TOPOLOGY, INTERSECTIONS AND FLAT MODULES

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Abstract. It is well-known that, in general, multiplication by an ideal I does not commute with the intersection of a family of ideals, but that this fact holds if I is flat and the family is finite. We generalize this result by showing that finite families of ideals can be replaced by compact subspaces of a natural topological space, and that ideals can be replaced by submodules of an epimorphic extension of a base ring. As a particular case, we give a new proof of a conjecture by Glaz and Vasconcelos.

1. Introduction

Let D be an integral domain with quotient field K. An *overring* of D is a ring between D and K. The set of all overrings of D is denoted by $\mathrm{Over}(D)$, and can be endowed with a natural topology (called the *Zariski topology*) whose basis of open sets consists of the sets of the form

$$\mathcal{B}(x_1,\ldots,x_n) := \{T \in \mathrm{Over}(D) : x_1,\ldots,x_n \in T\},$$

as $x_1,\ldots,x_n$ vary in K. Under this topology, $\mathrm{Over}(D)$ is a compact T_0 space with a unique closed point (D itself) and a generic point (the quotient field K). One of the clues that this topology is the most natural to be put on $\mathrm{Over}(D)$ is that it makes the localization map

$$\begin{aligned} \lambda\colon \mathrm{Spec}(D) &\longrightarrow \mathrm{Over}(D)\\ P &\longmapsto D_P \end{aligned}$$

a topological inclusion [4, Lemma 2.4].

This topology, whose origins can be traced back to Zariski's study of the space $\mathrm{Zar}(D)$ of the valuation overrings of an integral domain D [25, Chapter 6, §17] (what is now called the *Zariski space* or the *Riemann-Zariski space* of D), has recently been studied in greater detail (see for example [7, 8, 21, 20]). For example, it has been proved that $\mathrm{Over}(D)$ is a *spectral space*, meaning that there is a ring R such that $\mathrm{Spec}(R)$ is homeomorphic to $\mathrm{Over}(D)$ [5, Proposition 3.5]; the same can be proved of several distinguished subspaces of $\mathrm{Over}(D)$, like for

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example local overrings [7, Corollary 2.14] or integrally closed overrings [5, Proposition 3.6].

The aim of this paper is to prove a simple and very general result (Theorem 3, in the form of Corollary 5) which intertwines the Zariski topology on $\mathrm{Over}(D)$ with the algebraic properties of the overrings, namely the possibility to commute intersections and products in the case of compact spaces of overrings. In this way, we generalize [13, Lemma 1.1] (which deal with locally finite intersections) and [21, Theorem 3.5] (which proves the same for Noetherian collections of integrally closed overrings). As a consequence, we obtain a new proof of the Graz-Vasconcelos conjecture [12, page 340], independent from the one obtained in [23]. Since it poses no additional challenge, we also work in a more general setting, substituing to the extension $D \subseteq K$ any ring extension that is also an epimorphism, and using modules instead of only overrings.

2. Results

Let $A \subseteq B$ be a ring extension; we denote by $\mathcal{F}(B|A)$ the collection of all the A-submodules of B. The set $\mathcal{F}(B|A)$ becomes a T_0 topological space by declaring, as a basis of open sets, the family of the sets of the form $\mathcal{B}(x_1, \ldots, x_n) := \{G \in \mathcal{F}(B|A) : x_1, \ldots, x_n \in G\}$, for $x_1, \ldots, x_n$ varying in B. Note that, since $\mathcal{B}(x_1, \ldots, x_n) = \mathcal{B}(x_1) \cap \cdots \cap \mathcal{B}(x_n)$, a convenient subbasis for this topology is $\{\mathcal{B}(x) : x \in B\}$. We call this topology the *Zariski topology*, as it generalizes the Zariski topology on $\mathrm{Over}(D)$ defined in the Introduction. Note that, in particular, the set $\mathcal{I}(A)$ of all the integral ideals of A becomes then a subspace of $\mathcal{F}(B|A)$. On the set $\mathrm{Spec}(A)$ of the prime ideals of A, this topology does *not* coincide with the classical Zariski topology, but rather with the so-called *inverse topology* (see [14] and the discussion before Example 2.2 of [22]). This should, however, not cause any confusion; the only place where we will consider $\mathrm{Spec}(R)$ will be Proposition 11.

If X is any topological space and $Y \subseteq X$, we will denote by $\overline{Y}$ the closure of Y in X.

