Esquisabel, Oscar M.; **Raffo Quintana, Federico**
Fiction, possibility and impossibility: three kinds of mathematical fictions in Leibniz's work. (English) Zbl 07433736
Arch. Hist. Exact Sci. 75, No. 6, 613-647 (2021)

Mathematics is the reign of objects which do not exist in the physical world. For, no perfect square or circle exist in our daily experience. Therefore, every mathematical entity can be considered a fiction and the criteria according to which the property of existence is ascribed to a mathematical object have to be specified. Actually, the mathematicians have always shared the opinion that, e.g., a circle or "the number 3" exist, whereas something like "a round square" does not. However, there are objects whose ontological status was not completely clarified until the 19th century. Among them, someone is fundamental in the daily mathematical work, such as the imaginary numbers and the infinitesimal and infinite quantities. Along the history of mathematics, the term "fiction" has been assigned to these entities rather than to a square, a circle, or "the number 3". Among the scholars who developed a profound – though not always completely clear – speculation on the concept of mathematical fiction, Leibniz deserves a particular place because he was the inventor of several mathematical fictions and also thoroughly reflected on their nature. The literature on Leibniz's concept of mathematical fiction is vast and deep. Within this context, the paper under review by Esquisabel and Raffo Quintana is worth of a special mention for the care and refinement with which the authors have addressed this not easy subject.

In the Introduction they specify their analysis to be based on three different concepts of possibility/impossibility. The first one in given by the principle of contradiction: the already mentioned round square cannot exist because it is self-contradictory. Thus, it is not an admissible fiction. The two other criteria of not-existence are more subtle and concern respectively: 1) irrepresentability; 2) not-conformity with architectonical principles. In the second section, entitled "Symbolic knowledge, symbolic cognition and fictions", the authors specify important details of Leibniz's theory of ideas. In their context the crucial concept is that of symbolic notion: it is a sensible sign whose feature is given by its function as a support for cognition. The main goal of the paper is so expressed by Esquisabel and Raffo Quintana: "Using this framework, we propose to elucidate the notion of fiction in terms of a symbolic notion without denotation" (p. 617). In the third section "Fiction and symbolic notion", the authors, in the light of the examination they develop along the whole section, specify the thesis already anticipated in the Introduction: the actually infinitesimal and infinite quantities are impossible (namely without a denotation) fictions because they are geometrically irrepresentable or because they violate Leibniz's architectonical principles (p. 620). After these three general sections, starting from the fourth one "Leibniz and the fictionality of infinitary concepts", the authors address a more specific analysis of the way in which Leibniz conceived and used the infinitesimal and infinite quantities. First of all, they refer to an interesting letter addressed by Leibniz to Des Bosses on 11 March 1706 where Leibniz asserted that the infinitesimal quantities are, as the imaginary roots, useful fictions of the mind. For, they short the thinking process and the discovery, without implying any mistake. He also clarifies that such fictions (interpretable as actual infinitesimal) can be replaced by potentially infinitesimal quantities. The authors provide the reader with further evidences where Leibniz confirms this instrumental and pragmatic orientation, already pointed out in an interesting paper published by Arthur in 2009 (p. 622. For the notion of fiction in Leibniz the reader can also see [*P. Bussotti*, The complex itinerary of Leibniz's planetary theory. Physical convictions, metaphysical principles and Keplerian inspiration. Cham: Birkhäuser/Springer (2016; Zbl 1354.01002), pp. 50–53]. For the concept of fiction in Newton, also compared with Leibniz's, see [*R. Pisano* and *P. Bussotti*, "The Fiction of Infinitesimals in Newton's Works. On the Metaphorical use of Infinitesimals in Newton", in: Isonomia – Epistemologica, vol. 9. Special issue entitled Reasoning, Metaphors and Science, edited by F. Marcacci M. G. Rossi, pp. 141–160]). However, Leibniz expressed many doubts on the ontological status of such infinitesimal quantities (p. 622). The authors refer to two examples of the way in which Leibniz used the infinite and infinitesimal quantities. It is worth referring to the first one, at least. The situation (pp. 623–625) can be summarized like this: consider a general hyperbola $y^m x^n = a$. Then consider the rectangle obtained projecting a point of the hyperbola on the two coordinate axes. Imagine the projection on the abscissa y to be infinitely small and that on the ordinate x infinitely long. Leibniz argued that if $m < n$ the area of the rectangle is infinite, if $m > n$, it is infinitesimal and if $m = n$, it is finite. The authors refer here to the fundamental contributions of Knobloch. The second example concerns the rules of differentiation.

