs", i.e., bigraphs which allow for cyclic dependencies between controls, like in recursive processes; an example is in Fig. 1 (right).

Definition 3.6. *We define the category of* **DGraph**-hypergraphs (respectively **DAG**-hypergraphs) with interface **DHGraph** *(***DAGHGraph***) as the one in which objects are* 5*-tuples* $((E, T, s, t), V, X, f, e)$ *where* (E, T, s, t) *is in* **DGraph** *(in* **DAG***),* f *is a function* $X \to V$*, and* e *a function* $T \to V^\star$ *and as arrows triple* $((h_1, h_2), k, l) : ((E, T, s, t), V, X, f, e) \to ((F, T', s', t'), W, Y, g, e')$ *with* $(h_1, h_2) : (E, T, s, t) \to (F, T', s', t')$ *in* **DAG** *(in* **DGraph***),* $k : V \to W$ *and* $l : X \to Y$ *in* **Set** *such that the following squares commute*

$$\begin{array}{ccc} T & \xrightarrow{e} & V^\star \\ h_2\downarrow & & \downarrow k^\star \\ T' & \xrightarrow[e']{} & W^\star \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{f} & V \\ l\downarrow & & \downarrow k \\ Y & \xrightarrow[g]{} & W \end{array}$$

We can realise also **DHGraph** and **DAGHGraph** as comma categories: it is enough to take respectively the forgetful functors **DGraph** $\to$ **Set** and **DAG** $\to$ **Set** on one side and again the composition of the Kleene star with cod.

Theorem 3.5. **DHGraph** *is adhesive with respect to the classes*

$$\{((h_1,h_2),k,l) \in \mathsf{Mor}(\mathbf{DHGraph}) \mid (h_1,h_2) \in \mathsf{Reg}(\mathbf{DGraph}), k,l \in \mathsf{Mono}(\mathbf{Set})\}$$
$$\{((h_1,h_2),k,l) \in \mathsf{Mor}(\mathbf{DHGraph}) \mid (h_1,h_2) \in \mathsf{Mono}(\mathbf{DGraph})\}$$

while **DAGHGraph** *is adhesive with respect to the classes*

$$\{((h_1,h_2),k,l) \in \mathsf{Mor}(\mathbf{DAGHGraph}) \mid (h_1,h_2) \in \mathsf{dclosed}_{\mathsf{da}}, k,l \in \mathsf{Mono}(\mathbf{Set})\}$$
$$\{((h_1,h_2),k,l) \in \mathsf{Mor}(\mathbf{DHGraph}) \mid (h_1,h_2) \in \mathsf{Mono}(\mathbf{DAG})\}$$

3.4 Term graphs

The use of term graphs has been advocated as a tool for the optimal implementation of terms, with the intuition that the graphical counterpart of trees can allow for the sharing of sub-terms [26]. A brute force proof of quasiadhesivity of the category of terms graphs was given in [7]. In this section we recover that result by exploiting our new criterion for adhesivity.

Definition 3.7. *Let $\Sigma = (O, \mathsf{ar})$ be an algebraic signature (O is a set and $\mathsf{ar} : O \to \mathbb{N}$ a function called* arity function*). A* term graph over Σ *is a triple (V, l, s) where V is a set, $l : V \rightharpoonup O$, $s : V \rightharpoonup V^\star$ are partial functions such that*

- $\mathrm{dom}(l) = \mathrm{dom}(s)$*;*
- *for each $v \in \mathrm{dom}(l)$, $\mathsf{ar}(l(v)) = \mathsf{length}(s(a))$, where $\mathsf{length} : V^\star \to \mathbb{N}$ associates to each word its length.*

Elements of V are called nodes*, a node v not in $\mathrm{dom}(l)$ is called* empty*. A morphism $(V, l, s) \to (W, t, r)$ is a function $f : V \to W$ such that*

$$t(f(v)) = l(v) \qquad r(f(v)) = f^\star(s(v))$$

for every $v \in \mathrm{dom}(l)$. We will denote by $\mathbf{TG}_\Sigma$ the category of term graphs over Σ and their morphisms. We will use $\mathcal{U}$ to denote the forgetful functor $\mathbf{TG}_\Sigma \to \mathbf{Set}$ sending a term graph to the set of its nodes and that is the identity on arrows.

