PSPACE-Completeness of the Temporal Logic of Sub-Intervals and Suffixes

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Abstract

In this paper, we establish PSPACE-completeness of the finite satisfiability and model checking problems for the fragment of Halpern and Shoham interval logic with modality $\langle E \rangle$, for the "suffix" relation on pairs of intervals, and modality $\langle D \rangle$, for the "sub-interval" relation, under the homogeneity assumption. The result significantly improves the EXPSPACE upper bound recently established for the same fragment, and proves the rather surprising fact that the complexity of the considered problems does not change when we add either the modality for suffixes ($\langle E \rangle$) or, symmetrically, the modality for prefixes ($\langle B \rangle$) to the logic of sub-intervals (featuring only $\langle D \rangle$).

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1 Introduction

For a long time, in computer science, interval temporal logics (ITLs) have been considered an attractive, but impractical, alternative to standard point-based ones. On the one hand, they are a natural choice as a specification/representation language in a number of domains; on the other hand, the high undecidability of the satisfiability problem for the most well-known ITLs [8, 14, 16, 20, 30], such as Halpern and Shoham's modal logic of time intervals (HS for short) [14] and Venema's CDT [30], discouraged their extensive use (in fact, some restricted variants of them have been applied in formal verification and AI over the years [17, 24, 26]).

The present work finds its place in the framework of the logic HS, which features one modality for each of the 13 Allen's relations [1], apart from equality. In Table 1, we depict 6 Allen's relations for ordered pairs of intervals, together with the corresponding HS (existential) modalities; the other 7 relations are their inverses and the equality relation. The recent discovery of a significant number of expressive and computationally well-behaved fragments of HS changed the landscape of ITL research [11, 23]. Meaningful examples are the logic $A\overline{A}$ of the temporal neighborhood [10] (the HS fragment with modalities for the *meets* relation and its inverse) and the logic D of (temporal) sub-intervals [9] (the HS fragment with modality $\langle D \rangle$ for the *contains* relation only) over dense orderings.

Model checking (MC) of (finite) Kripke structures against HS and its fragments has been investigated in a series of papers [3, 5, 17, 18, 19, 21, 22] and shown to be decidable. In this setting, each finite path of a Kripke structure is interpreted as an interval whose

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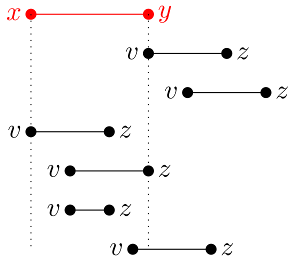
Table 1 Allen's relations and corresponding HS modalities.

Allen relation	HS	Definition w.r.t. interval structures	Example
MEETS	$\langle A \rangle$	$[x,y]\mathcal{R}_A[v,z] \iff y = v$	
BEFORE	$\langle L \rangle$	$[x,y]\mathcal{R}_L[v,z] \iff y < v$	
STARTED-BY	$\langle B \rangle$	$[x,y]\mathcal{R}_B[v,z] \iff x = v \wedge z < y$	
FINISHED-BY	$\langle E \rangle$	$[x,y]\mathcal{R}_E[v,z] \iff y = z \wedge x < v$	
CONTAINS	$\langle D \rangle$	$[x,y]\mathcal{R}_D[v,z] \iff [v,z] \subset [x,y]$	
OVERLAPS	$\langle O \rangle$	$[x,y]\mathcal{R}_O[v,z] \iff x < v < y < z$	



labeling satisfies the *homogeneity assumption* [27]: a proposition letter holds over an interval if and only if it holds over all its constituent points (states). MC against full HS is at least EXPSPACE-hard [5] and the only known upper bound is non-elementary [21, 6].[1] The known complexity bounds of MC for full HS coincide with those of MC for the linear-time fragment BE of HS which features modalities ⟨B⟩ and ⟨E⟩ for prefixes and suffixes. These complexity bounds easily transfer to finite satisfiability, that is, satisfiability over finite linear orders, of BE under the homogeneity assumption. Whether or not these problems can be solved elementarily is a difficult open question. On the other hand, MC and finite satisfiability under the homogeneity assumption for all the fragments of BE are known to be elementarily decidable [2, 3, 7]. In particular, for the fragment D of BE (note that the *contains* relation $\mathcal{R}_D$ can be expressed as $\mathcal{R}_B \cup \mathcal{R}_E \cup \mathcal{R}_B \cdot \mathcal{R}_E$), these problems are known to be PSPACE-complete [2].

In a recent contribution [7], we investigated finite satisfiability under the homogeneity assumption for the maximal fragment BD of BE that features modalities ⟨B⟩ and ⟨D⟩ (the other maximal fragment DE of BE with modalities ⟨D⟩ and ⟨E⟩ is completely symmetric, and thus all results for BD immediately transfer to it, and vice versa). The addition of modality ⟨B⟩ makes satisfiability checking for BD more complex than the one for D, as the two relations/modalities may interact in a non-trivial way. We proved EXPSPACE membership of the problem [7] by means of a purely model-theoretic argument, leaving the question of its exact complexity open. In this paper, we answer the question proving that, surprisingly, PSPACE-completeness of D is not affected by the addition of either ⟨B⟩ or ⟨E⟩, and the MC problem for DE (and symmetrically BD) is PSPACE-complete as well. In Figure 1, we add these new MC results to the picture of known MC complexities, showing that they enrich the set of HS "tractable" fragments with two meaningful members. We propose an automata-theoretic approach for solving MC and finite satisfiability under the homogeneity assumption of DE and BD which non-trivially generalizes the one for D [2] and the well-known one for standard LTL [29]. In particular, some important aspects that were not well understood in [2] are generalized by an elegant algebraic framework, which allows us to solve in an asymptotically optimal way the considered problems for DE and BD. In addition, we prove that, over finite linear orders and under the homogeneity assumption, D is less expressive than BD and DE, which in turn are less expressive than BE (in [4], we show that, under the homogeneity assumption, BE and LTL over finite words have the same expressive power).

[1] An expressive comparison of MC for HS and standard point-based temporal logics LTL [25], CTL, and CTL* [13] can be found in [4].

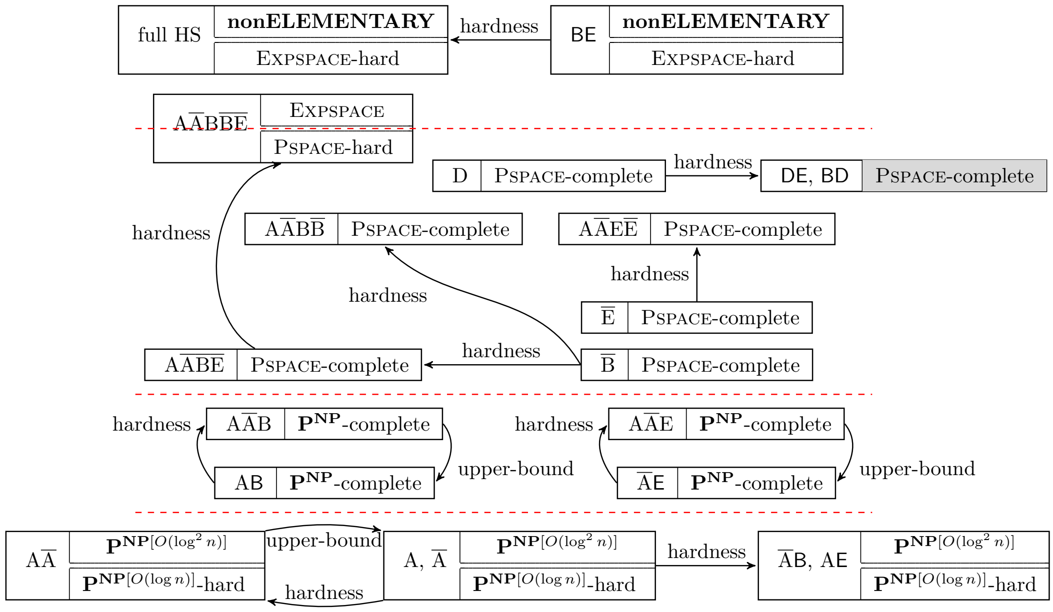


Figure 1 Complexity of the MC problem for HS and its fragments.

We conclude the introduction by recalling an interesting connection between the finite satisfiability problem for BE and its fragments, under the homogeneity assumption, and the non-emptiness problem for generalized $*$-free regular expressions [7]. The latter problem has been shown to be non-elementarily decidable by Stockmeyer in [28], and it can be easily proved to be equivalent to finite satisfiability for the interval temporal logic C of the chop modality [15, 24, 30] under the homogeneity assumption (the chop modality allows one to split the current interval in two parts and to state what is true over the first part and what over the second one). It can be shown that over finite linear orders and under the homogeneity assumption, BE (resp., its proper fragments BD and DE) is equivalent to the weakening of generalized $*$-free regular expressions where the concatenation operator is replaced by the weaker *prefix* and *suffix* ones (resp., *prefix* and *infix*, and *infix* and *suffix*) [7]. Note that the infix operator can be expressed in terms of the combination of the prefix and suffix operators. An immediate by-product of the results given in this paper is that the non-emptiness problem for $*$-free generalized regular expressions turns out to be elementarily decidable and, precisely, PSPACE-complete if one makes use of the *suffix* (resp., *prefix*) operator and the *infix* operator instead of the concatenation operator in the expressions. As for the fragment with both the *prefix* and the *suffix* operators, we only know that its non-emptiness problem is (non-elementarily decidable and) EXPSPACE-hard [5].

2 Preliminaries

We fix the following notation. For a finite word (or sequence) w over some finite alphabet Σ, we denote by $|w|$ the length of w. Moreover, for all $0 \leq i < |w|$, $w[i]$ denotes the $(i+1)^{th}$ letter of w. Given two non-empty finite words w, w' over Σ, we denote by $w \cdot w'$ the concatenation of w and w'. Moreover, if the last letter of w coincides with the first letter of w', we denote by $w \star w'$ the word $w \cdot w'[1] \ldots w'[n-1]$, where $n = |w|$ (i.e. the word obtained by concatenating w with the word obtained from w' by erasing the first letter). In particular, when $|w'| = 1$, then $w \star w' = w$.