After having offered the two mentioned examples, Esquisabel and Raffo Quintana in the fifth section entitled "Mathematical fictions and impossibility" (pp. 627–629) connect the problem of the ontological status of the mathematical fictions with that of mathematical existence. They recall that, according to Leibniz, the criterion of existence is possibility and the lack of contradiction indicates possibility. Therefore, though in nature, to follow Leibniz's example, no perfect pentagon exists, it is a possible object and, hence, it has a mathematical existence. Leibniz establishes, thence, a rather clear criterion. However, as to "the fictionality of infinitary objects [he] reveals a more prudent attitude" (p. 629). Here the concepts of paradoxicality and improbability come into play. They are far more nuanced and not easily interpretable concepts. The authors claim that "This suggests the idea that for Leibniz both the existence and the mathematical possibility or impossibility are not a matter of absolute opposition but they do to some extent admit degrees" (p. 629).

The following fundamental sixth section "Three concepts of mathematical possibility and impossibility" (pp. 629–637) is dedicated to clarify such assertion. In this section the authors clarify furtherly the nature of the couple possibility/impossibility beyond the concepts whose existence is not possible because of the contradiction principle. As already stressed, a second kind of impossibility is given by the lack of representability. So, for example, "the infinitely small abscissa of our first example can be represented only analogically by a finite abscissa" (p. 630). The imaginary roots are also affected by this kind of impossibility. The four following pages are devoted to clarify this particular notion of impossibility: objects such as the imaginary roots of an equation are impossible because they cannot be represented geometrically. Nonetheless, they are the result of operations which are mathematically thinkable and legitimate (p. 633). Therefore, if they are useful to solve specific problems, they must be used, which is not the case with self-contradictory objects.

The third kind of impossibility is the most subtle and connects Leibniz's conception of mathematics with his metaphysics. The authors reiterate that, according to Leibniz, a mathematical object is a fiction, is not really existing if its existence is not compatible with the architectonical principles of sufficient reason and of continuity. For, on more than one occasion, Leibniz claimed that infinitary objects do not exist because they are not consistent with the principle of sufficient reason (p. 635). The authors offer an interesting example in this regard, which they explain perspicuously (ibidem): Leibniz refused to consider the motion as a series of successive infinitesimal leaps because this conception would violate the principle of sufficient reason. This clarified, it is to point out that Leibniz, in this case too, recognised the utilization of these mathematical fictions to be useful and legitimate. Furthermore, they have to be regarded as not-existing unless a convincing proof of their existence be exhibited (p. 637). Therefore, his attitude was rather prudent.

