Closure Hyperdoctrines

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Abstract

(Pre)closure spaces are a generalization of topological spaces covering also the notion of neighbourhood in discrete structures, widely used to model and reason about spatial aspects of distributed systems.

In this paper we present an abstract theoretical framework for the systematic investigation of the logical aspects of closure spaces. To this end, we introduce the notion of *closure (hyper)doctrines*, i.e. doctrines endowed with inflationary operators (and subject to suitable conditions). The generality and effectiveness of this concept is witnessed by many examples arising naturally from topological spaces, fuzzy sets, algebraic structures, coalgebras, and covering at once also known cases such as Kripke frames and probabilistic frames (i.e., Markov chains). By leveraging general categorical constructions, we provide axiomatisations and sound and complete semantics for various fragments of logics for closure operators. Hence, closure hyperdoctrines are useful both for refining and improving the theory of existing spatial logics, and for the definition of new spatial logics for new applications.

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1 Introduction

Recently, much attention has been devoted in Computer Science to systems distributed in physical space; a typical example is provided by the so called *collective adaptive systems*, such as drone swarms, sensor networks, autonomous vehicles, etc. This begs the question of how to model and reason formally about spatial aspects of distributed systems. To this end, several researchers have advocated the use of *spatial logics*, i.e. modal logics whose modalities are interpreted using topological concepts of neighbourhood and connectivity.[1]

In fact, the interpretation of modal logics in topological spaces goes back to Tarski; we refer to [1] for a comprehensive discussion of variants and computability and complexity aspects. More recently, Ciancia *et al.* [10, 11] extended this approach to *preclosure spaces*, also called *Čech closure spaces*, which generalise topological spaces by not requiring idempotence of closure operator. This generalization unifies the notions of neighbourhood arising from topological spaces and from *quasi-discrete closure spaces*, like those induced by graphs and images. Building on this generalization, [10] introduced *Spatial Logic for Closure Spaces* (SLCS), a modal logic for the specification and verification on spatial concepts over preclosure spaces. This logic features a *closure* modality and a spatial *until* modality: intuitively $\phi\mathcal{U}\psi$ holds in an area where ϕ holds and it is not possible to "escape" from it unless passing

[1] Not to be confused with spatial logics for reasoning on the structure of agents, such as the Ambient Logic [8] or the Brane Logic [36].

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through an area where ψ holds. SLCS has been proved to be quite effective and expressive, as it has been applied to reachability problems, vehicular movement, digital image analysis (e.g., street maps, radiological images [5]), etc. The model checking problem for this logic over finite quasi-discrete structures is decidable in linear time [10].

Despite these results, an axiomatisation for SLCS is still missing; moreover, it is not obvious how to extend this logic to other spaces with closure operators, such as probabilistic automata (e.g. Markov chains). In fact, the main point is that we miss an abstract theoretical framework for investigating the logical aspects of (pre)closure spaces. Such a framework would be the basis for analysing spatial logics like SLCS, but also for developing further extensions and applications thereof.

In this paper, we aim to build such a framework. To this end, we introduce the new notion of *closure (hyper)doctrine* as the theoretical basis for studying the logical aspects of closure spaces. Doctrines were introduced by Lawvere [30] as a general way for endowing (the objects of) a category with logical notions from a suitable 2-category $\mathbf{E}$, which can be the category of Heyting algebras in the case of intuitionistic logic, of Boolean algebras in the case of classical logic, etc.. Along this line, in order to capture the logical aspects of closure spaces we introduce the notion of *closure operators* on doctrines, that is, families of inflationary morphisms over objects of $\mathbf{E}$ (subject to suitable conditions); a closure (hyper)doctrine is a (hyper)doctrine endowed with a closure operator. These structures arise from many common situations: we provide many examples ranging from topology to algebraic structures, from coalgebras to fuzzy sets. These examples cover the usual cases from literature (e.g., graphs, quasi-discrete spaces, (pre)topological spaces) but include also new settings, such as categories of coalgebras and probabilistic frames (i.e., Markov chains). Then, leveraging general machinery from categorical logic, we introduce a first order logic for closure spaces for which we provide an axiomatisation and a sound and complete categorical semantics. The propositional fragment corresponds to the SLCS from [10].

Overall, the importance of this work is twofold: on one hand, closure hyperdoctrines are useful for analysing and improving the theory of existing spatial logics; in particular, the proposed axiomatisation can enable both new proof methodologies and minimisation techniques. On the other, closure hyperdoctrines are useful for the definition of new logics when we have to deal with closure operators, connectivity, surroundedness, reachability, etc.

Synopsis. The paper is organized as follows. In Section 2 we recall (hyper)doctrines and introduce the key notion of closure doctrine. Many examples of closure doctrines are provided in Section 3. In Section 4 we introduce *logics for closure operators*, together with a sound and complete semantics in closure hyperdoctrines. Conclusions and directions for future work are in Section 5. Longer proofs are in Appendix A.

2 Closure (hyper)doctrines

2.1 Kinds of doctrines

In this section we recall the notion of elementary hyperdoctrine, due to Lawvere [30, 31]. The development of semantics of logics in this context or in the equivalent fibrational context is well established; we refer the reader to, e.g., [23, 35, 38].

▶ **Definition 2.1** ((Existential) Doctrine, Hyperdoctrine [28, 34, 37]). *A* primary doctrine *or simply a* doctrine *on a category* $\mathbf{C}$ *is a functor* $\mathcal{P} : \mathbf{C}^{op} \to \mathbf{InfSL}$ *where* $\mathbf{InfSL}$ *is the category of finite meet semilattices.*

A primary doctrine is existential *if:*

- $\mathbf{C}$ *has finite products;*
- *the image* $\mathcal{P}_{\pi_C}$ *of any projection* $\pi_C : C \times D \to C$ *admits a left adjoint* $\exists_{\pi_C}$*;*
- *for each pullback like aside, the* Beck-Chevalley condition $\exists_{\pi_{C'}} \circ \mathcal{P}_{1_D \times f} = \mathcal{P}_f \circ \exists_{\pi_C}$ *holds;*

$$\begin{array}{ccc} D \times C' & \xrightarrow{\pi_{C'}} & C' \\ {\scriptstyle 1_D \times f}\downarrow & & \downarrow{\scriptstyle f} \\ D \times C & \xrightarrow[\pi_C]{} & C \end{array}$$

- *for any* $\alpha \in \mathcal{P}(C)$ *and* $\beta \in \mathcal{P}(D \times C)$ *the* Frobenius reciprocity $\exists_{\pi_C}(\mathcal{P}_{\pi_C}(\alpha) \wedge \beta) = \alpha \wedge \exists_{\pi_C}(\beta)$ *holds.*

A hyperdoctrine *is an existential doctrine* $\mathcal{P}$ *such that:*

- $\mathcal{P}$ *factors through the category* **HA** *of Heyting algebras and Heyting algebras morphisms;*
- *for all projections* $\pi_C : D \times C \to C$, $\mathcal{P}_{\pi_C}$ *has a right adjoint* $\forall_{\pi_C} : \mathcal{P}(D \times C) \to \mathcal{P}(C)$ *satisfying the Beck-Chevalley condition:* $\forall_{\pi_{C'}} \circ \mathcal{P}_{1_D \times f} = \mathcal{P}_f \circ \forall_{\pi_C}$ *for any* $f : C' \to C$*.*

A primary doctrine, an existential doctrine or a hyperdoctrine, is elementary *if*

- $\mathbf{C}$ *has finite products;*
- *for each object* C *there exists a* fibered equality $\delta_C \in \mathcal{P}(C \times C)$ *such that*

$$\mathcal{P}_{(\pi_1,\pi_2)}(-) \wedge \mathcal{P}_{(\pi_2,\pi_3)}(\delta_C) \dashv \mathcal{P}_{1_D \times \Delta_C}$$

where π_1, π_2 *and* π_3 *are projections* $D \times C \times C \to D \times C$*. This left adjoint will be denoted by* $\exists_{1_D \times \Delta_C}$

▶ Remark 2.2. Usually $\mathbf{C}$ is required to having finite products even in the case of a primary doctrine (cfr. [37]), we will not ask it in order to get the coalgebraic examples in Section 3.

▶ Remark 2.3. Since $\mathbf{C}$ has a terminal object it follows that $\mathcal{P}_{\pi_1}(-) \wedge \delta_C \dashv \mathcal{P}_{\Delta_C}$. This left adjoint will be denoted by $\exists_{\Delta_C}$.

▶ Remark 2.4. In this paper, we work with hyperdoctrines over **HA**, the category of Heyting algebras and their morphisms; hence the resulting logic is inherently intuitionistic. Clearly, all the development still holds if we restrict ourselves to the subcategory of Boolean algebras **BA**, yielding a classical version of the logic.

▶ **Example 2.5.** Let $\mathbf{C}$ be a category with finite limits and $(\mathcal{E}, \mathcal{M})$ a stable and proper factorization system on it (see [27]). For every object $C \in \mathbf{C}$ we can define a relation on arrows in $\mathcal{M}$ with codomain C putting $m \leq n$ if and only if there exists t such that $n \circ t = m$. If we ignore size issues this gives us a preorder, from which we get a partial order $\mathcal{M}\text{-}\mathcal{S}ub_{\mathbf{C}}(C)$ by quotienting by the relation $m \simeq n$ if and only if $m \leq n$ and $n \leq m$. The top element is $[1_C]$, while meets are given by pullbacks, and we can pullback any m along any arrow $f : D \to C$ getting an arrow f^*m in $\mathcal{M}$ with codomain D. Summarizing we have a functor $\mathbf{C}^{op} \to \mathbf{InfSL}$ sending C to $\mathcal{M}\text{-}\mathcal{S}ub_{\mathbf{C}}(C)$. This is actually an elementary existential doctrine in which δ_C is the class of the diagonal $C \to C \times C$ (which can be shown to be an element of $\mathcal{M}$) and $\exists_{\pi_C}([m])$ is the $\mathcal{M}$-component of $\pi_C \circ m$, in the sense that it is the class of $n \in \mathcal{M}$ such that $n \circ e = \pi_C \circ m$ for some $e \in \mathcal{E}$ (see [22] for the correspondence between factorization systems and elementary existential doctrines). In general this functor is very far from having Heyting algebras as values but this is the case when $\mathbf{C}$ is a topos and $\mathcal{M}$ the class of all monomorphisms; in this case we get an elementary hyperdoctrine [32].

