UNIVERSITY OF UDINE
DEPARTMENT OF MATHEMATICS, COMPUTER SCIENCE AND PHYSICS
PH.D. IN COMPUTER SCIENCE, CYCLE XXVIII

DISSERTATION

COALGEBRAIC SEMANTICS OF SELF-REFERENTIAL BEHAVIOURS

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ACADEMIC YEAR 2015/2016

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V1.0 (3rd March 2017)

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ABSTRACT

In this thesis we investigate the semantics of systems which can refer to themselves, *e.g.*, by “passing around" systems of the same kind as values (hence potential observables). For this reason, we refer to these systems as *self-referential*. Instances of this scenario are higher-order calculi like the λ-calculus [16], the calculus of higher-order communicating systems (CHOCS) [143], the higher-order π-calculus (HOπ) [124], HOcore [93], *etc*. It is well known that higher-order systems pose unique challenges and are difficult to reason about. Many bisimulations and proof methods have been proposed also in recent works [25–28, 89, 91, 93, 94, 102, 126–128, 140–142, 151]. This ongoing active effort points out that a definition of abstract self-referential behaviour is still elusive.

We address these difficulties by providing an abstract characterisation of self-referential behaviours as *self-referential endofunctors*, *i.e*. functors whose definition depends on their own final coalgebra. The construction of these functors is not trivial, since they must be defined at once with their own final coalgebra and due to the presence of both covariant and contravariant dependencies (*e.g*. arising from higher-order inputs). We provide such a construction, where algebraically compact functors [17, 53, 54] are the key technicality, like other works dealing with mixed-variance dependencies of some kind [25, 26, 48].

Similarly defined endofunctors arise from considering as object systems (*i.e*., those which can be values) only certain subclasses of systems (usually via some syntactic restriction) or a syntactic representations (*cf*. higher-order process algebras): self-referential endofunctors are shown to be universal among them. Universality renders self-referential endofunctors a touchstone for similar behavioural functors and offers the mathematical structure for assessing soundness and completeness of other models via the associated universal morphisms.

As a further contribution we provide a construction capturing infinite trace semantics by finality whereas the state of the art characterisations are weakly final [43, 145, 146]. This result, together with existing accounts of finite traces [63, 64, 74, 117], allows the definition of self-referential behaviours with respect to (in)finite trace semantics.

CONTENTS

1 INTRODUCTION

In this thesis we investigate the behaviour of systems that operate on objects endowed with their own dynamics *i.e.* actual systems. Instances of this scenario are higher-order calculi like the λ-calculus [16], the calculus of higher-order communicating systems (CHOCS) [143], the higher-order π-calculus (HOπ) [124, 129], HOcore [93], and any other language where processes or systems can be passed around, as opposed to first-order ones which operate on static objects. It is well known that this sort of systems poses unique challenges and is difficult to reason about crucial properties like equivalence.

Bisimulation is an established and powerful operational method for proving behavioural equivalence of systems by coinduction. This is a general proof method pioneered by Milner [107] and later captured categorically by Aczel and Mendler [2] by means of coalgebras. This generalisation led to the flourishing of the coalgebraic framework for modelling concurrent and reactive systems, automata, and infinite data structures [121]. The coalgebraic methodology can be thought of as a "semantics first" approach to system reasoning: it is founded on the principle of abstracting from any specific concrete representation of systems. Instead, coalgebras focus on the definition of semantic models capturing the behaviour of systems under scrutiny—strong bisimulation for CCS processes is just bisimulation for (non-deterministic) labelled transition systems. In return, this level of abstraction enables a fruitful cross-fertilizing exchange of definitions, notions, and techniques with similar contexts and theories.

In the case of systems that operate on static objects (*e.g.* automata, labelled transition systems with values, names) there is a common consensus about what their behaviour is and how the associated behavioural equivalences are defined, and the related proof methods are rather mature. The situation is less clear when objects are actual systems as in the case of systems described by higher-order languages. In fact, there is a decades long and ongoing effort concerning bisimulations for higher-order calculi: many bisimulations and proof methods have been proposed, also in recent works such as [89, 91, 93, 94, 102, 126–128, 140, 141, 151]. Each of these works considers syntactic representation of systems given in some specific language and proposes bisimulations that, roughly speaking, test subject and object systems with challenges that are devised on the guise of contexts and are syntactically forged from knowledge about the observed computation. This approach can be traced back to the notion of contextual equivalence which these works take as their reference semantics. This notion of semantic equivalence is due to Morris and defines as equivalent all systems that "behave in the same manner" in any given "context" [112]. Clearly, the heart of this definition is understanding what "to behave in the same manner" means, what contexts are, and how they interact with system dynamics. These pieces of information depend on how an observer can monitor and test system computations via constructs of the language and available knowledge on the computation history. However fruitful, the approach followed by these works is tied to the specific constructs of the language used for representing systems. Moreover, there is no systematic way to derive the "right suite of tests" for a given higher-order language—see *e.g.* [75, 123, 124, 127, 141] for a discussion concerning testing of higher-order inputs and outputs and the correlation with contextual equivalence.

In contrast, we aim to develop a uniform methodology for the definition of semantic models for systems under scrutiny that abstracts from specific languages used for their representation. The main motivation behind this effort is that a characterisation of this kind enables the transfer between a wide range of systems of existing results and the development of general ones. In order to achieve this objective, we follow a "semantics first" approach:

- we take a completely semantic perspective on system dynamics and concentrate on their associated computational behaviours;

- we do not assume any programming language and avoid syntactic artefacts such as contexts or forged processes.

Reworded, we model systems whose objects are systems of the same type as coalgebras. This different point of view is the distinguishing trait of this work. As a consequence, we term the behaviour of systems considered in this work *self-referential* and avoid, as much as possible, the terminology usually associated to higher-order languages—for the sake of exposition, we sometimes relate, at the intuition level, to familiar examples of such languages.

In this introductory chapter we recall the coalgebraic approach to modelling of operational semantics, describe the main challenges posed by self-referential behaviours, and the overall methodology we put forward in order to solve them. In Section 1.1 we briefly discuss coalgebras and some basic notions relevant to the constructions described in the sequel of this chapter. In Section 1.2 we illustrate the semantics of self-referential systems by means of a simple example which we rephrase in Section 1.3 using the language of coalgebras. In Section 1.4 we outline the structure and the main contributions of the thesis.

1.1 COALGEBRAS

Coalgebras are a well established framework for modelling and studying the operational semantics of (abstract) computational devices such as automata, concurrent systems, and reactive ones; the methodology is termed *universal coalgebra* [121]. In this approach, the first step is to define a *behavioural endofunctor* F over **Set** (or other suitable category), modelling the computational aspects under scrutiny in the sense that, for X a set of states, FX is the set of possible behaviours over X. Then, a system is modelled by an F-coalgebra *i.e.* a pair $(X, h\colon X \to FX)$ where the set X (called *carrier*) is the state-space of the system and the map h (called *structure*) associates each state with its behaviour. Coalgebras are often identified with their structure; in this situation the carrier X of an F-coalgebra $h\colon X \to FX$ will be denoted by $|h|$. The definition of the endofunctor F constitutes the crucial step of this method, as it corresponds to specify the behaviours that the systems under scrutiny are meant to exhibit *i.e.* their *dynamics* and *observations*. Hence, F-coalgebras and the systems they model are said to be of *type* F. Once a behavioural endofunctor is defined, this canonically determines a notion of coalgebra homomorphism (*i.e.* structure preserving maps between carriers) and bisimulation (which is the abstract generalization of Milner's strong bisimulation, see *e.g.* [121, 138]). Moreover, under mild conditions on the behavioural endofunctor F, there exists a *final F-coalgebra* $\nu(F)$ which describes all abstract behaviours of type F. Final F-coalgebras are final objects in the category formed by F-coalgebras equipped with their homomorphisms [2, 149]. Finality means that every F-coalgebra h has a unique homomorphism $!_h$ into the final F-coalgebra. The morphism $|h| \to |\,\nu(F)|$ underlying $!_h\colon h \to \nu(F)$ uniquely associates each state in $|h|$ with an semantics, called *final semantics*, in the form of an abstract behaviour (see the example below). Final semantics uniquely associates bisimilar states to the same abstract behaviour. As a consequence, states of final coalgebras can only be bisimilar to themselves. This property is called *strong extensionality* and identifies final homomorphisms with the coinductive proof principle [73]. Because of its relation with coinduction, the unique homomorphism from a coalgebra h into the final one is also called *coinductive extension* of h.