Definition 3.8. *We define a functor $\Delta : \mathbf{Set} \to \mathbf{TG}_\Sigma$ putting*

$$\begin{array}{ccc} X & \longmapsto & (X, e_1, e_2) \\ f\downarrow & & \downarrow f \\ Y & \longmapsto & (Y, e'_1, e'_2) \end{array}$$

where the domains of the structural functions e_1, e_2 of $\Delta(X)$ are the empty set.

Lemma 3.2. *The following properties hold*

1. $\Delta \dashv \mathcal{U}$*;*
2. $\mathbf{TG}_\Sigma$ *has equalizers and binary products.*

Remark 3.9. Right adjoints preserves monomorphisms, so, by the first point of Lemma 3.2, if $f : (V, l, s) \to (W, t, r)$ is a monomorphism then its underlying function is injective. On the other hand $\mathcal{U}$ is faithful and thus reflects monomorphisms, i.e. also the other implication holds.

Remark 3.10. $\mathbf{TG}_{\Sigma}$ in general does not have terminal objects. Since $\mathcal{U}$ preserves limits, if a terminal object exists it must have the singleton as set of nodes. Now take as signature the one given by two operations $\{a, b\}$ both of arity 0, then we have three term graphs with only one node v: $\Delta(\{v\})$, $(\{v\}, l, s)$ and $(\{v\}, t, s)$ where $l(v) = a$, $t(v) = b$ and s sends v to the empty word. Clearly there are no morphisms between the last two and from the last two to the first one, and thus neither of them can be terminal.

Remark 3.11. $\mathbf{TG}_{\Sigma}$ is not an adhesive category. In particular it does not have pushouts along all monomorphisms. Take the signature of the previous remark, then we can use the identity $\{v\} \to \{v\}$ to form a span

$$(\{v\}, l, s) \xleftarrow{i} \Delta(\{v\}) \xrightarrow{i'} (\{v\}, t, s).$$

This span cannot be completed to commutative a square: if

$$\begin{array}{ccc} \Delta(\{v\}) & \xrightarrow{i} & (\{v\}, t, s) \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle g} \\ (\{v\}, l, s) & \xrightarrow[f]{} & (V, p, r) \end{array}$$

is commutative then $f(v) = g(v)$; therefore

$$a = l(v) = p(f(v)) = p(g(v)) = t(v) = b$$

and this is absurd.

Remark 3.12. It is worth to spell out the explicit construction of equalizers in $\mathbf{TG}_{\Sigma}$. Given two arrows $f, g : (V, l, s) \to (W, t, r)$, let

$$E = \{v \in V \mid f(v) = g(v)\}$$

be the equalizer of $\mathcal{U}(f)$ and $\mathcal{U}(g)$ in **Set**. We have a partial function $p : E \rightharpoonup O$ given by the restriction of l to E. Moreover, if $v \in E \cap \mathrm{dom}(s)$ then

$$f^{\star}(s(v)) = r(f(v)) = r(g(v)) = g^{\star}(s(v))$$

hence $s(v) \in E^{\star}$ (which is the equalizer of $f^{\star}$ and $g^{\star}$, see [5]), thus we can restrict s to $q : E \rightharpoonup E^{\star}$. In this way we get a term graph (E, p, q) with an arrow into (V, l, s) which clearly equalize f and g.

On the other hand, if $k : (U, a, b) \to (V, l, s)$ is such that

$$g \circ k = f \circ k$$

then the induced function $\bar{k} : U \to E$ is a morphism of $\mathbf{TG}_{\Sigma}$.