In the seventh section entitled "The reconsidered concept of mathematical fiction" (pp. 638–642) the authors summarize their view on the notion of mathematical fiction as an object lacking of denotation: in Leibniz, we have fiction1, referred to self contradictory entities; fiction2, referred to irrepresentable entities; fiction3 referred to entities lacking of denotation because they violate architectonical principles of Leibniz's metaphysics. After that Esquisabel and Raffo Quintana speak of the way in which mathematical fictions can be expressed (pp. 638–639): 1) verbal or written common language; 2) specific symbols as those of the infinitesimal calculus; 3) analogical and semiotic diagrammatic representation, such that the geometrical one through which the infinite and infinitesimal quantities are represented in a system of coordinates as if they were finite. As a conclusive step of their research, the authors analyse what Leibniz named "in appearance imaginary quantities". They enter the play while solving the irreducible algebraic equations of three degree, which have three real solutions but in whose resolutive formula imaginary numbers appear. In regard the authors write: "Thus, the concept of an 'in appearance imaginary quantity' or of an 'in appearance impossible quantity' seems to imply an objection against our interpretation, for these quantities seem ultimately to refer to real quantities, since imaginary quantities are put in equivalence with real quantities". (p. 639). In the light of two passages by Leibniz, Esquisabel and Raffo Quintana interpret the "in appearance imaginary quantities" as a synctatic procedure of symbolic nature. The expression of real roots through such quantities is a new mathematical operation (p. 640). Therefore, such quantities do not conflict with the interpretation offered of Leibniz's mathematical fictions. It seems to me that this brief part on the "in appearance imaginary quantities" is the less convincing of the whole paper. As a matter of fact, these quantities do not need a specific treatment within the general picture described by the authors. For, it is a usual procedure that mathematical fictions are used to obtain results concerning real quantities. This is extremely common in the infinitesimal calculus. Therefore, it is not completely clear to me the necessity to dedicate a specific consideration to the "in appearance imaginary quantities".

The "Concluding remarks" and the "References" follow.

This paper is very valuable. It is refined from a conceptual point of view and offers new insight in a difficult question such as the nature of mathematical fictions in Leibniz. The distinction between three kinds of fictions is noteworthy and surely deserves to be furtherly discussed and clarified. Thus, the authors also offer material for further researches on this subject.

Reviewer: Paolo Bussotti (Udine)

MSC:

01A45 History of mathematics in the 17th century

Full Text: DOI

References:

[1] Arthur, RTW; Kulstad, M.; Laerke, M.; Snyder, D., Actual infinitesimals in Leibniz's early thought, The philosophy of the young Leibniz, 11-28 (2009), Stuttgart: Franz Steiner Verlag, Stuttgart

[2] Arthur, RTW, Leibniz's syncategorematic infinitesimals, Archive for History of Exact Sciences, 67, 553-593 (2013) · Zbl 1273.01021 · doi:10.1007/s00407-013-0119-z

[3] Arthur, RTW; Nachtomy, O.; Winegar, R., Leibniz's syncategorematic actual infinite, Infinity in early modern philosophy, 155-179 (2018), Cham: Springer, Cham· doi:10.1007/978-3-319-94556-9$_1$0

[4] Bair, J.; Błaszczyk, Ely R.; Heinig, P.; Katz, M., Leibniz's well-founded fictions and their interpretations, MatematychniStudii, 49, 2, 186-224 (2018)

[5] Bos, H., Differentials, higher-order differentials and the derivative in the Leibnizian calculus, Archive for History of Exact Sciences, 14, 1-90 (1974) · Zbl 0291.01016 · doi:10.1007/BF00327456

[6] Brown, G., Who's afraid of infinite number?, The Leibniz Review, 8, 113-125 (1998)· doi:10.5840/leibniz199889

[7] Brown, G., Leibniz on wholes, unities and infinite number, The Leibniz Review, 10, 21-51 (2000)· doi:10.5840/leibniz2000102

[8] Crippa, D.; Pisano, R.; Fichant, M.; Bussotti, P.; Oliveira, ARE; Knobloch, E., Leibniz and the impossibility of squaring the circle, The dialogue between sciences, philosophy and engineering. New historical and epistemological insights. Homage to Gottfried W. Leibniz 1646-1716, 93-120 (2017), London: College Publications, London

[9] Debuiche, V.; Rabouin, D.; De Risi, V., On the plurality of spaces in Leibniz, Leibniz and the structure of sciences, 171-201 (2019), Cham: Springer, Cham · Zbl 1455.01007 · doi:10.1007/978-3-030-25572-5$_5$

[10] Duchesnau, F., Leibniz et la méthode de la science (1993), Paris: PUF, Paris

[11] Duchesnau, F., La dynamique de Leibniz (1994), Paris: Vrin, Paris

[12] Duchesnau, F., Le recours aux príncipes architectoniques dans la Dynamica de Leibniz, Revue d'Histoire des Sciences, 72, 1, 39-62 (2019)· doi:10.3917/rhs.721.0039