As an example, we discuss stream systems *i.e.* computational devices that perform deterministic transitions and outputs. Fix an alphabet of output symbols A, a stream systems on A is a triple (X, o, t) where X is the set of states forming the system, $o\colon X \to A$ is the output function associating each state with its output symbol, and $t\colon X \to X$ is the successor function associating each state with its successor state. For (X, o, t) and (X', o', t') stream systems on the same output alphabet A, a stream bisimulation on them is any relation $R \subseteq X \times X'$

with the property that it relates only states whose outputs are equal and whose successor states are related by R as well *i.e.*:

$$x\ R\ x' \implies o(x) = o'(x') \wedge t(x)\ R\ t'(x').$$

A stream system homomorphism from (X, o, t) into (X', o', t') is any function $f \colon X \to X'$ that preserves outputs and successors *i.e.*:

$$\forall x \in X\ \big(o(x) = (o' \circ f)(x) \wedge (f \circ t)(x) = (t' \circ f)(x)\big)\,.$$

Equivalently, homomorphisms are functional relations with the additional property of being stream bisimulations for their source and target systems. Because spans of maps are in bijective correspondence with maps into products, stream systems on an alphabet A are in bijective correspondence with coalgebras for the endofunctor $A \times Id$ over **Set**. In particular, a stream system (X, o, t) is modelled by the coalgebra $(X, \langle o, t\rangle)$ where $\langle o, t\rangle \colon X \to A \times X$ is the function universally induced by o and t that takes each $x \in X$ to the pair $(o(x), t(x))$. The coalgebraic notion of bisimulation for this functor instantiates to that of stream bisimulation. The functor $A \times Id$ admits a final coalgebra which consists in the set A^ω, of streams on A, together with (the map into $A \times A^\omega$ induced by the span of) the functions head and tail. Finality guarantees that for every system there is a unique function into the coalgebra of streams that is coherent with system dynamics and observations. Reworded, the coinductive extensions associate a state x with the unique sequence of output symbols generated by the deterministic computation starting at x. This correspondence exhibits coalgebras of type $A \times Id$ as an adequate model for the semantics of stream systems. The universal property of final semantics has an additional, and extremely useful, consequence: it allows the definition of streams and operations on them as constructions of suitable stream systems [62, 122] which in turn can be regarded as "compact" representations of streams.

Besides canonical notions of bisimulation and abstract behaviours, many important properties and general results can be readily instantiated: general automata determinisation and minimisation [8, 30, 32, 131], the construction of canonical trace semantics [33, 43, 64, 70, 74, 84], weak bisimulations [34, 35, 58, 104, 120, 134, 135], the notion of *abstract GSOS* [82, 144], *etc*. We stress the fact that behavioural functors are "syntax agnostic": they define the semantic behaviours, abstracting from any specific concrete representation of systems. In the wake of these important results, many functors have been defined for modelling a wide range of behaviours: deterministic and non-deterministic systems [121]; systems with I/O, with names, with resources [50, 51]; systems with quantitative aspects such as probabilities or stochastic rates [47, 85, 105, 106, 136]; systems with continuous states [13], *etc*. Indeed, the theory of coalgebras is under active development and constitutes an established approach to several disparate areas.

1.2 SELF-REFERENTIAL SYSTEMS AND THEIR BEHAVIOURS

In this section we present an introductory discussion about the main characteristics and challenges of self-referential systems. To this end we consider stream systems and their self-referential equivalent as a case study. Self-referential stream systems are represented as "ordinary" stream systems except that they output objects endowed with their own dynamics. These dynamics are again that of a self-referential system of the same type and can be given in the same way. Therefore, a possible representation of self-referential stream systems are pairs composed by:

- a stream system whose carrier is the actual state-space and whose structure define the system transitions and outputs;
- a stream system whose carrier is the output alphabet and whose structure describes the dynamics of each output symbol.

Dynamics of self-referential stream systems and ordinary ones are of the same form since devices of both classes perform deterministic transitions and outputs. What tells them apart is what can be observed about these computations:

- in the case of ordinary stream systems, outputs are indistinguishable to an external observer whenever they are the same symbol;
- in the case of self-referential stream systems, outputs are indistinguishable to an external observer whenever they generate observationally equivalent computations.

Reworded, outputs of self-referential stream systems are essentially streams in the eye of the observer. Assuming otherwise would mean to ignore the structure of alphabets and regard self-referential systems as ordinary ones. All in all, this would be like comparing programs for they source code: we know these strings can drive the computation of some device but *per se* they are just plain text *i.e.* a static object. Finally, we remark that regarding outputs of self-referential stream systems as streams is coherent with that interpretation of states from an ordinary stream systems as representations of streams.

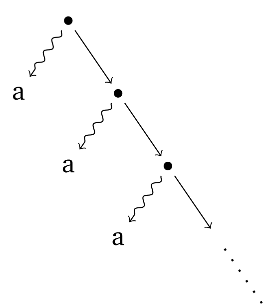
Recall from the previous section that the set of all streams for an alphabet carries the canonical stream system describing the behaviour of every other system on the same alphabet. It follows that to an external observer, all self-referential systems use as alphabets sets of streams equipped with their canonical system. Formally, these alphabets/sets of streams are exactly all the solutions to the following equation:

$$Z \cong Z^{\omega}. \tag{1.1}$$

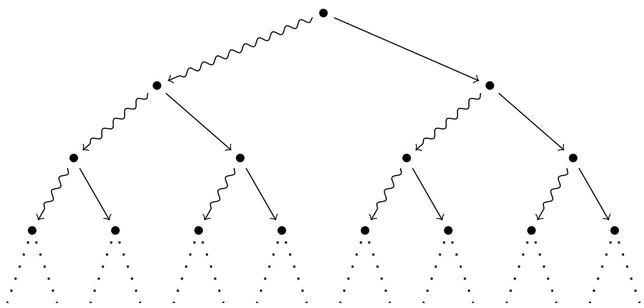
This equation has infinitely many (non-isomorphic) solutions: for Y any set, the set Y^{ω} solves (1.1). This situation poses the question of which solutions should be considered and under which circumstances. To this end we observe that functions between alphabets translate outputs of stream systems in a way that preserves behavioural equivalences. In this sense, a function $f\colon A \to B$ between alphabets that are solutions to (1.1) exhibits systems on B as complete with respect to those on A and *vice versa* it exhibits systems on A as sound with respect to those on B. If we organise solutions to (1.1) into a category whose morphisms are said functions, then the initial and final objects are characterised by the following property:

- the initial solution is sound with respect to any solution;
- the final solution is complete with respect to any solution.

The initial solution is the empty set 0 and corresponds to a degenerate case of stream systems where the alphabet is empty. The dynamics of stream systems prescribe computations to output a symbol at each transition and because there are no symbols no output can be performed: the codomain of the output function is the empty set and this forces the state-space to be empty as well. The final solution is the singleton 1 and corresponds to another degenerate case of stream systems where there is exactly one output symbol and hence exactly one stream. To understand why, consider the canonical stream system carried by the set of all streams. In the ordinary case, computations of the canonical system can be pictured as infinite sequences of output symbols.



In the self-referential case, computations of the canonical system can be pictured as binary trees since the state-space and output alphabet coincide.



These trees are all isomorphic since they have no leaves and are binary.

In order to present examples of non-degenerate final solutions, let us consider partial stream systems. A partial stream systems is a triple (X, o, t) whose components are defined as in the case of (total) stream systems except for the successor function t which can be partial. For (X, o, t) and (X', o', t') partial stream systems on A, a stream bisimulation on them is any relation $R \subseteq X \times X'$ with the property that it relates only states whose outputs are equal and whose successor states are either both in R or both undefined *i.e.*:

$$x \, R \, x' \implies o(x) = o(x') \wedge \left(\left(t(x)\uparrow \wedge t'(x')\uparrow \right) \vee t(x) \, R \, t'(x') \right)$$

where the predicate $t(x)\uparrow$ is true if and only if t is undefined on x. The semantics of these systems associates states with the non-empty finite or infinite sequences of symbols observed as output. For brevity we will refer to these sequences as "partial streams". Note that these definitions coincide with those for total stream systems whenever these are regarded as partial ones. Self-referential partial stream systems are defined as one would expect *i.e.* as partial stream systems whose alphabet carries a similar structure. To an external observer, outputs are partial streams which also form their alphabet. Formally, these alphabets/sets of partial streams are exactly all sets solution to the following equation:

$$Z \cong Z^{+} + Z^{\omega} \tag{1.2}$$

A solution to this equation is the set of all infinite trees with the property that each of their nodes has one or two children.

Stream bisimulation is oblivious to the additional structure carried by alphabets of self-referential stream systems and, in general, it fails to capture their semantics. We observe that the semantics of ordinary and self-referential stream systems coincide whenever their alphabets are solutions to (1.1) *i.e.* whenever their alphabets are composed by the all abstract behaviours of self-referential stream systems. This correspondence is a consequence of the strong extensionality property presented by the canonical stream systems carried by these alphabets.

Systems of this type have the remarkable property that final semantics coincides with the semantics of self-referential systems they model. This presents us with a situation somehow similar to that of timed automata and timed transition systems where the former are concrete representations of systems whose semantics is modelled by the latter [9, 24, 36]. Because of their rôle, timed transition systems are often called *semantic models* of timed automata dynamics. We adopt this terminology also for self-referential (partial) stream systems:

- concrete representations are the pairs of (partial) stream systems described at the beginning of this section;
- semantic models are (partial) stream systems on alphabets identified by solutions to equation (1.1) (equation (1.2)).

We remark that semantic models are intended for reasoning about the behaviour of self-referential systems and that in practice one should prefer concrete representations. In fact, semantic models require solutions to (1.1) and (1.2) to be known in advance whereas the dynamics of enough symbols from the output alphabet can be finitely expressed by a suitable stream system or other formalisms.