Remark 1. Let $A \subseteq B$ be a ring extension and let $\mathcal{F}(B|A)$ be endowed with the Zariski topology. The following properties hold.

(1) For any $F, G \in \mathcal{F}(B|A)$, we have $F \in \overline{\{G\}}$ if and only if $F \subseteq G$.
(2) Any compact nonempty subspace C of $\mathcal{F}(B|A)$ has minimal elements, with respect to the inclusion $\subseteq$. As a matter of fact, by Zorn's lemma it is enough to show that any chain (under inclusion) $\Sigma \subseteq C$ has a lower bound. By (1), the collection of sets $\mathcal{F} := \{\overline{\{F\}} \cap C : F \in \Sigma\}$ is a chain. Thus, in particular, given any finite subset $F_1, \ldots, F_n \in \Sigma$, if G is contained in all F_i, then $G \in \bigcap_{i=1}^n \overline{\{F_i\}} \cap C$. This proves that $\mathcal{F}$ is a collection of closed sets of C with the finite intersection property. By

compactness, there exists a submodule $F^\star \in \overline{\{F\}} \cap C$, for any $F \in \Sigma$, and applying again (1) we see that $F^\star$ is a lower bound of Σ in C.

Let now $\phi : A \longrightarrow B$ be a ring homomorphism. Then, ϕ is an *epimorphism* in the category of rings if, for every $\psi_1, \psi_2 : B \longrightarrow C$, the equality $\psi_1 \circ \phi = \psi_2 \circ \phi$ implies that $\psi_1 = \psi_2$. If the inclusion map $A \hookrightarrow B$ is an epimorphisms, we will call the ring extension $A \subseteq B$ an *epimorphic extension*.

Examples of epimorphisms are surjective maps and localizations; more generally, a map $\phi : A \longrightarrow B$ such that the induced homomorphism $\phi_{\mathfrak{p}} : A_{\mathfrak{p}} \longrightarrow B_{\mathfrak{p}}$ is surjective for every $\mathfrak{p} \in \mathrm{Spec}(A)$ such that $\phi(\mathfrak{p})B \neq B$ is an epimorphism (maps with this property are called *weakly surjective* [16, Chapter 1, §3]; on extensions, being an epimorphism and being weakly surjective are equivalent conditions [16, Theorem 4.4]). In particular, if D is an integral domain and K is its quotient field, the ring extension $D \subseteq K$ is an epimorphic extension. On the other hand, if X is an indeterminate over A, then the extension $A \subseteq A[X]$ is not epimorphic: indeed, for every $\alpha \in A$, we can build a ring homomorphism $\psi_\alpha : A[X] \longrightarrow A$ by defining $\psi_\alpha(a) := a$ if $a \in A$ and $\psi_\alpha(X) = \alpha$. In this case, we have $\psi_\alpha \neq \psi_\beta$ if $\alpha \neq \beta$, but every $i \circ \psi_\alpha$ is the identity on A.

The first step of our way is the following fact, which is a generalization of [1, Theorem 2].

Proposition 2. *Let $A \subseteq B$ be an epimorphic extension. Let I be a flat A-submodule of B, and let $G_1, \ldots, G_n \in \mathcal{F}(B|A)$. Then,*

$$I(G_1 \cap \ldots \cap G_n) = IG_1 \cap \ldots \cap IG_n.$$

Proof. With a small abuse of notation, for any $F, G \in \mathcal{F}(B|A)$, we will denote by $F \otimes G$ the submodule of $B \otimes B$ generated by the elements $f \otimes g$, as f varies in F and g varies in G. By induction, it suffices to show the statement for $n = 2$. Consider the map

$$\begin{aligned} \lambda \colon B \otimes_A B &\longrightarrow B \\ b_1 \otimes b_2 &\longmapsto b_1 b_2. \end{aligned}$$

Clearly, if $I, G \in \mathcal{F}(B|A)$, then $\lambda(I \otimes G) = IG$; therefore, by [18, Theorem 7.4]

$$I(G_1 \cap G_2) = \lambda(I \otimes (G_1 \cap G_2)) = \lambda((I \otimes G_1) \cap (I \otimes G_2)).$$

Since $A \subseteq B$ is an epimorphic extension, λ is an isomorphism (indeed, this property actually characterizes epimorphisms [17, Lemma 1.0]); in particular, λ is a bijection, and thus

$$\lambda((I \otimes G_1) \cap (I \otimes G_2)) = \lambda(I \otimes G_1) \cap \lambda(I \otimes G_2) = IG_1 \cap IG_2.$$

This completes the proof. $\square$

Note that this proposition does not hold if $A \subseteq B$ is not an epimorphism: for example, if X is an indeterminate over A, $B = A[X] = I$, $G_1 = A$, $G_2 = XA[X]$, then $G_1 \cap G_2 = (0)$ and so $I(G_1 \cap G_2) = (0)$, while $IG_1 \cap IG_2 = A[X] \cap XA[X] = XA[X]$.