[13] Esquisabel, OM; Lassalle Cassanave, A., Representing and abstracting. An analysis of Leibniz's concept of symbolic knowledge, Symbolic knowledge from Leibniz to Husserl, 1-49 (2012), London: College Publications, London

[14] Esquisabel, OM, Infinitesimales y conocimiento simbólico en Leibniz, Notae Philosophicae Scientiae Formalis, 1, 1, 66-79 (2012)

[15] Esquisabel, OM; Arroyo, G.; Sisto, M., Analogías e invención matemática en Leibniz. El caso de la matemática infinitesimal, La lógica de la analogía. Perspectivas actualessobre el rol de las analogías en ciencia y en filosofía (2020), Los Polvorines: General Sarmiento, Universidad Nacional de General Sarmiento, Los Polvorines

[16] Esquisabel, O.; Raffo Quintana, F., Leibniz in Paris: A discussion concerning the infinite number of all units, Revista Portuguesa de Filosofia, 73, 3-4, 1319-1342 (2017)· doi:10.17990/RPF/2017$_7$3$_3$$_1$319

[17] Esquisabel, OM; Raffo Quintana, F., Infinitos y filosofía natural en Leibniz (1672-1676), Anales del Seminario de Historia de la Filosofía, 37, 3, 425-435 (2020)· doi:10.5209/ashf.68281

[18] Esquisabel, O.M., and Raffo Quintana, Federico 2021. La doble perspectiva técnica y filosófica de Leibniz acerca de los infinitesimales: un camino hacia la idealidad de lo matemático. ÉNDOXA- Series filosóficas.

[19] Fazio, R., La crítica de Leibniz a los números infinitos y su repercusión en la metafísica de los cuerpos, Theoria, 31, 2, 164-169 (2016)

[20] Grosholz, E., Representation and productive ambiguity in mathematics and the sciences (2007), Oxford: Oxford University Press, Oxford · Zbl 1209.00016

[21] Hess, H.-J. 1986. Zur Vorgeschichte der 'Nova Methodus' (1676-1684). In 300 Jahre "Nova Methodus" von G. W. Leibniz (1684-1984). Symposion der Leibniz-Gesellschaft im Congresscentrum "Leewenhorst" in Noordwijkerhout (Niederlande), 28. Bis 30. August 1984, A. Heinekamp (comp.) Studia Leibnitiana, Sonderheft 14, 64-102.

[22] Ishiguro, H., Leibniz's philosophy of logic and language (1990), Cambridge: Cambridge University Press, Cambridge

[23] Jesseph, DM, Leibniz on the foundations of the calculus: The question of the reality of infinitesimal magnitudes, Perspectives

on Science, 6, 6-38 (1998) · Zbl 1330.01034

[24] Jesseph, DM; Jesseph, D.; Goldenbaum, U., Truth in fiction: Origins and consequences of Leibniz's doctrine of infinitesimal magnitudes, Infinitesimal differences: Controversies between Leibniz and his contemporaries, 215-233 (2008), Berlin: Walter de Gruyter, Berlin

[25] Jesseph, DM; Goethe, N.; Beeley, P.; Rabouin, D., Leibniz on the elimination of infinitesimals, G.W. Leibniz, interrelations between mathematics and philosophy, 189-205 (2015), Dordrecht: Springer, Dordrecht

[26] Jullien, V., Seventeenth-century indivisibles revisited (2015), Cham: Birkhäuser, Cham · Zbl 1322.01002

[27] Knobloch, E., Les courbesanalytiques simples chez Leibniz, Sciences et Techniques en Perspective, 6, 74-96 (1993)

[28] Knobloch, E.; Gavroglu, K.; Christianidisand, J.; Nicolaïdis, E., The infinite in Leibniz's mathematics—The historiographical method of comprehension in context, Trends in the historiography of science, 266-278 (1994), Dordrecht: Kluwer, Dordrecht

[29] Knobloch, E., Leibniz's rigorous foundation of infinitesimal geometry by means of Riemannian sums, Synthese, 133, 1-2, 43-57 (2002) · Zbl 1032.01011

[30] Leibniz, G.W. 1923 (A). Sämtliche Schriften und Briefe, editada por la Deutschen Akademie der Wissenschaften. Darmstadt (1923)-Leipzig (1938)—Berlin (1950 and ongoing): Akademie-Verlag.