If we want $\mathcal{M}$ to be the class of all monos we have the following theorem:

▶ **Theorem 2.6** ([23, Th. 4.4.4]). *If $\mathbf{C}$ has finite limits then $\mathcal{Sub}_{\mathbf{C}}$ is an elementary existential doctrine if and only if $\mathbf{C}$ is regular.*

▶ **Proposition 2.7.** *Let $\mathcal{P} : \mathbf{C}^{op} \to \mathbf{InfSL}$ be an existential doctrine, $\mathbf{D}$ a category with finite products and $\mathcal{F} : \mathbf{D} \to \mathbf{C}$ a product preserving functor. Then, $\mathcal{P} \circ \mathcal{F}^{op}$ is a existential doctrine. If $\mathcal{P}$ is elementary (resp., a hyperdoctrine) then $\mathcal{P} \circ \mathcal{F}^{op}$ is elementary (resp., a hyperdoctrine).*

Proof. See proof on page 18. ◀

▶ **Proposition 2.8.** *Let $\mathcal{P} : \mathbf{C}^{op} \to \mathbf{HA}$ be an elementary existential doctrine. For every arrow $f : C \to D$, the functor $\mathcal{P}_f$ has a left adjoint $\exists_f$ that satisfies the* Frobenius reciprocity*: $\exists_f(\mathcal{P}_f(\beta) \wedge \alpha) = \beta \wedge \exists_f(\alpha)$. If $\mathcal{P}$ is a hyperdoctrine then $\mathcal{P}_f$ has a right adjoint $\forall_f$ too.*

Proof. See proof on page 19. ◀

▶ Remark 2.9. In general these adjoints do not satisfy any form of Beck-Chevalley condition [12, 23, 33, 40].

▶ **Definition 2.10.** *Let $\mathcal{P} : \mathbf{C}^{op} \to \mathbf{InfSL}$, $\mathcal{S} : \mathbf{D}^{op} \to \mathbf{InfSL}$ be primary doctrines.*

A morphism $\mathcal{P} \to \mathcal{S}$ is a pair $(\mathcal{F}, \eta)$ where $\mathcal{F} : \mathbf{C} \to \mathbf{D}$ is a functor and $\eta : \mathcal{P} \to \mathcal{S} \circ \mathcal{F}^{op}$ is a natural transformation.

$(\mathcal{F}, \eta)$ is a morphism of elementary doctrines*, or* elementary*, if $\mathcal{F}$ preserves finite products and for any object C of $\mathbf{C}$, it is $\eta_{C\times C}(\delta_C) = \mathcal{S}_{(\mathcal{F}(\pi_1),\mathcal{F}(\pi_2))}(\delta_{\mathcal{F}(C)})$.*

$(\mathcal{F}, \eta)$ is a morphism of existential doctrine *if $\mathcal{F}$ preserves finite products and for any pair of objects C, D of $\mathbf{C}$ the diagram (a) below commutes.*

$$
\begin{array}{ccc}
\mathcal{P}(D\times C) & \xrightarrow{\exists_{\pi_C}} & \mathcal{P}(C) \\
{\scriptstyle \eta_{D\times C}}\downarrow & & \\
\mathcal{S}(\mathcal{F}(D\times C)) & & \downarrow{\scriptstyle \eta_C} \\
{\scriptstyle \mathcal{S}_{(\mathcal{F}(\pi_D),\mathcal{F}(\pi_C))}}\downarrow & & \\
\mathcal{S}(\mathcal{F}(D)\times\mathcal{F}(C)) & \xrightarrow{\exists_{\pi_{\mathcal{F}(C)}}} & \mathcal{S}(\mathcal{F}(D))
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{P}(D\times C) & \xrightarrow{\forall_{\pi_C}} & \mathcal{P}(C) \\
{\scriptstyle \eta_{D\times C}}\downarrow & & \\
\mathcal{S}(\mathcal{F}(D\times C)) & & \downarrow{\scriptstyle \eta_C} \\
{\scriptstyle \mathcal{S}_{(\mathcal{F}(\pi_D),\mathcal{F}(\pi_C))}}\downarrow & & \\
\mathcal{S}(\mathcal{F}(D)\times\mathcal{F}(C)) & \xrightarrow{\forall_{\pi_{\mathcal{F}(C)}}} & \mathcal{S}(\mathcal{F}(D))
\end{array}
$$

(a) *(b)*

$(\mathcal{F}, \eta)$ is a morphism of hyperdoctrines *if it is a morphism of existential doctrine, the diagram (b) above commutes too and each component of η preserves finite suprema and implication.*

If (F, η) is also elementary then we call it a morphism of elementary existential doctrines *or of* elementary hyperdoctrines*.*

Let $(\mathcal{F}, \eta), (\mathcal{G}, \epsilon) : \mathcal{P} \to \mathcal{S}$ be two morphisms; a 2-arrow $(\mathcal{F}, \eta) \to (\mathcal{G}, \epsilon)$ is a natural transformations $\theta : \mathcal{F} \to \mathcal{G}$ such that $\eta_C(\alpha) \le \mathcal{S}_{\theta_C}(\epsilon_C(\alpha))$.

This defines the 2-categories $\mathbf{PD}$, $\mathbf{ED}$, $\mathbf{HD}$ of primary doctrines, existential doctrines and hyperdoctrines, and the subcategories $\mathbf{EPD}$, $\mathbf{EED}$, $\mathbf{EHD}$ of their elementary variants.

2.2 Closure operators on doctrines

In this section we introduce the key notion of closure operators on doctrines.

▶ **Definition 2.11.** *Let $\mathcal{P}$ be a doctrine. A* closure operator *on $\mathcal{P}$ is a (possibly large) family $\mathfrak{c} = \{\mathfrak{c}_C\}_{C\in\mathbf{Ob(C)}}$ of functions $\mathfrak{c}_C : \mathcal{P}(C) \to \mathcal{P}(C)$ such that:*

- *for any object C, $\mathfrak{c}_C$ is monotone and* inflationary*, i.e., $1_{\mathcal{P}(C)} \leq \mathfrak{c}_C$*
- *any arrow $f : C \to D$ is* continuous*, i.e. $\mathfrak{c}_C \circ \mathcal{P}_f \leq \mathcal{P}_f \circ \mathfrak{c}_D$.*

A closure operator $\mathfrak{c}$ is said to be

- grounded *if $\mathfrak{c}_C(\bot) = \bot$ for all objects C such that $\mathcal{P}(C)$ has a minimum;*
- additive *if $\mathfrak{c}_C(\alpha \vee \beta) = \mathfrak{c}_C(\alpha) \vee \mathfrak{c}_C(\beta)$ for all objects C such that $\mathcal{P}(C)$ has binary suprema;*
- finitely additive *if it is grounded and additive;*
- full additive *if $\mathfrak{c}_C(\bigvee_{i\in I} \alpha_i) = \bigvee_{i\in I} \mathfrak{c}_C(\alpha_i)$ for all $I \neq \emptyset$ and C such that $\mathcal{P}(C)$ has I-indexed suprema;*
- idempotent *if $\mathfrak{c}_C \circ \mathfrak{c}_C = \mathfrak{c}_C$ for all object C.*

A closure doctrine *is a pair $(\mathcal{P}, \mathfrak{c})$ where $\mathcal{P}$ is a primary doctrine and $\mathfrak{c}$ a closure operator on it. We say that $(\mathcal{P}, \mathfrak{c})$ is* elementary, existential*, or a* hyperdoctrine*, if $\mathcal{P}$ is.*

▶ Remark 2.12. Continuity can be interpreted as a form of oplax naturality [20], even if $\mathfrak{c}_C$ is not an arrow of **InfSL** in general.

▶ **Example 2.13.** Lawvere-Tierney topologies on a topos provide examples of idempotent closure operators on the elementary hyperdoctrine of subobjects [6, 26, 32].

▶ Remark 2.14. Full additivity does not imply groundedness since we only ask for preservation of suprema indexed on non empty sets.

▶ **Proposition 2.15.** *Let $\mathcal{P}$ be a doctrine and $f : C \to D$ a morphism such that $\mathcal{P}_f$ has a left adjoint $\exists_f$, then for every closure operator $\mathfrak{c}$ on $\mathcal{P}$ continuity of f is equivalent to*

$$\exists_f \circ \mathfrak{c}_C \leq \mathfrak{c}_D \circ \exists_f$$

Proof. Let's compute:

$$\mathfrak{c}_C \circ \mathcal{P}_f \leq \mathcal{P}_f \circ \exists_f \circ \mathfrak{c}_C \circ \mathcal{P}_f \leq \mathcal{P}_f \circ \mathfrak{c}_D \circ \exists_f \circ \mathcal{P}_f \leq \mathcal{P}_f \circ \mathfrak{c}_D$$
$$\exists_f \circ \mathfrak{c}_C \leq \exists_f \circ \mathfrak{c}_C \circ \mathcal{P}_f \circ \exists_f \leq \exists_f \circ \mathcal{P}_f \circ \mathfrak{c}_D \circ \exists_f \leq \mathfrak{c}_D \circ \exists_f$$ ◀

If we think of a morphism of (primary, existential, elementary, hyper)doctrines $(\mathcal{F}, \eta) : \mathcal{P} \to \mathcal{Q}$ as a "translation" of "types" and "predicates" then, when closure operators are available, it is natural to ask for this "translation" to take place in a continuous way.

▶ **Definition 2.16.** *A* morphism of closure (elementary, existential, hyper)doctrines *$(\mathcal{F}, \eta) : (\mathcal{P}, \mathfrak{c}) \to (\mathcal{Q}, \mathfrak{d})$ is a morphism of (elementary, existential, hyper)doctrines $\mathcal{F} : \mathcal{P} \to \mathcal{Q}$ such that η is* continuous*, i.e. $\mathfrak{d}_{\mathcal{F}(C)} \circ \eta_C \leq \eta_C \circ \mathfrak{c}_C$ for all C. We say that $(\mathcal{F}, \eta)$ is* open *if equality holds for all the objects C. A 2-cell $\theta : (\mathcal{F}, \eta) \to (\mathcal{G}, \epsilon)$ is defined as in the case of doctrines. In this way we get the 2-categories* **cPD**, **cED**, **cHD** *of closure doctrines, closure existential doctrines, closure hyperdoctrines and the subcategories* **cEPD**, **cEED**, **cEHD** *of their elementary variants.*

3 Examples of closure hyperdoctrines

3.1 Topological examples

As a first class of examples, we introduce three closure hyperdoctrines starting from the usual category **Top** of topological spaces and continuous maps. The first one corresponds to the *closure spaces* used in, e.g., [10, 11, 18].

▶ **Definition 3.1.** *The category* **PrTop** *of* pretopological spaces *(or* closure spaces*) is the category in which:*

- *objects are pairs* $(X, \mathfrak{c})$ *of a set* X *and a monotone function* $\mathfrak{c} : \mathcal{P}(X) \to \mathcal{P}(X)$ *such that* $1_{\mathcal{P}(X)} \leq \mathfrak{c}$ *and* $\mathfrak{c}$ *preserves finite (even empty) suprema;*
- *an arrow* $f : (X, \mathfrak{c}_X) \to (Y, \mathfrak{c}_Y)$ *is a function* $f : X \to Y$ *such that* $f^{-1} : (\mathcal{P}(Y), \mathfrak{c}_Y) \to (\mathcal{P}(X), \mathfrak{c}_X)$ *is continuous.*

Another example is given by so called *convergence spaces* (cfr. [14]).