1.3 SELF-REFERENTIAL ENDOFUNCTORS

Stream systems on an alphabet A are modelled by coalgebras for the endofunctor $A \times Id$ over **Set**. Actually, the definition of these behavioural endofunctors is *generic* where the term "generic" has to be interpreted in the sense of parametric polymorphisms: these endofunctors use symbols from their alphabet in the same way. In fact, this definition schema corresponds to a functor:

$$S \colon \mathbf{Set} \to \mathbf{End}(\mathbf{Set})$$

associating alphabets and substitutions (*i.e.* functions) with endofunctors modelling stream systems and natural transformations between as formalised by the following mappings:

$$A \mapsto A \times Id \qquad s \mapsto s \times Id.$$

These in turn extends to functors between categories of coalgebras that "implement" the alphabet substitution:

$$(X, \langle o, t\rangle) \mapsto (X, \langle s \circ o, t\rangle) \qquad f \mapsto f.$$

In this work, we refer to functors like S as (covariant) *behavioural schemata*.

Behavioural schemata offer the base for rephrasing the concrete notions of representations and semantic models of self-referential stream systems into the coalgebraic language. Concrete representations of these systems are pairs of coalgebras $(h\colon X \to S_A(X), d\colon A \to S_A(A))$ where h describes the state-space and transitions of the system and d the dynamics of each output symbol. Semantic models of self-referential stream systems are coalgebras whose type is obtained instantiating the schema S on an alphabet of all streams of the same type *i.e.* a solution to (1.1). These behavioural endofunctors are precisely the solutions of the equation below.

$$F \cong |\nu F| \times Id \tag{1.3}$$

Likewise, semantic models of self-referential partial stream systems are coalgebras for endofunctors that are solutions to the equation below.

$$F \cong |\nu F| \times (Id + 1). \tag{1.4}$$

The correspondence with (1.1) and (1.2) is immediate once the equations are reformulated in the unknowns F and Z representing the behavioural endofunctor and the carrier of its final coalgebra:

$$\begin{cases} F \cong Z \times Id \\ Z \cong |\nu F| \end{cases} \qquad \begin{cases} F \cong Z \times (Id + 1) \\ Z \cong |\nu F| \end{cases}$$

This is enough to prove that definitions based on (1.1) and (1.3) (resp. (1.2) and (1.4)) are equivalent. In fact, final coalgebras for these behavioural endofunctors correspond to the canonical system on the sets of all (partial) streams. Endofunctors modelling self-referential systems present us with the challenging characteristic of being defined in terms of their own final coalgebra, which can be defined (if it exists) only after the endofunctor is defined—a circularity!

This circularity is the gist of self-referential systems and behaviours: any attempt to escape it would be restricting and distorting. One may be tempted to take as values some representation of behaviours (*e.g.*, states of a stream system, processes, terms), but this would fall short:

- first, the resulting behaviours would not be really self-referential, but rather behaviours manipulating some *ad hoc* representation of behaviours;
- second, we would need some mechanism for moving between behaviours and their representations–which would hardly be complete;
- third, the resulting functor would not be abstract and independent from the syntax of processes, thus hindering the possibility of reasoning about the computational aspect on its own, and comparing different models that share the same kind of behaviour.

This fundamental shift from terms/processes as values to behaviours as values is at the hearth of this work.

To avoid the circularity of (1.3), one may consider to model the semantics of self-referential stream systems by means of coalgebras like $X \to X \times X$. This point of view aligns with the interpretation of outputs as forked computations and still captures the initial and final solutions to (1.1): in fact 0 and 1 carry the initial and final invariants for $Id \times Id$, respectively. Nonetheless, this strategy is bound to fail due to the following issues:

- system outputs are restricted to the system own state-space since outputs are modelled by the first projection of $X \to X \times X$;
- this approach cannot be used to model systems with inputs since elements of their state-space would occur in contravariant position *e.g.* $X \to X^X$.

Both issues are non-trivial problems and, although the first one may be object of debate, the second is not. The modelling of self-referential behaviours with inputs prompts us to maintain inputs and outputs as parameters in the definition of behavioural endofunctors and thus to consider (mixed-variance) behavioural schemata. This approach allows us to define self-referential endofunctors also in presence of inputs and outputs via recursive equations akin to (1.3). Solving these equations is non-trivial, especially because unknowns may occur in both covariant and contravariant positions, initial and final invariants for behavioural endofunctors may be used in combination and potentially nested, and said invariants may not exist. We equip solutions with a notion of morphism that capture soundness in the following sense: solution morphisms induce functors between categories of coalgebras that preserve state-spaces and bisimulations. In this setting we identify as initial and final solutions as *canonical solutions* since:

- final solutions identify behavioural endofunctors that are complete with respect to any other endofunctor within the given schema;
- initial solutions identify behavioural endofunctors that are sound with respect to any other endofunctor within the given schema.

These canonical solutions support reasoning about self-referential systems (for the same schema) even if they may have semantic models of different type. Solution morphism into the final solution induce functors into a shared category of coalgebras such that they preserve bisimulations and carriers. Dually, solution morphisms from the initial solution induce functors from a shared category of coalgebras and such that they preserve bisimulations and carriers.

Algebraically compact functors and categories [17, 53, 54] are the technical foundation enabling most of our constructions. These notions were initially introduced by Freyd and Barr as part of an abstract framework for developing category theoretic domains. Since these seminal works, two main classes of algebraically compact functors have emerged: that of locally continuous functors and that of locally contractive functors. Historically, the former is the first non-trivial class of algebraically compact functors to be identified (see [17]) and was initially studied as part of a categorical generalisation of order-theoretic constructions used in domain theory, especially Scott's limit-colimit coincidence result [130]. The class of locally contractive functors was introduced more recently as the technical foundation of guarded (co)recursion and guarded type theory [25–28, 142]. In the thesis we instantiate our abstract construction for computing self-referential endofunctors to both classes of algebraically compact functors and, whenever this is not possible, we provide alternative constructions for computing approximated semantics of self-referential behaviours using techniques akin to abstract interpretation and forcing.

1.4 STRUCTURE AND MAIN CONTRIBUTIONS OF THE THESIS

In Chapter 2 we recall preliminary notions on sheaf categories and algebraically compact functors; the chapter is mainly aimed at fixating the notation and the terminology that will be used in the rest of the thesis. All the material in this chapter is not original and can be found in the referred works and any textbook on sheaf theory. The only content we were not able to directly find in the literature is limited to a short section about categories enriched over categories of sheaves. Nonetheless, basic definitions contained in that section are instances of standard notions from enriched category theory.

In Chapter 3 we focus on linear and trace semantics for coalgebras. This chapter is mainly aimed at presenting an alternative semantics that can be used in the definition of self-referential behaviours. To this end we firstly recall how linear semantics, and especially trace semantics, are captured via final semantics in Kleisli categories. Then we introduce a novel and general construction for defining coalgebras whose final semantics captures (possibly) infinite trace semantics for the systems under scrutiny. To this end, we use ingredients from sheaf, Kleisli, and algebraically compact categories in a combination that allows us to perform guarded (co)recursion in the context of Kleisli categories of suitable monads. We term this technique *guarded Kleisli (co)recursion*. This construction, together with the account of infinite trace semantics it underpins, are the main contributions of the chapter. As a further contribution, an intermediate step of our construction identifies behavioural endofunctors whose final semantics coincides with certain behavioural pseudo-metrics due to Barr and Adámek [3, 18]. This result leads to a modest generalisation of these behavioural metrics and points to a possible application to richer metrics.

In Chapter 4 we present the main contribution of this work: a coalgebraic account of self-referential behaviours. We start from systems characterised by covariant behavioural schemata such as stream systems and then proceed towards the general situation of systems characterised by mixed-variance behavioural schemata like those performing higher-order inputs and outputs. For each of these classes of self-referential systems:

- we identify endofunctors whose coalgebras are models of self-referential systems;
- we study endofunctors characterising canonical models;
- we provide general constructions for computing such endofunctors.

We conclude the chapter considering "relaxed" models of self-referential systems. As the keyword "relaxed" suggests, these models do not fully capture self-referential behaviours but are only an approximation of them. Nonetheless, these are of interests because they can be defined in a wider array of settings.

2 PRELIMINARIES

2.1 CATEGORY-VALUED SHEAVES OVER SITES

In this section we recall basic concepts of sheaf theory; we refer the reader to [19, 59, 76, 77, 101, 137] for a thorough introduction to the topic.

2.1.1 Sites

Sites are a categorical generalisation of topological spaces and locales. Roughly speaking, sites are categories equipped with additional data describing how their objects can be "covered" by families of objects: on one hand the firsts provide "well-behaved quotients" of the seconds and, on the other hand, the latter provide "localizations" of the former.

DEFINITION 2.1. *A* coverage *on a category* **S** *consists of a rule J assigning to each object U in* **S** *a collection of families of morphisms $\{p_i \colon U_i \to U\}_{i \in I}$ called* covering families *with the following property: if $\{p_i \colon U_i \to U\}_{i \in I}$ is a covering family and $g \colon V \to U$ is a morphism in* **S**, *then there exists a covering family $\{q_k \colon V_k \to V\}_{k \in K}$ such that each composite $g \circ q_k$ factors through some p_i as depicted in the diagram below.*

$$\begin{array}{ccc} V_k & \longrightarrow & U_i \\ {\scriptstyle q_k}\downarrow & & \downarrow{\scriptstyle p_i} \\ V & \underset{g}{\longrightarrow} & U \end{array} \tag{2.1}$$

A good source of examples are topologies and topological bases. A topological base for a set S is a collection $\mathcal{B}_S$ of subsets of S (called basic open sets) subject to the following two requirements:

1. $S = \bigcup \mathcal{B}_S$;
2. for any U_1, U_2 in $\mathcal{B}_S$, if $s \in U_1 \cap U_2$ then there is $U_0 \in \mathcal{B}_S$ such that $s \in U_0$ and $U_0 \subseteq U_1 \cap U_2$.

