An ultrapower analogue of the Kronecker function ring

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AN ULTRAPOWER ANALOGUE OF THE KRONECKER FUNCTION RING

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Abstract. We introduce an analogue of the Kronecker function ring construction in the ultrapower setting, and study when it gives a Bézout domain.

1. Introduction

The ultraproduct construction is an extremely powerful technique in logic and model theory: in fact, by Łoś's theorem, a first-order formula is satisfied in an ultraproduct if and only if it is satisfied for almost all the factors (see [11], [5, Chapter 5, Theorem 2.1] or [6, Theorem 4.1.9]), and this allows a rather simple proof of the compactness theorem for first-order logic (see e.g. [6, Corollary 4.1.11]). In algebra, the use of ultraproducts has been pioneered by Ax and Kochen [2, 3, 4], and has grown considerably, for example as a way to transfer results from rings of positive characteristic to rings of characteristic 0 (see [18]).

In general, the algebraic structure of the ultrapower of a family of rings is very complicated, and this construction does not preserve all properties of the factors: for example, the ultraproduct of a family of Noetherian rings is very rarely Noetherian. In particular, ultraproducts and ultrapowers gain many new prime ideals: for example, under some mild hypotheses every non-zero prime ideal of the ultrapower of an integral domain has infinite height [13, Proposition 6.2], and to describe the set of maximal ideals one needs to consider ultrafilters on the set of ideals that are induced by maximal ideals of the factors (see [13, Section 4] and [14, Theorem 4.3]).

In this paper, we study a generalization of the Kronecker function ring to the ultrapower setting. The Kronecker function ring is a classical construction that associates to every integrally closed integral domain D a new domain $\mathrm{Kr}(D)$, contained between $D[X]$ and $K(X)$ (where K is the quotient field of D) that, while being an extension of D (in the sense that $\mathrm{Kr}(D) \cap K = D$), gains several strong properties, among them that of being

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a Bézout domain, meaning that every finitely generated ideal of $\mathrm{Kr}(D)$ is principal. One of the equivalent definitions of $\mathrm{Kr}(D)$ is as the intersection of a family of valuation rings, each one extending a valuation overring of D; this idea leads also to the definition of the Kronecker function ring $\mathrm{Kr}(D, \Delta)$ associated to a subset Δ of the Zariski space of D.

In the ultraproduct setting, this construction admits a straightforward generalization: given a set Δ of valuation overrings of D, and an ultrafilter $\mathscr{U}$ on an index set I, we can consider for every $V \in \Delta$ the ultrapower $V^\star$ of V (with respect to $\mathscr{U}$), and then intersect all the $V^\star$ (considering all $V^\star$ as subsets of the ultrapower $K^\star$ of the quotient field K of V). We call the ring obtained in this way the *Kronecker-ultrafilter ring* $\mathrm{KU}(D, \Delta, \mathscr{U})$ of D (with respect to Δ and $\mathscr{U}$); when Δ is the whole Zariski space of D, and $\mathscr{U}$ is understood from the context, we set $D^\sharp := \mathrm{KU}(D, \Delta, \mathscr{U})$.

Comparing these two settings, we see that, for D integrally closed, the polynomial extension $D[X]$ will have many more maximal ideals than D and an abundance of valuation overrings which are not trivial extensions of valuation overrings of D. Then, in this larger environment, the collection of trivial extensions of the valuation overrings of D has a *thin* character, which leads to their intersection (the Kronecker function ring) being a Bézout domain. The notion of thinness of a collection of valuation domains resulting in a Prüfer domain is made explicit in several different settings in [15]. Similarly, the ultrapower of an integral domain acquires many new prime ideals and many new valuation overrings, and hence will have a lot of valuation overrings which are not ultrapowers of valuation overrings of D. It would seem natural then that the *thinness* of this collection of valuation domains would lead to the intersection being a Bézout domain: the main purpose of this paper is to understand how much the Kronecker-ultrafilter ring mirrors the Kronecker function ring, and in particular if the former construction always gives a Bézout domain.

The main setting in which we work is when the index set I is countable: under this hypothesis, we show in Section 3 that $D^\sharp$ is larger than the ultrapower $D^\star$ unless D is a semilocal Prüfer domain, while in Section 4 we give a few sufficient conditions for $\mathrm{KU}(D, \Delta, \mathscr{U})$ to be a Bézout domain: for example, we show this when Δ is countable (Theorem 4.1) or when D is a unique factorization domain and Δ is the set of localizations of D at the height-one primes (Corollary 4.4).

In Section 5, we consider uncountable index sets, and show that in this case the properties of the ultrafilter play an important role in the algebraic properties of the Kronecker-ultrafilter rings. For example, we show that if $\mathscr{U}$ is κ-complete and the Zariski space of D has cardinality at most κ, then $D^\star = D^\sharp$ (and in particular $D^\sharp$ may not be a Bézout domain; Proposition 5.1(c)), but that if the ultrafilter is regular then the methods of the countable case can be generalized (Theorem 5.10).

2. Notation and preliminaries

2.1. **Ultrafilters and ultraproducts.** Let I be a set and $\mathscr{U}$ be a family of subsets of I. Then, $\mathscr{U}$ is an *ultrafilter* on I if the following properties hold:

- $\emptyset \notin \mathscr{U}$;
- if $X, Y \in \mathscr{U}$, then $X \cap Y \in \mathscr{U}$;
- if $X \subseteq Y$ and $X \in \mathscr{U}$, then $Y \in \mathscr{U}$;
- for every $X \subseteq I$, one of X and $I \setminus X$ is in $\mathscr{U}$.

A family that satisfies the first three properties if said to be a *filter*; an ultrafilter is exactly a maximal filter.

It is easy to see that, if $i \in I$, the family of subsets containing i is an ultrafilter; such ultrafilters are said to be *principal*, while those that are not of this form are said to be *free*.

Let $\{R_i\}_{i\in I}$ be a collection of commutative rings and $\mathscr{U}$ be an ultrafilter on I. The *ultraproduct* of the R_i with respect to $\mathscr{U}$ is the ring of all equivalence classes of the direct product $\prod_{i\in I} R_i$ by the equivalence relation $\sim$ defined by

$$(a_i)_{i\in I} \sim (b_i)_{i\in I} \iff \{i \in I \mid a_i = b_i\} \in \mathscr{U}.$$

We denote by $[a_i]$ the class of the sequence $(a_i)_{i\in I}$, and by $\prod_{\mathscr{U}} R_i$ the ultraproduct of the R_i. When all the R_i are equal (say $R_i = R$), we also write $R^\star$ for the ultraproduct, and we call it the *ultrapower* of R with respect to $\mathscr{U}$.