Finite automata over finite words. A nondeterministic finite automaton (NFA) is a tuple $\mathcal{N} = \langle \Sigma, Q, Q_0, \delta, F \rangle$, where Σ is a finite alphabet, Q is a finite set of states, $Q_0 \subseteq Q$ is the set of *initial* states, $\delta : Q \times \Sigma \to 2^Q$ is the transition function, and $F \subseteq Q$ is the set of *accepting* states. Given a finite word w over Σ, with $|w| = n$, a run of $\mathcal{N}$ over w is a finite sequence of states $q_0, \ldots, q_n$ such that $q_0 \in Q_0$, and for all $i \in [0, n-1]$, $q_{i+1} \in \delta(q_i, w[i])$. The language $\mathcal{L}(\mathcal{N})$ accepted by $\mathcal{N}$ consists of the finite words w over Σ such that there is a run over w ending in some accepting state. A deterministic finite automaton (DFA) is an NFA $\mathcal{D} = \langle \Sigma, Q, \{q_0\}, \delta, F \rangle$ such that for all $(q, \sigma) \in Q \times \Sigma$, $\delta(q, \sigma)$ is a singleton.

Finite Kripke structures. We fix a finite set $\mathcal{AP}$ of proposition letters which represent predicates over the states of the given system. A *finite Kripke structure* over $\mathcal{AP}$ is a tuple $\mathcal{K} = \langle W, s_0, E, \mu \rangle$, where W is a finite set of states, $s_0 \in W$ is the initial state, $E \subseteq W \times W$ is a left-total relation between states, and $\mu : W \to 2^{\mathcal{AP}}$ is a labelling function assigning to each state the set of propositions that hold at it.

A *path of* $\mathcal{K}$ is a non-empty finite sequence of states $\rho = s_1 \cdots s_n$ such that (i) the first state s_1 coincides with the initial state s_0 of $\mathcal{K}$, and (ii) $(s_i, s_{i+1}) \in E$ for $i = 1, \ldots, n-1$. We extend the labeling μ to paths of $\mathcal{K}$ in the usual way: for a path $\rho = s_1 \ldots s_n$, $\mu(\rho)$ denotes the word over $2^{\mathcal{AP}}$ of length n given by $\mu(s_1) \ldots \mu(s_n)$. A *trace* of $\mathcal{K}$ is a non-empty finite word over $2^{\mathcal{AP}}$ of the form $\mu(\rho)$ for some path ρ of $\mathcal{K}$.

3 The logics DE and BD under the homogeneity assumption

In this section, we recall the logic BE of prefix and suffixes corresponding to the linear-time fragment of HS, and we focus our attention on the fragments DE and BD of BE interpreted over finite linear orders under the homogeneity assumption.

Let $\mathbb{S} = \langle S, < \rangle$ be a linear order over the nonempty set $S \neq \emptyset$, and $\leq$ be the reflexive closure of $<$. Given $x, y \in S$ such that $x \leq y$, we denote by $[x, y]$ the (closed) *interval* over S given by the set of elements $z \in S$ such that $x \leq z$ and $z \leq y$. We denote the set of all intervals over $\mathbb{S}$ by $\mathbb{I}(\mathbb{S})$. We focus our attention on three Allen's relations over intervals:

1. the *proper prefix (or started-by)* relation $\mathcal{R}_B$ defined as follows: $[x, y]\mathcal{R}_B[x', y']$ if $x = x'$ and $y' < y$.
2. the *proper sub-interval (or contains)* relation $\mathcal{R}_D$ defined as follows: $[x, y]\mathcal{R}_D[x', y']$ if $x' \geq x$, $y' \leq y$, and $[x, y] \neq [x', y']$ (the proper subset relation over intervals), and
3. the *proper suffix (or finished-by)* relation $\mathcal{R}_E$ defined as follows: $[x, y]\mathcal{R}_E[x', y']$ if $x < x'$ and $y' = y$.

BE formulas φ are defined by the following syntax:

$$\varphi ::= \top \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle \mathrm{B} \rangle \varphi \mid \langle \mathrm{D} \rangle \varphi \mid \langle \mathrm{E} \rangle \varphi$$

where $p \in \mathcal{AP}$, and $\langle \mathrm{B} \rangle$ (resp, $\langle \mathrm{D} \rangle$, resp., $\langle \mathrm{E} \rangle$) is the existential temporal modality for the Allen's relation $\mathcal{R}_B$ (resp., $\mathcal{R}_D$, resp., $\mathcal{R}_E$). We also exploit the conjunction connective $\wedge$, and for any temporal modality $\langle \mathrm{X} \rangle$, with $X \in \{B, D, E\}$, the dual universal modality $[\mathrm{X}]$ defined as: $[\mathrm{X}]\,\psi := \neg \langle \mathrm{X} \rangle \neg\psi$. The size $|\varphi|$ of a formula φ is the number of distinct sub-formulas of φ. We focus on the fragments DE (logic of sub-intervals and suffixes) and BD (logic of sub-intervals and prefixes) of BE obtained by disallowing the temporal modalities for the Allen's relations $\mathcal{R}_B$ and $\mathcal{R}_E$, respectively. We also consider the fragments B, D, and E defined in the obvious way.

The semantics of the logic BE is given in terms of *interval models*. An interval model $\mathcal{M}$ is a pair $\langle \mathbb{I}(\mathbb{S}), \mathcal{V}\rangle$, where $\mathcal{V} : \mathcal{AP} \mapsto 2^{\mathbb{I}(\mathbb{S})}$ is a *valuation function* that assigns to every proposition letter p the set of intervals $\mathcal{V}(p)$ over which p holds. Given an interval model $\mathcal{M} = \langle \mathbb{I}(\mathbb{S}), \mathcal{V}\rangle$, an interval $[x, y] \in \mathbb{I}(\mathbb{S})$, and a formula φ, the *satisfaction relation* $\mathcal{M}, [x, y] \models \varphi$, meaning that φ holds over the interval $[x, y]$ of $\mathcal{M}$, is inductively defined as follows:

- for every proposition letter $p \in \mathcal{AP}$, $\mathcal{M}, [x, y] \models p$ *if* $[x, y] \in \mathcal{V}(p)$;
- $\mathcal{M}, [x, y] \models \neg\varphi$ *if* $\mathcal{M}, [x, y] \not\models \varphi$;
- $\mathcal{M}, [x, y] \models \varphi_1 \vee \varphi_2$ *if* $\mathcal{M}, [x, y] \models \varphi_1$ or $\mathcal{M}, [x, y] \models \varphi_2$;
- $\mathcal{M}, [x, y] \models \langle \mathrm{X}\rangle \varphi$ for $X \in \{B, D, E\}$ *if* there is an interval $[x', y'] \in \mathbb{I}(\mathbb{S})$ such that $[x, y]\mathcal{R}_X[x', y']$ and $\mathcal{M}, [x', y'] \models \varphi$.

A BE-formula is *satisfiable* if it holds over some interval of an interval model. In this paper, we restrict our attention to the finite satisfiability problems, that is, satisfiability over the class of finite linear orders, for the fragments BD and DE. The problems are known to be undecidable [20] in the general case, but decidability can be recovered by restricting to the class of *homogeneous* interval models [21]. Formally, an interval model $\mathcal{M} = \langle \mathbb{I}(\mathbb{S}), \mathcal{V}\rangle$ is *homogeneous* if for every interval $[x, y] \in \mathbb{I}(\mathbb{S})$ and every $p \in \mathcal{AP}$, it holds that $[x, y] \in \mathcal{V}(p)$ if and only if $[x', x'] \in \mathcal{V}(p)$ for every $x' \in [x, y]$.

We observe that homogeneous interval models over finite linear orders correspond to non-empty finite words over $2^{\mathcal{AP}}$. In particular, each non-empty finite word w over $2^{\mathcal{AP}}$ induces the homogeneous interval model $\mathcal{M}(w) = \langle \mathbb{I}(\mathbb{S}), \mathcal{V}\rangle$ over the finite linear order induced by w defined as follows:

- $\mathbb{S} = \langle\{0, \ldots, |w| - 1\}, <\rangle$, and
- for every interval $[i, j]$ of $\mathbb{S}$ (note that $0 \leq i \leq j < |w|$) and $p \in \mathcal{AP}$, $[i, j] \in \mathcal{V}(p)$ if and only if $p \in w[h]$ for all $h \in [i, j]$.

Any fragment $\mathcal{F}$ of BE interpreted over homogeneous models is denoted by $\mathcal{F}_{\mathcal{H}om}$. A non-empty finite word w over $2^{\mathcal{AP}}$ *satisfies an* $\mathcal{F}_{\mathcal{H}om}$ *formula* φ, denoted by $w \models \varphi$, if $\mathcal{M}(w), [0, |w| - 1] \models \varphi$. A finite Kripke structure $\mathcal{K}$ over $\mathcal{AP}$ is *a model of* φ, written $\mathcal{K} \models \varphi$, if each trace w of $\mathcal{K}$ satisfies φ. We also consider the *model checking problem* against $\mathsf{DE}_{\mathcal{H}om}$ (resp., $\mathsf{BE}_{\mathcal{H}om}$) that is the problem of deciding for a given finite Kripke structure $\mathcal{K}$ and $\mathsf{DE}_{\mathcal{H}om}$ (resp., $\mathsf{BD}_{\mathcal{H}om}$) formula φ, whether $\mathcal{K} \models \varphi$.

Expressiveness issues. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two logics interpreted over non-empty finite words over $2^{\mathcal{AP}}$. Given $\psi_1 \in \mathcal{F}_1$ and $\psi_2 \in \mathcal{F}_2$, ψ_1 and ψ_2 are *equivalent* if ψ_1 and ψ_2 are satisfied by the same non-empty finite words over $2^{\mathcal{AP}}$. We say that $\mathcal{F}_1$ *is subsumed by* $\mathcal{F}_2$, denoted $\mathcal{F}_1 \preceq_f \mathcal{F}_2$, if for each $\mathcal{F}_1$ formula there is an equivalent $\mathcal{F}_2$ formula. $\mathcal{F}_1$ and $\mathcal{F}_2$ *have the same expressiveness* (resp.,*are expressively incomparable*) if $\mathcal{F}_1 \preceq_f \mathcal{F}_2$ and $\mathcal{F}_2 \preceq_f \mathcal{F}_1$ (resp., $\mathcal{F}_1 \not\preceq_f \mathcal{F}_2$ and $\mathcal{F}_2 \not\preceq_f \mathcal{F}_1$). Finally, $\mathcal{F}_1$ *is less expressive than* $\mathcal{F}_2$, denoted by $\mathcal{F}_1 \prec_f \mathcal{F}_2$, if $\mathcal{F}_1 \preceq_f \mathcal{F}_2$ and $\mathcal{F}_2 \not\preceq_f \mathcal{F}_1$. It is known [4] that $\mathsf{BE}_{\mathcal{H}om}$ has the same expressiveness as standard LTL over finite words. Here, we show that over finite words, the fragment $\mathsf{D}_{\mathcal{H}om}$ is less expressive than the fragments $\mathsf{BD}_{\mathcal{H}om}$ and $\mathsf{DE}_{\mathcal{H}om}$, which in turn are less expressive than $\mathsf{BE}_{\mathcal{H}om}$ or, equivalently, LTL. In particular, the following hold (a proof is in Appendix A).