Given a topological base $\mathcal{B}_S$, let **S** be the poset $(\mathcal{B}_S, \subseteq)$ regarded as a thin category[1] and define the coverage J on it as the function mapping every basic

[1] A category is called thin or posetal whenever all parallel morphisms are equal.

open set to the set of its basic open covers:

$$J(U) = \left\{\{U_i \to U\}_{i\in I} \,\middle|\, U = \bigcup_{i\in I} U_i\right\}.$$

In order to prove that this function is indeed a coverage, we construct for each covering family on U and $V \subseteq U$, a suitable covering family on V. For $\{U_i \to U\}_{i\in I} \in J(U)$ and $V \to U$, consider a family $\{V_{i,x} \to V\}$ such that for each $i \in I$ and $x \in V \cup U_i$, $V_{i,x}$ is a basic open with the property that $x \in V_{i,x}$ and $V_{i,x} \subseteq V \cup U_i$ (which exists by definition of base). Because $V \cap U_i = \cup_{x\in V\cap U_i} V_{i,x}$ and $V = \cup_{i\in I} V \cap U_i$, the family $\{V_{i,x} \to V\}$ belongs to $J(V)$. Finally, note that each $V_{i,k} \to V$ satisfies (2.1) by construction. Coverages for topological spaces are obtained via the same construction.

A special case of bases are those closed under finite intersections such as the set $\left\{\{s\}^{\downarrow} \,\middle|\, s \in S\right\}$ of *cones* in a preorder $(S, \leq)$. From the point of view of coverages, these topological bases are closed under finite intersections whenever their coverages have pullbacks.

DEFINITION 2.2. *A coverage J on a category* **S** *with enough pullbacks is said to* have pullbacks *whenever it has the following property: if $\{p_i \colon U_i \to U\}_{i\in I}$ is a covering family and $g \colon V \to U$ is a morphism, then the family of pullbacks of g along each p_i is a covering family of V.*

Coverages associated to topological spaces and complete Heyting algebras are instances of a stronger notion of coverage known as *Grothendieck coverage* or *Grothendieck topology*. These are more conveniently presented in terms of particular covering families called (covering) sieves.

DEFINITION 2.3. *For U an object in a category* **S***, a* sieve *S on U is a covering family on U that is closed by post-composition* i.e.*, for all p and q with suitable domain and codomain it holds that:*

$$p \in S \implies p \circ q \in S.$$

The set $\{p \mid \mathrm{cod}(p) = U\}$ is the maximal sieve *on U. For S a sieve in U and $g \colon V \to U$ a morphism in* **S***, $g^*(S)$ is the sieve on V consisting of all morphisms h such that $g \circ h$ factors through some morphism in S.*

DEFINITION 2.4. *A* Grothendieck coverage *(or* Grothendieck topology*) on a category* **S** *is a rule J mapping each object U of* **S** *to a collection $J(U)$ of sieves called* covering sieves *on U with the following properties:*

- *the maximal sieve on U belongs to the collection $J(U)$;*
- *if $S \in J(U)$, then $g^*(S) \in J(V)$ for any arrow $g \colon V \to U$ in* **S***;*

- *if $S \in J(U)$ and S' is any sieve on U such that for all $g\colon V \to U$ in S the sieve $g^*(S')$ belongs to $J(V)$, then $S' \in J(U)$.*

Akin to how topological bases canonically induce topologies, coverages canonically induce Grothendieck topologies. For $\{p_i\colon U_i \to U\}_{i\in I}$ a family of morphisms of **S**, define the *sieve on U generated by K* as the smallest sieve S on U such that each $p_i\colon U_i \to U$ belongs to S. For J a coverage on a category **S**, define $\tilde{J}$ as the rule assigning each object U in **S** to the collection of sieves $\tilde{J}(U)$ on U with the following property: there exists a covering family $K = \{p_i\colon U_i \to U\}_{i\in I}$ in $J(U)$ such that the sieve on U generated by K is contained in $\tilde{J}(U)$.

PROPOSITION 2.1. *For J a coverage on **S**, $\tilde{J}$ is the smallest Grothendieck coverage on **S** with the property that every covering family of J generates a covering sieve.*

In virtue of this result, for J a coverage $\tilde{J}$ is called its *associated Grothendieck coverage*.

DEFINITION 2.5. *A* site *is pair $(\mathbf{S}, J)$ where **S** is a category (referred as the category* underlying *the site) and J a coverage for **S**. A site is said to* have pullbacks *whenever its coverage has pullbacks. A site is called a* Grothendieck site *whenever its coverage is a Grothendieck coverage. A site is called* small *if its underlying category is small.*

Sites are equipped with a suitable notion of morphisms given by functors between their underlying categories that are well-behaved with respect to the associated coverages.

DEFINITION 2.6. *For $(\mathbf{S}, J)$ and $(\mathbf{S}', J')$ sites, a* morphism of sites $f\colon (\mathbf{S}, J) \to (\mathbf{S}', J')$ *is a functor, denoted by f as well, between their underlying categories and subject to the following conditions:*

- *It is covering-flat* i.e. *for every finite category **I**, diagram $D\colon \mathbf{I} \to \mathbf{S}$ and cone C for $f \circ D$ in $\mathbf{S}'$ with vertex U, the family of morphisms (actually a sieve) with target U*

$$\{p\colon V \to U \mid p \text{ factors through the } f\text{-image of some cone for } D\}$$

belongs to the collection $J'(U)$.

- *It preserves covering families* i.e. *for $\{p_i\colon U_i \to U\}$ a covering family in J, $\{f(p_i)\colon f(U_i) \to f(U)\}$ is a covering family in J'.*

The functor $f\colon \mathbf{S} \to \mathbf{S}'$ is called underlying.

For $\mathcal{B}$ a topological base on a set S and $\mathcal{O}$ its generated topology, let $(\mathbf{S}_{\mathcal{B}}, J_{\mathcal{B}})$ and $(\mathbf{S}_{\mathcal{O}}, J_{\mathcal{O}})$ their corresponding sites. The coverage $J_{\mathcal{B}}$ extends to a coverage

$J'_{\mathcal{B}}$ on $\mathbf{S}_{\mathcal{O}}$ given on basic opens as $J_{\mathcal{B}}$:

$$J'_{\mathcal{B}}(U) = \begin{cases} J_{\mathcal{B}}(U) & \text{if } U \in \mathcal{B} \\ \emptyset & \text{otherwise} \end{cases}$$

and such that the Grothendieck coverage $\tilde{J}'_{\mathcal{B}}$ associated to $J'_{\mathcal{B}}$ is $J_{\mathcal{O}}$. Then, there is a morphism of sites $i\colon (\mathbf{S}_{\mathcal{B}}, J_{\mathcal{B}}) \to (\mathbf{S}_{\mathcal{O}}, J_{\mathcal{O}})$ whose underlying functor is given by the inclusion $\mathcal{B} \subseteq \mathcal{O}$.

The *Alexandrov topology* on a set S equipped with a preorder relation $\leq$ is the topology $\mathcal{A}(S, \leq)$ whose open sets are all the subsets of S that downward closed: $\mathcal{A}(S, \leq) = \left\{X^{\downarrow} \,\middle|\, X \subseteq S\right\}$ where $X^{\downarrow} = \{s \mid \exists x \in X \text{ s.t. } s \leq x\}$ is the downward closure of the subset S of S. The smallest topological base for $\mathcal{A}(S, \leq)$ is the set $\left\{\{s\}^{\downarrow} \,\middle|\, s \in S \land \forall X \subseteq S \setminus \{s\}(s \bigvee X)\right\}$ of primitive cones in $(S, \leq)$. In several examples we will consider ordinal numbers (seen as sets of ordinals) equipped with the standard ordering (whence omitted) and their Alexandrov topology. In particular, for α an ordinal, $\mathcal{A}(\alpha)$ is the successor ordinal $\alpha+1$ and, since ordinals are already cones, the smallest base for $\mathcal{A}(\alpha)$ is the set $\{\beta + 1 \mid \beta + 1 < \alpha\}$ of all successor ordinals in α. We refer to this topological base as the *successors base* for α. For instance, consider the first infinite ordinal ω, that is the set of natural numbers under natural ordering, $\mathcal{A}(\omega)$ is $\omega+1$ the set of natural numbers together with ω itself and the successor base for ω is given by all natural numbers except 0 (for it is a limit ordinal).