Theorem 3. *Let $A \subseteq B$ be an epimorphic extension, let I be a flat A-submodule of B and let Y be a (nonempty) compact subspace of $\mathcal{F}(B|A)$. Then, the following equality holds:*

$$I\left(\bigcap_{J\in Y} J\right) = \bigcap_{J\in Y} IJ$$

Proof. The $(\subseteq)$ containment is obvious. Take now an element $x \in \bigcap\{IJ : J \in Y\}$. For any $J \in Y$, by definition, there exist a positive integer n_J and elements $i_1^{(J)}, \ldots, i_{n_J}^{(J)} \in I$, $t_1^{(J)}, \ldots, t_{n_J}^{(J)} \in J$ such that

$$x = i_1^{(J)} t_1^{(J)} + \cdots + i_{n_J}^{(J)} t_{n_J}^{(J)} = \sum_{h=1}^{n_J} i_h^{(J)} t_h^{(J)}.$$

Consider the open neighborhood $\Omega_J := \mathcal{B}(\{t_1^{(J)}, \ldots, t_{n_J}^{(J)}\})$ of J. Then the collection of sets $\mathscr{A} := \{\Omega_J : J \in Y\}$ is an open cover of Y. By compactness, $\mathscr{A}$ admits a finite subcover, say $\{\Omega_{J_1}, \ldots, \Omega_{J_r}\}$, for suitable $J_1, \ldots, J_r \in Y$. For any $l = 1, \ldots, r$, set $Y_l := \Omega_{J_l} \cap Y$. By Proposition 2, we have

$$I\left(\bigcap_{J\in Y} J\right) = I\left(\bigcap_{J\in Y_1} J \cap \ldots \cap \bigcap_{J\in Y_r} J\right) = I\left(\bigcap_{J\in Y_1} J\right) \cap \ldots \cap I\left(\bigcap_{J\in Y_r} J\right),$$

and thus it suffices to show that $x \in I\left(\bigcap_{J\in Y_l} J\right)$, for each $l = 1, \ldots, r$. However, the elements $t_1^{(J_l)}, \ldots, t_{n_{J_l}}^{(J_l)}$ belong to J for every $J \in Y_l$, and thus they belong to the intersection $\bigcap\{J : J \in Y_l\}$; hence, the representation $x = \sum_{h=1}^{n_{J_l}} i_h^{(J_l)} t_h^{(J_l)}$ shows that $x \in I(\bigcap_{J\in Y_l} J)$. □

Before giving some corollaries of independent interest, we state the following useful lemma.

Lemma 4. *Let $A \subseteq B$ be a ring extension and let $\mathcal{F}(B|A)$ be endowed with the Zariski topology. Fix a submodule $I \in \mathcal{F}(B|A)$. Then, the maps*

$$\begin{aligned} s_I \colon \mathcal{F}(B|A) &\longrightarrow \mathcal{F}(B|A) \\ J &\longmapsto I + J \end{aligned} \quad \text{and} \quad \begin{aligned} m_I \colon \mathcal{F}(B|A) &\longrightarrow \mathcal{F}(B|A) \\ I &\longmapsto IJ \end{aligned}$$

are continuous.

Proof. Let $\mathcal{B}(x)$ be a subbasic open set of $\mathcal{F}(B|A)$, with $x \in B$. If $J_0 \in s_I^{-1}(\mathcal{B}(x))$, then $x = i + j$ for some $i \in I$, $j \in J_0$; therefore, $\mathcal{B}(j)$ is an open neighborhood of J_0 contained in $s_I^{-1}(\mathcal{B}(x))$, and thus s_I is continuous. Similarly, if $J_0 \in m_I^{-1}(\mathcal{B}(x))$, then $x = i_1 j_1 + \cdots + i_n j_n$ for some $j_1, \ldots, j_n \in J_0$ and $i_1, \ldots, i_n \in I$. Then, $J_0 \in \mathcal{B}(j_1, \ldots, j_n) \subseteq m_I^{-1}(\mathcal{B}(x))$, and this shows that m_I is continuous. $\square$

Note that the continuity of m_I make it possible to shorten the proof of [21, Lemma 3.7].