If $\mathscr{U}$ is the principal ultrafilter induced by a $j \in I$, then $\prod_{\mathscr{U}} R_i \simeq R_j$. For this reason, *throughout the paper, we shall assume that all ultrafilters are free.*

For general properties of ultraproducts and ultrapowers, the reader may consult [5] or [6].

2.2. **Valuations and the Zariski space.** All results on valuation, Prüfer and Bézout domains we will use are standard and can be found, for example, in [8]. For Kronecker function rings, the Zariski topology and their relationship, see for example [7].

A *valuation domain* is an integral domain V whose ideals (equivalently, whose principal ideals) are linearly ordered. Every valuation domain is local, and we denote the maximal ideal of V as $\mathfrak{m}_V$. A *Bézout domain* is a domain such that every finitely generated ideal is principal.

A *Prüfer domain* is an integral domain D such that every finitely generated ideal is invertible, i.e., such that for every finitely generated ideal I there is a fractional ideal J such that $IJ = D$. Every valuation domain is a Bézout domain and every Bézout domain is a Prüfer domain; conversely, every local Prüfer domain is a valuation domain, and every semilocal Prüfer domain is Bézout. Furthermore, D is a Prüfer domain if and only if all its localizations are valuation domains. If D is a Prüfer (respectively, Bézout) domain and T is a ring between D and its quotient field, then T is Prüfer (resp., Bézout).

If $\{R_i\}$ is a family of valuation (resp., Bézout, Prüfer) domains, then the ultraproduct $\prod_{\mathscr{U}} R_i$ is a valuation (resp., Bézout, Prüfer) domain.

Given an integral domain D, the *Zariski space* $\mathrm{Zar}(D)$ of D is the set of all rings contained between D and its quotient field K that are valuation domains. The Zariski space is always nonempty; more precisely, for every prime ideal $\mathfrak{p}$ of D there is a $V \in \mathrm{Zar}(D)$ such that $\mathfrak{m}_V \cap D = \mathfrak{p}$. The Zariski space can also be endowed with a natural topology (the *Zariski topology*) which is generated by the sets of the form $\mathcal{B}(x_1, \dots, x_n) := \{V \in \mathrm{Zar}(D) \mid x_1, \dots, x_n \in V\}$, as $x_1, \dots, x_n$ range in K. Under this topology, $\mathrm{Zar}(D)$ is a compact space; furthermore, it is a *spectral space*, i.e., there is a ring R such that $\mathrm{Zar}(D) \simeq \mathrm{Spec}(R)$. An example of such a ring is the *Kronecker function ring* of D:

$$\mathrm{Kr}(D) := \left\{ \frac{f}{g} \in K(X) \mid f, g \in K[X], \mathbf{c}(f)V \subseteq \mathbf{c}(g)V \text{ for all } V \in \mathrm{Zar}(D) \right\},$$

where $\mathbf{c}(f)$ is the *content* of f, i.e., the ideal of D generated by the coefficients of f. The Kronecker function ring can also be defined as

$$\mathrm{Kr}(D) := \bigcap_{V \in \mathrm{Zar}(D)} V^b,$$

where V^b is the *Gaussian extension* v_G of V, i.e., it is the valuation domain of $K(X)$ associated to the valuation

$$v_G\left(\sum_i f_i X^i\right) := \min_i v(f_i),$$

where v is the valuation associated to V.

The *constructible topology* on $\mathrm{Zar}(D)$ is the topology generated by the Zariski topology and the complements of open and compact subsets of the

Zariski topology. The constructible topology is still spectral, but it also becomes Hausdorff.

3. When $D^\sharp$ is big

The main object of study of this paper is the following.

Definition 3.1. Let D be an integral domain, $\Delta \subseteq \mathrm{Zar}(D)$, and let $\mathscr{U}$ be an ultrafilter over an index set I. The *Kronecker-ultrafilter ring of D with respect to Δ and $\mathscr{U}$* is

$$\mathrm{KU}(D, \Delta, \mathscr{U}) := \bigcap_{V \in \Delta} \prod\nolimits_{\mathscr{U}} V.$$

When $\mathscr{U}$ is understood from the context, we set

$$D^\sharp := \mathrm{KU}(D, \mathrm{Zar}(D), \mathscr{U}).$$

The terminology "Kronecker-ultrafilter ring" is chosen to highlight the similarity between the definition of $D^\sharp$ (or, more generally, of $\mathrm{KU}(D, \Delta, \mathscr{U})$) with the definition of the Kronecker function ring of D (and the more general construction $\mathrm{Kr}(D, \Delta)$): we replace the Gaussian extension V^b with the ultrapower $V^\star$.

In this and in the following section, we shall assume that the index set I is countable; the uncountable case will be studied in Section 5.

The main purpose of this section is to show that, in almost all cases, $D^\sharp$ is larger than the ultrapower $D^\star$. It is not immediately obvious that this is ever true: however, a simple example shows how they can be different.

Example 3.2. Let $D := \mathbb{Z}$ be the ring of integers, and let $\{p_1, \ldots, p_n, \ldots\}$ be the set of prime numbers of $\mathbb{Z}$. Let $\mathbf{x}$ be the element

$$\mathbf{x} := \left[\frac{1}{p_1}, \frac{1}{p_2}, \ldots, \frac{1}{p_n}, \ldots\right].$$

Then, $\mathbf{x} \notin \mathbb{Z}^\star$ since $x_i = 1/p_i \notin \mathbb{Z}$ for every i. On the other hand, if $M = p\mathbb{Z}$ is a maximal ideal of $\mathbb{Z}$, then $1/p_i \in \mathbb{Z}_M$ for all i such that $p_i \neq p$; hence, if $\mathscr{U}$ is not principal then $\mathbf{x} \in (\mathbb{Z}_M)^\star$. Therefore, $\mathbf{x}$ belongs to the intersection of all the $(\mathbb{Z}_M)^\star$, which are the ultrapowers of the minimal valuations of $\mathbb{Z}$; thus, $\mathbf{x} \in \mathbb{Z}^\sharp \setminus \mathbb{Z}^\star$.