▶ **Theorem 1.** *There exists a* $\mathsf{B}_{\mathcal{H}om}$ *(resp.,* $\mathsf{E}_{\mathcal{H}om}$*) formula which cannot be expressed in* $\mathsf{DE}_{\mathcal{H}om}$ *(resp.,* $\mathsf{BD}_{\mathcal{H}om}$*) over finite linear orders. Hence,* $\mathsf{D}_{\mathcal{H}om} \prec_f \mathsf{BD}_{\mathcal{H}om} \prec_f \mathsf{BE}_{\mathcal{H}om}$*,* $\mathsf{D}_{\mathcal{H}om} \prec_f \mathsf{DE}_{\mathcal{H}om} \prec_f \mathsf{BE}_{\mathcal{H}om}$*, and* $\mathsf{BD}_{\mathcal{H}om}$ *and* $\mathsf{DE}_{\mathcal{H}om}$ *are expressively incomparable.*

In order to illustrate the succinctness of the logics $\mathsf{D}_{\mathcal{H}om}$, $\mathsf{BD}_{\mathcal{H}om}$, and $\mathsf{DE}_{\mathcal{H}om}$, we consider a combinatorial requirement. For each $n \geq 1$, let $\mathcal{AP}_n = \{p_1, \ldots, p_n, \overline{p}_1, \ldots, \overline{p}_n\}$. The property that "there is a proper infix such for each $i \in [1, n]$, *exclusively* either p_i holds at some position, or $\overline{p}_i$ holds at some position" can be expressed by the following $\mathsf{D}_{\mathcal{H}om}$ formula ψ_n.

$$\psi_n := \langle \mathrm{D} \rangle \bigwedge_{i=1}^{i=n} ((\langle \mathrm{D} \rangle\, p_i \wedge [D]\neg \overline{p}_i) \vee (\langle \mathrm{D} \rangle\, \overline{p}_i \wedge [D]\neg p_i))$$

We conjecture that there is no LTL formula equivalent to ψ_n of size polynomial in n.

Encoding $\mathsf{DE}_{\mathcal{H}om}$ and $\mathsf{BE}_{\mathcal{H}om}$ in fragments of generalized $*$-free regular expressions. In the following, we show how to encode in $\mathsf{DE}_{\mathcal{H}om}$ (resp., $\mathsf{BE}_{\mathcal{H}om}$) over finite linear orders the fragment of generalized $*$-free regular expressions where the concatenation operator is replaced by the *infix* and *suffix* ones (resp., the *infix* and *prefix* ones). Recall that a *generalized $*$-free regular expression* (hereafter, simply called *general expression*) e over a finite alphabet Σ is a term of the form:

$$e ::= \emptyset \mid a \mid \neg e \mid e + e \mid \; e \cdot e, \text{ for any } a \in \Sigma.$$

We exclude the empty word ϵ from the syntax as it makes more direct the correspondence between restricted expressions and $\mathsf{BE}_{\mathcal{H}om}$ fragments (such a simplification is quite common in the literature). A general expression e of the above form defines the language $\mathcal{L}(e) \subseteq \Sigma^+$, which is inductively defined as follows: (i) $\mathcal{L}(\emptyset) = \emptyset$; (ii) $\mathcal{L}(a) = \{a\}$; (iii) $\mathcal{L}(\neg e) = \Sigma^+ \setminus \mathcal{L}(e)$; (iv) $\mathcal{L}(e_1 + e_2) = \mathcal{L}(e_1) \cup \mathcal{L}(e_2)$; (v) $\mathcal{L}(e_1 \cdot e_2) = \{w_1 \cdot w_2 : w_1 \in \mathcal{L}(e_1), w_2 \in \mathcal{L}(e_2)\}$.

In [28], Stockmeyer proves that the problem of deciding non-emptiness of $\mathcal{L}(e)$, for a given general expression e, is non-elementary hard. Here, we focus our attention on the following class of restricted expressions, called *prefix/suffix expressions*:

$$e ::= \emptyset \mid a \mid \neg e \mid e + e \mid \; \mathrm{Pre}(e) \mid \; \mathrm{Suf}(e) \mid \mathrm{Inf}(e), \text{ for any } a \in \Sigma,$$

where $\mathrm{Pre}(e)$ and $\mathrm{Suf}(e)$ are, respectively, a shorthand for $e \cdot (\neg\emptyset)$ and $(\neg\emptyset) \cdot e$, while $\mathrm{Inf}(e)$ is a shorthand for $\mathrm{Pre}(e) + \mathrm{Suf}(e) + \mathrm{Pre}(\mathrm{Suf}(e))$. An *infix/suffix* (resp., *infix/prefix*) expression is obtained by a prefix/suffix expression by disallowing the prefix operator Pre (resp., the suffix operator Suf). Assuming that $\Sigma = 2^{\mathcal{AP}}$, every suffix/prefix (resp., infix/suffix, resp., infix/prefix) expression e can be mapped into an equivalent formula ψ_e of $\mathsf{BE}_{\mathcal{H}om}$ (resp., $\mathsf{DE}_{\mathcal{H}om}$, resp., $\mathsf{BD}_{\mathcal{H}om}$) by applying the usual constructions for empty language, letters, negation, and union, plus the following three rules: (i) $\psi_{\mathrm{Pre}(e)} = \langle \mathrm{B} \rangle\, \psi_e$, (ii) $\psi_{\mathrm{Suf}(e)} = \langle \mathrm{E} \rangle\, \psi_e$, and (iii) $\psi_{\mathrm{Inf}(e)} = \langle \mathrm{D} \rangle\, \psi_e$. It is well known that LTL over finite words characterizes the class of languages defined by general expressions [12]. Since over finite words, LTL and $\mathsf{BE}_{\mathcal{H}om}$ have the same expressiveness [4], prefix/suffix expressions and general expressions have the same expressiveness as well. On the other hand, by Theorem 1, infix/suffix (resp., infix/prefix) expressions are less expressive than general expressions.

4 Satisfiability and model checking of $\mathsf{DE}_{\mathcal{H}om}$ over finite linear orders

In this section, we provide an automata-theoretic approach for solving satisfiability and model checking for $\mathsf{DE}_{\mathcal{H}om}$-formulas over finite linear orders. The proposed approach generalizes in a non-trivial manner the classical automata construction [29] for standard LTL over finite words based on the notion of Hintikka sequence. Given a $\mathsf{DE}_{\mathcal{H}om}$-formula φ and a non-empty finite word w over $2^{\mathcal{AP}}$, we associate to each interval $[i, j]$ of w a maximal propositionally consistent set of formulas (φ-*atom*) in the syntactical closure $\mathrm{CL}(\varphi)$ of φ which, intuitively, represents the set of formulas in $\mathrm{CL}(\varphi)$ which hold at the interval $[i, j]$. The syntactical definition of φ-atom locally captures the semantics of the Boolean connectives. In order to capture the semantics of the temporal modalities and the homogeneity assumption, we define syntactical "semi-local" rules which allow (i) to specify in a functional way the atom associated to a non-singleton interval I in terms of the atoms associated to the two proper maximal sub-intervals of I, and (ii) to enforce "termination" conditions on the atoms associated with singleton intervals of w. Next, for each prefix w_p of w, we consider the sequence of φ-atoms, called *row*, associated with the suffixes of w_p ordered for increasing values of the length (note that in the automata-theoretic approach for LTL, the notion of row collapses to the atom associated to the current position of the given finite word). The previous syntactical rules guarantee monotonicity properties on the atoms of a row and the existence of a functional relation that given the row of a proper prefix w_p of w and the uniquely determined atom of the singleton interval associated to position $|w_p|$ of w, provides the row for the prefix of w leading to position $|w_p|$ (see Section 4.1). As a main technical step (see Section 4.2), by exploiting the monotonicity of rows, we deduce for the given $\mathsf{DE}_{\mathcal{H}om}$-formula, the existence of an equivalence relation on the set of rows of exponential-size index satisfying three fundamental properties: (i) the equivalence relation preserves the set of atoms visited by a row and their relative ordering along the row, (ii) each equivalence class has a minimal representative whose length is polynomial in the size of the given formula, and (iii) the functional relation crucially preserves the equivalence between rows. The previous three properties (i)–(iii) lead to the construction in singly exponential time of a DFA having as states the set of *minimal rows* and accepting the non-empty finite words over $2^{\mathcal{AP}}$ which satisfy the given formula (see Section 4.3). We now proceed with the technical details.

Given a DE-formula φ, we define the *closure of* φ, denoted by $\mathrm{CL}(\varphi)$, as the set of all sub-formulas ψ of φ and of their negations $\neg\psi$ (we identify $\neg\neg\psi$ with ψ). A φ*-atom* A is a subset of $\mathrm{CL}(\varphi)$ such that:

- for every $\psi \in \mathrm{CL}(\varphi)$, $\psi \in A$ if and only if $\neg\psi \notin A$, and
- for every $\psi_1 \vee \psi_2 \in \mathrm{CL}(\varphi)$, $\psi_1 \vee \psi_2 \in A$ if and only if $A \cap \{\psi_1, \psi_2\} \neq \emptyset$.

We denote the set of all φ-atoms by $\mathcal{A}_\varphi$; its cardinality is clearly bounded by $2^{|\varphi|}$. We now consider non-empty finite words over $2^{\mathcal{AP}}$ equipped with a mapping assigning to each interval a φ-atom.

▶ **Definition 2** (φ-word structures and fulfilling φ-word structures)**.** *Let φ be a* DE*-formula. A* φ-word structure $\mathcal{W}$ *is a pair $\mathcal{W} = (w, \mathcal{L})$ consisting of a non-empty finite word over $2^{\mathcal{AP}}$ and a mapping $\mathcal{L}$ assigning to each interval of w (i.e., an interval in the homogeneous interval model $\mathcal{M}(w)$) a φ-atom such that for each position $0 \leq i < |w|$, $\mathcal{L}([i,i]) \cap \mathcal{AP} = w[i]$. The φ-word structure $\mathcal{W} = (w, \mathcal{L})$ is* fulfilling *if for each interval I of w (we also say that I is an interval of $\mathcal{W}$) and for each $\psi \in \mathrm{CL}(\varphi)$ it holds that $\psi \in \mathcal{L}(I)$ if and only if $\mathcal{M}(w), I \models \psi$.*

Evidently, for each non-empty finite word w over $2^{\mathcal{AP}}$, there exists a unique fulfilling φ-word structure associated with w. Let $\mathcal{W} = (w, \mathcal{L})$ be a φ-word structure. For each interval $[i,j]$ of $\mathcal{W}$, we write $\mathcal{L}(i,j)$ to mean $\mathcal{L}([i,j])$. For each $0 \leq i < |w|$, the *i-row of* $\mathcal{W}$ is the sequence row_i of φ-atoms given by $row_i = \mathcal{L}(i,i) \cdots \mathcal{L}(0,i)$, i.e., the sequence of atoms labeling the suffixes of the prefix of w until position i ordered for increasing values of the length.