Corollary 5. *Let D be an integral domain, let I and T be D-submodules of the quotient field K of D, and let Δ be a compact subset of* $\mathrm{Over}(D)$*, with respect to the Zariski topology. If T is flat over D, then*

$$\left(\bigcap_{U\in\Delta} IU\right)T = \bigcap_{U\in\Delta}(IUT).$$

Proof. By Lemma 4, the collection $\{IU : U \in \Delta\}$ is compact, since it is the continuous image of Δ via m_I. The conclusion is now an immediate consequence of Theorem 3. $\square$

As a particular case of the main results, we provide now a new topological proof of the Glaz-Vasconcelos conjecture.

Corollary 6. [23, Theorem 1.7] *Let D be an integrally closed integral domain, and let I be a D-submodule of its quotient field K. If I is flat over D, then $I = \bigcap\{IV : V \in \mathrm{Zar}(D)\}$.*

Proof. The space $\mathrm{Zar}(D)$ is compact in the Zariski topology [25, Chapter 6, Theorem 40]; moreover, since D is integrally closed, $D = \bigcap\{V : V \in \mathrm{Zar}(D)\}$ [3, Corollary 5.22]. Hence, by Theorem 3,

$$I = ID = I\left(\bigcap_{V\in\mathrm{Zar}(D)} V\right) = \bigcap_{V\in\mathrm{Zar}(R)} IV,$$

as claimed. $\square$

Another immediate consequence of the main results deals with intersections of localizations of integral domains.

Corollary 7. *Let D be an integral domain, let Y be a compact nonempty subspace of* $\mathrm{Over}(D)$ *such that $D = \bigcap\{R : R \in Y\}$, and let S be a multiplicative subset of D. Then, $S^{-1}D = \bigcap\{S^{-1}R : R \in Y\}$.*

Proof. It suffices to use Theorem 3, keeping in mind that $S^{-1}D$ is a flat D-module. $\square$

Corollary 8. [11, Proposition 43.5] *Let D be an integral domain, let Y be a locally finite subspace of* $\mathrm{Over}(D)$ *(i.e., any nonzero element of D is noninvertible only in finitely many members of Y) such that $D = \bigcap\{R : R \in Y\}$, and let S be a multiplicative subset of D. Then, $S^{-1}D = \bigcap\{S^{-1}R : R \in Y\}$*

Proof. By Corollary 7, it is enough to show that a locally finite collection of overrings of D is compact, with respect to the Zariski topology of Over(D).

Let $\mathscr{A}$ be an open cover of Y. By Alexander's Subbasis Theorem (see e.g. [15, Chapter 5, Theorem 6, page 139]), we can assume, without loss of generality, that $\mathscr{A}$ consists of subbasic open sets of Over(D), say $\mathscr{A} = \left\{\mathcal{B}\left(\frac{a_i}{b_i}\right) : i \in I\right\}$, where $a_i, b_i \in D, b_i \neq 0$, for any $i \in I$. Fix now an index $i' \in I$ and note that, by assumption, the set $Y' := \{R \in Y : b_{i'}^{-1} \notin R\}$ is finite, say $Y' = \{R_1, \dots R_n\}$. Thus, any member of $Y - Y'$ belongs to $\mathcal{B}\left(\frac{a_{i'}}{b_{i'}}\right)$ and any $R_j \in Y'$ belongs to some $\mathcal{B}\left(\frac{a_{i_j}}{b_{i_j}}\right)$. The proof is now complete. $\square$

Note that the main part of the proof of the previous corollary is also a consequence of [8, Proposition 2.9], where it was proved in the more general context of semistar operations; we inserted the proof here for the reader's convenience. Moreover, the proof of the previous corollary also extends [9, Remark 4.7], where the authors proved that any locally finite family of localizations is compact.