▶ **Definition 3.2.** *For any set* X *let* $\mathbf{Fil}(X)$ *be the set of proper filters (i.e.,* $\emptyset$ *is not among them) on it. The category* **FC** *of* filter convergence spaces *is the category in which:*

- *an object is a pair* (X, q_X) *given by a set* X *and a function* $q_X : X \to \mathcal{P}(\mathbf{Fil}(X))$ *such that, for any* $x \in X$*,* $q_X(x)$ *is upward closed and* $\dot{x} := \{A \subset X \mid x \in A\}$ *belongs to* $q_X(x)$*.*
- *an arrow* $f : (X, q_x) \to (Y, q_Y)$ *is a function* $f : X \to Y$ *such that the filter* $f(F)$ *generated by the images of* F*'s elements belongs to* $q_Y(f(x))$ *whenever* $F \in q_X(x)$*.*

▶ **Proposition 3.3.** *The obvious forgetful functors from* **Top***,* **PrTop** *and* **FC** *to* **Set** *preserve finite products.*

Proof. For **Top** it is clear, for the other two categories see [14, Ch.3]. ◀

By Proposition 2.7 and the previous one, we have three elementary hyperdoctrines

$$\mathcal{P}^t : \mathbf{Top}^{op} \to \mathbf{HA} \quad \mathcal{P}^p : \mathbf{PrTop}^{op} \to \mathbf{HA} \quad \mathcal{P}^f : \mathbf{FC}^{op} \to \mathbf{HA}$$

which we now endow with closure operators.

▶ **Definition 3.4.** *We define the following closure operators:*

1. *the* Kuratowski closure operator $k = \{k_{(X,\theta)}\}_{(X,\theta)\in\mathbf{Ob}(\mathbf{Top})}$ *on* $\mathcal{P}^t$ *where* $k_{(X,\theta)}$ *is the closure operator associated with the topology* θ*;*
2. *the* Čech closure operator $c = \{c_{(X,\mathfrak{c})}\}_{(X,\mathfrak{c})\in\mathbf{Ob}(\mathbf{PrTop})}$ *on* $\mathcal{P}^p$ *where* $c_{(X,\mathfrak{c})}$ *is just* $\mathfrak{c}$*;*
3. *the* Katětov closure operator $\mathfrak{k} = \{\mathfrak{k}_{(X,q_X)}\}_{(X,\mathfrak{q}_X)\in\mathbf{Ob}(\mathbf{FC})}$ *on* $\mathcal{P}^f$ *where*

$$\mathfrak{k}_{(X,q_X)} : \mathcal{P}(X) \to \mathcal{P}(X) \qquad A \mapsto \{x \in X \mid \exists F \in q_X(x).A \in F\}$$

▶ **Proposition 3.5** ([14, Ch. 3])**.**

1. k*,* c *and* $\mathfrak{k}$ *are grounded and additive closure operators, moreover* k *is idempotent.*
2. *There exists a sequence of inclusion functors* $\mathbf{Top} \xrightarrow{i} \mathbf{PrTop} \xrightarrow{j} \mathbf{FC}$ *each of which has a left adjoint.*
3. *We have a sequence* $(\mathcal{P}^t, k) \xrightarrow{(i,\eta)} (\mathcal{P}^p, c) \xrightarrow{(j,\epsilon)} (\mathcal{P}^f, \mathfrak{k})$ *of morphisms in* **cEHD** *where* η *and* ϵ *have identities as components.*

Proof.

1. For k and c the proposition is obvious, let us examine $\mathfrak{k}$: since $\dot{x} \in q_X(x)$ then $A \subset \mathfrak{k}_X(A)$, if $A \subset B$ then any filters that contains the former contains the latter too and this implies monotonicity, groundedness follows from the fact that $\emptyset$ does not belong to any proper filter, for additivity we can complete any filter $\mathcal{F}$ to which $A \cup B$ belong to an ultrafilter $\mathcal{U}$ that belongs to $q_X(x)$ since the latter is upward closed, either A or B must belong to $\mathcal{U}$ and we are done.
2. i sends a topological space to the pretopological space given by the closure operator associate to its topology, j sends $(X, \mathfrak{c})$ to $(X, q_X^{\mathfrak{c}})$ where

$$q_X^{\mathfrak{c}} : X \to \mathcal{P}(\mathbf{Fil}(X)) \qquad x \mapsto \{\mathcal{F} \in \mathbf{Fil}(X) \mid \mathcal{V}_x \subset \mathcal{F}\}$$

where $\mathcal{V}_x := \{S \subset X \mid x \notin \mathfrak{c}(X \smallsetminus S)\}$. For the left adjoints see [14].
3. This is obvious. ◀

For other examples of closure operators on topological spaces we refer the reader to [14].

3.2 Algebraic examples

▶ **Proposition 3.6.** *Let* **Grp** *be the category of groups and* **CRing** *that of commutative, unital rings (where we require that $f(1_A) = 1_B$ for any $f : A \to B$). Then, $\mathcal{S}ub_{\mathbf{Grp}}$ and $\mathcal{S}ub_{\mathbf{CRing}}$ are elementary existential doctrines.*

Proof. This follows at once from Theorem 2.6. ◀

▶ Remark 3.7. Notice that, even if $\mathcal{S}ub_{\mathbf{Grp}}(G)$ and $\mathcal{S}ub_{\mathbf{CRing}}(A)$ admit finite suprema for any group G or commutative ring A with unity, preimages do not preserve them in general: for instance they do not preserve the bottom subobject. Then $\mathcal{S}ub_{\mathbf{Grp}}$ or $\mathcal{S}ub_{\mathbf{CRing}}$ cannot be universal doctrines.

The following examples are taken from [14].

▶ **Definition 3.8** (Groups). *The* normal closure *on a group G is given by*

$$\nu_G : \mathcal{S}ub_{\mathbf{Grp}}(G) \to \mathcal{S}ub_{\mathbf{Grp}}(G) \qquad H \mapsto \bigcap\{N \leq G \mid H \leq N \trianglelefteq G\}$$

where we have chosen the image of a monomorphism as a canonical representative of it.

▶ **Proposition 3.9.** *The family previous defined forms a closure operators ν on $\mathcal{S}ub_{\mathbf{Grp}}$ that is idempotent, fully additive and grounded.*

Proof. Since the preimage of a normal subgroup is normal we have that the ν actually exists as a closure operator. The three poperties of it follow immediately by the fact that $\{0\}$ is normal and so are the arbitrary intersections or sums of normal subgroups. ◀

▶ **Definition 3.10** (Rings). *Let A be a unital commutative ring and B a subring, we define $\mathsf{int}_A(B)$ to be the* integral closure of *B:*

$$\mathsf{int}_A(B) := \{a \in A \mid p(a) = 0 \text{ for some } p \in B[x]\}$$

Again we are denoting a subobject by the image of any representative of it.

▶ **Proposition 3.11.** *For any A int_A is a function $\mathcal{S}ub_{\mathbf{CRing}}(A) \to \mathcal{S}ub_{\mathbf{CRing}}(A)$, moreover the family of this functions forms an idempotent closure operator* int.

Proof. To show that $\mathsf{int}_A(B)$ is a subring of A and idempotency we refer to [2, Cor. 5.3, 5.5]. Let us show that int is actually a closure operator. Consider $f : A \to B$ and C a subring of B, let $a \in A$ such that $p(a) = 0$ for some $p \in f^{-1}(C)[X]$ with coefficients $\{p_i\}_{i=0}^{\deg(p)}$, then $q(f(a)) = 0$ where $q \in C[X]$ has coefficients $\{f(p_i)\}_{i=0}^{\deg(p)}$ and we are done. ◀

3.3 Contact algebras

▶ **Definition 3.12** ([15, 16]). *A* contact algebra *is an Heyting algebra H equipped with a symmetric binary relation C such that*

- *if xCy then x and y are different from $\bot$;*
- *if $x \neq \bot$ then xCx;*
- *if $xC(y \vee z)$ if and only if xCy or xCz.*

(H, C) is complete *if H is so. A morphism $f : (H, C) \to (K, D)$ is a morphism of Heyting algebras $f : H \to K$ such that $(f \times f)(C) \subset D$.* **CA** *denotes the category of contact algebras and* **CCA** *its subcategory of complete contact algebras and morphisms preserving all suprema.*

For a complete contact algebra (H, C) and $x \in H$, we define C_x to be the set $\{y \in H \mid xCy\}$ and the contact closure *on (H, C) as*

$$\mathfrak{c}_{(H,C)} : H \to H \qquad x \mapsto x \vee \sup(C_x)$$

▶ Remark 3.13. Clearly the third condition of the definition of contact algebra can be rephrased as $C_{x \vee y} = C_x \cup C_y$.

▶ Remark 3.14. Let $f : (H, C) \to (K, D)$ be an arrow of **CCA**, by the adjoint functor theorem ([7]) it has a right adjoint f^*. If we regard H and K as meet-semilattices then $f^* : K \to K$, being a right adjoint, is an arrow of **InfSL**.

▶ **Proposition 3.15.** *$\mathfrak{c}$ is a closure operator on the doctrine $\mathcal{U} : \mathbf{CCA}^{op} \to \mathbf{InfSL}$ sending*

$$\begin{array}{ccc} (H,C) & \longmapsto & H \\ f \downarrow & & \uparrow f^* \\ (K,D) & \longmapsto & K \end{array}$$

Proof. Let (H, C) be a contact algebra and $x, y \in H$. Clearly $x \leq \mathfrak{c}_{(H,C)}(x)$, if $x \leq y$ then $y = x \vee y$ and, by the previous remark $C_x \leq C_y$ so that $\mathfrak{c}_{(H,C)}(x) \leq \mathfrak{c}_{(H,C)}(y)$. Let $f : (H, C) \to (K, D)$ be an arrow of **CCA**, then, for every $x \in H$, $f(\sup(C_x)) = \sup(f(C_x))$. Now, if $y \in f(C_x)$ then $y = f(z)$ for some z such that zCx, so $yDf(x)$ and thus $f(C_x) \subset D_{f(x)}$, we can conclude that $f(\sup(C_x)) \leq \sup(D_{f(x)})$ from which

$$f(\mathfrak{c}_{(H,C)}(x)) \leq \mathfrak{c}_{(K,D)}(f(x))$$

and we can conclude by Proposition 2.15. ◀

3.4 A representable example

▶ **Theorem 3.16.** *For any complete Heyting algebra H, the functor $\mathbf{Set}(-, H) : \mathbf{Set}^{op} \to \mathbf{HA}$ is an elementary hyperdoctrine.*

Proof. See, for instance, [39, Section 2.2]. ◀

▶ **Corollary 3.17.** $\mathbf{Set}(-, [0,1]) : \mathbf{Set}^{op} \to \mathbf{HA}$ *is an elementary hyperdoctrine on* **Set**.

▶ **Definition 3.18.** *For any fixed $\epsilon \in [0, 1]$, and any set X, we define, for an $f : X \to [0, 1]$:*

$$\mathfrak{c}_{X,\epsilon}(f) : X \to [0,1] \qquad x \mapsto f(x) \dot{+} \epsilon$$

where

$$\dot{+} : [0,1] \times [0,1] \to [0,1] \qquad (t, s) \mapsto \max(t + s, 1)$$

In this way we get a function

$$\mathfrak{c}_{X,\epsilon} : \mathbf{Set}(X, [0,1]) \to \mathbf{Set}(X, [0,1]) \qquad f \mapsto \mathfrak{c}_{X,\epsilon}(f)$$

▶ **Proposition 3.19.** *For any* $\epsilon \geq 0$*, the collection* $\mathfrak{c}_\epsilon$ *of all the functions* $\mathfrak{c}_{X,\epsilon}$ *is a closure operator.*

Proof. Clearly $f \leq \mathfrak{c}_{X,\epsilon}(f)$ for any $f : X \to [0,1]$, monotonicity is clear, let's check continuity of any function $g : X \to Y$:

$$\mathfrak{c}_{X,\epsilon}(f \circ g)(x) = (f \circ g)(x) \dot{+} \epsilon = f(g(x)) \dot{+} \epsilon = \mathfrak{c}_{x,\epsilon}(f)(g(x)) = (\mathfrak{c}_{x,\epsilon}(f) \circ g)(x)$$ ◀

▶ Remark 3.20. $\mathfrak{c}_\epsilon$ is not grounded if $\epsilon \neq 0$ (in that case it reduces to the discrete closure operator) but it is additive.