[31] Leibniz, G.W. 1846 (HOCD). Historia et origo calculi differentialis (ed. by C. I. Gerhardt). Hannover: Hahn.

[32] Leibniz, G.W. 1849-1863 (GM). Mathematische Schriften (ed. by C. I. Gerhardt). Vol. 7. Berlin: A. H.W. Schmidt.

[33] Leibniz, G.W. 1855. Die Geschichte der höheren Anlysis (C. I. Gerhardt). Halle: H.W. Schmidt.

[34] Leibniz, G.W. 1875-1890 (GP). Die philosophischen Schriften von Gottfried Wilhelm Leibniz (ed. by C. I. Gerhardt), Vol. 7. Berlin: Weidmann.

[35] Leibniz, G.W. 1920 (Child). The early mathematical manuscripts of Leibniz (translated from the latin texts published by Carl Immanuel Gerhardt with critical and historical notes by J. M. Child). Chicago: The Open Court Publishing Company.

[36] Leibniz, GW, De quadraturaarithmeticacirculiellipseoset hyperbolae cujuscorollariumestrignonometria sine tabulis (kritischherausgegeben und kommentiert von Eberhard Knobloch) (1992), Göttingen: Vandenhoedk \& Ruprecht, Göttingen

[37] Leibniz, GW, De summa rerum. Metaphysical Papers, 1675-1676 (translated with an introduction and notes by G. H. R. Parkinson) (1992), New Haven: Yale University Press, New Haven· doi:10.2307/j.ctt2250wvx

[38] Leibniz, GW, Naissance du calculdifférentiel. 26 articles des Actaeruditorum (introduction, traductionet notes par Marc Parmentier) (1995), Paris: Vrin, Paris

[39] Leibniz, GW, New essays on human understanding (translated and edited by Peter Remnant and Jonathan Bennett) (1996), Cambridge: Cambridge University Press, Cambridge

[40] Leibniz, GW, The labyrinth of the continuum. Writings on the continuum problem, 1672-1686 (translated, edited, and with an introduction by Richard T. W. Arthur) (2001), New Haven: Yale University Press, New Haven

[41] Leibniz, GW, Quadrature airthmétique du cercle, de l'ellipse et de l'hyperbole et la trigonométrie sans tables trigonométriques qui en est le corollaire (introduction, traduction et notes de Marc Parmentier, textelatinédité par Eberhard Knobloch) (2004), Paris: Vrin, Paris · Zbl 1315.01068

[42] Leibniz, GW, Confessio philosophi. Papers concerning the problem of evil, 1671-1678 (translated, edited, and with an introduction by Robert C. Sleigh, Jr., additional contributions from Brandon Look and James Stam) (2005), New Haven: Yale University Press, New Haven

[43] Leibniz, GW, The Leibniz-Des Bosses correspondence (translated, edited, and with an introduction by Brandon C. Look and Donald Rutherford) (2007), New Haven: Yale University Press, New Haven

[44] Leibniz, GW, Obras Filosóficas y Científicas. 7A Escritos matemáticos (edited by Mary Sol de Mora Charles) (2014), Comares: Granada, Comares

[45] Leibniz, GW, De quadratura arithmetica circuli ellipseos et hyperbolae cujus corollarium es trignonometria sine tabulis (herausgegeben und mit einem Nachwort versehen von Eberhard Knobloch, aus dem Lateinisch übersetzt von Otto Hamborg) (2016), Berlin: Springer, Berlin