2.1.2 Sheaves

Let **C** and **S** be categories. A *(***C***-valued) presheaf on* **S** is any contravariant functor from **S** to **C**. Presheaves and natural transformations form the category $\mathbf{PSh}_{\mathbf{C}}(\mathbf{S})$ *i.e.* the category of (covariant) functors $\mathbf{Fun}(\mathbf{S}^{op}, \mathbf{C})$. Objects of **S** are called *stages* and will be usually denoted by letter U, V and variations thereof. For X a presheaf on **S**, the object $X(U)$ (written also as X_U) is called the *object of sections* or *value* of X at stage U and the morphism $X(r)\colon X(V) \to X(U)$ (written also as X_r) is called the *restriction* morphism from V to U. When values are taken in a concrete category, $x \in X(U)$ is called *section* of X at U. If **S** is thin, then we denote any $U \to V$ as $\iota_{U,V}$ and, if in addition **C** is concrete, we write $x|_U$, instead of $X_{\iota_{U,V}}(x)$, for action of restriction maps on sections.

Sheaves on a site are presheaves that are well-behaved with respect to the coverage for their site where "well-behaved" means values at any stage U are given by families of values on any covering on U. Reworded, they are presheaves that transport covering families to limits.

DEFINITION 2.7. *A presheaf X on a site $(\mathbf{S}, J)$ with values on a category* **C** *is called a* sheaf *if for every covering family $\{p_i\colon U_i \to U\}_{i \in I}$ the family of restriction* *morphisms* $\{X(p_i)\colon X(U) \to X(U_i)\}_{i \in I}$ *induce an isomorphism*

$$X(U) \cong \varprojlim_{i \in I} X(U_i).$$

Sheaves on a site $(\mathbf{S}, J)$ *form the full subcategory* $\mathbf{Sh_C}(\mathbf{S}, J) \hookrightarrow \mathbf{PSh_C}(\mathbf{S})$.

In the sequel we adopt the standard convention of omitting the category of values for (pre)sheaves of sets and thus write $\mathbf{PSh}(\mathbf{S})$ and $\mathbf{Sh}(\mathbf{S}, J)$ instead of $\mathbf{PSh_{Set}}(\mathbf{S})$ and $\mathbf{Sh_{Set}}(\mathbf{S}, J)$, respectively.

Sheaves on topological spaces Sites defined from topological spaces are such that their covering families are colimiting cones. It follows that sheaves on sites of this kind are presheaves transporting colimits induced by open covers to limits.

PROPOSITION 2.2. *Let* $(\mathbf{S}, J)$ *be a site induced by some topological space. A presheaf* X *of* $\mathbf{PSh_C}(\mathbf{S})$ *is a sheaf of* $\mathbf{Sh_C}(\mathbf{S}, J)$ *if, and only if, for every complete full subcategory* $I\colon \mathbf{U} \hookrightarrow \mathbf{S}$*:*

$$X \varinjlim I \cong \varprojlim X \circ I.$$

Akin to topologies and topological bases, several properties of sheaves on topological spaces can be reduced to statements about sheaves on a base generating that topology. Formally, this situation corresponds to an equivalence of categories induced by the inclusion of a base into its generated topology, as stated by Proposition 2.3.

PROPOSITION 2.3. *For* $\mathcal{B}$ *a topological base on a set* S*,* $\mathcal{O}$ *the topology generated by* $\mathcal{B}$*, and* $i\colon : (\mathbf{S}_\mathcal{B}, J_\mathcal{B}) \to (\mathbf{S}_\mathcal{O}, J_\mathcal{O})$ *the inclusion morphisms for their associated sites, the following is an equivalence of categories:*

$$\mathbf{Sh_C}(\mathbf{S}_\mathcal{O}, J_\mathcal{O}) \xrightarrow{(-\circ i)} \mathbf{Sh_C}(\mathbf{S}_\mathcal{B}, J_\mathcal{B}).$$

Examples of this situation are sheaves on Alexandrov topologies for partial orders and sheaves on their bases of primitive cones. In particular, for α an ordinal, $\mathcal{B}(\alpha)$ the set $\{\beta + 1 \mid \beta + 1 < \alpha\}$ of all successor ordinals in α, the inclusion $\mathcal{B}(\alpha) \subseteq \mathcal{A}(\alpha)$ yields an equivalence for the categories $\mathbf{Sh_C}(\mathcal{A}(\alpha))$ and $\mathbf{Sh_C}(\mathcal{B}(\alpha))$. We will rely on this equivalence in several occasions, especially in Chapter 3. For the sake of conciseness, we will write $\mathbf{Sh_C}(\alpha)$ for the category of sheaves on $\mathcal{A}(\alpha)$ and on $\mathcal{B}(\alpha)$ when confusion is unlikely.

Sheaves on sites with pullbacks If the category of values has enough products, sheaves on sites that have pullbacks are precisely presheaves that satisfy the (familiar) *descent condition*.

PROPOSITION 2.4. *Let* $(\mathbf{S}, J)$ *be a site that has pullbacks and* **C** *be a category with products. A* **C***-valued presheaf* X *on* $(\mathbf{S}, J)$ *is a sheaf if, and only if, for* $\{p_i \colon U_i \to U\}_{i \in I}$ *a covering family, the following diagram is an equaliser:*

$$X(U) \to \prod_{i \in I} X(U_i) \rightrightarrows \prod_{i,j \in I} X(U_i) \times_U X(U_j).$$

Sheaves of sets Sheaves taking sets as values are often introduced by the following component-wise definition based on *matching families of sections i.e.* families of sections that are pair-wise compatible with respect to restrictions sharing their target stage.

PROPOSITION 2.5. *Let* $(\mathbf{S}, J)$ *be a small site. A presheaf* X *on* $(\mathbf{S}, J)$ *and with values in* **Set** *is a sheaf if, and only if, for every covering family* $\{p_i \colon U_i \to U\}$ *and for every family of sections* $\{x_i\}_{i \in I}$ *with the property (called* matching*) that* $x_i \in X(U_i)$ *and* $X(g)(x_i) = X(h)(x_j)$ *for all* $g \colon V \to U_i$, $h \colon V \to U_j$, *and* $i, j \in I$, *then there is a unique element* $x \in X(U)$ *such that* $X(p_i)(x) = x_i$.

2.1.3 Associated and constant sheaves

Let $(\mathbf{S}, J)$ be a site and **C** a category. For a presheaf X in $\mathbf{PSh_C}(\mathbf{S})$ let X^+ denote, when defined, the presheaf given on each stage as:

$$X^+(U) \cong \varinjlim_{K \in J(U)} \varprojlim_{V \to U \in K} X(V)$$

where the colimit is taken over covering families on U and the limits over morphism in the covering. This construction is usually known as the *plus construction* and, in general, it is not always possible for it relying on the existence of enough (co)limits in the category of values. A sufficient condition on **C** is that it is a (co)complete category but weaker conditions can be derived for particular sites of interest. For instance, if $(\mathbf{S}, J)$ corresponds to the Alexandrov topology for an ordinal α, then it suffices to assume **C** has limits of γ-sequences for $\gamma \leq \alpha$.

Iterating the plus construction always results in sheaves for the given site thus universally associating sheaves to presheaves. For X any presheaf, the sheaf X^{++} is called the *associated sheaf* or *sheafification* of X.

PROPOSITION 2.6. *For* $(\mathbf{S}, J)$ *a site and* **C** *a category, the following are equivalent:*

- *every presheaf in* $\mathbf{PSh_C}(\mathbf{S})$ *admits sheafification;*
- *the inclusion* $\mathbf{i} \colon \mathbf{Sh_C}(\mathbf{S}, J) \to \mathbf{PSh_C}(\mathbf{S})$ *exhibits* $\mathbf{Sh_C}(\mathbf{S}, J)$ *as a reflective subcategory of* $\mathbf{PSh_C}(\mathbf{S})$.

Whenever it exists, the left adjoint to $\mathbf{i}$ is denoted as $\mathbf{a}$ and called *associated sheaf functor* or *sheafification functor*. For this functor to exists it suffices to assume $(\mathbf{S}, J)$ small and all small (co)limits in $\mathbf{C}$.

There is an inclusion of the category of values $\mathbf{C}$ into $\mathbf{PSh_C}(\mathbf{S})$ by means of $(-\circ !_\mathbf{S}) \colon \mathbf{C} \to \mathbf{PSh_C}(\mathbf{S})$ where $!_\mathbf{S} \colon \mathbf{S} \to \mathbf{1}$ is the final morphism in $\mathbf{Cat}$. This functor is denoted by Δ and called *constant presheaf functor* for it assigns each object Y in $\mathbf{C}$ to the constant presheaf of Y:

$$\Delta(Y)(U) = Y \qquad \Delta(Y)(r) = id_Y$$

and each morphism $f \colon Y \to Y'$ to the natural transformation whose components are all f. Whenever $\mathbf{C}$ has limits for diagrams of type $\mathbf{S}^{op}$, the constant presheaf functor Δ has a right adjoint $\Gamma \colon \mathbf{PSh_C}(\mathbf{S}) \to \mathbf{C}$ called *global sections functor* for it maps each presheaf to the object of its global sections *i.e.* it maps each presheaf X the limit of a diagram $X \colon \mathbf{S}^{op} \to \mathbf{C}$. By composition with the sheafification adjunction $(\mathbf{a} \dashv \mathbf{i}) \colon \mathbf{Sh_C}(\mathbf{S}, J) \to \mathbf{PSh_C}(\mathbf{S})$, the constant presheaf adjunction $(\Delta \dashv \Gamma)$ restricts to the subcategory of sheaves $\mathbf{Sh_C}(\mathbf{S}, J)$. As common practice, we will abuse the notation and just write Δ and Γ instead of $\mathbf{a} \circ \Delta$ and $\Gamma \circ \mathbf{i}$. The terminology for Δ and Γ is extended accordingly to sheaves: for X an object of $\mathbf{C}$, the sheaf $\Delta(X)$ is called constant sheaf of X and Δ constant sheaf functor, for Y a $\mathbf{C}$-valued sheaf on $(\mathbf{S}, J)$, $\Gamma(Y)$ is the object of global sections of Y and Γ is called the global sections functor.