3.5 Fuzzy sets

We can refine the previous example considering *fuzzy sets*.

▶ **Definition 3.21.** *[41, 42] The category* **Fzs** of fuzzy sets *has:*
- *pairs* (A, α) *with* $\alpha : A \to [0,1]$ *as objects;*
- *as arrows* $f : (A, \alpha) \to (B, \beta)$ *functions* $f : A \to B$ *such that* $\alpha(x) \leq \beta(f(x))$*.*

▶ **Definition 3.22.** *A* fuzzy subset *of* (A, α) *is a function* $\xi : A \to [0,1]$ *such that* $\xi(x) \leq \alpha(x)$ *for all* $x \in A$*.*
Let us summarize some results about **Fzs**.

▶ **Proposition 3.23.** **1.** **Fzs** *is a quasitopos;*
2. *there exists a proper and stable factorization system given by strong monomorphisms and epimorphisms;*
3. *fuzzy subsets of* (A, α) *correspond to equivalence of strong monomorphisms of codomain* (A, α)*;*
4. *the functor* $\mathbf{Fzs}^{op} \to \mathbf{HA}$ *assigning to each* (A, α) *the set of its fuzzy subsets and* $f : (A, \alpha) \to (B, \beta)$ *to the function* f^* *defined by:*

$$f^*(\xi) : A \to [0,1] \qquad x \mapsto \alpha(x) \wedge \xi(f(x))$$

is an elementary hyperdoctrine.

Proof. See [41, Ch. 8]. Explicitly the hyperdoctrine structure is given by:

$$\exists_f(\xi) : B \to [0,1] \qquad\qquad \forall_f(\xi) : B \to [0,1]$$
$$y \mapsto \bigvee_{x \in f^{-1}(y)} \xi(x) \qquad\qquad y \mapsto \beta(y) \wedge \bigwedge_{x \in f^{-1}(y)} (\alpha(x) \Rightarrow \xi(x))$$

for any $f : (A, \alpha) \to (B, \beta)$ and $\xi \in \mathcal{FzSub}(A, \alpha)$. ◀

▶ **Proposition 3.24.** *Let* $\mathcal{E} = \{\epsilon_{(A,\alpha)}\}_{(A,\alpha) \in \mathbf{Ob(Fzs)}}$ *be a family of functions* $\epsilon_{(A,\alpha)} : (A, \alpha) \to [0,1]$ *such that* $\epsilon_{(A,\alpha)}(x) \leq \epsilon_{(B,\beta)}(f(x))$ *for any* $f : (A, \alpha) \to (B, \beta)$*. Then*

$$\mathfrak{c}^{\mathcal{E}}_{(A,\alpha)} : \mathcal{FzSub}(A, \alpha) \to \mathcal{FzSub}(A, \alpha) \qquad \xi \mapsto (\xi + \epsilon_{(A,\alpha)}) \wedge \alpha$$

gives us an additive closure operator on $\mathcal{FzSub}$*.*

Proof. See proof on page 20. ◀

▶ Remark 3.25. $\mathfrak{c}^{\mathcal{E}}$ is not grounded in general.

The condition on the elements of $\mathcal{E}$ is very restrictive. In fact, it can be eased restricting to a suitable subclass of arrows and using the following lemma.

▶ **Lemma 3.26.** *Let $\mathcal{P} : \mathbf{C}^{op} \to \mathbf{InfSL}$ be a doctrine, and $\mathfrak{c} = \{\mathfrak{c}_C : \mathcal{P}(C) \to \mathcal{P}(C)\}_{C \in \mathbf{Ob(C)}}$ be a family of monotone and inflationary operators. Let $\mathcal{A}$ be a (possibly large) family of $\mathbf{C}$-arrows such that:*

- *$\mathcal{A}$ is closed under composition;*
- *if $f \in \mathcal{A}$ then $1_{\mathrm{dom}(A)}$ and $1_{\mathrm{cod}(A)}$ are in $\mathcal{A}$;*
- *$f : C \to D$ in $\mathcal{A}$ implies $\mathfrak{c}_C \circ \mathcal{P}_f \leq \mathcal{P}_f \circ \mathfrak{c}_D$.*

Then $\mathcal{P}$ induces a doctrine $\mathcal{P}^{\mathcal{A}}$ on the subcategory $\mathbf{C}_{\mathcal{A}}$ induced by $\mathcal{A}$ for which $\mathfrak{c} = \{\mathfrak{c}_C\}_{C \in \mathbf{Ob}(\mathbf{C}_{\mathcal{A}})}$ is a closure operator. Moreover, if for all f, g in $\mathcal{A}$ also (f, g) and the projections from $\mathrm{cod}(f) \times \mathrm{cod}(g)$ are in $\mathcal{A}$, then $\mathcal{P}^{\mathcal{A}}$ is existential, elementary or an hyper-doctrine if $\mathcal{P}$ is.

Proof. This is almost tautological since the condition on $\mathcal{A}$ guarantee that the inclusion functor $\mathbf{C}_{\mathcal{A}}$ preserves limits and we can use Proposition 2.7. ◀

3.6 Coalgebraic examples

▶ **Definition 3.27** ([24, 29])**.** *Let $\mathbf{C}$ be a category with finite products and $\mathcal{F} : \mathbf{C} \to \mathbf{C}$ an endofunctor. The category $\mathbf{CoAlg}(\mathcal{F})$ of* coalgebras for *$\mathcal{F}$ has*

- *arrows $\gamma_C : C \to \mathcal{F}(C)$ as objects;*
- *arrows $f : C \to D$ such that $\gamma_D \circ f = \mathcal{F}(f) \circ \gamma_C$ as morphisms $f : \gamma_C \to \gamma_D$.*

Notice that in general $\mathbf{CoAlg}(\mathcal{F})$ is not complete and products in it can be very different from products in $\mathbf{C}$ [21], so it does not make much sense to look for an existential doctrine on it. However, for $\mathbf{Set}$-based coalgebras we get a primary doctrine $\mathcal{P}^c : \mathbf{CoAlg}(\mathcal{F})^{op} \to \mathbf{InfSL}$ composing the contravariant power object $\mathcal{P} : \mathbf{Set}^{op} \to \mathbf{InfSL}$ with the opposite of the obvious forgetful functor $\mathbf{CoAlg}(\mathcal{F}) \to \mathbf{Set}$.

▶ **Definition 3.28.** *Let $\mathcal{F} : \mathbf{C} \to \mathbf{C}$ be a functor and $\mathcal{P}$ a primary doctrine on $\mathbf{C}$. A* predicate lifting *is a natural transformation $\Box : \mathcal{U} \circ \mathcal{P} \to \mathcal{U} \circ \mathcal{P} \circ \mathcal{F}^{op}$ where $\mathcal{U}$ is the forgetful functor $\mathbf{InfSL} \to \mathbf{Poset}$.*

▶ Remark 3.29. A similar notion can be found in [25]. In particular, the predicate liftings of Examples 3.32 and 3.34 below fit Jacobs and Sokolova's framework.

▶ **Definition 3.30.** *For any predicate lifting $\Box$, we define two closure operators on $\mathcal{P}^c$.*

1. *For any coalgebra $\gamma_X : X \to \mathcal{F}(X)$, notice that $\mathcal{P}^c(\gamma_X) = \mathcal{P}(X)$; hence we can define*

$$\mathsf{pre}_{\gamma_X} : \mathcal{P}(X) \to \mathcal{P}(X) \qquad \alpha \mapsto \alpha \vee \mathcal{P}_{\gamma_X}(\Box_X(\alpha))$$

2. *Suppose that $\mathcal{P}$ admits arbitrary meets; for $\gamma_X : X \to \mathcal{F}(X)$ and $\alpha \in \mathcal{P}(X)$ we define*

$$\mathcal{N}_{\gamma_X}(\alpha) := \{\beta \in \mathcal{P}(X) \mid \alpha \leq \mathcal{P}_{\gamma_X}(\Box_X(\beta))\} \quad \mathsf{s}_{\gamma_X}(\alpha) := \inf(\mathcal{N}_{\gamma_X}(\alpha))$$

   and

$$\mathsf{suc}_{\gamma_X} : \mathcal{P}(X) \to \mathcal{P}(X) \qquad \alpha \mapsto \alpha \vee \mathsf{s}_{\gamma_X}(\alpha)$$

▶ **Lemma 3.31.** *Let $\mathcal{F} : \mathbf{C} \to \mathbf{C}$ be a functor and $\Box$ a predicate lifting, then:*

1. *$\{\mathsf{pre}_{\gamma_X}\}_{\gamma_X \in \mathbf{Ob}(\mathbf{CoAlg}(\mathcal{F}))}$ defines a closure operator* pre *on $\mathcal{P}^c$.*
2. *$\mathsf{s}_{\gamma_X}(\alpha)$ is the minimum of $\mathcal{N}_{\gamma_X}(\alpha)$ whenever $\mathcal{P}$ has arbitrary meets and, for any coalgebra $\gamma_X : X \to \mathcal{F}(X)$, $\mathcal{P}_{\gamma_X}$ and $\Box_X$ commute with them;*
3. *in the hypothesis above if $\mathcal{P}_f$ commutes with arbitrary infima for all arrows f then $\{\mathsf{suc}_{\gamma_X}\}_{\gamma_X \in \mathbf{Ob}(\mathbf{CoAlg}(\mathcal{F}))}$ defines a closure operators* suc *on $\mathcal{P}^c$.*

Proof. See proof on page 21. ◀

The previous result provides us with many examples with practical applications.

▶ **Example 3.32.** Let $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ be the covariant powerset functor, and $\mathscr{P} : \mathbf{Set}^{op} \to \mathbf{InfSL}$ be the controvariant one, seen as primary doctrine. We can define a predicate lifting $\Box$ taking as components:

$$\Box_X : \mathscr{P}(X) \to \mathscr{P}(\mathcal{P}(X)) \qquad A \mapsto {\downarrow}A$$

where ${\downarrow}A$ denotes the set of downward-closed subsets of A. In this case for any coalgebra $\gamma_X : X \to \mathcal{P}(X)$ we have

$$x \in \gamma_X^{-1}(\Box_X(A)) \iff \gamma_X(x) \subset A$$
$$B \in \mathcal{N}_{\gamma_X}(A) \iff \gamma_X(a) \subset B \text{ for any } a \in A$$

so $\mathsf{s}_{\gamma_X}(A) = \bigcup_{a\in A} \gamma_X(a)$ and $\mathsf{suc}_{\gamma_X}(A) = A \cup \bigcup_{a\in A} \gamma_X(a)$.