[46] Leibniz, GW, MathesisUniversalis. Écritssur la mathématiqueuniverselle. Textesintroduits, traduitsetannotés sous la direction de David Rabouin (2018), Paris: Vrin, Paris

[47] Levey, S., Leibniz on mathematics and the actually infinite division of matter, The Philosophical Review, 107, 1, 49-96 (1998)· doi:10.2307/2998315

[48] Levey, S.; Jesseph, D.; Goldenbaum, U., Archimedes, infinitesimals and the law of continuity: On Leibniz's fictionalism, Infinitesimal differences: Controversies between Leibniz and his contemporaries, 107-133 (2008), Berlin: Walter de Gruyter, Berlin

[49] Lison, E., The philosophical assumptions underlying Leibniz's use of the diagonal paradox in 1672, StudiaLeibnitiana, 38, 2, 197-208 (2006)

[50] Lison, E., What does God know but can't say? Leibniz on infinity, fictitious infinitesimals and a possible solution of the labyrinth of freedom, Philosophia, 48, 261-288 (2020)· doi:10.1007/s11406-019-00086-4

[51] Luna Alcoba, M., La ley de continuidad en G W. Leibniz (1996), Sevilla: Universidad de Sevilla, Sevilla · Zbl 0874.01002

[52] Mancosu, P., Philosophy of mathematics and mathematical practice in the seventeenth century (1996), New York: Oxford University Press, New York · Zbl 0939.01004

[53] Nicolás, JA, Razón, verdad y libertad en Leibniz (1993), Granada: Universidad de Granada, Granada

[54] Poser, H., Signum, Notio und Idea. Elemente der Leibnizschen Zeichentheorie, Semiotik, 1, 309-324 (1979)

[55] Poser, H., Leibniz' Philosophie. Über die Einheit von Metaphysik und Wissenschaft (herausgegeben von Wenchao Li) (2016), Hamburg: Felix Meiner, Hamburg

[56] Rabouin, D.; Jullien, V., Leibniz's rigorous foundations of the method of indivisibles, Seventeenth-century indivisibles revisited, 347-364 (2015), Cham: Birkhäuser, Cham · Zbl 1326.01030 · doi:10.1007/978-3-319-00131-9$_1$5

[57] Rabouin, D.; Arthur, RTW, Leibniz's syncategorematic infinitesimals II: Their existence, their use and their role in the justification of the differential calculus, Archive for History of Exact Sciences, 75, 401-443 (2020) · Zbl 1448.01009 · doi:10.1007/s00407-020-00249-w

[58] Raffo Quintana, F., Leibniz on the requisites of an exact arithmetical quadrature, Studies in History and Philosophy of Science, 67, 65-73 (2018)· doi:10.1016/j.shpsa.2017.12.003

[59] Raffo Quintana, F., Continuo e infinito en el pensamiento leibniziano de juventud (2019), Comares: Granada, Comares

[60] Raffo Quintana, F., Sobre compendios y ficciones en el pensamiento juvenil de Leibniz, Revista Latinoamericana de Filosofía, 46, 131-150 (2020)· doi:10.36446/rlf2020203

[61] Rescher, N.; Rescher, N., Leibniz and the plurality of space-time frameworks, Leibniz's metaphsycs of nature, 84-100 (1981), Dordrecht: Reidel Publishing Company, Dordrecht· doi:10.1007/978-94-009-8445-5$_6$

[62] Sherry, D.; Katz, M., Infinitesimals, imaginaries, ideals, and fictions, Studia Leibnitiana, 44, 166-192 (2012)

[63] Sonar, T., Die Geschichte des Prioritätsstreitszwischen Leibniz und Newton. Geschichte-Kulturen-Menschen (2016), Berlin: Springer, Berlin · Zbl 1343.01005

[64] Swoyer, C., Structural representation and surrogative reasoning, Synthese, 87, 449-508 (1991)· doi:10.1007/BF00499820

[65] Swoyer, C., Leibnizian expression, Journal of the History of Philosophy, 33, 1, 65-99 (1995)· doi:10.1353/hph.1995.0009