2.1.4 Enrichment over sheaf categories

In this subsection we state some basic definitions about categories enriched over categories of sheaves which will be needed in order to introduce locally contractive functors and in the remaining of this thesis as well. Although we were not able to find published work that explicitly introduces this blend of enriched categories and functors, all definitions are obtained as instance of standard notions from enriched category theory; we refer the interested reader to Kelly's body of work, especially [79].

Let $(\mathbf{S}, J)$ be a site. A category enriched over $\mathbf{Sh}(\mathbf{S}, J)$ $\mathbf{C}$ is characterised by the following data:

- a collection $\mathrm{obj}(\mathbf{C})$ of objects,
- a sheaf $\mathbf{C}(X, Y)$ over $(\mathbf{S}, J)$ for each pair of objects $X, Y \in \mathrm{obj}(\mathbf{C})$,
- a point $id_X \colon 1 \to \mathbf{C}(X, X)$ in $\mathbf{Sh}(\mathbf{S}, J)$ determining the identity for each $X \in \mathrm{obj}(\mathbf{C})$,

- a morphism $\circ_{X,Y,Z} \colon \mathbf{C}(Y,Z) \times \mathbf{C}(X,Y) \to \mathbf{C}(X,Z)$ in $\mathbf{Sh}(\mathbf{S},J)$ determining composition for each $X,Y,Z \in \mathbf{C}$ such that it is associative

$$
\begin{array}{ccc}
(\mathbf{C}(Y,Z) \times \mathbf{C}(X,Y)) \times \mathbf{C}(W,X) & \xrightarrow{\cong} & (\mathbf{C}(Y,Z) \times (\mathbf{C}(X,Y) \times \mathbf{C}(W,X)) \\
\downarrow{\scriptstyle \circ_{X,Y,Z} \times id_{\mathbf{C}(W,X)}} & & \downarrow{\scriptstyle id_{\mathbf{C}(Y,Z)} \times \circ_{W,X,Y}} \\
\mathbf{C}(X,Z) \times \mathbf{C}(W,X) & & \mathbf{C}(Y,Z) \times \mathbf{C}(W,Y) \\
& \searrow^{\circ_{W,X,Z}} \quad {}^{\circ_{W,Y,Z}}\swarrow & \\
& \mathbf{C}(W,Z) &
\end{array}
$$

and has the points from above as identities

$$
\begin{array}{ccc}
& & \mathbf{C}(Y,Y) \times \mathbf{C}(X,Y) \\
& \nearrow^{id_Y \times id_{\mathbf{C}(X,Y)}} & \downarrow{\scriptstyle \circ_{X,Y,Y}} \\
1 \times \mathbf{C}(X,Y) & \xrightarrow[\cong]{} & \mathbf{C}(X,Y)
\end{array}
\qquad
\begin{array}{ccc}
\mathbf{C}(X,Y) \times \mathbf{C}(X,X) & & \\
\downarrow{\scriptstyle \circ_{X,X,Y}} & \nwarrow^{id_{\mathbf{C}(X,Y)} \times id_X} & \\
\mathbf{C}(X,Y) & \xleftarrow[\cong]{} & \mathbf{C}(X,Y) \times 1
\end{array}
$$

Note that sheaf enriched categories do not have "morphisms" for their hom-objects do not have proper elements. It is convenient however convenient to be able to speak about "morphisms" especially in order to express "diagrams" in sheaf enriched categories in a convenient way. For an example of what a "diagram" looks like in this setting, see (2.2) which expresses naturality condition. besides these practical conveniences, the ability to express what a "collection of morphisms" is in sheaf enriched settings is of relevance to the definition of natural transformations. Elements of a sheaf X are understood as its points *i.e.* morphisms from the final objects $x \colon 1 \to X$. These are defined as the "morphisms" of a sheaf enriched category. It follows that there are several distinct enriched categories that present us with the same morphisms; this happens because restriction maps of hom-sheaves might not be subjections and hence hom-sheaves are not defined by their points. Nonetheless, there is a "minimal" category presenting a given collection of morphism: this category has the property that its hom-sheaves are subobjects of those of any other category with the same morphisms. If one is interested in morphisms per se, then restriction to these minimal categories does not introduce any loss of generality.

For $\mathbf{C}$ a category enriched over $\mathbf{Sh}(\mathbf{S},J)$, we can associate to $\mathbf{C}$ a category $\lfloor \mathbf{C} \rfloor$, called its *externalisation* or its *underlying* category, defined by the following data:

- objects $\mathrm{obj}(\lfloor \mathbf{C} \rfloor) \triangleq \mathrm{obj}(\mathbf{C})$;
- hom-set $\lfloor \mathbf{C} \rfloor(X,Y) \triangleq \mathbf{Sh}(\mathbf{S},J)(1,\mathbf{C}(X,Y))$ for each pair of objects X,Y.

Note that hom-sets are defined as the sets of points into hom-sheaves *i.e.* the notion of morphisms in sheaf enriched categories. From this perspective, the "minimal" category mentioned at the end of the previous paragraph is the initial object in the thin category whose objects of categories with the same externalisation and whose morphisms are inclusions. As common practice, we will often abuse notation and terminology by referring to an enriched category and its externalisation as if they were the same entity—provided the distinction is clear from the context. In practice, we will call a (genuine) category $\mathbf{Sh}(\mathbf{S}, J)$-enriched if it is the externalisation of some $\mathbf{Sh}(\mathbf{S}, J)$-enriched category and write $\mathbf{C}$, instead of $\lfloor \mathbf{C} \rfloor$, for the externalisation of $\mathbf{C}$.

Examples of categories enriched over categories of **Set**-valued sheaves are sheaf categories themselves. Formally, this means that for any $\mathbf{Sh}_{\mathbf{C}}(\mathbf{S}, J)$ there is a $\mathbf{Sh}(\mathbf{S}, J)$-enriched category such that its externalisation is $\mathbf{Sh}_{\mathbf{C}}(\mathbf{S}, J)$.

LEMMA 2.7. *For* $\mathbf{C}$ *a category and* $(\mathbf{S}, J)$ *a site, the category* $\mathbf{Sh}_{\mathbf{C}}(\mathbf{S}, J)$ *is enriched over* $\mathbf{Sh}(\mathbf{S}, J)$.

PROOF. For $X, Y \in \mathbf{Sh}_{\mathbf{C}}(\mathbf{S}, J)$, the hom-sheaf $\mathbf{Sh}_{\mathbf{C}}(\mathbf{S}, J)(X, Y)$ takes each object U in $\mathbf{S}$ to the value

$$\mathbf{Sh}_{\mathbf{C}}(\alpha)(X, Y)(U) = \{(f, f_U) \mid f \in \mathbf{Sh}_{\mathbf{C}}(\mathbf{S}, J)(X, Y)\}$$

and each morphism $p \colon U \to V$ in $\mathbf{S}$ to the restriction

$$\mathbf{Sh}_{\mathbf{C}}(\alpha)(X, Y)(p)(f, f_V) = (f, f_U).$$

We remark that the use of pairs morphism-component is crucial to the definition restriction maps: the fact that two morphisms share their component at a given stage, say U, does not imply that they do the same for components at each stage V such that $V \to U \in \mathbf{S}$ whence information about which transformation a component belongs to has to be included. The sheaf condition follows from definition of morphism of $\mathbf{C}$-valued sheaves. For $X \in \mathbf{Sh}_{\mathbf{C}}(\mathbf{S}, J)$, the point $id_X \colon 1 \to \mathbf{Sh}_{\mathbf{C}}(\alpha)(X, X)$ determines at each stage U to the identity on the object of sections at U:

$$id_{X,U}(*) = (id_X, id_{X(U)}).$$

For $X, Y, Z \in \mathbf{Sh}_{\mathbf{C}}(\mathbf{S}, J)$, components of $(- \circ_{X,Y,Z} -)$ apply composition of arrows in $\mathbf{C}$ as follows:

$$(f \circ_{X,Y,Z} g)_U = (f \circ g, f_U \circ g_U).$$

Diagrams for associativity and identities readily follow from the analogous properties of composition in $\mathbf{C}$. □