By this description it is clear that suc is grounded and fully additive. pre is grounded too but it is not even finitely additive: take $4 := \{0, 1, 2, 3\}$ with stuctural map γ_4 given by

$$0 \mapsto \{3\} \quad 1 \mapsto \{2, 3\} \quad 2 \mapsto \{2\} \quad 3 \mapsto \{3\}$$

Now take $A := \{2, 3\}$, it is immediate to see that $\mathsf{pre}_{\gamma_4}(A) = 4$, on the other hand $\mathsf{pre}_{\gamma_4}(\{2\}) = \{2\}$ and $\mathsf{pre}_{\gamma_4}(\{3\}) = \{0, 3\}$.

▶ Remark 3.33. In this case, pre and suc meanings (and notation) become clearer: if we think to the value of $\gamma_X(x)$ as the family of points accessible from $x \in X$ then pre_{γ_X} adds to a subset A the set of its *predecessors*, i.e. points from which some $a \in A$ is accessible, while suc_{γ_X} adds the set of *successors*, i.e. points which are accessible from some point of A.

▶ **Example 3.34** (Probabilistic frames [3, 4, 19])**.** Let $\mathbf{Meas}$ be the category of measurable space and measurable functions; then we can take as primary doctrine $\mathscr{P}$ the functor

$$\begin{array}{ccc} (X, \Omega_X) & \longmapsto & \Omega_X \\ f \big\downarrow & & \big\downarrow f^{-1} \\ (Y, \Omega_Y) & \longmapsto & \Omega_Y \end{array}$$

As endofunctor we can take the *Giry monad* $\mathcal{G} : \mathbf{Meas} \to \mathbf{Meas}$:

- given an object (X, Ω_X), $\mathcal{G}(X, \Omega_X)$ is the set of all probability measures on Ω_X equipped with the smallest σ-algebra for which all the *evaluation functions*

  $$\mathsf{ev}_A : \mathcal{G}(X, \Omega_X) \to [0, 1] \qquad \mu \mapsto \mu(A)$$

  with $A \in \Omega_X$, are Borel-measureable.
- for a measurable $f : (X, \Omega_X) \to (Y, \Omega_Y)$ we can define

  $$\mathcal{G}(f) : \mathcal{G}(X, \Omega_X) \to \mathcal{G}(Y, \Omega_Y) \qquad \mu \mapsto \mu \circ f^{-1}$$

Given a coalgebra $\gamma_{(X,\Omega_X)}$ and $p \in [0, 1]$ we can now define

$$\Box_{(X,\Omega_X),p} : \Omega_X \to \mathcal{P}(\mathcal{G}(X)) \qquad A \mapsto \{\mu \in \mathcal{G}(X, \Omega_X) \mid \mu(A) \geq p\}$$

Notice that the set on the right is $\mathsf{ev}_A^{-1}([p, 1])$ and so $\Box_{(X,\Omega_X),p}$ is well defined. In this situation we have

$$\mathsf{pre}_{\gamma_{(X,\Omega_X)}}(A) := A \cup \{x \in X \mid p \leq \gamma_{(X,\Omega_X)}(x)(A)\}$$

▶ Remark 3.35. If we think of a coalgebra $\gamma_{(X,\Omega_X)}$ as describing how likely is a transition from a state to the various $A \in \Omega_X$ then, given a $p \in [0, 1]$, $\mathsf{pre}_{\gamma_{(X,\Omega_X)}}(A)$ is the set of points which access A with probability at least p.

4 Logics for Closure Operators

In this section, we provide a sound and complete logic for closure hyperdoctrines. This logic is a (first order) version of Spatial Logic for Closure Spaces (SLCS) [11], although with a slightly different presentation.

4.1 Syntax and derivation rules

We briefly recall the categorical presentation of signatures, as in [23].

▶ **Definition 4.1.** *A* signature Σ *is a triple* $(|\Sigma|, \Gamma, \Pi)$ *where*

- $|\Sigma|$ *is a set, called the set of* basic types*;*
- Γ *is a functor*[2] $|\Sigma|^\star \times |\Sigma| \to \mathbf{Sets}$*. We will call* function symbol *an element* f *of* $\Gamma((\sigma_1, \ldots, \sigma_n), \sigma_{n+1})$ *and we will write* $f : \sigma, \ldots, \sigma_n \to, \sigma_{n+1}$*;*
- Π *is a functor* $|\Sigma|^\star \to \mathbf{Set}$*, we will call* predicate symbol *an element* P *of* $\Pi(\sigma_1, \ldots, \sigma_n)$ *and we will write* $P : \sigma_1, \ldots, \sigma_n$*.*

A morphism of signatures $\phi : \Sigma_1 \to \Sigma_2$ *is a triple* (ϕ_1, ϕ_2, ϕ_3) *such that*

- ϕ_1 *is a function* $|\Sigma_1| \to |\Sigma_2|$*;*
- ϕ_2 *is a natural transformation* $\Gamma_1 \to \Gamma_2 \circ (\phi_1^\star \times \phi_1)$*;*
- ϕ_3 *is a natural transformation* $\Pi_1 \to \Pi_2 \circ \phi_1^\star$*.*

For any $\sigma \in |\Sigma|$ *we fix an countably infinite set* X_σ *of* variables*; definition of terms is straightforward ([23]).*

▶ **Definition 4.2.** *Given a signature* Σ*, its* classifying category $\mathbf{Cl}(\Sigma)$ *is such that*

- *objects are contexts;*
- *Given* $\Gamma := [x_i : \sigma_i]_{i=1}^n$ *and* $\Delta = [y_i : \tau_i]_{i=1}^m$ *an arrow* $\Gamma \to \Delta$ *is a* m*-uple of terms* $(T_1, ..., T_m)$ *such that* $\Gamma \vdash T_i : \tau_i$ *for any* i*;*
- *composition is given by substitution.*

▶ **Proposition 4.3.** $\mathbf{Cl}(\Sigma)$ *is a category with finite products for any signature* Σ*.*

Proof. Associativity of composition and the fact that $(x_1, ..., x_n)$ is the identity for $[x_i : \sigma_i]_{i=1}^n$ follows from a straightforward computation. The empty context is clearly terminal while, given two contexts $\Gamma := [x_i : \sigma_i]_{i=1}^n$ and $\Delta = [y_i : \tau_i]_{i=1}^m$ we can take their concatenation as a product $\Gamma \times \Delta$, the universal property follows immediately. ◀

Now we can introduce the rules for context and closure operators of the Spatial Logic for Closure Spaces, over any given signature.

As usual, we denote by $\Gamma \vdash t : \tau$ the judgment "t has type τ in context Γ", and by $\Gamma \vdash \phi : \mathsf{Prop}$ the judgment "ϕ is a well-formed formula in context Γ".

▶ **Definition 4.4.** *The rules for contexts and well-formed formulae for the closure operators for a signature* Σ *are the usual ones for a first order signature (see [23]) plus:*

$$\frac{\Gamma \vdash \phi : \mathsf{Prop}}{\Gamma \vdash \mathcal{C}(\phi) : \mathsf{Prop}}\ \mathcal{C}\text{-}F \qquad \frac{\Gamma \vdash \phi : \mathsf{Prop} \qquad \Gamma \vdash \psi : \mathsf{Prop}}{\Gamma \vdash \phi\,\mathcal{U}\,\psi : \mathsf{Prop}}\ \mathcal{U}\text{-}F$$

For any context Γ *we define* $\mathbf{Form}_\Sigma(\Gamma)$ *to be the set of formulae* ϕ *such that* $\Gamma \vdash \phi : \mathsf{Prop}$.

[2] $|\Sigma|$ and $|\Sigma|^\star$ are viewed here as discrete categories.

Then, we can introduce the rules for the logical judgments of the form $\Gamma \mid \Phi \vdash \phi$, where Φ is a finite set of propositions well-formed in Γ.

▶ **Definition 4.5.** *We define four rules for the well-formed formulae previously defined:*

- *$\mathcal{C}$'s rules:*

$$\frac{\Gamma \mid \Phi \vdash \psi}{\Gamma \mid \Phi \vdash \mathcal{C}(\psi)}\ \textsc{Cl}\text{-}1 \qquad \frac{\Gamma \mid \Phi, \psi \vdash \phi}{\Gamma \mid \Phi, \mathcal{C}(\psi) \vdash \mathcal{C}(\phi)}\ \textsc{Cl}\text{-}2$$

- *$\mathcal{U}$'s rules*

$$\frac{\Gamma \mid \Phi, \varphi \vdash \phi \qquad \Gamma \mid \Phi, \mathcal{C}(\varphi), \neg\phi \vdash \psi}{\Gamma \mid \Phi, \varphi \vdash \phi\mathcal{U}\psi}\ \mathcal{U}\text{-}I \qquad \frac{\{\Gamma \mid \Phi, \varphi \vdash \theta \mid \varphi \in \mathsf{u}_{(\Gamma,\Phi)}(\phi,\psi)\}}{\Gamma \mid \Phi, \phi\mathcal{U}\psi \vdash \theta}\ \mathcal{U}\text{-}E$$

where

$$\mathsf{u}_{(\Gamma,\Phi)}(\phi,\psi) := \{\varphi \text{ such that } \Gamma \vdash \varphi : \mathsf{Prop}, \Gamma \mid \Phi, \varphi \vdash \phi, \Gamma \mid \Phi, \mathcal{C}(\varphi), \neg\varphi \vdash \psi\}$$

The Propositional Logic for Closure Operators on Σ *(PLCO) is given by the usual propositional rules (i.e., without the quantifiers) for the typed (intuitionistic) sequent calculus (see e.g. [23]), extended with the four rules above.*

The First Order Logic for Closure Operators on Σ *(FOLCO) is given by the four rules above added to the usual rules for first order logic. Similarily with equality.*

Derivability of sequents is defined in the usual way [38].

▶ Remark 4.6. PLCO corresponds to the Spatial Logic for Closure Spaces considered in [10].

▶ Remark 4.7. Rules $\mathcal{U}$-I and $\mathcal{U}$-E come from the intended meaning of $\phi\mathcal{U}\psi$. In fact, this formula must be interpreted as the "largest region" for which there is no escape from ϕ without passing through ψ.

▶ Remark 4.8. Notice that $\mathcal{U}$-E is an *infinitary* rule saying that a formula θ can be derived from $\phi\mathcal{U}\psi$ if it can be derived from *all* the formulae φ satisfying precise conditions. Thus, this rule shows the second-order nature of the $\mathcal{U}$ operator.

4.2 Categorical semantics of closure logics

In this section we provide a sound and complete categorical semantics of the logics for the closure operators defined above.