This example can be easily generalized.

Proposition 3.3. *Let D be an integral domain, and let $x_1, \ldots, x_n, \ldots$ a sequence of nonunits of D such that $(x_i, x_j)D = D$ for all $i \neq j$. Then,*

$$\mathbf{y} := \left[\frac{1}{x_1}, \ldots, \frac{1}{x_n}, \ldots\right] \in D^\sharp \setminus D^\star,$$

and so $D^\star \subsetneq D^\sharp$.

Proof. Since each x_i is a nonunit, $1/x_i \notin D$ and thus $\mathbf{y} \notin D^\star$. On the other hand, for each maximal ideal M there is at most one i such that $x_i \in M$; hence, $\mathbf{y} \in (D_M)^\star$. If now V is a minimal valuation overring of D, then $V^\star$ contains $(D_M)^\star$ (where $M := \mathfrak{m}_V \cap D$), and thus $\mathbf{y} \in D^\sharp$. In particular, $D^\sharp \neq D^\star$. □

Proposition 3.4. *Let D be an integral domain, and suppose there is a non-maximal prime ideal P of D such that* $\mathrm{Jac}(D) \subseteq P$*. Then, $D^\star \neq D^\sharp$.*

Proof. Let M be a maximal ideal containing P, and let $x_1 \in M \setminus P$. Suppose we have constructed a sequence $x_1, \ldots, x_{n-1}$ of nonunits such that $(x_i, x_j)D = D$ for $i < j < n$ and such that $x_i \notin P$ for all $i < n$: then, $\widetilde{x} := x_1 \cdots x_{n-1} \notin P$, and thus $\widetilde{x} \notin \mathrm{Jac}(D)$. Hence, there is a y such that $x_n := y\widetilde{x}-1$ is not a unit of D. Clearly, $(x_i, x_n)D = D$ for all $i < n$; in particular, $x_n \notin P$, since otherwise $(x_1, x_n)D \subseteq M$. The sequence $x_1, \ldots, x_n, \ldots,$ satisfies the hypothesis of Proposition 3.3, and thus $D^\star \neq D^\sharp$. □

The hypothesis of the previous proposition can be restated as requiring that $D/\mathrm{Jac}(D)$ has dimension greater than 0; in particular, an important case that is left out is when D is a local ring. To analyze this situation, we use a similar method, but based on polynomials.

Proposition 3.5. *Let D be an integrally closed domain, and let $\boldsymbol{\lambda} := \{\lambda_n\}_{n\geq 1}$ be a sequence of monic polynomials on D such that $(\lambda_i, \lambda_j)D[X] = D[X]$ for all $i \neq j$.*

- *(a) For every nonzero $t \in K$, the element*
$$\boldsymbol{\lambda}^{-1}(t) := \left[\frac{1}{\lambda_i(t)}\right]_{i\in\mathbb{N}}$$
belongs to $D^\sharp$.
- *(b) If V is a valuation overring of D, then $\boldsymbol{\lambda}^{-1}(t) \in \mathfrak{m}_{V^\star}$ if and only if $t \notin V$.*
- *(c) If V, W are noncomparable valuation overrings of D, then $\mathfrak{m}_{V^\star} \cap D^\sharp \neq \mathfrak{m}_{W^\star} \cap D^\sharp$.*

Proof. Let V be any valuation overring of D, and let v be the valuation relative to V. Note that if the constant term of $\lambda \in D[X]$ is not a unit in V, then $\lambda \in (\mathfrak{m}_V, X)V[X]$; in particular, no two polynomials with this property can be coprime in $V[X]$ (and thus also in $D[X]$). Furthermore, since the λ_i are coprime, for any t there is at most one i such that $\lambda_i(t) = 0$, and so $\boldsymbol{\lambda}^{-1}(t)$ is well-defined.

We distinguish three cases.

- If $t \notin V$, then $v(t) < 0$: hence, $v(\lambda_i(t))$ is equal to the valuation of its leading term, which is equal to $n_i v(t) < 0$ (where n_i is the degree of λ_i). Hence, $1/\lambda_i(t) \in \mathfrak{m}_V$, and thus $\boldsymbol{\lambda}^{-1}(t) \in \mathfrak{m}_{V^\star}$.
- If $t \in \mathfrak{m}_V$, i.e., if $v(t) > 0$, then (since the constant term of λ_i is a unit for all but at most one i), we have $v(\lambda_i(t)) = 0$ (again, for all but at most one i), and so $1/\lambda_i(t)$ is a unit of V, i.e., $\boldsymbol{\lambda}^{-1}(t)$ is a unit of $V^\star$.
- If $v(t) = 0$ and $v(\lambda_i(t)) > 0$, then t is a zero of λ_i (when t and λ_i are seen over $V/\mathfrak{m}_V$). Since the λ_i are coprime in $D[X]$, they are also coprime in $V/\mathfrak{m}_V[X]$; hence, t cannot be a zero of more than one polynomial. Thus, $v(\lambda_i(t)) = 0$ for all but at most one i, and so $\boldsymbol{\lambda}^{-1}(t)$ is a unit of $V^\star$.

In particular, $\boldsymbol{\lambda}^{-1}(t) \in V^\star$ for every V, and so $\boldsymbol{\lambda}^{-1}(t) \in D^\sharp$; furthermore, $\boldsymbol{\lambda}^{-1}(t) \in \mathfrak{m}_{V^\star}$ if and only if $t \notin V$.

If V and W are non-comparable, we can find $t \in V \setminus W$; then, $\boldsymbol{\lambda}^{-1}(t) \in \mathfrak{m}_{W^\star} \setminus \mathfrak{m}_{V^\star}$, and since $\boldsymbol{\lambda}^{-1}(t) \in D^\sharp$ we have $\mathfrak{m}_{V^\star} \cap D^\sharp \neq \mathfrak{m}_{W^\star} \cap D^\sharp$. □

Lemma 3.6. *Let D be an integral domain. If I, J are D-submodules of K, then $(I \cap J)^\star = I^\star \cap J^\star$.*

Proof. Clearly $(I \cap J)^\star \subseteq I^\star \cap J^\star$. If $\mathbf{x} := [x_i] \in I^\star \cap J^\star$, then

$$\{i \mid x_i \in I \cap J\} = \{i \mid x_i \in I\} \cap \{i \mid x_i \in J\} \in \mathscr{U}$$

being the intersection of two subsets belonging to $\mathscr{U}$. Hence, $\mathbf{x} \in (I \cap J)^\star$, as claimed. □