Corollary 9. *Let D be a Prüfer domain with quotient field K, let $\mathfrak{a}$ be an ideal of D, and let $Y \subseteq \mathcal{I}(D)$ be compact. Then,*

$$\mathfrak{a} + \bigcap_{\mathfrak{b}\in Y} \mathfrak{b} = \bigcap_{\mathfrak{b}\in Y} (\mathfrak{a} + \mathfrak{b}).$$

Proof. It suffices to prove that, for every prime ideal $\mathfrak{p}$, the equality

$$\mathfrak{a}D_\mathfrak{p} + \left(\bigcap_{\mathfrak{b}\in Y} \mathfrak{b}\right) D_\mathfrak{p} = \left(\bigcap_{\mathfrak{b}\in Y} (\mathfrak{a} + \mathfrak{b})\right) D_\mathfrak{p}$$

holds. Fix thus a prime ideal $\mathfrak{p}$, and let $V := D_\mathfrak{p}$; since D is a Prüfer domain, V is a valuation domain.

Since V is flat over D and $\{\mathfrak{a} + \mathfrak{b} : \mathfrak{b} \in Y\}$ is compact (Lemma 4), we have, by Theorem 3,

$$\left(\bigcap_{\mathfrak{b}\in Y} (\mathfrak{a} + \mathfrak{b})\right) V = \bigcap_{\mathfrak{b}\in Y} ((\mathfrak{a} + \mathfrak{b})V) = \bigcap_{\mathfrak{b}\in Y} (\mathfrak{a}V + \mathfrak{b}V) \tag{1}$$

Observe now that, since V is a valuation domain, the collection of ideals $Y' := \{\mathfrak{b}V : \mathfrak{b} \in Y\}$ is totally ordered and compact, by Lemma 4. Thus, since by compactness Y' has minimal elements under inclusion (Remark 1), it follows that Y' has a minimum. Then, there is an ideal $\mathfrak{b}_0 \in Y$ such that $\mathfrak{b}_0 V \subseteq \mathfrak{b}V$, for any $\mathfrak{b} \in Y$. It follows that the last member of the equality (1) becomes

$$\bigcap_{\mathfrak{b}\in Y} (\mathfrak{a}V + \mathfrak{b}V) = \mathfrak{a}V + \mathfrak{b}_0 V = \mathfrak{a}V + \left(\bigcap_{\mathfrak{b}\in Y} \mathfrak{b}V\right) = \mathfrak{a}V + \left(\bigcap_{\mathfrak{b}\in Y} \mathfrak{b}\right) V,$$

where the last equality is again a consequence of Theorem 3. The proof is now complete. □

Remark 10. The previous corollary is closely related to the dual AB-5* of Grothendieck AB-5 (see, for example, [2]). Precisely, if D is a Prüfer domain and any filter base of ideals of D is compact, with respect to the Zariski topology of $\mathcal{I}(D)$, then D is AB-5* (as D-module).

In the case of Prüfer domains, we can also prove a partial converse of Theorem 3. Recall that a prime ideal P of a Prüfer domain D is *branched* if the set of prime ideals of D properly contained in P has a maximum (see e.g. [11, Theorem 17.3]). If the dimension of D is finite, every prime ideal is branched.

Proposition 11. *Let D be a Prüfer domain with quotient field K, and let $\Delta \subseteq \mathrm{Spec}(D)$ be a nonempty set.*

(a) *Δ is compact (in the "classical" Zariski topology of $\mathrm{Spec}(D)$) if and only if, for every flat D-submodule I of K, $\bigcap_{\mathfrak{p}\in\Delta} ID_{\mathfrak{p}} = I\left(\bigcap_{\mathfrak{p}\in\Delta} D_{\mathfrak{p}}\right)$.*

(b) *Suppose that every prime ideal of D is branched. Then, Δ is compact (in the "classical" Zariski topology of $\mathrm{Spec}(R)$), if and only if $\bigcap_{\mathfrak{p}\in\Delta} D_{\mathfrak{q}}D_{\mathfrak{p}} = D_{\mathfrak{q}}\left(\bigcap_{\mathfrak{p}\in\Delta} D_{\mathfrak{p}}\right)$ for every $\mathfrak{q} \in \mathrm{Spec}(D)$.*

Proof. In both points, one implication follows from Corollary 7 and the fact that the map $\lambda : \mathrm{Spec}(D) \longrightarrow \mathrm{Over}(D)$, $P \mapsto D_P$, is a topological inclusion. Suppose Δ is not compact, and let $T := \bigcap_{\mathfrak{p}\in\Delta} D_{\mathfrak{p}}$; note that, without loss of generality, we can suppose that $\Delta = \Delta^{\downarrow} = \{\mathfrak{q} \in \mathrm{Spec}(D) : \mathfrak{q} \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \Delta\}$, since Δ is compact if and only if $\Delta^{\downarrow}$ is compact.