▶ **Definition 4.9.** *Two formulae $\phi, \psi \in \mathbf{Form}_\Sigma(\Gamma)$ are* provably equivalent *if $\Gamma \mid \psi \vdash \phi$ and $\Gamma \mid \phi \vdash \psi$. We will denote the quotient of $\mathbf{Form}_\Sigma(\Gamma)$ by this relation with $\mathcal{L}(\Sigma)(\Gamma)$, $[\phi]$ will denote the class of ϕ in it.*

▶ **Proposition 4.10.** *For any signature Σ the following are true:*

1. *$\mathcal{L}(\Sigma)(\Gamma)$ equipped with the order $[\phi] \leq [\psi]$ if and only if $\Gamma \mid \phi \vdash \psi$ is derivable is:*
   - *a meet semilattice in the case we are considering regular logic;*
   - *a Heyting algebra if we are considering propositional or first order logic;*
2. *$[\phi\mathcal{U}\psi]$ is the supremum of the set*

$$\mathsf{u}_\Gamma(\phi,\psi) := \{[\varphi] \in \mathcal{L}(\Sigma)(\Gamma) \text{ such that } \Gamma \mid \varphi \vdash \phi, \Gamma \mid \mathcal{C}(\varphi), \neg\varphi \vdash \psi\}$$

3. *there exists a (elementary) closure or existential doctrine or a (elementary) hyperdoctrine $(\mathcal{L}(\Sigma), \mathfrak{c}_\Sigma)$ on $\mathbf{Cl}(\Sigma)$ sending Γ to $\mathcal{L}(\Sigma)(\Gamma)$.*

Proof.

1. The logical connectives induce a Heyting algebra or a meet semilattice structure on $\mathcal{L}(\Sigma)(\Gamma)$ which has precisely $\leq$ as associated order.
2. From $\mathcal{U}$-I follows that $[\phi\mathcal{U}\psi]$ is an upper bound for u_Γ while $\mathcal{U}$-E implies that $[\phi\mathcal{U}\psi]$ is the least of them.
3. For any morphism $(T_1, ..., T_n) : \Gamma \to \Delta$ substitution of terms gives us a morphism of Heyting algebras/meet semilattices $\mathcal{L}(\Sigma)(\Delta) \to \mathcal{L}(\Sigma)(\Gamma)$; quantifiers gives us the existential doctrine/hyperdoctrine structure (cfr. [38] for the details). In any case have to define a preclosure operator $\mathfrak{c}_{\Sigma,\Gamma}$ on each $\mathcal{L}(\Sigma)(\Gamma)$ but this is easily done defining

$$\mathfrak{c}_{\Sigma,\Gamma} : \mathcal{L}(\Sigma)(\Gamma) \to \mathcal{L}(\Sigma)(\Gamma) \qquad [\phi] \mapsto [\mathcal{C}(\phi)]$$

The $\mathcal{C}$'s rules assure us that $\mathfrak{c}_\Sigma$ is well defined, inflationary and monotone, while an easy induction shows that $\mathcal{L}(\Sigma)_{(T_1,...,T_n)}([\mathcal{C}(\phi)]) = \mathfrak{c}_{\Sigma,\Gamma}(\mathcal{L}(\Sigma)_{(T_1,...,T_n)}(\phi))$ for any $(T_1, ..., T_n) : \Gamma \to \Delta$. We can add fibered equalities, given $\Gamma := [x_i : \sigma_i]$ putting:

$$\delta_{\Gamma\times\Gamma} := \bigwedge_{i=1}^{n} [x_i =_{\sigma_i} y_i]$$

where $\{y_i\}_{i=1}^n$ is a set of fresh variables such that $y_i : \sigma_i$ for any i. ◀

Let us prove the soundness and completeness of the categorical semantics wrt. the various logical fragments.

▶ **Definition 4.11.** *Let* $(\mathcal{P}, \mathfrak{c}) : \mathbf{C}^{op} \to \mathbf{InfSL}$ *be an (elementary) closure doctrine (existential doctrine/hyperdoctrine) then a morphism of* **cPD** *(***cED***,* **cEED***,* **cEHD***,* **cHD***)* $(\mathcal{M}, \mu) : (\mathcal{L}(\Sigma), \mathcal{C}) \to (\mathcal{P}\mathfrak{c})$ *is a* model of the propositional (first-order) logic (with equality) of closure operators in $(\mathcal{P}, \mathfrak{c})$ *if it is open.*

A sequent $\Gamma \mid \Phi \vdash \psi$ *is* satisfied by $(\mathcal{M}, \mu)$ *if*

$$\bigwedge_{\phi\in\Phi} \mu_\Gamma(\phi) \leq \mu_\Gamma(\psi)$$

▶ **Theorem 4.12.** *A sequent* $\Gamma \mid \Phi \vdash \psi$ *is satisfied by the* generic model $(1_{\mathbf{Cl}(\Sigma)}, 1_{\mathcal{L}(\Sigma)})$ *if and only if it is derivable.*

Proof. By definition, $\Gamma \mid \Phi \vdash \psi$ is satisfied if and only if $\bigwedge_{\phi\in\Phi}[\phi] \leq [\psi]$ in $\mathcal{L}(\Sigma)(\Gamma)$, but this is equivalent to the derivability of $\Gamma \mid \bigwedge_{\phi\in\Phi} \phi \vdash \psi$ which in turn is equivalent (applying the conjunction rules a finite number of times) to the derivability of $\Gamma \mid \Phi \vdash \psi$ and we are done. ◀

▶ **Corollary 4.13.** *The above defined categorical semantics for PLCO or FOLCO (with or without equality) is sound and complete.*

Proof. The only thing left to show is soundness for an arbitrary $(\mathcal{P}, \mathfrak{c})$ but this follows at once since each component μ_Γ of μ is monotone. ◀

4.3 Approximating $\mathcal{U}$ in continuous models

As we have remarked before, the rule $\mathcal{U}$-E for the operator $\mathcal{U}$ is infinitary. Although in general this is needed, in this section we will define a class of hyperdoctrines in which the semantics of $\mathcal{U}$ can be given as a supremum of approximants.

▶ **Definition 4.14.** *Let* $(\mathcal{P}, \mathfrak{c}) : \mathbf{C}^{op} \to \mathbf{InfSL}$ *be a closure doctrine that factors through the category of Heyting algebras. For any object* C *define the* external boundary*:*

$$\partial^+_C : \mathcal{P}(C) \to \mathcal{P}(C) \qquad \alpha \mapsto \mathfrak{c}_C(\alpha) \wedge \neg\alpha$$

For ϕ *and* $\psi \in \mathcal{P}(C)$*, we define* $\phi\mathfrak{U}_C\psi \in \mathcal{P}(C)$ *as the supremum, if it exists, of the set*

$$\mathfrak{u}_C(\phi, \psi) := \{\varphi \in \mathcal{P}(C) \mid \varphi \leq \phi \text{ and } \partial^+_C(\varphi) \leq \psi\}$$

▶ Remark 4.15. If $\mathcal{P}$ is $\mathcal{L}(\Sigma)$ then $[\phi]\mathfrak{U}_\Gamma[\psi] = [\phi\mathcal{U}\psi]$ for any $[\phi]$ and $[\psi] \in \mathcal{L}(\Sigma)(\Gamma)$.

▶ Remark 4.16. If $(\mathcal{M}, \mu)$ is a model then $\mu_\Gamma(\mathsf{u}_\Gamma(\phi, \psi)) \subset \mathfrak{u}_{\mathcal{M}(\Gamma)}(\mu_\Gamma([\phi]), \mu_\Gamma([\psi]))$ for any Γ.

▶ **Example 4.17.** Let $(X, \mathfrak{c})$ be a pretopological space and $S, T \in \mathcal{P}^p(X, \mathfrak{c})$, then

$$S\mathfrak{U}_{(X,\mathfrak{c})}T = \bigcup\{W \subset S \mid \partial^+_{(X,\mathfrak{c})}(W) \subset T\}$$

i.e. $x \in S\mathfrak{U}_{(X,\mathfrak{c})}T$ if and only if there exists $W \subset S$ such that $X \in W$ and $\partial^+_{(X,\mathfrak{c})}(W) \subset T$.

▶ **Example 4.18.** Let us consider the closure operator $\mathfrak{c}_\epsilon$ on $\mathbf{Set}(-, [0, 1])$ (see Section 3.4). For any $f : X \to [0, 1]$, it is $(\neg f)(x) = 1$ if and only if $f(x) = 0$. So,

$$(\mathfrak{c}_{X,\epsilon}(f) \wedge \neg f)(x) = \begin{cases} \epsilon & f(x) = 0 \\ 0 & f(x) \neq 0 \end{cases},$$

hence, given $g, h : X \to [0, 1]$, $f \in \mathsf{u}_\Gamma(g, h)$ if and only if $f \leq g$ and $h(x) \geq \epsilon$ for any $x \in f^{-1}(0)$.

▶ Remark 4.19. If $(\mathcal{M}, \mu)$ is a model then for any $[\varphi] \in \mathcal{L}(\Sigma)(\Gamma)$ such that $\varphi \in \mathsf{u}_\Gamma(\phi, \psi)$ we have $\mu_\Gamma([\varphi]) \leq \mu_\Gamma([\phi\mathcal{U}\psi])$.

▶ **Definition 4.20.** *Let* $(\mathcal{P}, \mathfrak{c})$ *be as in Definition 4.14. A model* $(\mathcal{M}, \mu) : \mathcal{L}(\Sigma) \to (\mathcal{P}, \mathfrak{c})$ *is said* continuous *if the equality*

$$\mu_\Gamma([\phi\mathcal{U}\psi]) = \mu_\Gamma([\phi])\mathfrak{U}_{\mathcal{M}(\Gamma)}\mu_\Gamma([\psi])$$

holds for any context Γ *and* $[\phi], [\psi] \in \mathcal{L}(\Sigma)(\Gamma)$.

▶ **Proposition 4.21.** *Let* Σ *be a signature and* $(\mathcal{P}, \mathfrak{c})$ *a complete (elementary, existential, or hyper)doctrine, i.e.* $\mathcal{P}(C)$ *is complete for any object* C *of* $\mathbf{C}$*; then, for any product preserving functor:* $\mathcal{M} : \mathbf{Cl}(\Sigma) \to \mathbf{C}$ *and functions*

$$\mu^*_\Gamma : \Pi(\sigma_1, ..., \sigma_n) \to \mathcal{P}(\mathcal{M}(\Gamma))$$

for all $\Gamma = [x_i : \sigma_i]^n_{i=1}$*, there exists a unique continuous model* $(\mathcal{M}, \mu)$ *in* $(\mathcal{P}, \mathfrak{c})$ *such that*

$$\mu_\Gamma([P(x_1, ..., x_n)]) = \mu^*_\Gamma(P)$$

Proof. By induction over n. ◀

▶ **Example 4.22.** Let $\mathcal{X} = \{(X_i, \mathfrak{c}_i)\}_{i\in I}$ be a small family of pretopological spaces and let us define Σ as follows:

$$|\Sigma| := \mathcal{X} \qquad \Gamma(((X_{i_1}, c_{i_1}), ..., (X_{i_n}, c_{i_n})), (X_j, c_j)) := \mathbf{PrTop}(\prod_{k=1}^{n}(X_{i_k}, c_{i_k}), (X_j, c_j))$$

$$\Pi((X_{i_1}, c_{i_1}), ..., (X_{i_n}, c_{i_n})) := \mathcal{P}(\prod_{k=1}^{n} X_{i_k})$$

We can take as $\mathcal{M}$ the unique product preserving functor $\mathbf{Cl}(\Sigma) \to \mathbf{PrTop}$ sending contexts to products and lists of terms to the corresponding product arrow. We can define μ^* sending each predicate $P : (X_{i_1}, \mathfrak{c}_{i_1}), ..., (X_{i_n}, \mathfrak{c}_{i_n})$ to corresponding subset of $\prod_{k=1}^{n} (X_{i_k}, \mathfrak{c}_{i_k})$. Example 4.17 guarantees that this semantics is the same as the one developed in [10].