Remark 3.13. Lemma 3.2 implies that $\mathbf{TG}_\Sigma$ has pullbacks. In the following we will need their explicit description. The pullback of a cospan

$$(V, l, s) \xrightarrow{f} (W, t, r) \xleftarrow{g} (U, a, b)$$

is given by (P, p, q) where

$$P = \{(v, u) \in V \times U \mid f(u) = g(v)\}$$

is the pullback of f along g in **Set** and

$$p : P \rightharpoonup O \qquad (v, u) \mapsto \begin{cases} l(v) & v \in \mathrm{dom}(l), w \in \mathrm{dom}(t) \\ \text{undefined} & \text{otherwise} \end{cases}$$

$$q : P \rightharpoonup P^\star \qquad (v, u) \mapsto \begin{cases} [(s(v)_i, r(u)_i)]_{i=1}^{\mathsf{ar}(l(v))} & v \in \mathrm{dom}(l), w \in \mathrm{dom}(t) \\ \text{undefined} & \text{otherwise} \end{cases}$$

where, given $x \in X^\star$, x_i denotes its i^{th} letter and, given $x_1, \dots, x_n \in X$, $[x_i]_{i=1}^n$ denotes the element in $X^\star$ such that $([x_i]_{i=1}^n)_i$ is exactly x_i.

Now, notice that q is the unique partial function $P \rightharpoonup P^\star$ that makes the projections arrows of $\mathbf{TG}_\Sigma$. Moreover even p has a uniqueness property: it is the unique partial function $P \rightharpoonup O$ such that the projections are arrows of $\mathbf{TG}_\Sigma$ and $p(x)$ is undefined if and only if at least one of its image is undefined. In particular this implies the following result.

Proposition 3.2. *$\mathcal{U}$ creates pullbacks along arrows which preserves empty nodes.*

This is especially useful when paired with the following result from [7].

Proposition 3.3 ([7], Prop. 4.3). *An arrow $f : (V, l, s) \to (W, t, r)$ in $\mathbf{TG}_\Sigma$ is a regular mono if and only if f is injective and preserves empty nodes.*

Proof. ($\Rightarrow$) Follows by the construction of equalizers given in Remark 3.12.
($\Leftarrow$) Consider (U, a, b) where $U = W \sqcup (W \smallsetminus f(V))$. Let i_1 and i_2 be the inclusions of W and $W \smallsetminus f(V)$ into U, we can define

$$a : U \rightharpoonup O \qquad u \mapsto \begin{cases} t(w) & u = i_1(w), w \in \mathrm{dom}(t) \\ t(w) & u = i_2(w), w \in (W \smallsetminus f(V)) \cap \mathrm{dom}(t) \\ \text{undefined} & \text{otherwise} \end{cases}$$

while for $b : U \rightharpoonup U^\star$, we put $b(u) = r(w)$ if $u = i_1(w), w \in \mathrm{dom}(r)$, while if $u = i_2(w)$ with $w \in \mathrm{dom}(r)$ we define $b(u) = [u_i]_{i=1}^{\mathsf{ar}(a(u))}$ where

$$u_i = \begin{cases} i_2(r(w)_i) & r(w) \in W \smallsetminus f(V) \\ i_1(r(w)_1) & r(w) \in f(V) \end{cases}$$

We have two functions $(V, t, r) \to (U, a, b)$: one is just i_1, while the other one is given by

$$g : W \to U \qquad w \mapsto \begin{cases} i_1(w) & w \in f(V) \\ i_2(w) & w \notin f(V) \end{cases}$$

Now, $i_1 \circ f$ and $g \circ f$ both send v to $i_1(f(v))$, therefore

$$i_1 \circ f = g \circ f$$

Suppose that $h : (P, p, q) \to (W, t, r)$ equalizes i_1 and g, thus $h(x) \in f(V)$ for every $x \in P$, and we have a unique function $h' : P \to V$ such that $f \circ h' = h$. For every $x \in \mathrm{dom}(p)$, $t(h(x)) = p(x)$, thus $h(x) = f(h'(x)) \in \mathrm{dom}(t)$. Since f preserves the empty nodes, $h'(x)$ belongs to $\mathrm{dom}(l)$, so:

$$p(x) = t(h(x)) = t(f(h'(x))) = l(h'(x))$$

Preservation of successors follows at once, while uniqueness follows from the uniqueness of the function h' in **Set**. ☐

Lemma 3.3. *$\mathcal{U}$ preserves and lifts pushouts along regular monomorphisms, moreover it reflects all pushout squares*