The set of prime ideals $\mathfrak{p}$ such that $\mathfrak{p}T \neq T$ is the image of $\mathrm{Spec}(T)$ under the canonical map $\mathrm{Spec}(T) \longrightarrow \mathrm{Spec}(D)$; since it contains Δ, and Δ is not compact, it must also contain a prime ideal $\mathfrak{q} \notin \Delta$. Since D is a Prüfer domain, $\mathfrak{q}$ is a flat D-module; however,

$$\bigcap_{\mathfrak{p}\in\Delta} \mathfrak{q}D_{\mathfrak{p}} = \bigcap_{\mathfrak{p}\in\Delta} D_{\mathfrak{p}} = T \neq \mathfrak{q}T = \mathfrak{q}\left(\bigcap_{\mathfrak{p}\in\Delta} D_{\mathfrak{p}}\right),$$

against the hypothesis. Therefore, part (a) is proved.

If every prime ideal of D is branched, so is $\mathfrak{q}$; therefore, there is a prime ideal $\mathfrak{q}_0$ directly below $\mathfrak{q}$. No ideal $\mathfrak{p} \in \Delta$ contains $\mathfrak{q}$; therefore, $D_{\mathfrak{p}}D_{\mathfrak{q}} \supsetneq D_{\mathfrak{q}}$, and in particular $D_{\mathfrak{q}_0} \subseteq D_{\mathfrak{p}}D_{\mathfrak{q}}$. Hence,

$$\bigcap_{\mathfrak{p}\in\Delta} D_{\mathfrak{q}}D_{\mathfrak{p}} \supseteq D_{\mathfrak{q}_0} \supsetneq D_{\mathfrak{q}} = D_{\mathfrak{q}}T = D_{\mathfrak{q}}\left(\bigcap_{\mathfrak{p}\in\Delta} D_{\mathfrak{p}}\right),$$

against the hypothesis. Part (b) is proved. □

Note that part (b) of the previous proposition does not hold without the hypothesis that the prime ideals are branched: indeed, if V is a valuation domain with maximal ideal $\mathfrak{m}$ unbranched, and $\Delta := \mathrm{Spec}(V) \setminus \{\mathfrak{m}\}$, then

$$V_{\mathfrak{m}}V = V = \bigcap_{\mathfrak{p}\in\Delta} V_{\mathfrak{p}} = \bigcap_{\mathfrak{p}\in\Delta} V_{\mathfrak{p}}V_{\mathfrak{m}},$$

despite Δ not being compact.

Another question arising from Theorem 3 is if the equality $I\left(\bigcap_{J\in Y} J\right) = \bigcap_{J\in Y} IJ$, for all compact families Y of submodules of an epimorphic extension $A \subseteq B$, implies that I is flat. This is true if the base ring A is a domain, but fails in general (see [1, Theorem 2] and the subsequent discussion).

Remark 12. While Theorem 3 is quite general, it may be in general hard, or at least not easy, to find examples of compact subspaces to which it can be applied, or to prove that a given family is actually compact.

Some examples can be constructed using the fact that, under the Zariski topology, $\mathcal{F}(B|A)$ is a spectral space, i.e., it is homeomorphic to the prime spectrum of a ring [22, Example 2.2(2)]. For example, it follows form Remark 1(1) and either [24, Proposition 2.3] or [19, Proposition 2.2] that a subset Y of $\mathcal{F}(B|A)$ is compact if and only if every element of the closure of Y, with respect to the constructible topology, contains a point of Y, where the *constructible topology* on $\mathcal{F}(B|A)$ is the coarsest topology on $\mathcal{F}(B|A)$ for which any open and compact subspace of $\mathcal{F}(B|A)$ is both open and closed.

Another class of examples comes from the domination map $d : \mathrm{Zar}(A) \longrightarrow \mathrm{Spec}(A)$ of the Zariski space of a domain A (i.e., the set of valuation overrings of A). For example, if S is a compact subspace of $\mathrm{Spec}(A)$, then $d^{-1}(S)$ is compact, by [19, Proposition 2.2 and Lemma 2.7(3)].

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