▶ **Proposition 4.23.** *For any signature Σ a sequent is derivable if and only if it is satisfied by any continuous model.*

Proof. This follows from the fact that the generic model is continuous. ◀

5 Conclusions and future work

In this paper we have introduced *closure (hyper)doctrines* as a theoretical framework for studying the logical aspects of closure spaces. First we have shown the generality of this notion with a range of examples arising naturally from topological spaces, fuzzy sets, algebraic structures, coalgebras, and covering at once also known cases such as Kripke frames and probabilistic frames. Then, we have applied this framework to provide axiomatisations and sound and complete categorical semantics for various fragments of a logic for closure doctrines. In particular, the propositional fragment corresponds to the Spatial Logic for Closure Spaces [10], a modal logic for the specification and verification on spatial properties over preclosure spaces. But the flexibility of our approach allows us to readily obtain closure logics for a wide range of cases (including all the examples presented above). A possible extension is given by *closure triposes* [39]. Triposes are the categorical setting for higher order logic, so these would provide the categorical setting for higher order logic for closure spaces.

Albeit already quite general, the theory presented in this paper paves the way for several extensions. Due to lack of space, we have not been able to present the constructions for modeling logical operators concerning *surroundedness*. To this end, we need to endow doctrines with an object representing the "type of paths"; for more details we refer to the extended version of this work [9].

We can enrich the logic with other spatial modalities, e.g., the spatial counterparts of the various temporal modalities of CTL* [17]. It could be interesting to investigate a spatial logic with fixed points *a la* μ-calculus; to interpret such a logic, we could consider closure hyperdoctrines over Löb algebras [13]. Moreover, it would be interesting to develop some "generic" model checking algorithm for spatial logic. The abstraction provided by the categorical approach can guide the generalization of existing model checking algorithms, such as [10], and suggest new proof methodologies and minimisation techniques.

On a different direction, we are interested in the type theory induced by closure hyperdoctrines. A Curry-Howard isomorphism would yield a functional programming language with constructors for spatial aspects, which would be very useful in *collective spatial programming*, e.g. for collective adaptive systems.

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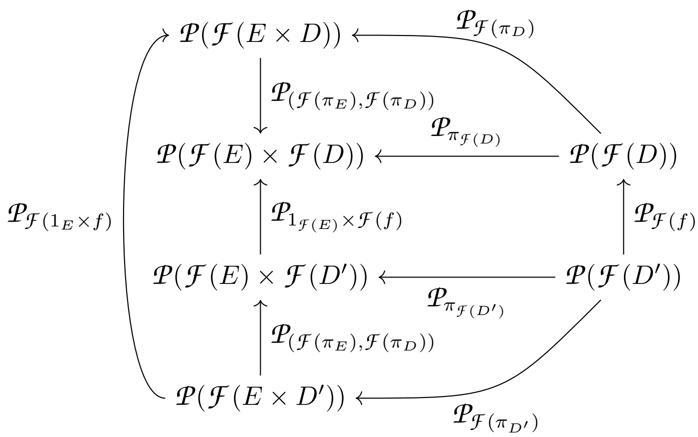
A Omitted proofs

Proof of Proposition 2.7. We have to show that $\mathcal{P}_{\mathcal{F}(\pi_D)}$ has a left adjoint for any projection $\pi_D : E \times D \to D$ but this follows at once since the diagonal arrow in the diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{F}(D)) & \xrightarrow{\mathcal{P}_{\pi_{\mathcal{F}(D)}}} & \mathcal{P}(\mathcal{F}(E)\times\mathcal{F}(D)) \\ {\scriptstyle \mathcal{P}_{\mathcal{F}(\pi_D)}}\downarrow & & \\ \mathcal{P}(\mathcal{F}(E\times D)) & & \end{array}$$

$$\mathcal{P}_{(\mathcal{F}(\pi_E),\mathcal{F}(\pi_D))} : \mathcal{P}(\mathcal{F}(E\times D)) \to \mathcal{P}(\mathcal{F}(E)\times\mathcal{F}(D))$$

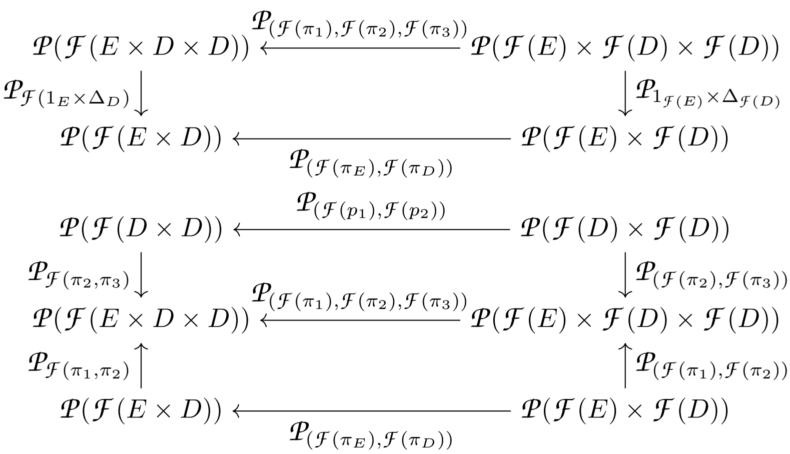
is an isomorphism, hence we can define $\exists_{\pi_D}$ as the composition $\exists_{\pi_{\mathcal{F}(D)}} \circ \mathcal{P}_{(\mathcal{F}(\pi_E),\mathcal{F}(\pi_D))}$. The same argument shows that $\forall_{\pi_{\mathcal{F}(D)}} \circ \mathcal{P}_{(\mathcal{F}(\pi_E),\mathcal{F}(\pi_D))}$ is the right adjoint to $\mathcal{P}_{\mathcal{F}(\pi_D)}$ whenever $\forall_{\pi_{\mathcal{F}(D)}}$ exists. Let now $f : D' \to D$ be an arrow in $\mathbf{D}$, the two Beck-Chevalley conditions follow from the commutativity of



and the fact that both the upper and the lower vertical arrow are isomorphisms since $\mathcal{F}$ preserves products. For Frobenius reciprocity:

$$\exists_{\mathcal{F}(\pi_D)}(\mathcal{P}_{\mathcal{F}(\pi_D)}(\alpha)\wedge\beta)=\exists_{\pi_{\mathcal{F}(D)}}(\mathcal{P}_{(\mathcal{F}(\pi_E),\mathcal{F}(\pi_D))}(\mathcal{P}_{\mathcal{F}(\pi_D)}(\alpha)\wedge\beta))$$
$$=\exists_{\pi_{\mathcal{F}(D)}}(\mathcal{P}_{\pi_{\mathcal{F}(D)}}(\alpha)\wedge\mathcal{P}_{(\mathcal{F}(\pi_E),\mathcal{F}(\pi_D))}(\beta))=\alpha\wedge\exists_{\pi_{\mathcal{F}(D)}}(\mathcal{P}_{\mathcal{F}(\pi_D)}(\alpha)\wedge\beta)=\alpha\wedge\exists_{\mathcal{F}(\pi_D)}(\beta)$$

So we're left with the fibered equalities, but the commutativity of



entails that $\mathcal{P}_{(\mathcal{F}(p_1),\mathcal{F}(p_2))}(\delta_{\mathcal{F}(D)})$ has the property of a fibered equality. ◀

Proof of Proposition 2.8. (Cfr. [23, 31] and lemma 1.5.8 of [26], vol. 1 for the hyperdoctrine case). It is enough to define

$$\exists_f(\alpha):=\exists_{\pi_D}(\mathcal{P}_{f\times 1_D}(\delta_D)\wedge\mathcal{P}_{\pi_C}(\alpha))\qquad\forall_f(\alpha):=\forall_{\pi_D}(\mathcal{P}_{f\times 1_D}(\delta_D)\to\mathcal{P}_{\pi_C}(\alpha))$$

Let us now show that $\exists_f \dashv \mathcal{P}_f$.

If $\exists_f(\alpha) \leq \beta$	If $\alpha \leq \mathcal{P}_f(\beta)$
$\alpha = \alpha \wedge \top_C = \alpha \wedge \exists_{\pi_2}(\delta_C)$	$\exists_f(\alpha) \leq \exists_f(\mathcal{P}_f(\beta))$
$\leq \alpha \wedge \exists_{\pi_2}(\mathcal{P}_{f\times f}(\delta_D))$	$= \exists_{\pi_D}(\mathcal{P}_{f\times 1_D}(\delta_D) \wedge \mathcal{P}_{\pi_C}(\mathcal{P}_f(\beta))$
$= \exists_{\pi_2}(\mathcal{P}_{f\times f}(\delta_D) \wedge \mathcal{P}_{p_2}(\alpha))$	$= \exists_{\pi_D}(\mathcal{P}_{f\times 1_D}(\delta_D) \wedge \mathcal{P}_{1_D\times f}(\mathcal{P}_{q_2}(\beta)))$
$= \exists_{\pi_2}(\mathcal{P}_{1_C\times f}(\mathcal{P}_{f\times 1_D}(\delta_D) \wedge \mathcal{P}_{\pi_C}(\alpha))$	$= \exists_{\pi_D}(\mathcal{P}_{1_D\times f}(\delta_D) \wedge \mathcal{P}_{1_D\times f}(\mathcal{P}_{q_2}(\beta)))$
$= \mathcal{P}_f(\exists_{\pi_D}(\mathcal{P}_{f\times 1_D}(\delta_D) \wedge \mathcal{P}_{\pi_C}(\alpha)))$	$= \exists_{\pi_D}(\mathcal{P}_{1_D\times f}(\delta_D \wedge \mathcal{P}_{q_2}(\beta))) = \exists_{\pi_D}(\mathcal{P}_{1_D\times f}(\exists_{\Delta_D}(\beta)))$
$= \mathcal{P}_f(\exists_f(\alpha)) \leq \mathcal{P}_f(\beta)$	$\leq \exists_{\pi_D}(\mathcal{P}_{1_D\times f}(\mathcal{P}_{q_1}(\beta))) = \exists_{\pi_D}(\mathcal{P}_{\pi_D}(\beta)) \leq \beta$

where p_2 is the second projection $C \times C \to C$ and q_1 and q_2 those $D \times D \to D$. For the adjunction $\mathcal{P}_f \dashv \forall_f$ we already know that $\exists_{\pi_C} \dashv \mathcal{P}_{\pi_C}$, $\mathcal{P}_{f\times 1_D}(\delta_D) \wedge (-) \dashv \mathcal{P}_{f\times 1_D}(\delta_D) \to (-)$ and $\mathcal{P}_{\pi_D} \dashv \forall_{\pi_D}$, so it is enough to show that $\mathcal{P}_f(\beta) = \exists_{\pi_C}(\mathcal{P}_{f\times 1_D}(\delta_D) \wedge \mathcal{P}_{\pi_D}(\beta))$ for all $\beta \in \mathcal{P}(D)$. But this is easily done:

$$\exists_{\pi_C}(\mathcal{P}_{f\times 1_D}(\delta_D) \wedge \mathcal{P}_{\pi_D}(\beta)) = \exists_{\pi_C}(\mathcal{P}_{1_D\times f}(\delta_D) \wedge \mathcal{P}_{1_D\times f}(\mathcal{P}_{\pi_1}(\beta)))$$
$$= \exists_{\pi_C}(\mathcal{P}_{1_D\times f}(\delta_D \wedge \mathcal{P}_{\pi_1}(\beta))) = \mathcal{P}_f(\exists_{\pi_2}(\exists_{\Delta_D}(\beta))) = \mathcal{P}_f(\beta)$$