4.1 Characterization of fulfilling φ-word structures

In this section, for the given $\mathsf{DE}_{\mathcal{H}om}$-formula φ, we provide a characterization of fulfilling φ-word structures $\mathcal{W}$ in terms of a "syntactical" functional relation between adjacent $\mathcal{W}$-rows. For a φ-atom A and $X \in \{D, E\}$, we consider the following sets:

- $Req_X(A) := \{\psi \in \mathrm{CL}(\varphi) : \langle \mathrm{X} \rangle\, \psi \in A\}$ (*temporal requests* of A);
- $Obs_X(A) := \{\psi \in A : \langle \mathrm{X} \rangle\, \psi \in \mathrm{CL}(\varphi)\}$ (*observables* of A).

The next proposition, stating that, once the proposition letters of A and its temporal requests have been fixed, A gets unambiguously determined, can be easily proved by induction.

▶ **Proposition 3.** *Let φ be a* DE*-formula. Given a set $R_D \subseteq \{\psi \mid \langle \mathrm{D} \rangle \psi \in \mathrm{CL}(\varphi)\}$, a set $R_E \subseteq \{\psi \mid \langle \mathrm{E} \rangle \psi \in \mathrm{CL}(\varphi)\}$, and a set $P \subseteq \mathrm{CL}(\varphi) \cap \mathcal{AP}$, there exists a unique φ-atom A that satisfies $Req_D(A) = R_D$, $Req_E(A) = R_E$, and $A \cap \mathcal{AP} = P$.*

▶ **Definition 4.** *Let A_B and A_E be two φ-atoms. We denote by $succ_\varphi(A_B, A_E)$ the unique φ-atom A whose sets of propositions and (sub-interval and suffix) temporal requests satisfy:*
(i) $A \cap \mathcal{AP} = A_B \cap A_E \cap \mathcal{AP}$,
(ii) $Req_D(A) = Req_D(A_B) \cup Obs_D(A_B) \cup Req_D(A_E) \cup Obs_D(A_E)$, *and*
(iii) $Req_E(A) = Req_E(A_E) \cup Obs_E(A_E)$.

Definition 4 can be exploited to label a fulfilling φ-word structure $\mathcal{W}$, namely, to determine the φ-atoms labeling all the intervals $[i, j]$ of $\mathcal{W}$, starting from the singleton ones. The idea is the following: if two φ-atoms A_B and A_E label respectively the greatest proper prefix $[i, j-1]$ and the greatest proper suffix $[i+1, j]$ of the same non-singleton interval $[i, j]$, then the atom A labeling interval $[i, j]$ is unique, and it is precisely the one given by $succ_\varphi(A_B, A_E)$. The next lemma proves that this is the general rule for labeling fulfilling φ-word structures (a proof of Lemma 5 is in Appendix B).

▶ **Lemma 5.** *Let $\mathcal{W} = (w, \mathcal{L})$ be a φ-word structure. Then $\mathcal{W}$ is fulfilling* if and only if *for each interval $[i, j]$ of $\mathcal{W}$, it holds that (i) $\mathcal{L}(i, j) = succ_\varphi(\mathcal{L}(i, j-1), \mathcal{L}(i+1, j))$, if $i < j$, and (ii) $Req_D(\mathcal{L}(i, j)) = \emptyset$ and $Req_E(\mathcal{L}(i, j)) = \emptyset$, if $i = j$.*

We now introduce the abstract notion of *φ-rows*, finite sequences of φ-atoms satisfying "syntactical" adjacency requirements which capture the behaviour of $\mathcal{W}$-rows in fulfilling φ-word structures $\mathcal{W}$.

▶ **Definition 6.** *A φ-*row *row is a non-empty finite sequence of φ-atoms such that for all $0 \le i < |row| - 1$: (i) $(row[i] \cap \mathcal{AP}) \supseteq (row[i+1] \cap \mathcal{AP})$, (ii) $Req_D(row[i+1]) \supseteq Req_D(row[i]) \cup Obs_D(row[i])$, and (iii) $Req_E(row[i+1]) = Req_E(row[i]) \cup Obs_E(row[i])$. The φ-row row is* initialized *if $Req_D(row[0]) = \emptyset$ and $Req_E(row[0]) = \emptyset$.*

We denote by $\mathcal{R}ows_\varphi$ the set of all possible φ-rows. We observe that the sequence of atoms along a φ-row $row = A_0 \cdots A_n$ has a monotonic behaviour, and the number of distinct atoms in row is linearly bounded in the size of φ. Indeed, by Definition 6, row induces three monotonic sequences: (i) one concerns the atomic propositions, is decreasing and is given by $(A_0 \cap \mathcal{AP}) \supseteq (A_1 \cap \mathcal{AP}) \supseteq \ldots \supseteq (A_n \cap \mathcal{AP})$, (ii) the second and the third are increasing, concern the temporal requests, and are given by $Req_D(A_0) \subseteq Req_D(A_1) \subseteq \ldots \subseteq Req_D(A_n)$ and $Req_E(A_0) \subseteq Req_E(A_1) \subseteq \ldots \subseteq Req_E(A_n)$. The number of distinct elements in each sequence is bounded by $|\varphi|$ (w.l.o.g, we assume that $|\mathcal{AP}| \le |\varphi|$, i.e. we can consider only the propositional letters actually occurring in φ). Since a set of requests and a set of proposition letters uniquely determine a φ-atom, any φ-row may feature at most $3|\varphi|$ distinct atoms, i.e., $n \le 3|\varphi|$. Since in a fulfilling φ-word structure there are no temporal requests in the atoms labeling the singleton intervals, by Lemma 5, we have the following result.

▶ **Lemma 7.**
1. *The number of distinct atoms in a φ-row row is at most $3|\varphi|$. Moreover, for all $0 \le i < j < |row|$, if $A_i = A_j$, then $A_k = A_i$ for all $k \in [i, j]$.*
2. *Each $\mathcal{W}$-row of a fulfilling φ-word structure $\mathcal{W}$ is an initialized φ-row.*

We now generalize the successor function $succ_\varphi$ to φ-rows: given a φ-row row and a φ-atom A, we consider the φ-row of length $|row|+1$ and first atom A obtained by a component-wise application of $succ_\varphi$ starting from A and the first atom of row.

▶ **Definition 8.** *Given a φ-atom A and a φ-row row with $|row| = n$, the A-*successor of *row, denoted by $succ_\varphi(row, A)$, is the sequence $B_0 \dots B_n$ of φ-atoms defined as follows: $B_0 = A$ and $B_{i+1} = succ_\varphi(row[i], B_i)$ for all $i \in [0, n-1]$.*

By Definitions 4 and 8, we can easily derive the following lemma.

▶ **Lemma 9.** *Let row be a φ-row and A be a φ-atom. Then, $succ_\varphi(row, A)$ is a φ-row. Moreover, if row is of the form $row = row_1 \cdot row_2$, then $succ_\varphi(row, A) = succ_\varphi(row_1, A) \star succ_\varphi(row_2, A_1)$, where A_1 is the last φ-atom of $succ_\varphi(row_1, A)$.*

By Lemma 5, consecutive rows in fulfilling φ-word structures respect the successor function. In particular, by Lemmata 5 and 7, we obtain the following characterization result.

▶ **Corollary 10** (Characterization of fulfilling φ-word structures). *Let $\mathcal{W} = (w, \mathcal{L})$ be a φ-word structure such that for all $0 \le i < |w|$, $Req_D(\mathcal{L}(i,i)) = \emptyset$ and $Req_E(\mathcal{L}(i,i)) = \emptyset$. Then $\mathcal{W}$ is fulfilling* if and only if*, for each $0 \le j < |w|-1$, $row_{j+1} = succ_\varphi(row_j, row_{j+1}[0])$, where row_i is the i-row of $\mathcal{W}$ for all $0 \le i < |w|$.*

4.2 Finite abstractions of rows

We describe now the core of the proposed automata-theoretic approach to the satisfiability and model checking problems for $\mathsf{DE}_{\mathcal{H}om}$. Given a $\mathsf{DE}_{\mathcal{H}om}$ formula φ, we introduce an equivalence relation $\sim_\varphi$ of finite index over $\mathcal{R}ows_\varphi$, whose number of equivalence classes is singly exponential in the size of φ and such that each equivalence class has a representative whose length is polynomial in the size of φ. As a crucial result we show that the successor function preserves the equivalence between φ-rows. The equivalence relation $\sim_\varphi$ is based on the notion of *uniform factorization of φ-rows* and *rank of φ-atoms*. In the following, we denote by $N_{D,\varphi}$ the number of sub-interval temporal requests in $\mathrm{CL}(\varphi)$ plus one, i.e., $|\{\psi \mid \langle \mathrm{D} \rangle\, \psi \in \mathrm{CL}(\varphi)\}| + 1$, and by $N_{E,\varphi}$ the number of suffix temporal requests in $\mathrm{CL}(\varphi)$ plus one, i.e., $|\{\psi \mid \langle \mathrm{E} \rangle\, \psi \in \mathrm{CL}(\varphi)\}| + 1$. Note that $1 \le N_{D,\varphi} \le |\varphi|$ and $1 \le N_{E,\varphi} \le |\varphi|$.

▶ **Definition 11** (Uniform φ-rows). *A φ-row row is* uniform *if for all $0 \le i < |row| - 1$, $(row[i] \cap \mathcal{AP}) = (row[i+1] \cap \mathcal{AP})$ and $Req_D(row[i+1]) = Req_D(row[i])$.*

Thus, in a uniform φ-row row, all the atoms occurring in row have the same propositional letters and the same sub-interval temporal requests. We represent an arbitrary φ-row row in the form $row = row_1 \cdot \ldots \cdot row_k$ (*uniform factorization of row*) where $row_1, \ldots, row_k$ are uniform φ-rows and $row_i \cdot row_{i+1}[0]$ is not uniform for all $1 \le i < k$. By the monotonicity properties of a φ-row row (see Definition 6 and Lemma 7(1)), the number k of uniform segments in the factorization of row is linearly bounded in the size of φ.