A $\mathbf{Sh}(\mathbf{S}, J)$-enriched functor $F\colon \mathbf{C} \to \mathbf{D}$ between $\mathbf{Sh}(\mathbf{S}, J)$-enriched categories is a functorial mapping such that for every pair of objects X, Y in $\mathbf{C}$ the assignment $F_{X,Y}\colon \mathbf{C}(X, Y) \to \mathbf{D}(FX, FY)$ is a morphism of sheaves in $\mathbf{Sh}(\mathbf{S}, J)$. Unless otherwise stated, we implicitly assume domain and codomain of an enriched functor to be similarly enriched. Categories and functors enriched over $\mathbf{Sh}(\mathbf{S}, J)$ form the category $\mathbf{Sh}(\mathbf{S}, J)$**-Cat**. For $F, G\colon \mathbf{C} \to \mathbf{D}$ functors enriched over $\mathbf{Sh}(\mathbf{S}, J)$, an enriched natural transformation $\rho\colon F \to G$ is a family of morphisms

$$\{\rho_X \colon 1 \to \mathbf{D}(FX, GX)\}_{X \in \mathbf{C}}$$

indexed over $\mathrm{obj}(\mathbf{C})$ and such that for any pair of objects $X, Y \in \mathbf{C}$, the following naturality diagram commutes:

$$\begin{array}{ccccc}
 & & \mathbf{C}(X,Y) & & \\
 & \overset{\cong}{\swarrow} & & \overset{\cong}{\searrow} & \\
1 \times \mathbf{C}(X,Y) & & & & \mathbf{C}(X,Y) \times 1 \\
{\scriptstyle \rho_Y \times F_{X,Y}} \downarrow & & & & \downarrow {\scriptstyle G_{X,Y} \circ \rho_X} \\
\mathbf{D}(FY,GY) \times \mathbf{D}(FX,FY) & & & & \mathbf{D}(GX,GY) \times \mathbf{D}(FX,FX) \\
 & \underset{- \circ_{FX,FY,GY} -}{\searrow} & & \underset{- \circ_{FX,GX,GY} -}{\swarrow} & \\
 & & \mathbf{D}(FX,GY) & &
\end{array} \tag{2.2}$$

For $\mathbf{C}$ and $\mathbf{D}$ enriched over $\mathbf{Sh}(\mathbf{S}, J)$, enriched functors and natural transformations form the $\mathbf{Sh}(\mathbf{S}, J)$-enriched category $\mathbf{Sh}(\mathbf{S}, J)$-$\mathbf{Fun}(\mathbf{C}, \mathbf{D})$ whose hom-sheaves are given on each pair of functors $F, G\colon \mathbf{C} \to \mathbf{D}$ as the sheaf taking value $\{(\rho, \{\rho_{X,U}\}_{X \in \mathbf{C}}) \mid \rho\colon F \to G\}$ at stage $U \in \mathbf{S}$. In order to keep the notation concise, we shall write $\mathbf{Fun}(\mathbf{C}, \mathbf{D})$ instead of $\mathbf{Sh}(\mathbf{S}, J)$-$\mathbf{Fun}(\mathbf{C}, \mathbf{D})$, provided enrichment is clear from the context.

2.2 ALGEBRAICALLY COMPACT FUNCTORS AND CATEGORIES

In [18, 53] it is shown that for any given endofunctor there is a unique and canonical morphism from its initial to final invariants *i.e.* (co)algebras—provided both exists. In [53], Freyd termed *algebraically compact* categories for which that morphism always exists and is an isomorphism. In [17] Barr observed that a category rarely is algebraically compact in Freyd's sense. Instead, he suggested to treat algebraic compactness as a property of individual functors, or classes of functors, since in several occasions it might be worth restricting the attention to classes of functors that are "relevant" to specific situations *e.g.*, when working in enriched settings. In [17, 53, 54] the term *algebraically complete* is introduced to indicate existence of initial invariants; clearly algebraic compactness implies algebraic completeness.

DEFINITION 2.8. *An endofunctor F is called:*

- algebraically complete *if there is an initial F-algebra;*
- coalgebraically cocomplete *if there is a final F-coalgebra;*
- algebraically compact *if there is an initial F-algebra, a final F-coalgebra, and they are canonically isomorphic.*

The terminology is extended to classes of functors and categories in the obvious way.

DEFINITION 2.9. *For a category* **E** *of endofunctors over* **C**, *the category* **C** *is said:*

- algebraically complete with respect to **E** *if every functor in* **E** *is algebraically complete;*
- coalgebraically cocomplete with respect to **E** *if every functor in* **E** *is coalgebraically cocomplete;*
- algebraically compact with respect to **E** *if every functor in* **E** *is algebraically compact.*

The vast majority of works and results about algebraic compactness (or that rely on it) consider two main classes of functors: *locally continuous functors* and *locally contractive functors*. Historically, the former is the first non-trivial class of algebraically compact functors identified (see [17]) and was initially studied as part of a categorical generalisation of order-theoretic constructions used in domain theory, especially Scott's limit-colimit coincidence result [130].Nowadays locally continuous functors are fruitfully applied to a broad class of problems besides domain theoretic ones, see *e.g.* [11, 41, 60, 61, 64, 67, 146]. The class of locally contractive functors was introduced more recently as the technical foundation of guarded recursion and guarded type theory [10, 25–28, 99, 108, 116, 142]. In Sections 2.2.1 and 2.2.2 we recall preliminary notions and results about locally continuous and locally contractive functor, respectively.

2.2.1 Locally continuous functors

Let **Cpo** be the category whose objects are (small) ω-complete partial orders and whose morphisms are continuous maps and let $\mathbf{Cpo}_{\bot}$ be its subcategory whose objects have bottoms and whose morphisms take bottoms to bottoms *i.e.* are bottom-strict.

A **Cpo**-*enriched category* (or simply **Cpo**-category) **C** is a locally small category whose hom-sets $\mathbf{C}(X, Y)$ come equipped with an ω-complete partial order $\leq_{X,Y}$

such that composition $(- \circ -)\colon \mathbf{C}(Y,Z) \times \mathbf{C}(X,Y) \to \mathbf{C}(X,Z)$ is a continuous operation. A special case of **Cpo**-categories are those enriched over $\mathbf{Cpo}_\bot$ *i.e.* any **Cpo**-category **C** whose hom-sets $\mathbf{C}(X,Y)$ are additionally equipped with a bottom element $\bot_{X,Y}$ and whose composition operation is also strict. We shall drop subscripts from $\leq_{X,Y}$ and $\bot_{X,Y}$ when possible. Forgetting the order structure from a category **C** enriched over **Cpo** (or $\mathbf{Cpo}_\bot$) leaves us with the structure of a genuine category called *underlying* or *externalisation* and denoted as $\lfloor \mathbf{C} \rfloor$. In the vein of the usual convention of using the same notation for cpos and their underlying sets we often write **C**, instead of $\lfloor C \rfloor$, for the externalisation of **C**. In general, we adopt the conventions on notation and terminology described for sheaf enriched categories also for order enriched ones.

In the following let **V** stand for either **Cpo** or $\mathbf{Cpo}_\bot$.

EXAMPLE 2.1. *The category* **V** *is enriched over itself. The single object category* **1** *is trivially* $\mathbf{Cpo}_\bot$*-enriched. The dual of a* **V***-category* **C** *is the* **V***-category* $\mathbf{C}^{op}$ *such that* $\mathrm{obj}(\mathbf{C}^{op}) = \mathrm{obj}(\mathbf{C})$ *and* $\mathbf{C}^{op}(X,Y) = \mathbf{C}(Y,X)$*. The product of* **V***-categories* **C** *and* **D** *is the* **V***-category* $\mathbf{C} \times \mathbf{D}$ *such that* $(\mathbf{C} \times \mathbf{D})((X,X'),(Y,Y')) = \mathbf{C}(X,Y) \times \mathbf{D}(X',Y')$*. The category of relations* $\mathbf{Rel} \cong \mathbf{Kl}(\mathcal{P})$ *is a* $\mathbf{Cpo}_\bot$*-category where the order structure is defined by pointwise extension of the inclusion order created by the powerset monad—see* e.g. *[35, 36, 64] for more behavioural functors of endowed with monadic structures yielding* **V***-enriched Kleisli categories.*

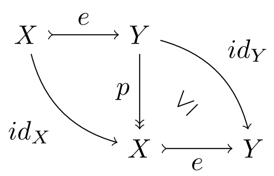
A **V**-*enriched functor* $F\colon \mathbf{C} \to \mathbf{D}$ between **V**-categories is a functorial mapping with the property that for every pair of objects X, Y in **C** the assignment $F_{X,Y}\colon \mathbf{C}(X,Y) \to \mathbf{D}(FX,FY)$ is continuous and, in the case of $\mathbf{Cpo}_\bot$-functors, strict. Functors enriched over **Cpo** are often called *locally continuous functors* (*e.g.* [64, 145, 146]). Unless otherwise stated, we implicitly assume that domain and codomain of an enriched functor are similarly enriched. Categories and functors enriched over **V** form the category **V-Cat**. We denote the functor underlying a **V**-functor $F\colon \mathbf{C} \to \mathbf{D}$ as $\lfloor F \rfloor\colon \lfloor \mathbf{C} \rfloor \to \lfloor \mathbf{D} \rfloor$ or simply F, when confusion seems unlikely. For **V**-categories **C** and **D**, the functor category $\mathbf{V\text{-}Fun}(\mathbf{C},\mathbf{D})$ is the **V**-category whose objects are **V**-functors and such that $\mathbf{V\text{-}Fun}(\mathbf{C},\mathbf{D})(X,Y)$ is the complete partial order on the set $\mathbf{Nat}(X,Y)$ of natural transformations given by pointwise extension of the order on their components. When clear from the context we shall write $\mathbf{Fun}(\mathbf{C},\mathbf{D})$ instead of $\mathbf{V\text{-}Fun}(\mathbf{C},\mathbf{D})$. A **Cpo**-*adjunction* $\chi\colon L \dashv R\colon \mathbf{C} \to \mathbf{D}$ is given by a natural isomorphism:

$$\chi\colon \mathbf{C}(L-,-) \cong \mathbf{D}(-,R-)\colon \mathbf{D}^{op} \times \mathbf{C} \to \mathbf{Pos}$$

where **Pos** is the category of posets and monotonic maps. Actually, the above statement defines a **Pos**-adjunction but, since the inclusion functor $\mathbf{Cpo} \to \mathbf{Pos}$ creates isomorphisms, any **Pos**-adjunction involving **Cpo**-categories yields a

Cpo-adjunction.