Corollary 3.7. *Let V be a valuation overring of D. Then, $\mathfrak{m}_{V^\star} \cap D^\star = (\mathfrak{m}_V \cap D)^\star$*

Proof. It is enough to note that $\mathfrak{m}_{V^\star} = (\mathfrak{m}_V)^\star$ and apply Lemma 3.6. □

Proposition 3.8. *Let D be an integrally closed integral domain that is not Prüfer. Then, $D^\star \neq D^\sharp$.*

Proof. Let λ_1 be any monic polynomial and, for $n > 1$, let $\lambda_n := \lambda_1 \cdots \lambda_{n-1} - 1$. Then, all λ_i are monic non-constant polynomials, and $(\lambda_i, \lambda_j)D[X] = D[X]$ whenever $i \neq j$. Let $\boldsymbol{\lambda} := \{\lambda_n\}_{n \geq 1}$: then, $\boldsymbol{\lambda}$ satisfies the hypothesis of Proposition 3.5, and thus $\mathfrak{m}_{V^\star} \cap D^\sharp \neq \mathfrak{m}_{W^\star} \cap D^\sharp$ for all noncomparable valuation overrings V, W of D.

However, if D is not a Prüfer domain, there is a maximal ideal M of D such that D_M is not a valuation domain; in particular, there are two different minimal valuation overrings of D_M, say V and W, and both V and W have the same center over D, namely M. By Corollary 3.7, $V^\star$ and $W^\star$ have the same center over $D^\star$, namely $M^\star$, i.e., $\mathfrak{m}_{V^\star}\cap D^\star = M^\star = \mathfrak{m}_{W^\star}\cap D^\star$. By the previous reasoning, this is impossible if $D^\star = D^\sharp$. Hence, $D^\star \neq D^\sharp$, as claimed. □

The only case left is when D is a Prüfer domain such that the minimal primes of the Jacobson radical are all maximal. We shall use a topological lemma.

Lemma 3.9. *Let X be an topological space that is compact and totally disconnected, and suppose that $|X| = \infty$. Then, there is a descending chain $X = X_0 \supsetneq X_1 \supsetneq \cdots \supsetneq X_n \supsetneq \cdots$ such that every X_i is compact and open in X.*

Proof. Since X is totally disconnected, there is a proper subset U of X that is both open and closed. Since $|X| = \infty$, at least one of U and $X \setminus U$ is infinite; let it be X_1. Then, X_1 is both open and closed; since X is compact, X_1 is compact as well. Since X_1 is also totally disconnected, we can apply the same reasoning, finding an X_2 which is compact and open in X_1; since X_1 is open in X, it follows that X_2 is also open in X. Repeating the process we have the sequence. □

Proposition 3.10. *Let D be a Prüfer domain such that* $\mathrm{Max}(D)$ *is infinite and such that every minimal prime of* $\mathrm{Jac}(D)$ *is maximal. Then, $D^\star \neq D^\sharp$.*

Proof. By the hypothesis, $D/\mathrm{Jac}(D)$ has dimension 0 and $\mathrm{Spec}(D/\mathrm{Jac}(D))$ is homeomorphic to $\mathrm{Max}(D)$. Hence, $\mathrm{Max}(D)$ is compact, Hausdorff and totally disconnected; since it is infinite, we can apply Lemma 3.9 and find a sequence $\mathrm{Max}(D) = X_0 \supsetneq X_1 \supsetneq \cdots$ of open and compact subsets of $\mathrm{Max}(D)$; since $\mathrm{Max}(D)$ is Hausdorff, each X_i is also closed. Let $\Omega_i := X_i \setminus X_{i-1}$, for each $i > 0$. Then, $\Omega_i = X_i \cap (\mathrm{Max}(D) \setminus X_{i-1})$ is open and closed in $\mathrm{Max}(D)$; in particular, since $\mathrm{Max}(D)$ is closed in $\mathrm{Spec}(D)$ (being equal to $V(\mathrm{Jac}(D))$), then Ω_i is closed in $\mathrm{Spec}(D)$. Furthermore, since it is open, there is an ideal J_i such that $V(J_i) \cap \mathrm{Max}(D) = \mathrm{Max}(D) \setminus \Omega_i$.

Then, $V(J_i)$ and Ω_i are disjoint closed subsets of $\mathrm{Spec}(D)$; by [9, Lemma 1.1], we can find

$$x_i \in \bigcap_{P\in\Omega_i} P \setminus \bigcup_{Q\in V(J_i)} Q.$$

In particular, $V(x_i) \cap \mathrm{Max}(D) = \Omega_i$; since $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$, we have $(x_i, x_j)D = D$ for all $i \neq j$. Hence, we can apply Proposition 3.3, and $D^\sharp \neq D^\star$. □

The following theorem recaps the results of this section.

Theorem 3.11. *Let D be an integral domain, and suppose that the index set is countably infinite. Then, $D^\star = D^\sharp$ if and only if D is a semilocal Prüfer domain.*

Proof. If D is a semilocal Prüfer domain, say $\mathrm{Max}(D) = \{M_1, \dots, M_n\}$, then

$$D^\star = (D_{M_1} \cap \cdots \cap D_{M_n})^\star = \bigcap_{i=1}^{n} (D_{M_i})^\star = \bigcap_{i=1}^{n} (D_{M_i})^\sharp = D^\sharp$$

using Lemma 3.6 and the fact that each D_{M_i} is a valuation domain.

Suppose that D is a Prüfer domain that is not semilocal. Then, either $\dim(D/\mathrm{Jac}(D)) = 0$ or $\dim(D/\mathrm{Jac}(D)) > 0$. In the latter case, $D^\star \neq D^\sharp$ by Proposition 3.4; in the former, $D^\star \neq D^\sharp$ by Proposition 3.10. If D is not a Prüfer domain, then $D^\star \neq D^\sharp$ by Proposition 3.8. □

4. Bézout domains

One of the most important properties of the Kronecker function ring $\mathrm{Kr}(D)$ of D is that it is a Bézout domain; in particular, the spectrum and the Zariski space of $\mathrm{Kr}(D)$ are homeomorphic. There does not seem to be a simple way to extend this result to the Kronecker-ultrafilter ring of D: indeed, when the index set is uncountable this is in general not true (see the next section), and thus the Bézoutness of $D^\sharp$ depends, at least in part, on cardinality issues. Nevertheless, we advance the following

Conjecture. If the index set I is countable, then $D^\sharp$ is a Bézout domain.