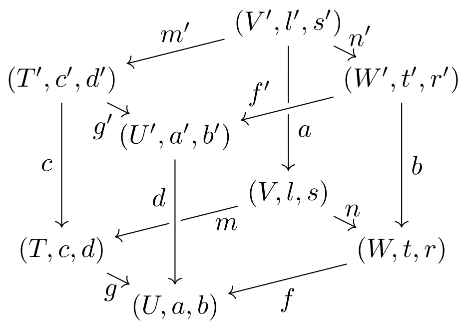
$$\begin{array}{ccc} \mathcal{U}(P, p, q) & \xrightarrow{\mathcal{U}(f)} & \mathcal{U}(W, t, r) \\ {\scriptstyle \mathcal{U}(m)}\downarrow & & \downarrow{\scriptstyle \mathcal{U}(n)} \\ \mathcal{U}(V, l, s) & \xrightarrow{\mathcal{U}(g)} & \mathcal{U}(U, a, b) \end{array}$$

in which n is regular. In addition $\mathsf{Reg}(\mathbf{TG}_\Sigma)$ *is closed under pushouts.*

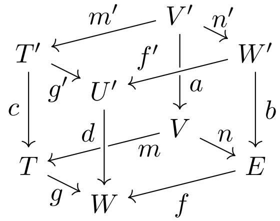
We can now use the first point of Theorem 2.2 to get half of the following result.

Theorem 3.6 ([7, Thm. 4.2]). *The category $\mathbf{TG}_\Sigma$ is quasi-adhesive.*

Proof. We already know by Lemmas 3.2 and 3.3 and Theorem 2.2 that pushouts along regular monos are stable. So, let us take a cube



in which m is regular, the top and bottom faces are pushouts and the back faces pullbacks. Applying $\mathcal{U}$ we get another cube



with pushouts along monos as top and bottom faces and pullbacks as vertical ones. By Proposition 3.2 $\mathcal{U}$ creates pullbacks along regular monos and $f \in \mathsf{Reg}(\mathbf{TG}_{\Sigma})$, then we can conclude that the front right face of the starting cube is a pullback as well. We have to show that the front left face of the starting cube is a pullback too. Suppose it is not, then, by the explicit description of pullbacks, there must be a node $t \in T'$ which is empty in (T', c', d') and such that $g'(t)$ and $c(t)$ are non empty. By the computation of pushouts along regular monos we can deduce that $g'(t) \in \mathrm{dom}(a')$ implies the existence of $v \in V'$, necessarily empty, such that $m'(v) = t$ and $f'(n'(v)) = g'(t)$, thus $n'(v)$ is non empty since f' is regular. Moreover, $c(m'(v)) = m(a(v))$ and the left hand side is non empty, therefore even $a(v)$ is non empty by the regularity of m, but this contradicts the hypothesis that the back right face is a pullback. □

4 Conclusions

In this paper we have introduced a new criterion for $\mathcal{M},\mathcal{N}$-adhesivity, based on the verification of some properties of functors connecting the category of interest to a family of suitably adhesive categories. This criterion can be seen as a distilled abstraction of many *ad hoc* proofs of adhesivity found in literature. This criterion allows us to prove in a uniform and systematic way some previous results about the adhesivity of categories built by products, exponents, and comma construction. We have applied the criterion to several significant examples, such as term graphs and directed (acyclic) graphs; moreover, using the modularity of our approach, we have readily proved suitable adhesivity properties to categories constructed by combining simpler ones. In particular, we have been able to tackle the adhesivity problem for several categories of hierarchical (hyper)graphs, including Milner's bigraphs, bigraphs with sharing, and a new version of bigraphs with recursion.

As future work, we plan to analyse other categories of graph-like objects using our criterion; an interesting case is that of *directed bigraphs* [13,3,4]. Moreover, it is worth to verify whether the $\mathcal{M},\mathcal{N}$-adhesivity that we obtain from the results of this paper is suited for modelling specific rewriting systems, e.g. based on the DPO approach. As an example, $\mathbf{TG}_{\Sigma}$ is quasiadhesive but this does not suffice in most applications, because the rules are often spans of monomorphisms, and not of regular monos [7].

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