▶ **Lemma 12.** *The following statements hold:*

1. *Let row be a φ-row with uniform factorization $row_1 \cdot \ldots \cdot row_k$. Then k is at most $3|\varphi|$.*
2. *Let row be a uniform φ-row such that $|row| > N_{E,\varphi}$. Then row is of the form $row = row' \cdot B^m$ where $|row'| = N_{E,\varphi}$, $m \ge 1$, and B is the last atom of row'.*
3. *Given a φ-atom A and an integer $n \ge 1$, there is at most one uniform φ-row row such that $row[0] = A$ and $|row| = n$.*

Proof. By Lemma 7(1), the number k of uniform segments in the uniform factorization of the φ-row *row* is at most the number of distinct atoms in *row*. Hence, k is at most $3|\varphi|$, and Property (1) directly follows. As for Property (2), let *row* be a uniform φ-row such that $|row| > N_{E,\varphi}$. By definitions of φ-row and uniform φ-row, for all $0 \leq i < |row| - 1$, *either* $Req_E(row[i]) \subset Req_E(row[i+1])$ *or* $row[j] = row[i]$ for all $i \leq j < |row|$. Thus, since for a φ-atom A, $0 \leq |Req_E(A)| < N_{E,\varphi}$, we obtain that *row* is of the form $row = row' \cdot B^m$ where $|row'| = N_{E,\varphi}$, $m \geq 1$, and B is the last atom of row'.

As for Property (3), it suffices to observe that in a uniform φ-row all the atoms have the same propositional letters and the same sub-interval temporal requests. Thus, since the suffix temporal requests of a non-first atom in a φ-row are completely specified by the previous atom along the row, the result follows. ◀

We now introduce the notion of *rank* of a φ-atom. We first define the *D-rank of a φ-atom A*, written $rank_D(A)$, as $N_{D,\varphi} - |Req_D(A)|$. Clearly, $1 \leq rank_D(A) \leq |\varphi|$. The *rank of a φ-atom A*, written $rank(A)$, is defined as $rank_D(A) \cdot N_{E,\varphi}$ (i.e. the product of the D-rank of A with the increment of the overall number of suffix temporal requests in φ). Clearly, $1 \leq rank(A) \leq |\varphi|^2$. Intuitively, we can see the rank of an atom as the "number of degrees of freedom" that it gives to the atoms that stay "above it". In particular, by Definition 6, for every φ-row $row = A_0 \cdots A_n$, we have $rank_D(A_0) \geq \ldots \geq rank_D(A_n)$ and $rank(A_0) \geq \ldots \geq rank(A_n)$. Moreover, in a uniform φ-row *row*, all the atoms occurring in *row* have the same rank, and we write $rank(row)$ for such a rank.

▶ **Definition 13** (Equivalence relation). *Given two* uniform *φ-rows row and row', we say that row and row' are* equivalent, *written $row \sim_\varphi row'$, if the following conditions hold:*

- *$row[0] = row'[0]$ (hence $rank(row) = rank(row')$), and*
- *either $|row| = |row'|$ or both $|row|$ and $|row'|$ are strictly greater than $rank(row)$.*

Two arbitrary φ-rows row and row' with uniform factorizations $row_1 \cdot \ldots row_k$ and $row'_1 \cdot \ldots row'_{k'}$, respectively, are equivalent, *written $row \sim_\varphi row'$, if $k = k'$ and $row_i \sim_\varphi row'_i$ for all $i \in [1, k]$. A* minimal φ-row *is a φ-row whose uniform factorization $row_1 \cdot \ldots row_k$ is such that $|row_i| \in [1, rank(row_i) + 1]$, for each $1 \leq i \leq k$.*

By construction and Lemma 12, the number of minimal φ-rows is finite and each equivalence class of $\sim_\varphi$ contains a unique minimal φ-row. Thus, the equivalence relation $\sim_\varphi$ has finite index coinciding with the number of minimal φ-rows. This number is roughly bounded by the number of all the possible uniform factorizations of the form $row_1 \cdot \ldots \cdot row_k$ where $k \leq 3|\varphi|$ and for all $i \in [1, k]$, $|row_i|$ ranges from 1 to $|\varphi|^2$ and row_i is the unique uniform φ-row of length $|row_i|$ having as first atom $row_i[0]$. Since the number of possible φ-atoms is $2^{|\varphi|}$, the number of distinct equivalence classes of $\sim_\varphi$ is bounded by $(2^{|\varphi|} \cdot |\varphi|^2)^{3|\varphi|} \leq 2^{9|\varphi|^2}$, which is exponential in the length of the input formula φ. Moreover, each minimal φ-row has length at most $3|\varphi|^3$. Hence, we obtain the following result.

▶ **Lemma 14.** *Each equivalent class of $\sim_\varphi$ contains a unique minimal φ-row. The length of a minimal φ-row is at most $3|\varphi|^3$, and the number of minimal φ-rows is at most $2^{9|\varphi|^2}$.*

We observe that if we replace a segment (sub-row) of a φ-row by an equivalent one, we obtain a φ-row which is equivalent to the original one (for a proof, see Appendix C).

▶ **Lemma 15.** *Let $row_1, row'_1, row_2, row'_2$ be φ-rows such that $row_1 \sim_\varphi row'_1$ and $row_2 \sim_\varphi row'_2$. If $row_1 \star row_2$ and $row'_1 \star row'_2$ are defined, then $row_1 \star row_2 \sim_\varphi row'_1 \star row'_2$.*

We now show that the successor function $succ_\varphi$ on φ-rows preserves the equivalence of φ-rows. We first show (Lemma 16) that the result holds for uniform φ-rows (the proof is provided in Appendix D), and then we generalize Lemma 16 to arbitrary φ-rows.

▶ **Lemma 16.** *Let A be a φ-atom. Then the following statements hold:*

1. *let row be a uniform φ-row such that $|row| > rank(row)$. Then the φ-row $succ_\varphi(row, A)$ is of the form $A \cdot row_1 \cdot \ldots \cdot row_k$ for some $k \geq 1$ such that $row_1, \ldots, row_k$ are uniform φ-rows and $|row_k| > rank(row_k)$.*
2. *Let row and row' be two uniform φ-rows such that $row \sim_\varphi row'$. Then $succ_\varphi(row, A) \sim_\varphi succ_\varphi(row', A)$.*

▶ **Lemma 17.** *Let A be a φ-atom and row and row' be two φ-rows such that $row \sim_\varphi row'$. Then for the φ-rows $succ_\varphi(row, A)$ and $succ_\varphi(row', A)$, it holds that $succ_\varphi(row, A) \sim_\varphi succ_\varphi(row', A)$.*

Proof. The proof is by induction on the number $N(row)$ of distinct uniform segments in the uniform factorization of row. Being row and row' equivalent, $N(row') = N(row)$.

Base step: $N(row) = N(row') = 1$, i.e. row and row' are uniform. In this case, the result directly follows from Lemma 16.

Inductive step: $N(row) = N(row') > 1$. Hence, being $row \sim_\varphi row'$, row (resp., row') can be written in the form $row = row_1 \cdot row_2$ (resp., $row' = row'_1 \cdot row'_2$) such that $row_1 \sim_\varphi row'_1$, $row_2 \sim_\varphi row'_2$, $N(row_1) = N(row'_1) < N(row) = N(row')$, and $N(row_2) = N(row'_2) < N(row) = N(row')$. Let A_1 (resp., A'_1) be the last atom in $succ_\varphi(row_1, A)$ (resp., $succ_\varphi(row'_1, A)$). By the induction hypothesis, $succ_\varphi(row_1, A) \sim_\varphi succ_\varphi(row'_1, A)$, $A_1 = A'_1$, and $succ_\varphi(row_2, A_1) \sim_\varphi succ_\varphi(row'_2, A'_1)$ (note that by Lemma 12, two equivalent φ-rows have the same last atom). By Lemma 9, $succ_\varphi(row, A) = succ_\varphi(row_1, A) \star succ_\varphi(row_2, A_1)$ and $succ_\varphi(row', A) = succ_\varphi(row'_1, A) \star succ_\varphi(row'_2, A'_1)$. Thus, by applying Lemma 15, we obtain that $succ_\varphi(row, A) \sim_\varphi succ_\varphi(row', A)$, and we are done. ◀

4.3 Optimal upper bounds for $\mathsf{DE}_{\mathcal{H}om}$ satisfiability and model-checking

In this subsection, by exploiting Corollary 10 and Lemma 17, we derive an asymptotically optimal automata-theoretic approach for satisfiability and model checking of $\mathsf{DE}_{\mathcal{H}om}$ over finite linear orders. Given a $\mathsf{DE}_{\mathcal{H}om}$-formula φ, we show that it is possible to construct a deterministic finite automaton (DFA) $\mathcal{D}_\varphi$ over the alphabet $2^{\mathcal{AP}}$ having as states the initialized minimal φ-rows which accepts the non-empty finite words over $2^{\mathcal{AP}}$ which satisfy formula φ.

▶ **Definition 18.** *Let row be a minimal φ-row and A an atom. We denote by $succ_\varphi^{min}(row, A)$ the unique minimal φ-row in the equivalence class of $\sim_\varphi$ containing $succ_\varphi(row, A)$. Moreover, for a set $P \subseteq \mathcal{AP}$ of proposition letters, we denote by $A(P)$ the unique φ-atom such that $A(P) \cap \mathcal{AP} = P$, $Req_D(A(P)) = \emptyset$, and $Req_E(A(P)) = \emptyset$.*

We associate with the formula φ the DFA $\mathcal{D}_\varphi = \langle 2^{\mathcal{AP}}, \mathcal{R}ows_\varphi^{min} \cup \{q_0\}, \{q_0\}, \delta, F\rangle$, where $\mathcal{R}ows_\varphi^{min}$ is the set of initialized minimal φ-rows, and δ and F are defined as follows:

- *$\delta(q_0, P) = A(P)$ for all $P \in 2^{\mathcal{AP}}$,*
- *$\delta(row, P) = succ_\varphi^{min}(row, A(P))$ for all $P \in 2^{\mathcal{AP}}$ and $row \in \mathcal{R}ows_\varphi^{min}$;*
- *F is the set of φ-rows $row \in \mathcal{R}ows_\varphi^{min}$ such that $\varphi \in row[n-1]$, where $n = |row|$.*

We now establish the main technical result of this paper.

▶ **Theorem 19.** *Given a $\mathsf{DE}_{\mathcal{H}om}$-formula φ, the DFA $\mathcal{D}_\varphi$ accepts all and only the non-empty finite words over $2^{\mathcal{AP}}$ which satisfy φ.*

Proof. Let w be a non-empty finite word over $2^{\mathcal{AP}}$ and $n = |w| - 1$. We need to show that for the homogeneous interval model $\mathcal{M}(w)$, $\mathcal{M}(w), [0, n] \models \varphi$ if and only if $w \in \mathcal{L}(\mathcal{D}_\varphi)$.