A first evidence in favor of this conjecture is Proposition 3.5(c): the ultrapowers $V^\star$, in the Zariski space of $D^\sharp$, are spread out so much that their centers on $D^\sharp$ (and thus on each $\mathrm{KU}(D, \Delta, \mathscr{U})$) are distinct. In particular, every localization of $D^\sharp$ at its prime ideals is dominated by at most one $V^\star$.

In this section we use a few different approaches to prove some special cases of the conjecture. We still assume, throughout the section, that the index set is countable.

The first idea is to approximate the set Δ of valuation rings.

Theorem 4.1. *Let $\Delta \subseteq \mathrm{Zar}(D)$ be a countable set. Then, $\mathrm{KU}(D, \Delta, \mathscr{U})$ is a Bézout domain.*

Proof. Let $\mathcal{D} := \mathrm{KU}(D, \Delta, \mathscr{U})$. Write $\Delta := \{V_1, V_2, \dots\}$, and let $T_n := V_1 \cap \cdots \cap V_n$: then, T_n is a semilocal Prüfer domain (and thus it is Bézout) for every n. Take $\mathbf{a} := [a_i], \mathbf{b} := [b_i] \in K^\star$. For every i, the ideal $(a_i, b_i)T_i$

is principal, and thus is generated by some $c_i \in K$. Let $\mathbf{c} := [c_i]$: we claim that $\mathbf{c}$ generates $(\mathbf{a}, \mathbf{b})\mathcal{D}$.

Indeed, let $V_n \in \Delta$: then, for every $i \geq n$, $T_i \subseteq V_n$ and thus $(a_i, b_i)V_n = c_iV_n$. Hence, the set $\{i \mid c_i \in (a_i, b_i)V_n\}$ contains $[n, \infty)$ and thus belongs to $\mathscr{U}$, and so $\mathbf{c} \in (\mathbf{a}, \mathbf{b})D^\sharp$. In the same way, the sets $\{i \mid a_i \in c_iV_n\}$ and $\{i \mid b_i \in c_iV_n\}$ contain $[n, \infty)$ and belong to $\mathscr{U}$, so that both $\mathbf{a}$ and $\mathbf{b}$ belong to $\mathbf{c}\mathcal{D}$. The claim is proved. □

Corollary 4.2. *If* $\mathrm{Zar}(D)$ *is countable, then* $D^\sharp$ *is a Bézout domain.*

We shall generalize Theorem 4.1 in Theorem 5.10.

A second way of constructing a generator for $(\mathbf{a}, \mathbf{b})D^\sharp$ is by using factorization properties. For the definitions and properties of GCD domains, PvMDs, and t-maximal ideals, see for example [1]. A domain has *t-finite character* if every nonzero nonunit is contained in only finitely many t-maximal ideals.

Proposition 4.3. *Let D be a GCD domain that has t-finite character, and let Δ be the set of localizations of D at the t-maximal ideals. Then,* $\mathrm{KU}(D, \Delta, \mathscr{U})$ *is a Bézout domain.*

Proof. A GCD domain is a PvMD [1, Theorem 4.1], and thus if P is a t-maximal ideal then D_P is a valuation domain [1, Theorem 3.1]; hence, it makes sense to consider $\mathrm{KU}(D, \Delta, \mathscr{U})$.

Let $\mathbf{x} := [x_i]$ and $\mathbf{y} := [y_i]$ be two elements of $K^\star$. Since D is a GCD domain, for every i there is a $g_i \in D$ such that $(x_i, y_i)^v = g_i$; dividing both $\mathbf{x}$ and $\mathbf{y}$ by $\mathbf{g} := [g_i]$, we can suppose that $(x_i, y_i)^v = D$, i.e., that, for every i, x_i and y_i are coprime elements of D.

We claim that we can find two sequences $\{a_n\}_{n\geq 1}, \{b_n\}_{n\geq 1}$ of elements of D such that $a_ix_i + b_iy_i$ and $a_jx_j + b_jy_j$ are coprime for every $i \neq j$. Indeed, start with $a_1 = b_1 = 1$, suppose we have found the sequences up to $n - 1$, and let $z_k := a_kx_k + b_ky_k$ for $k < n$. Let S_k be the set of elements that are not coprime with z_k: then, S_k is just the union of all t-maximal primes containing z_k, and thus it is the union of finitely many prime ideals; hence, also $S := \bigcup_{k<n} S_k$ is the union of finitely many primes.

Suppose that, for every $\alpha, \beta \in D$, the element $z := \alpha x_n + \beta y_n$ is not coprime with some z_k: then, $(x_n, y_n)D$ is contained in S. However, by prime avoidance, it would follow that $(x_n, y_n)D$ is contained in some t-maximal ideal, contradicting the fact that x_n and y_n are coprime; thus, we can find $a_n, b_n \in D$ such that $a_nx_n + b_ny_n$ is coprime with every z_k.

Now let $\mathbf{a} := [a_i]$ and $\mathbf{b} := [b_i]$, and let $\mathbf{z} := \mathbf{ax} + \mathbf{by} = [a_ix_i + b_iy_i]$. Then, $\mathbf{z} \in (\mathbf{x}, \mathbf{y})D^\star \subseteq D^\star$. Let Λ be the set of t-maximal ideals: then, every $P \in \Lambda$ contains at most one z_i, and thus $1 \in \mathbf{z}(D_P)^\star$. It follows that

$$1 \in \bigcap_{P\in\Lambda} \mathbf{z}(D_P)^\star = \mathbf{z} \bigcap_{P\in\Lambda} (D_P)^\star.$$

Hence, $\mathbf{x}$ and $\mathbf{y}$ generate a principal ideal in $\bigcap_{P\in\Lambda}(D_P)^\star = \mathrm{KU}(D, \Delta, \mathscr{U})$; since $\mathbf{x}$ and $\mathbf{y}$ were arbitrary, $\mathrm{KU}(D, \Delta, \mathscr{U})$ is a Bézout domain. □

Corollary 4.4. *Let D be a unique factorization domain, and let $\Delta := \{D_P \mid h(P) = 1\}$. Then, $\mathrm{KU}(D, \Delta, \mathscr{U})$ is a Bézout domain.*

Proof. If D is a unique factorization domain, then it is a GCD domain and the t-maximal ideals are exactly the height-1 prime ideals. The claim follows from Proposition 4.3. □

The last result of this section shows that, under some hypothesis on the units of D, we can find a Kronecker function ring of D inside $D^\sharp$.