Where π_2 is the second projection $D \times D \to D$. We're left with Frobenius reciprocity: the inequality $\exists_f(\mathcal{P}_f(\beta) \wedge \alpha) \leq \beta \wedge \exists_f(\alpha)$ follows from adjointness, let's show the other. If π_1 and π_2 are the projections from $D \times D$, then

$$\exists_f(\mathcal{P}_f(\beta) \wedge \alpha) = \exists_{\pi_D}(\mathcal{P}_{f\times 1_D}(\delta_D) \wedge \mathcal{P}_{\pi_C}(\mathcal{P}_f(\beta) \wedge \alpha))$$
$$= \exists_{\pi_D}(\mathcal{P}_{f\times 1_D}(\delta_D) \wedge \mathcal{P}_{\pi_C}(\mathcal{P}_f(\beta)) \wedge \mathcal{P}_{\pi_C}(\alpha)) = \exists_{\pi_D}(\mathcal{P}_{f\times 1_D}(\delta_D) \wedge \mathcal{P}_{f\times 1_D}(\mathcal{P}_{\pi_1}(\beta)) \wedge \mathcal{P}_{\pi_C}(\alpha))$$
$$= \exists_{\pi_D}(\mathcal{P}_{f\times 1_D}(\delta_D \wedge \mathcal{P}_{\pi_1}(\beta)) \wedge \mathcal{P}_{\pi_C}(\alpha)) \leq \exists_{\pi_D}(\mathcal{P}_{f\times 1_D}(\delta_D \wedge \mathcal{P}_{\pi_2}(\beta)) \wedge \mathcal{P}_{\pi_C}(\alpha))$$
$$= \exists_{\pi_D}(\mathcal{P}_{f\times 1_D}(\delta_D) \wedge \mathcal{P}_{f\times 1_D}(\mathcal{P}_{\pi_2}(\beta)) \wedge \mathcal{P}_{\pi_C}(\alpha)) = \exists_{\pi_D}(\mathcal{P}_{f\times 1_D}(\delta_D) \wedge \mathcal{P}_{\pi_D}(\beta) \wedge \mathcal{P}_{\pi_C}(\alpha))$$
$$= \exists_{\pi_D}(\mathcal{P}_{f\times 1_D}(\delta_D) \wedge \mathcal{P}_{\pi_C}(\alpha)) \wedge \beta = \exists_f(\alpha) \wedge \beta$$

where we have used $\delta_D \wedge \mathcal{P}_{\pi_1}(\beta) \leq \mathcal{P}_{\pi_2}(\beta)$ which follows from the definition of $\exists_{\Delta_D}$. ◀

Proof of Proposition 3.24. We have to show continuity of all arrows $f : (A, \alpha) \to (B, \beta)$. Let $\xi \in (B, \beta)$ and $x \in A$, we have four cases:

1. $f^*(\xi)(x) + \epsilon_{(A,\alpha)}(x) < \alpha(x)$ and $\xi(x) + \epsilon_{(B,\beta)}(x) < \beta(x)$.

$$(\mathfrak{c}^{\mathcal{E}}_{(A,\alpha)}(f^*(\xi)))(x) = (f^*(\xi) + \epsilon_{(A,\alpha)})(x) = (\alpha(x) \wedge \xi(f(x))) + \epsilon_{(A,\alpha)}(x)$$
$$= \alpha(x) \wedge (\xi(f(x)) + \epsilon_{(A,\alpha)}(x)) \leq \alpha(x) \wedge (\xi(f(x)) + \epsilon_{(B,\beta)}(f(x))) = f^*(\mathfrak{c}^{\mathcal{E}}_{(B,\beta)}(\xi))(x)$$

2. $f^*(\xi)(x) + \epsilon_{(A,\alpha)}(x) < \alpha(x)$ and $\xi(f(x)) + \epsilon_{(B,\beta)}(f(x)) \geq \beta(f(x))$. Notice that $\alpha(x) \leq \beta(f(x))$ so $f^*(\mathfrak{c}^{\mathcal{E}}_{(B,\beta)}(\xi))(x) = \alpha(x)$ and thus

$$(\mathfrak{c}^{\mathcal{E}}_{(A,\alpha)}(f^*(\xi)))(x) = (f^*(\xi) + \epsilon_{(A,\alpha)})(x) = (\alpha(x) \wedge \xi(f(x))) + \epsilon_{(A,\alpha)}(x)$$
$$= \alpha(x) \wedge (\xi(f(x)) + \epsilon_{(A,\alpha)}(x)) = \alpha(x) = f^*(\mathfrak{c}^{\mathcal{E}}_{(B,\beta)}(\xi))(x)$$

3. $f^*(\xi)(x) + \epsilon_{(A,\alpha)}(x) \geq \alpha(x)$ and $\xi(x) + \epsilon_{(B,\beta)}(x) < \beta(x)$.

$$(\mathfrak{c}^{\mathcal{E}}_{(A,\alpha)}(f^*(\xi)))(x) = \alpha(x) = \alpha(x) \wedge (\xi(f(x)) + \epsilon_{(A,\alpha)}(x))$$
$$\leq \alpha(x) \wedge (\xi(f(x)) + \epsilon_{(B,\beta)}(f(x))) = f^*(\mathfrak{c}^{\mathcal{E}}_{(B,\beta)}(\xi))(x)$$

4. $f^*(\xi)(x) + \epsilon_{(A,\alpha)}(x) \geq \alpha(x)$ and $\xi(x) + \epsilon_{(B,\beta)}(x) \geq \beta(x)$.

$$(\mathfrak{c}^{\mathcal{E}}_{(A,\alpha)}(f^*(\xi)))(x) = \alpha(x) = \alpha(x) \wedge \beta(f(x)) = f^*(\mathfrak{c}^{\mathcal{E}}_{(B,\beta)}(\xi))(x)$$

We are left with additivity, but this follows immediately since, for ξ and $\zeta \in \mathcal{FzSub}(A,\alpha)$ and $x \in A$, $(\xi \vee \zeta)(x)$ is $\xi(x)$ or $\zeta(x)$. ◀

Proof of Lemma 3.31. **1.** Clearly $\alpha \leq \mathsf{pre}_{\gamma_X}(\alpha)$; if $\alpha \leq \beta$ we also have $\mathcal{P}_{\gamma_X}(\Box_X(\alpha)) \leq \mathcal{P}_{\gamma_X}(\Box_X(\beta))$ from which monotonicity follows. Tanke now an arrow f between $\gamma_X : X \to \mathcal{F}(X)$ and $\gamma_Y : Y \to \mathcal{F}(Y)$, thus $\mathcal{F}(f) \circ \gamma_X = \gamma_Y \circ f$ so

$$\begin{aligned}\mathsf{pre}_{\gamma_X}(\mathcal{P}_f(\alpha)) &= \mathcal{P}_f(\alpha) \vee \mathcal{P}_{\gamma_X}(\Box_X(\mathcal{P}_f(\alpha))) = \mathcal{P}_f(\alpha) \vee \mathcal{P}_{\gamma_X}(\mathcal{P}_{\mathcal{F}(f)}(\Box_Y(\alpha)))\\ &= \mathcal{P}_f(\alpha) \vee \mathcal{P}_f(\mathcal{P}_{\gamma_Y}(\Box_Y(\alpha))) = \mathcal{P}_f(\alpha \vee \mathcal{P}_{\gamma_Y}(\Box_Y(\alpha))) = \mathcal{P}_f(\mathsf{pre}_{\gamma_Y}(\alpha))\end{aligned}$$

2. For any $\alpha \in \mathcal{P}(X)$

$$\mathcal{A}_{\gamma_X}(\alpha) := \{\Box_X(\beta) \mid \beta \in \mathcal{N}_{\gamma_X}(\alpha)\} \qquad \mathcal{B}_{\gamma_X}(\alpha) := \{\mathcal{P}_{\gamma_X}(\Box_X(\beta)) \mid \beta \in \mathcal{N}_{\gamma_X}(\alpha)\}$$

then by the hypothesis on $\mathcal{P}_{\gamma_X}$ and the previous point we have

$$\alpha \leq \inf(\mathcal{B}_{\gamma_X}(\alpha)) = \mathcal{P}_{\gamma_X}(\inf(\mathcal{A}_{\gamma_X}(\alpha))) = \mathcal{P}_{\gamma_X}(\Box_X(\mathsf{s}_{\gamma_X}(\alpha)))$$

3. The inequality $\alpha \leq \mathsf{suc}_{\gamma_X}(\alpha)$ follows at once, if $\alpha \leq \beta$ we have $\mathcal{P}_{\gamma_X}(\Box_X(\alpha))$ as in the first point but this implies that $\mathcal{N}_{\gamma_X}(\beta) \subset \mathcal{N}_{\gamma_X}(\alpha)$. Hence, $\bigwedge_{\theta \in \mathcal{N}_{\gamma_X}(\alpha)} \theta \leq \bigwedge_{\theta \in \mathcal{N}_{\gamma_X}(\beta)} \theta$, from which we deduce the monotonicity of suc_{γ_X}. For any morphism $f : \gamma_X \to \gamma_Y$ of coalgebras we have

$$\mathcal{P}_f(\alpha) \leq \mathcal{P}_f(\mathcal{P}_{\gamma_Y}(\Box_Y(\theta))) = \mathcal{P}_{\gamma_X}(\mathcal{P}_{\mathcal{F}(f)}(\Box_Y(\theta))) = \mathcal{P}_{\gamma_X}(\Box_X(\mathcal{P}_f(\theta)))$$

for all $\theta \in \mathcal{N}_Y(\alpha)$. Hence $\mathcal{P}_f(\theta) \in \mathcal{N}_X(\mathcal{P}_f(\alpha))$, then $\mathsf{s}_{\gamma_X}(\mathcal{P}_f(\alpha)) \leq \inf(\mathcal{P}_f(\mathcal{N}_Y(\alpha)))$:

$$\begin{aligned}&\mathsf{suc}_{\gamma_X}(\mathcal{P}_f(\alpha)) = \mathcal{P}_f(\alpha) \vee \mathsf{s}_{\gamma_X}(\mathcal{P}_f(\alpha)) \leq \mathcal{P}_f(\alpha) \vee \inf(\mathcal{P}_f(\mathcal{N}_Y(\alpha)))\\ &\leq \mathcal{P}_f(\alpha) \vee \mathcal{P}_f(\inf(\mathcal{N}_Y(\alpha))) = \mathcal{P}_f(\alpha \vee \mathsf{s}_{\gamma_Y}(\alpha)) = \mathcal{P}_f(\mathsf{suc}_{\gamma_Y}(\alpha))\end{aligned}$$

and we are done. ◀