($\Rightarrow$). Assume that $\mathcal{M}(w), [0,n] \models \varphi$. Let $\mathcal{W} = (w, \mathcal{L})$ be the unique fulfilling φ-word structure associated with the word w and for all $i \in [0,n]$, let row_i be the initialized φ-row corresponding to the i-row of $\mathcal{W}$. By hypothesis $\varphi \in row_n[n]$, and by construction $|row_0| = 1$ and $row_i[0] = A(w[i])$ for all $i \in [0,n]$. Moreover, by Corollary 10, $row_{i+1} = succ_\varphi(row_i, row_{i+1}[0])$ for all $i \in [0, n-1]$. For each $i \in [0,n]$, let row_i^{min} be the unique minimal φ-row in the equivalence class $[row_i]_{\sim_\varphi}$. Note that $row_0^{min} = row_0$, the last atom of row_n^{min} contains φ, row_i^{min} is initialized and $row_i^{min}[0] = row_i[0] = A(w[i])$ for all $i \in [0,n]$. By applying Lemma 17, $succ_\varphi(row_i^{min}, row_{i+1}[0])$ is equivalent to $row_{i+1} = succ_\varphi(row_i, row_{i+1}[0])$ for all $i \in [0, n-1]$. Hence, by the definition of $succ_\varphi^{min}$, we obtain that $row_{i+1}^{min} = succ_\varphi^{min}(row_i^{min}, A(w[i+1]))$ for all $i \in [0, n-1]$. By Definition 18, it follows that there is an accepting run of $\mathcal{D}_\varphi$ over w, i.e. $w \in \mathcal{L}(\mathcal{D}_\varphi)$.

($\Leftarrow$). Let us assume that w is accepted by $\mathcal{D}_\varphi$. By Definition 18, there exist $n+1$ minimal initialized minimal φ-rows $row_0^{min}, \ldots, row_n^{min}$ such that $row_0^{min} = A(w[0])$, φ belongs to the last atom of row_n^{min}, $row_i^{min}[0] = A(w[i])$ for all $i \in [0,n]$, and $row_{i+1}^{min} = succ_\varphi^{min}(row_i^{min}, row_{i+1}^{min}[0])$ for all $i \in [0, n-1]$. Let $row_0, \ldots, row_n$ be the sequence of φ-rows defined as follows: $row_0 = row_0^{min}$ and $row_{i+1} = succ_\varphi(row_i, row_{i+1}^{min}[0])$ for all $i \in [0, n-1]$. By Lemma 17, we have $row_i \sim_\varphi row_i^{min}$ for all $i \in [0,n]$. Hence, row_i is initialized for all $i \in [0,n]$, and $\varphi \in row_n[n]$. Let us define the φ-word structure $\mathcal{W} = (w, \mathcal{L})$ where $\mathcal{L}(i,j) = row_j[j-i]$ for every $0 \leq i \leq j \leq n$. By Corollary 10, $\mathcal{W}$ is fulfilling. Thus, since $\varphi \in \mathcal{L}(0,n)$, we obtain that $\mathcal{M}(w), [0,n] \models \varphi$ and the result follows. ◀

By Theorem 19, satisfiability of a $\mathsf{DE}_{\mathcal{H}om}$-formula φ reduces to checking non-emptiness of the DFA $\mathcal{D}_\varphi$ in Definition 18 whose number of states is singly exponential in the size of φ (Lemma 14). For the model-checking problem, given a finite Kripke structure $\mathcal{K}$, for checking that $\mathcal{K}$ is a model of φ, we apply the standard model-checking approach taking the synchronous product $\mathcal{K} \times \mathcal{D}_{\neg\varphi}$ of $\mathcal{K}$ with the automaton associated with the negation of the formula φ: the NFA $\mathcal{K} \times \mathcal{D}_{\neg\varphi}$ accepts all and only the traces of $\mathcal{K}$ which violate property φ. Hence, $\mathcal{K} \not\models \varphi$ if and only if the language accepted by $\mathcal{K} \times \mathcal{D}_{\neg\varphi}$ of $\mathcal{K}$ is not empty. Note that the number of states in $\mathcal{K} \times \mathcal{D}_{\neg\varphi}$ is linear in the number of $\mathcal{K}$-states and singly exponential in the size of φ, and the automata $\mathcal{D}_\varphi$ and $\mathcal{K} \times \mathcal{D}_{\neg\varphi}$ can be constructed "on the fly". Thus, since non-emptiness of NFA is in NLOGSPACE, the complexity classes NPSPACE =PSPACE and NLOGSPACE are closed under complement, and satisfiability and model checking against the fragment $\mathsf{D}_{\mathcal{H}om}$ are known to be PSPACE-complete [2], we obtain the following result.

▶ **Theorem 20.** *Finite satisfiability and model checking for $\mathsf{DE}_{\mathcal{H}om}$-formulas are both* PSPACE-*complete. Moreover, for $\mathsf{DE}_{\mathcal{H}om}$-formulas of fixed size, model checking is in* NLOGSPACE.

As for the logic $\mathsf{BD}_{\mathcal{H}om}$ over finite linear orders, we obtain results similar to Theorem 20. Let $\mathsf{DE}(\varphi)$ be the $\mathsf{DE}_{\mathcal{H}om}$ formula obtained from a $\mathsf{BD}_{\mathcal{H}om}$ formula φ by replacing each occurrence of modality $\langle \mathrm{B} \rangle$ with $\langle \mathrm{E} \rangle$. For each non-empty finite word w over $2^{\mathcal{AP}}$, $w \models \varphi$ iff $w^R \models DE(\varphi)$ (w^R is the reverse of w). Hence, the automaton $\mathcal{N}_\varphi$ accepting the models w of φ corresponds to the "reverse" of the DFA $\mathcal{D}_{DE(\varphi)}$ of Definition 18 associated with $DE(\varphi)$. Note that $\mathcal{N}_\varphi$ has the same states as $\mathcal{D}_{DE(\varphi)}$ but it is not deterministic. This is an important difference between $\mathsf{BD}_{\mathcal{H}om}$ and $\mathsf{DE}_{\mathcal{H}om}$ in the proposed automata-theoretic approach.

5 Results for $\mathsf{BD}_{\mathcal{H}om}$ and concluding remarks

For the logic $\mathsf{BD}_{\mathcal{H}om}$ over finite linear orders, we obtain results similar to Theorem 20. For a $\mathsf{BD}_{\mathcal{H}om}$ formula φ, let $\mathsf{DE}(\varphi)$ be the $\mathsf{DE}_{\mathcal{H}om}$ formula obtained by replacing each occurrence of modality $\langle \mathrm{B} \rangle$ with $\langle \mathrm{E} \rangle$. Evidently, for each non-empty finite word w over $2^{\mathcal{AP}}$, $w \models \varphi$ iff

$w^R \models DE(\varphi)$ (w^R is the reverse of w). Hence, the automaton $\mathcal{N}_\varphi$ accepting the models w of φ corresponds to the "reverse" of the DFA $\mathcal{D}_{DE(\varphi)}$ of Definition 18 associated with $DE(\varphi)$. Note that $\mathcal{N}_\varphi$ has the same states as $\mathcal{D}_{DE(\varphi)}$ but it is not deterministic. On the other hand, $\mathcal{N}_\varphi$ is deterministic in the *backward-direction*. Thus, for the $\mathsf{DE}_{\mathcal{H}om}$ formulas, the associated automata are deterministic in the *forward-direction* but non-deterministic in the *backward-direction*. Dually, for the $\mathsf{BD}_{\mathcal{H}om}$ formulas, the equivalent automata are deterministic in the *backward-direction* but non-deterministic in the *forward-direction*.

The results obtained for $\mathsf{DE}_{\mathcal{H}om}$ and $\mathsf{BD}_{\mathcal{H}om}$ are particularly interesting when compared with known results for $\mathsf{BE}_{\mathcal{H}om}$, where the latter includes $\mathsf{DE}_{\mathcal{H}om}$ and $\mathsf{BD}_{\mathcal{H}om}$ as proper fragments and, apparently, is quite close to $\mathsf{DE}_{\mathcal{H}om}$ and $\mathsf{BD}_{\mathcal{H}om}$. The complexity of MC for $\mathsf{BE}_{\mathcal{H}om}$ is still unknown: the problem is at least EXPSPACE-hard [5], while the only known upper bound is nonelementary [21]. Whether or not this problem can be solved elementarily is a difficult open question. Being DE and BD the most significant fragments of BE, the proved results provide a better insight into such an open question. The exact complexity of finite satisfiability for $\mathsf{BE}_{\mathcal{H}om}$ is also an open issue: the same upper/lower bounds can be shown to hold by linear-time reductions to/from the MC problem.

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A Proof of Theorem 1

In this section we provide a proof of Theorem 1. We establish the following result. Hence, Theorem 1 directly follows.

▶ **Proposition 21.** *Over finite linear orders, the following holds:*

1. *there exists a* $\mathsf{B}_{\mathcal{H}om}$ *formula which cannot be expressed in* $\mathsf{DE}_{\mathcal{H}om}$*;*
2. *there exists an* $\mathsf{E}_{\mathcal{H}om}$ *formula which cannot be expressed in* $\mathsf{BD}_{\mathcal{H}om}$*.*

We prove Proposition 21(2). The proof of Proposition 21(1) is similar and we omit the details here. Let $\mathcal{AP} = \{p\}$. We consider the $\mathsf{E}_{\mathcal{H}om}$ formula φ_E over $\mathcal{AP}$ defined as follows:

$$\varphi_E := \langle \mathrm{E} \rangle (p \wedge \langle \mathrm{E} \rangle \top)$$

which asserts the existence of a proper suffix of length at least 2 where p holds. In order to prove that no formula in $\mathsf{BD}_{\mathcal{H}om}$ is equivalent to φ_E over finite linear orders, we define two families $(w_n)_{n\geq 1}$ and $(w'_n)_{n\geq 1}$ of non-empty finite words over $2^{\{p\}}$ such that

- φ_E distinguishes between w_n and w'_n for each $n \geq 1$, and
- for every $\mathsf{BD}_{\mathcal{H}om}$ formula ψ, there is $n \geq 1$ such that ψ does not distinguish between w_n and w'_n.