Lemma 4.5. *Let D be an integral domain and let $u_1, u_2, \ldots,$ be a sequence of units of D such that $u_i - u_j$ is a unit for every $i \neq j$. For every $f \in K[X]$ and every valuation overring V of D we have $v(f(u_i)) = v_G(f)$ for all but finitely many i, where v is the valuation relative to V and v_G is the Gaussian valuation of v.*

Proof. Let $L := V/\mathfrak{m}_V$: the hypothesis implies that the images $\overline{u_1}, \overline{u_2}, \ldots,$ of the u_i are distinct elements of L.

Let $f(X) := \sum_i f_iX^i$, and let $s \in K$ be an element of value $v_G(f)$; then, all coefficients of $\frac{1}{s}f := \sum_i \frac{f_i}{s}X^i$ are in V and some of them are units of V, so that its image $g(X)$ in $L[X]$ is well-defined and not the zero polynomial. Hence, $g(\overline{u_i}) = 0$ for only finitely many i; for all others, $\frac{1}{s}f(u_i)$ is a unit of V, and thus $v(f(u_i)) = v(s) = v_G(f)$. The claim is proved. □

Proposition 4.6. *Let D be an integral domain and let $u_1, u_2, \ldots,$ be a sequence of units of D such that $u_i - u_j$ is a unit for every $i \neq j$; let $\mathbf{u} := [u_i]$. Then, the Kronecker function ring of D in $K(\mathbf{u})$ is contained in $D^\sharp$; in particular, for every $a, b \in D$ the ideal $(a, b)D^\sharp$ is principal.*

Proof. Note first that $\mathbf{u}$ is transcendental over K, so it makes sense to construct the Kronecker function ring T of D in $K(\mathbf{u})$. Let $\phi \in T$: then, we can write ϕ as $f(\mathbf{u})/g(\mathbf{u})$, where $f, g \in K[X]$ are polynomials with $v_G(f) \geq v_G(g)$ for all v. In the ultraproduct representation, $f(\mathbf{u})/g(\mathbf{u}) = [f(u_i)/g(u_i)]$ (at least for all i such that $g(u_i) \neq 0$; however, $g(u_i) = 0$ for only finitely many i, and thus for these indexes we can just set $f(u_i)/g(u_i) = 0$). By Lemma 4.5, for all but finitely many i we have $v(f(u_i)) = v_G(f)$ and $v(g(u_i)) = v_G(g)$; hence, for all but finitely many i we have

$$v(f(u_i)) = v_G(f) \geq v_G(g) = v(g(u_i)),$$

and so $f(u_i)/g(u_i) \in V$. Therefore, $f(\mathbf{u})/g(\mathbf{u}) \in V^\star$, i.e., $\phi \in V^\star$. Since V was arbitrary, it follows that $\phi \in \bigcap_V V^\star = D^\sharp$ and so $T \subseteq D^\sharp$, as claimed.

The last claim follows since T is a Bézout domain. □

Note that the properties of the Kronecker function ring show that a generator of $(a, b)T$ is $a+\mathbf{u}b$, which thus also generates $(a, b)D^\sharp$. This claim can also be proved directly. See Proposition 5.12 for an extension to uncountable index sets.

5. When the index set is uncountable

In this section, we analyze what happens when the index set I is not countable; this case is more delicate, since it hits on cardinality problems. As a first example, we show that the equality $D^\star = D^\sharp$ may hold also outside the semilocal Prüfer domain case.

Let κ be a cardinal number. An ultrafilter $\mathscr{U}$ is said to be *κ-complete* if the intersection of any family of at most κ elements of $\mathscr{U}$ belongs to $\mathscr{U}$; equivalently, $\mathscr{U}$ is *not* κ-complete (or *κ-incomplete*) if there is a partition of the index set into (at most) κ subsets none of which belong to $\mathscr{U}$.

An uncountable cardinal κ such that there is an ultrafilter on an index set of cardinality κ that is α-complete for every $\alpha < \kappa$ is said to be *measurable*. It is consistent with ZFC that measurable cardinals do not exist (see [19] or [5, Chapter 14, §6]). In our context, complete ultrafilters give rise to situations where $D^\star = D^\sharp$, showing that Theorem 3.11 cannot be extended beyond the countable case.

Proposition 5.1. *Suppose $\mathscr{U}$ is κ-complete, and let Δ be a family of subsets of K with $|\Delta| \leq \kappa$. Then, the following hold.*

(a) $\left(\bigcap_{J\in\Delta} J\right)^\star = \bigcap_{J\in\Delta} J^\star$.

(b) If $\Delta \subseteq \mathrm{Zar}(D)$ and $\bigcap_{V\in\Delta} V = D$ then $\mathrm{KU}(D, \Delta, \mathscr{U}) = D^\star$.

(c) If D is integrally closed and $|\mathrm{Zar}(D)| \leq \kappa$, then $D^\star = D^\sharp$.

Proof. Part (a) follows in the same way of Lemma 3.6, using the completeness of $\mathscr{U}$. The other two points are immediate consequences of the first one. □

Remark 5.2. If $\mathscr{U}$ is a κ-complete ultrafilter, and $\mathscr{U}$ is not principal, then the cardinality of the index set is strictly greater than κ: otherwise, $\{I \setminus \{i\} \mid i \in I\}$ would be a family of at most κ subsets in the ultrafilter with empty intersection, which would imply that the empty set is in $\mathscr{U}$, a contradiction.

In particular, if κ is an infinite cardinal and the index set I is countable, then every non-principal ultrafilter is countably incomplete (i.e., it is $\aleph_0$-incomplete). Since we are considering only non-principal ultrafilters, Proposition 5.1 does non apply (non-trivially) to the case of countable index set considered in Sections 3 and 4.