For each $n \geq 1$, the words w_n and w'_n over $2^{\{p\}}$ are defined as follows:

$$w_n = (\emptyset\{p\}^n)^{n+2}\{p\} \text{ and } w'_n = (\emptyset\{p\}^n)^{n+2}\emptyset\{p\}$$

Note that for each $n \geq 1$, the property of having a suffix of length at least 2 where p holds is satisfied by w_n but not by w'_n. Hence, the following holds:

▶ **Lemma 22.** *For each $n \geq 1$, $w_n \models \varphi_E$ and $w'_n \not\models \varphi_E$.*

For a $\mathsf{BD}_{\mathcal{H}om}$ formula ψ, we denote by $d(\psi)$ the joint nesting depth of the temporal modalities in ψ. Proposition 21(2) directly follows from Lemma 22 and the following lemma.

▶ **Lemma 23.** *Let $n \geq 1$. Then, for each $\mathsf{BD}_{\mathcal{H}om}$ formula ψ such that $d(\psi) < n$, it holds that $w_n \models \psi$ if and only if $w'_n \models \psi$*

Proof. Fix $n \geq 1$. In order to prove Lemma 23, we need some preliminary results. The following claim can be proved by a straightforward induction on $k \geq 1$.

▷ Claim 1. Let $k \geq 1$. Then for each $\mathsf{BD}_{\mathcal{H}om}$ formula ξ such that $d(\xi) < k$, it holds that (i) $\{p\}^k \models \xi$ iff $\{p\}^{k+1} \models \xi$ and (ii) $\{p\}^j\emptyset\{p\}^k \models \xi$ iff $\{p\}^j\emptyset\{p\}^{k+1} \models \xi$ for each $j \geq 0$.

▷ Claim 2. Let $i, j \geq 0$, $k \geq 1$, and $\nu(j, i, k) = \{p\}^j(\emptyset\{p\}^n)^i\emptyset\{p\}^k$. Then for each $\mathsf{BD}_{\mathcal{H}om}$ formula ξ such that $d(\xi) < k$, $\nu(j, i, k) \models \xi$ iff $\nu(j, i, k) \cdot \{p\} \models \xi$.

Proof of Claim 2. The proof is by induction on $k \geq 1$. For the base case ($k = 1$), being $d(\xi) < 1$ (hence, ξ does not contain temporal modalities), the result trivially follows. Now, assume that $k > 1$. We proceed by a double induction on $i \geq 0$. If $i = 0$, the result directly follows from Claim 1. Now, let us assume that $i > 0$. By construction, for each proper prefix (resp., proper infix) v of $\nu(j, i, k) \cdot \{p\}$, there is a proper prefix (resp., proper infix) v' of $\nu(j, i, k)$ such that either (i) $v' = v$, or (ii) $v = \{p\}^{k+1}$ and $v' = \{p\}^k$, or (iii) $v = \nu(j', i', k') \cdot \{p\}$ and $v' = \nu(j', i', k')$ for some $j', i' \geq 0$ and $k' \geq 1$ such that either $k' = k$ and $i' < i$, or $k' = k - 1$ and $i' \leq i$. Hence, the result easily follows from the induction hypothesis and Claim 1. ◁

▷ Claim 3. Let $h \geq 1$, $i, j \geq 0$, $\nu(j, h) = \{p\}^j(\emptyset\{p\}^n)^h$ and $\nu(j, h, i) = \{p\}^j(\emptyset\{p\}^n)^h\emptyset\{p\}^i$. Then, for each $\mathsf{BD}_{\mathcal{H}om}$ formula ξ such that $d(\xi) < h$, it holds that (i) $\nu(j, h) \models \xi$ iff $\nu(j, h + 1) \models \xi$, and (ii) $\nu(j, h, i) \models \xi$ iff $\nu(j, h + 1, i) \models \xi$.

Proof of Claim 3. The proof is by induction on $h \geq 1$. The base case ($h = 1$) trivially follows, since being $d(\xi) < 1$, ξ does not contains temporal modalities. Now, assume that $h > 1$. Let $w = \nu(j, h)$ (resp., $w = \nu(j, h, i)$) and $w' = \nu(j, h + 1)$ (resp., $w' = \nu(j, h + 1, i)$). We need to prove that $w \models \xi$ iff $w' \models \xi$. By construction, the following holds:

- for each proper infix of v (resp., v') of w (resp., w), there exists a proper infix of v' (resp., v) of w' (resp., w) such that v' (resp., v) is a prefix of w' (resp., w) if v (resp., v') is a prefix of w (resp., w') and either (i) $v' = v$, or (ii) $v' = \nu(j', h)$ and $v = \nu(j', h-1)$ for some $j' \geq 0$, or (iii) $v' = \nu(j', h, i')$ and $v = \nu(j', h-1, i')$ for some $i', j' \geq 0$.

Hence, the result easily follows from the induction hypothesis. ◁

By Claim 3, we easily deduce the following result.

▷ Claim 4. Let $j \geq 0$. Then for each $\mathsf{BD}_{\mathcal{H}om}$ formula ξ such that $d(\xi) < n$, $\{p\}^j(\emptyset\{p\}^n)^{n+1} \models \xi$ iff $\{p\}^j(\emptyset\{p\}^n)^{n+1}\emptyset\{p\} \models \xi$.

We now prove Lemma 23. The proof is by induction on n. The base case ($n = 1$) trivially follows, since being $d(\psi) < n$, ψ does not contains temporal modalities. Now, assume that $n > 1$. We need to show that for each $\mathsf{BE}_{\mathcal{H}om}$ formula ξ such that $d(\xi) < n - 1$, the following holds:

- for each proper infix (resp., proper prefix) v of w_n, there exists a proper infix (resp., proper prefix) v' of w'_n such that $v \models \xi$ iff $v' \models \xi'$;
- for each proper infix (resp., proper prefix) v' of w'_n, there exists a proper infix (resp., proper prefix) v of w_n such that $v \models \xi$ iff $v' \models \xi'$.

The result easily follows from the definition of w_n and w'_n and Claims 1–4. ◀

B Proof of Lemma 5

▶ **Lemma 5.** *Let $\mathcal{W} = (w, \mathcal{L})$ be a φ-word structure. Then $\mathcal{W}$ is fulfilling* if and only if *for each interval $[i, j]$ of $\mathcal{W}$, it holds that (i) $\mathcal{L}(i, j) = succ_\varphi(\mathcal{L}(i, j-1), \mathcal{L}(i+1, j))$, if $i < j$, and (ii) $Req_D(\mathcal{L}(i, j)) = \emptyset$ and $Req_E(\mathcal{L}(i, j)) = \emptyset$, if $i = j$.*

Proof.
($\Rightarrow$). Assume that $\mathcal{W}$ is fulfilling. Hence, for each interval $[i, j]$ of $\mathcal{W}$, $\mathcal{L}(i, j)$ is the set of formulas $\psi \in \mathrm{CL}(\varphi)$ such that $\mathcal{M}(w), [i, j] \models \psi$ (recall that $\mathcal{M}(w)$ is the homogeneous interval model associated with the word w). Thus, if $i = j$, then $Req_D(\mathcal{L}(i, j)) = \emptyset$ and $Req_E(\mathcal{L}(i, j)) = \emptyset$. Otherwise, $i < j$ and being $\mathcal{M}(w)$ homogeneous, we have that $\mathcal{L}(i, j) \cap \mathcal{AP} = \mathcal{L}(i, j-1) \cap \mathcal{L}(i+1, j) \cap \mathcal{AP}$. Moreover, by the semantics of DE, the following holds:

- for each $\langle \mathrm{D} \rangle \psi \in \mathrm{CL}(\varphi)$, $\langle \mathrm{D} \rangle \psi \in \mathcal{L}(i, j)$ if and only if either $\langle \mathrm{D} \rangle \psi \in \mathcal{L}(i, j-1)$, or $\psi \in \mathcal{L}(i, j-1)$, or $\langle \mathrm{D} \rangle \psi \in \mathcal{L}(i+1, j)$, or $\psi \in \mathcal{L}(i+1, j)$;
- for each $\langle \mathrm{E} \rangle \psi \in \mathrm{CL}(\varphi)$, $\langle \mathrm{E} \rangle \psi \in \mathcal{L}(i, j)$ if and only if either $\langle \mathrm{E} \rangle \psi \in \mathcal{L}(i+1, j)$ or $\psi \in \mathcal{L}(i+1, j)$.

This means that $\mathcal{L}(i, j) = succ_\varphi(\mathcal{L}(i, j-1), \mathcal{L}(i+1, j))$, and the result follows.

($\Leftarrow$). Assume that for every interval $[i, j]$ of $\mathcal{W}$, we have $\mathcal{L}(i, j) = succ_\varphi(\mathcal{L}(i, j-1), \mathcal{L}(i+1, j))$ if $i < j$, and $Req_D(\mathcal{L}(i, j)) = \emptyset$ and $Req_E(\mathcal{L}(i, j)) = \emptyset$ if $i = j$. We have to prove that $\mathcal{W}$ is fulfilling. Let $[i, j]$ be an interval of $\mathcal{W}$ and $\psi \in \mathrm{CL}(\varphi)$. We prove by induction on the structure of ψ that $\psi \in \mathcal{L}(i, j)$ if and only if $\mathcal{M}(w), [i, j] \models \psi$. Hence, the result follows.

- $\psi = p$ with $p \in \mathcal{AP}$: we have to show that $\mathcal{L}(i, j) \cap \mathcal{AP} = \bigcap_{h \in [i, j]} \mathcal{L}(h, h) \cap \mathcal{AP}$. The proof is by a double induction on $j - i \geq 0$. If $i = j$, the property trivially holds. Let us assume now that $j - i > 0$. Since $\mathcal{L}(i, j) = succ_\varphi(\mathcal{L}(i, j-1), \mathcal{L}(i+1, j))$, by Condition (i) of Definition 4 and the induction hypothesis, we obtain that $\mathcal{L}(i, j) \cap \mathcal{AP} = \bigcap_{h \in [i+1, j]} \mathcal{L}(h, h) \cap \bigcap_{h \in [i, j-1]} \mathcal{L}(h, h) \cap \mathcal{AP}$. Hence, the result directly follows.

- $\psi = \neg\psi_1$ or $\psi = \psi_1 \vee \psi_2$: for these cases, the result directly follows from the induction hypothesis and the definition of φ-atom (recall that $\mathcal{L}(i,j)$ is a φ-atom).
- $\psi = \langle \mathrm{D} \rangle\, \psi_1$ or $\psi = \langle \mathrm{E} \rangle\, \psi_1$: the proof is by a double induction on $j - i \geq 0$. If $i = j$, then $\mathcal{M}(w), [i,j] \not\models \psi$, $Req_D(\mathcal{L}(i,j)) = \emptyset$, and $Req_E(\mathcal{L}(i,j)) = \emptyset$. Hence, the result follows. Now, assume that $j - i > 0$. First, let us consider the case where $\psi = \langle \mathrm{D} \rangle\, \psi_1$. Since $\mathcal{L}(i,j) = succ_\varphi(\mathcal{L}(i,j-1), \mathcal{L}(i+1,j))$, by Condition *(ii)* of Definition 4 and the induction hypothesis, we have that $\langle \mathrm{D} \rangle\, \psi_1 \in \mathcal{L}(i,j)$ if and only if either $\mathcal{M}(w), [i,j-1] \models \langle \mathrm{D} \rangle\, \psi_1$, or $\mathcal{M}(w), [i,j-1] \models \psi_1$, or $\mathcal{M}(w), [i+1,j] \models \langle \mathrm{D} \rangle\, \psi_1$, or $\mathcal{M}(w), [i+1,j] \models \psi_1$ if and only if $\mathcal{M}(w), [i,j] \models \langle \mathrm{D} \rangle\, \psi_1$.
  Now, let us consider the case where $\psi = \langle \mathrm{E} \rangle\, \psi_1$. By Condition *(iii)* of Definition 4 and the induction hypothesis, we have that $\langle \mathrm{E} \rangle\, \psi_1 \in \mathcal{L}(i,j)$ if and only if either $\mathcal{M}(w), [i+1,j] \models \langle \mathrm{E} \rangle\, \psi_1$ or $\mathcal{M}(w), [i+1,j] \models \psi_1$ if and only if $\mathcal{M}(w), [i,j] \models \langle \mathrm{E} \rangle\, \psi_1$, and the result follows. ◀

C Proof of Lemma 15

▶ **Lemma 15.** *Let $row_1, row'_1, row_2, row'_2$ be φ-rows such that $row_1 \sim_\varphi row'_1$ and $row_2 \sim_\varphi row'_2$. If $row_1 \star row_2$ and $row'_1 \star row'_2$ are defined, then $row_1 \star row_2 \sim_\varphi row'_1 \star row'_2$.*

Proof. We consider the case where row_1 and row_2 are uniform, hence, row'_1 and row'_2 are uniform as well. The general case easily follows from the considered case. By hypothesis $row_1 \star row_2$ and $row'_1 \star row'_2$ are defined. This entails that $row_1 \star row_2$ and $row'_1 \star row'_2$ are uniform as well. Thus since $row_1 \sim_\varphi row'_1$ and $row_2 \sim_\varphi row'_2$, by Definition 13, we obtain that $row_1 \star row_2$ and $row'_1 \star row'_2$ have the same first atom A and indicated by m (resp., m') the length of $row_1 \star row_2$ (resp., $row'_1 \star row'_2$), it holds that either $m = m'$, or both $m > rank(A)$ and $m' > rank(A)$. Hence, the result follows. ◀

D Proof of Lemma 16

In order to prove Lemma 16, we need a preliminary technical result (Lemma 5) that considers uniform φ-rows of the form B^m for some φ-atom B.

▶ **Lemma 5.** *Let A and B be two φ-atoms such that $rank_D(succ_\varphi(B,A)) = rank_D(B) - h$ for some $h \geq 0$ (note that $h < rank_D(B)$). Given $m > (rank_D(B) - h) \cdot N_{E,\varphi}$, if B^m is a φ-row than the φ-row $succ_\varphi(B^m, A)$ is of the form $A \cdot row_1 \cdot \ldots \cdot row_k$ for some $k \geq 1$ such that $row_1, \ldots, row_k$ are uniform φ-rows and*
- *$rank_D(row_i) > rank_D(row_{i+1})$ for each $1 \leq i < k$,*
- *$|row_k| > rank(row_k)$.*

Proof. Let $rank_D(succ_\varphi(B,A)) = rank_D(B) - h$ for some $0 \leq h < rank_D(B)$, $m > (rank_D(B) - h) \cdot N_{E,\varphi}$, and row be the φ-row of length $m+1$ given by $succ_\varphi(B^m, A)$. Since $row[0] = A$ and $row[i+1] = succ_\varphi(B, row[i])$ for all $i \in [0, m-1]$, by Definition 4, for all $i \in [1, m]$, the following holds:
- $row[i] = B \cap A \cap \mathcal{AP}$;
- $rank_D(row[i-1]) \geq rank_D(row[i])$ and $Req_E(row[i-1]) \subseteq Req_E(row[i])$;
- if $i < m$ and $row[i] = row[i+1]$, then $row[j] = row[i]$ for all $j \geq i$.

Since by hypothesis $rank_D(row[1]) = rank_D(B) - h$, we easily deduce that $row = succ_\varphi(B^m, A)$ is of the form

$$A \cdot row_1 \cdot \ldots \cdot row_k$$

for some $k \geq 1$ such that $row_1, \ldots, row_k$ are uniform φ-rows and

- $rank_D(row_1) = rank_D(B) - h$,
- $rank_D(row_i) > rank_D(row_{i+1})$ for each $1 \leq i < k$, and
- $|row_i| \leq N_{E,\varphi}$ for each $1 \leq i < k$.

It remains to show that $|row_k| > rank(row_k)$. By the previous points, we have that $rank_D(B)-h = rank_D(row_1) > \ldots > rank_D(row_k)$. Hence, $rank_D(B)-h \geq rank_D(row_k)+k-1$. Since $m > (rank_D(B)-h) \cdot N_{E,\varphi}$ and $|row_i| \leq N_{E,\varphi}$ for each $1 \leq i < k$, we obtain $|row_k| = m - \sum_{i=1}^{i=k-1} |row_i| \geq m - \sum_{i=1}^{i=k-1} N_{E,\varphi} > (rank_D(B)-h) \cdot N_{E,\varphi} - (k-1)N_{E,\varphi} \geq rank_D(row_k) \cdot N_{E,\varphi} = rank(row_k)$. ◀

By exploiting Lemma 5, we now prove Lemma 16.

▶ **Lemma 16.** *Let A be a φ-atom. Then the following statements hold:*

1. *let row be a uniform φ-row such that $|row| > rank(row)$. Then the φ-row $succ_\varphi(row, A)$ is of the form $A \cdot row_1 \cdot \ldots \cdot row_k$ for some $k \geq 1$ such that $row_1, \ldots, row_k$ are uniform φ-rows and $|row_k| > rank(row_k)$.*
2. *Let row and row' be two uniform φ-rows such that $row \sim_\varphi row'$. Then $succ_\varphi(row, A) \sim_\varphi succ_\varphi(row', A)$.*

Proof.

Proof of Property (1). Let A be a φ-atom and row be a uniform φ-row such that $|row| > rank(row)$. We need to show that the length $|\eta|$ of the last uniform segment η in the uniform factorization of $succ_\varphi(row, A)$ satisfies $|\eta| > rank(\eta)$. Since $|row| > rank(row)$ and $rank(row) \geq N_{E,\varphi}$, by Lemma 12(2), row is of the form $row = row_1 \cdot B^m$ where $m \geq 1$, $|row_1| = N_{E,\varphi}$ and B is the last atom of row_1. Let row' be the φ-row given by $succ_\varphi(row, A)$. Then row' can be written in the form

$$row' = (A \cdot row'_1) \star succ_\varphi(B^m, B')$$

where $A \cdot row'_1 = succ_\varphi(row_1, A)$ and B' is the last atom of row'_1. In particular, $|row'_1| = N_{E,\varphi}$. Let $B'' = succ_\varphi(B, B')$. By Definition 4, we have that $rank_D(B'') \leq rank_D(row'_1[0]) \leq rank_D(row)$. We distinguish two cases:

- $rank_D(B'') = rank_D(row)$. In this case, we have that all the atoms in $row'_1 \cdot B''$ have the same sub-interval temporal requests. Moreover, since row is uniform, by Definition 4, all the atoms in $row'_1 \cdot B''$ have the same propositional letters. Hence, $row'_1 \cdot B''$ is a uniform φ-row. Since $|row'_1| = N_{E,\varphi}$, by Lemma 12(2), B'' coincides with the last atom B' of row'_1. Thus, $B' = succ_\varphi(B, B')$ and $row' = A \cdot row'_1 \cdot (B')^m$ where $row'_1 \cdot (B')^m$ is a uniform φ-row having the same length and the same rank as row. Thus, since $|row| > rank(row)$, the result in this case holds.
- $rank_D(B'') < rank_D(row) = rank_D(B)$. We have that $m = |row| - N_{E,\varphi} > rank(row) - N_{E,\varphi} = (rank_D(B) - 1) \cdot N_{E,\varphi} \geq rank(B'')$. Since $B'' = succ_\varphi(B, B')$, by Lemma 5, the length $|\eta|$ of the last uniform segment η in the uniform factorization of $succ_\varphi(B^m, B')$ satisfies $|\eta| > rank(\eta)$, and the result follows.

Proof of Property (2). Let A be a φ-atom and row and row' be two uniform φ-rows such that $row \sim_\varphi row'$. We need to show that $succ_\varphi(row, A) \sim_\varphi succ_\varphi(row', A)$. By hypothesis and Definition 13, there are two cases:

- $row[0] = row'[0]$ and $|row| = |row'|$. Since row and row' are uniform, by Lemma 12(3), $row = row'$, and the result obviously follows.

- $row[0] = row'[0]$, $|row| \neq |row'|$, $|row| > rank(row)$ and $|row'| > rank(row')$. Assume that $|row| < |row'|$ (the case where $|row'| < |row|$ being similar). Since row and row' are uniform and $row[0] = row'[0]$, it holds that $rank(row) = rank(row')$. Moreover, $|row| > rank(row) \geq N_{E,\varphi}$. Applying Lemma 12(2) and Lemma 12(3), we deduce that row is of the form $row = row_1 \cdot B^2$ and $row' = row_1 \cdot B^{k+2}$ where B is a φ-atom and $k = |row'| - |row|$. By Property (1) of Lemma 16 the last uniform segment η of $succ_\varphi(row, A)$ satisfies $|\eta| > rank(\eta) \geq N_{E,\varphi}$. Thus, by Lemma 12(2), $succ_\varphi(row, A)$ is of the form $row' \cdot (B')^2$ for a φ-atom B' such that $B' = succ_\varphi(B, B')$. Since $succ_\varphi(row', A) = (row' \cdot (B')^2) \star succ_\varphi(B^k, B')$, we obtain that $succ_\varphi(row', A) = row' \cdot (B')^{2+k}$. Thus, since the last uniform segment η in $row' \cdot (B')^2$ satisfies $|\eta| > rank(\eta)$, we deduce that $succ_\varphi(row, A)$ and $succ_\varphi(row', A)$ are equivalent. ◀