In particular, if $\mathscr{U}$ is κ-complete, then $D^\sharp$ may not be a Bézout domain, or even a Prüfer domain. For example, if L is a countable field, X, Y indeterminates, and $D := L + YL(X)[[Y]]$, then $\mathrm{Zar}(D)$ is countable (as it is composed by $L(X)((Y))$, $L(X)[[Y]]$, $L[X]_{(1/X)} + YL(X)[[Y]]$ and the rings $L[X]_{(f)} + YL(X)[[Y]]$, as f ranges among the irreducible polynomials of $L[X]$) so $D^\star = D^\sharp$; however, D is not a Prüfer domain, and thus neither is $D^\star$. Therefore, the fact that $D^\sharp$ is a Bézout domain for arbitrary index sets would imply that κ-complete ultrafilters (and thus measurable cardinals) cannot exist.

If we step outside the complete case, however, the situation becomes much better. The following "approximation" method can be seen as a generalization of the proof of Theorem 4.1.

Proposition 5.3. *Let D be an integral domain, and let $\{T_i\}_{i\in I}$ be a set of overrings of D. Let*

$$\Delta := \{V \in \mathrm{Zar}(D) \mid \{i \in I \mid T_i \subseteq V\} \in \mathscr{U}\}.$$

Then, $\prod_{\mathscr{U}} T_i \subseteq \mathrm{KU}(D, \Delta, \mathscr{U})$. In particular, if each T_i is a Prüfer (respectively, Bézout) domain then $\mathrm{KU}(D, \Delta, \mathscr{U})$ is a Prüfer (resp., Bézout) domain.

Proof. Let $\mathbf{x} := [x_i] \in \prod_{\mathscr{U}} T_i$, and let $V \in \Delta$. Then,

$$\{i \mid x_i \in V\} \supseteq \{i \mid x_i \in T_i\} \cap \{i \mid T_i \subseteq V\}$$

and both sets on the right hand side are in $\mathscr{U}$ (the first one since $\mathbf{x} \in \prod_{\mathscr{U}} T_i$, the second one by the definition of Δ). Thus, $\mathbf{x} \in \mathrm{KU}(D, \Delta, \mathscr{U})$ and $\prod_{\mathscr{U}} T_i \subseteq \mathrm{KU}(D, \Delta, \mathscr{U})$.

The "in particular" statement follows since if each T_i is a Prüfer (resp., Bézout) domain then so is their ultraproduct, and an overring of a Prüfer (resp., Bézout) domain is still Prüfer (resp., Bézout). □

Before showing how to extend Theorem 4.1, we use this criterion together with a result of Roquette. Recall that a field F is *real closed* if it is elementary equivalent to the field of real numbers; that is, if every first-order property in the language of fields is true in F if and only if it is true in $\mathbb{R}$.

Lemma 5.4. *Let F be a field that is not algebraically closed nor real closed. Then, there are irreducible polynomials over F of arbitrary large degree.*

Proof. Let $\overline{F}$ be the algebraic closure of D. If F is not algebraically closed nor real closed, then $[\overline{F} : F] = \infty$ [10, Corollary 9.3]. If F is perfect, the claim follows. If F is not perfect, and it has characteristic p, then there is an element $a \in F \setminus F^p$, and for every l the polynomial $X^{p^l} - a$ is irreducible [10, Corollary 9.2]; the claim follows again. □

Proposition 5.5. *Let $(D, \mathfrak{m})$ be a local domain, and let $F := D/\mathfrak{m}$. Suppose that F is not algebraically closed nor real closed. Let Δ be the set of valuation overrings V of D such that the algebraic closure of F in $V/\mathfrak{m}_V$ is finite over F. Then,* $\mathrm{KU}(D, \Delta, \mathscr{U})$ *is a Prüfer domain.*

Proof. Since F is not algebraically closed nor real closed, by Lemma 5.4 we can find a sequence $\{\lambda_n\}_{n \geq 1}$ of irreducible polynomials over F of increasing degree. Let $T_i := \bigcap\{V \in \Delta \mid \lambda_i \text{ has no roots in } V/\mathfrak{m}_V\}$. By [16, Theorem 1], each T_i is a Prüfer domain.

Let V be a valuation overring of D. By hypothesis, the degree of the algebraic closure of F in $V/\mathfrak{m}_V$ over F is finite, say equal to n: since the degrees of the λ_i are increasing, only finitely many λ_i can have a root in $V/\mathfrak{m}_V$. Therefore, $T_i \subseteq V$ for all but finitely many i; in particular, $\{i \in I \mid T_i \subseteq V\} \in \mathscr{U}$ for all $V \in \Delta$. By Proposition 5.3, it follows that $\mathrm{KU}(D, \Delta, \mathscr{U})$ is a Prüfer domain, as claimed. □

In particular, the set Δ of Proposition 5.5 contains all valuation overrings whose residue field is F and those whose residue field is purely transcendental over F.

Corollary 5.6. *Let $(D, \mathfrak{m})$ be a local domain, and let Δ be the set of valuation overrings of D with finite residue field. If $\Delta \neq \emptyset$, then* $\mathrm{KU}(D, \Delta, \mathscr{U})$ *is a Prüfer domain.*

We now want to apply Proposition 5.3 more directly.

Proposition 5.7. *Let D be an integral domain and $\Delta \subseteq \mathrm{Zar}(D)$, and suppose there is an injection $\psi : \Delta \longrightarrow \mathscr{U}$. For every $i \in I$, let $T_i := \bigcap\{V \in \Delta \mid i \in \psi(V)\}$. Then, $\prod_{\mathscr{U}} T_i \subseteq \mathrm{KU}(D, \Delta, \mathscr{U})$.*

Proof. For every $V \in \Delta$, the set $\{i \in I \mid T_i \subseteq V\}$ contains $\psi(V)$, and thus is in $\mathscr{U}$. The claim follows from Proposition 5.3. □

Since we are trying to show that the Kronecker-ultrafilter ring is Bézout, we want the T_i of the previous proposition to be Bézout; the easiest way to guarantee this property is to require them to be a finite intersection of valuation rings.

Definition 5.8. An ultrafilter $\mathscr{U}$ on I is *regular* if there is a family $E \subseteq \mathscr{U}$ such that:

- $|E| = |I|$;
- each $i \in I$ belongs to only finitely many $X \in E$.

Remark 5.9.

(1) If $\mathscr{U}$ is regular, then every element of $\mathscr{U}$ has the same cardinality of I.
(2) If I is countable, every free ultrafilter is regular (if $I = \mathbb{N}$, take E formed by the sets $[n, \infty)$).

Theorem 5.10. *Let D be an integral domain. Suppose that $|I| \geq |\Delta|$ and that $\mathscr{U}$ is a regular ultrafilter. Then,* $\mathrm{KU}(D, \Delta, \mathscr{U})$ *is a Bézout domain.*

Proof. Take a family $E \subseteq \mathscr{U}$ that makes $\mathscr{U}$ into a regular ultrafilter: then, there is an injection $\psi : \Delta \longrightarrow E \subseteq \mathscr{U}$. Define T_i as in Proposition 5.7. Since $\mathscr{U}$ is regular, each T_i is a semilocal Bézout domain; hence, $\prod_{\mathscr{U}} T_i$ is a Bézout domain, and thus also $\mathrm{KU}(D, \Delta, \mathscr{U})$ (which is an overring of $\prod_{\mathscr{U}} T_i$) is Bézout. □

Note that, using Remark 5.9(2), this theorem can be seen as a generalization of Theorem 4.1.

Furthermore, suppose that φ is a first-order property such that:

- φ holds for semilocal Prüfer domains;
- if φ holds for the Bézout domain T, then it holds also for all overrings of T.

Then, under the hypothesis of Theorem 5.10 φ holds for every T_i, and thus holds also for the ultraproduct $\prod_{\mathscr{U}} T_i$ and for the Kronecker-ultrafilter ring $\mathrm{KU}(D, \Delta, \mathscr{U})$. Examples of this phenomenon is when φ is "being an elementary divisor domain" or when φ is "having stable range 1" (see [17] for the definitions).

For regular ultrafilters, we can actually say more about the set $\Delta^\star$.

Proposition 5.11. *Let D be an integral domain. Suppose that $|I| \geq |\Delta|$ and that $\mathscr{U}$ is a regular ultrafilter. Then, $\Delta^\star := \{V^\star \mid V \in \Delta\}$ is discrete in the constructible topology of* $\mathrm{Zar}(D^\sharp)$.

Proof. Fix a valuation overring W of D. We shall show that $\{W^\star\}$ is equal to $\mathcal{B}(\mathbf{x}) \cap (\Delta^\star \setminus \mathcal{B}(\mathbf{y}))$ for some $\mathbf{x}, \mathbf{y} \in K^\star$; this will show that $\{W^\star\}$ is open in the constructible topology.

As in the previous proof, we denote by E a subfamily of $\mathscr{U}$ that makes $\mathscr{U}$ into a regular ultrafilter.

We first construct $\mathbf{x}$. Let $\Delta_1 := \{V \in \Delta \mid V \supseteq W\}$; then, there is an injection $\psi_1 : \Delta \setminus \Delta_1 \longrightarrow E$. For every i, define R_i as the intersection of all $V \in \Delta \setminus \Delta_1$ such that $i \in \psi_1(V)$; then, every R_i is a semilocal Bézout domain that does not contain W. Hence, we can find an $x_i \in W \setminus R_i$; let $\mathbf{x} := [x_i]$. By construction, $\mathbf{x} \in W^\star$. On the other hand, if $V \in \Delta \setminus \Delta_1$, then

$$\{i \in I \mid x_i \notin V\} \supseteq \psi_1(V) \in \mathscr{U}$$

and thus $\mathbf{x} \notin V$; hence, $\mathcal{B}(\mathbf{x}) \cap \Delta^\star = \Delta_1^\star$.

To construct $\mathbf{y}$, we use essentially the same method: let $\Delta_2 := \{V \in \Delta \mid V \subseteq W\}$, and take an injection $\psi_2 : \Delta \setminus \Delta_2 \longrightarrow E$. For every i, define T_i as the intersection of all $V \in \mathrm{Zar}(D) \setminus \Delta_2$ such that $i \in \psi_2(V)$; then, every T_i is a semilocal Bézout domain that is not contained in W. Hence, we can find an $y_i \in T_i \setminus W$; let $\mathbf{y} := [y_i]$. By construction, $\mathbf{y} \notin W^\star$, while if $V \in \Delta \setminus \Delta_2$, then

$$\{i \in I \mid y_i \in V\} \supseteq \psi_2(V) \in \mathscr{U}$$

and thus $\mathbf{y} \notin V$; hence, $\mathcal{B}(\mathbf{y}) \cap \Delta^\star = \Delta^\star \setminus \Delta_2^\star$, i.e., $\Delta^\star \setminus \mathcal{B}(\mathbf{y}) = \Delta_2^\star$.

Therefore

$$\mathcal{B}(\mathbf{x}) \cap (\Delta^\star \setminus \mathcal{B}(\mathbf{y})) = \Delta_1^\star \cap \Delta_2^\star = (\Delta_1 \cap \Delta_2)^\star = \{W^\star\},$$

and so $\{W^\star\}$ is open in the constructible topology. Since this happens for every W, $\Delta^\star$ is discrete in the constructible topology. □

As a last application, we generalize Proposition 4.6.

Proposition 5.12. *Suppose $\mathscr{U}$ is a regular ultrafilter. Let D be an integral domain containing an infinite field F. If $|F|^{|I|} > |\Delta|$, then $\mathrm{KU}(D, \Delta, \mathscr{U})$ is a Bézout domain.*

Proof. By [5, Chapter 6, Corollary 3.21], the field $F^\star$ has cardinality at least $|F|^{|I|}$; furthermore, $F^\star \subseteq D^\star$ and so $F^\star \subseteq V^\star$ for every $V \in \mathrm{Zar}(D)$. By [12, Theorem 6.6], the intersection of any set of $\kappa < |F|^{|I|}$ valuation rings containing $F^\star$ and contained in $K^\star$ is a Bézout domain; in particular, we can apply this result to $\Delta^\star := \{V^\star \mid V \in \Delta\}$. Thus, $\mathrm{KU}(D, \Delta, \mathscr{U})$ is a Bézout domain. □

Remark 5.13.

(1) The proof of Theorem 6.6 of [12] does not actually use the fact that F is a field, but rather that F is a set of units such that $u - u'$ is a unit for every $u \neq u'$ in F. This property is conserved by passing from F to $F^\star$; thus, we can weaken the hypothesis of Proposition 5.12 in the same way.

(2) Since $|F|^{|I|} > |I|$, the hypothesis that $|F|^{|I|} > |\Delta|$ of Proposition 5.12 is weaker than the hypothesis $|I| \geq |\Delta|$ of Theorem 5.10; however, the latter theorem also covers the cases where we cannot find an infinite field F.

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