Two morphisms $e\colon X \to Y$ and $p\colon Y \to X$ in a **Cpo**-category **C** form an *embedding-projection pair* (written $e \lhd p\colon X \to Y$) whenever $p \circ e = id_X$ and $e \circ p \leq id_Y$ or, diagrammatically:



The components e and p are called *embedding* (of X in Y) and *projection* of (Y in X), respectively, and uniquely determine each other. Since complete partial orders are small categories, embedding-projection pairs are *coreflections*;[2] henceforth we use the two terms interchangeably in the context of **Cpo**-categories.

Coreflections in a **Cpo**-category **C** form a sub-**Cpo**-category of **C** whose objects are those of **C** and whose arrows are embedding-projections with the order on hom-sets given by the ordering on the embeddings (note that $e \leq e' \iff p \geq p'$). We write $\mathbf{C}^{\lhd}$ for such category. By forgetting either the projection or embedding part of a coreflection we get the categories $\mathbf{C}^e$ and $\mathbf{C}^p$ (of embeddings and projections), respectively, and such that $\mathbf{C}^e \cong \mathbf{C}^{\lhd} \cong (\mathbf{C}^p)^{op}$.

PROPOSITION 2.8 (Limit-colimit coincidence). *Assume* **C** *enriched over the category* **Cpo**. *For an ω-chain of coreflections $(e_n \lhd p_n\colon X_n \to X_{n+1})_{n<\omega}$ and a cone of coreflections $(f_n \lhd q_n\colon X \to X_n)_{n<\omega}$ for it the following are equivalent:*

1. *the cocone $(f_n\colon X \to X_n)_{n<\omega}$ is a colimit for the ω-chain of embeddings $(e_n\colon X_n \to X_{n+1})_{n<\omega}$;*
2. *the cone $(q_n\colon X_n \to X)_{n<\omega}$ is a limit for the ω-chain of projections $(p_n\colon X_{n+1} \to X_n)_{n<\omega}$.*

The above is a slight reformulation of the limit-colimit coincidence result used in [133] to solve recursive domain equations with unknowns occurring in covariant and contravariant positions like the well-known domain equation:

$$D \cong (D \to D) + At$$

and in general to find invariant objects for functors that were contravariant or of mixed variance [20].

Algebraic compactness was developed by Freyd and Barr as an abstract framework for addressing the same question in a more principled and general

[2] A coreflection is an adjunction whose unit is a pseudo-cell.

way. Results developed in the order enriched setting considered in [133] are shown to follow from axioms of **Cpo**-algebraically compact categories [17]. These categories are characterised by a limit-colimit coincidence property for initial and final sequences of **Cpo**-endofunctors as formalised in the definition below.

DEFINITION 2.10. *A category* **C** *enriched over* **Cpo** *is said:*

- **Cpo**-algebraically complete *whenever it is algebraically complete with respect to the class of* **Cpo**-*enriched functors;*

- **Cpo**-coalgebraically cocomplete *whenever it is algebraically complete with respect to the class of* **Cpo**-*enriched functors;*

- **Cpo**-algebraically compact *whenever it is algebraically compact with respect to the class of* **Cpo**-*enriched functors;*

Note that there are **Cpo**-enriched categories that are not **Cpo**-algebraically compact, for instance **Cpo**. Proposition 2.9 below provides mild and easily verifiable assumptions on categories that are sufficient for **Cpo**-algebraic compactness.

PROPOSITION 2.9. *The following statements are true.*

1. *A* **Cpo**-*category with an embedding-initial object and colimits of ω-chains of embeddings is* **Cpo**-*algebraically complete [48].*

2. *A* **Cpo**-*algebraically complete* $\mathbf{Cpo}_\perp$-*enriched category is* **Cpo**-*algebraically compact [54].*

The class of **Cpo**-algebraically compact categories is closed under products and dualisation. In particular, if **C** is **Cpo**-algebraically compact then so is $\mathbf{C}^{op} \times \mathbf{C}$.

COROLLARY 2.10. *Assume* **C** *and* **D** **Cpo**-*algebraically compact,* $\mathbf{C}^{op}$ *and* $\mathbf{C} \times \mathbf{D}$ *are* **Cpo**-*algebraically compact.*

REMARK 2.2. Algebraic compactness is at the core of several works on categorical domain theory, especially by Fiore [48] who refined and extended the theory. We restricted ourselves to **Cpo**-algebraic compactness in order to simplify the exposition but results presented in this work can be formulated in the general setting of pseudo-algebraically compact 2-categories [42].

2.2.2 Locally contractive functors

Let A be a complete Heyting algebra and write $\mathbf{PSh}_\mathbf{C}(A)$ and $\mathbf{Sh}_\mathbf{C}(A)$ for the categories of **C**-valued (pre)sheaves on the Grothendieck site associated to A. This topology is often called *sup topology* for its sieves are such to cover their supremum: the coverage on A (regarded as a thin category) is the function mapping an element $a \in A$ to the set $\left\{(A')^{\downarrow} \,\middle|\, A' \subseteq A \wedge a = \bigvee A'\right\}$.

Following [26, 55] define the *predecessor map* on A $\mathbf{p}\colon A \to A$ as:

$$\mathbf{p}(a) \triangleq \bigvee \{b \in B \mid b < a\}.$$

where B is a base for A *i.e.* any subset of A such that each element is a supremum for the set of elements from the base that are less or equal to it:

$$a \in A \implies a = \bigvee \{b \in B \mid b \leq a\}.$$

The predecessor map induces an endofunctor $\mathbf{p}^*$ over the presheaf category $\mathbf{PSh_C}(A)$ and defined as its inverse image $\mathbf{p}^*(X) = X \circ \mathbf{p}$. Restriction morphisms induce a natural transformation $next^{\mathbf{p}}\colon Id \to \mathbf{p}^*$ whose components are given, on each presheaf X and stage a, as $next^{\mathbf{p}}_{X,a} = X_{\iota_{\mathbf{p}(a),a}}$ and such that $(\mathbf{p}^*, next^{\mathbf{p}})$ is well-pointed[3]. Assume $(\mathbf{a} \dashv \mathbf{i})\colon \mathbf{PSh_C}(A) \to \mathbf{Sh_C}(A)$, then the predecessor endofunctor over $\mathbf{PSh_C}(A)$ induces an endofunctor $\blacktriangleright$ over $\mathbf{Sh_C}(A)$ as the restriction:

$$\blacktriangleright = \mathbf{a} \circ \mathbf{p}^* \circ \mathbf{i}.$$

This endofunctor is called *later* or *delay* in contexts where stages describe future words. Similarly to the predecessor endofunctor $\mathbf{p}^*$, $\blacktriangleright$ is well-pointed when equipped with a point $next\colon Id \to \blacktriangleright$ defined as the composite $\eta \bullet next^{\mathbf{p}}$ where $\bullet$ denotes vertical composition in the 2-category **Cat** and $\eta\colon Id \to \mathbf{a} \circ \mathbf{i}$ is the unit of the associated sheaf adjunction. The functor $\blacktriangleright$ preserves all limits in $\mathbf{Sh_C}(A)$.

For **C** $\mathbf{Sh}(A)$-enriched define ${}_{\blacktriangleright}\mathbf{C}$ as the $\mathbf{Sh}(A)$-enriched category given by the following data:

- the objects of **C**, $\mathrm{obj}({}_{\blacktriangleright}\mathbf{C}) = \mathrm{obj}(\mathbf{C})$,
- for any pair of objects $X, Y \in \mathrm{obj}(\mathbf{C})$, the sheaf ${}_{\blacktriangleright}\mathbf{C}(X,Y) = \blacktriangleright\mathbf{C}(X,Y)$,
- for each object $X \in \mathrm{obj}(\mathbf{C})$, the point $next_{\mathbf{C}(X,Y)} \circ id_X\colon 1 \to {}_{\blacktriangleright}\mathbf{C}(X,X)$ where $id_X\colon 1 \to \mathbf{C}(X,X)$ is the identity on X in **C**,
- for each $X, Y, Z \in \mathbf{C}$, the morphism

  $$\blacktriangleright\mathbf{C}(Y,Z) \times \blacktriangleright\mathbf{C}(X,Y) \xrightarrow{\cong} \blacktriangleright(\mathbf{C}(Y,Z) \times \mathbf{C}(X,Y)) \xrightarrow{\blacktriangleright(-\circ_{X,Y,Z}-)} \blacktriangleright\mathbf{C}(X,Z)$$

  where $\circ_{X,Y,Z}\colon \mathbf{C}(Y,Z) \times \mathbf{C}(X,Y) \to \mathbf{C}(X,Z)$ is composition in **C**.

[3] A well-pointed functor is any endofunctor F equipped with a natural transformation $\eta\colon Id \to F$ called point and such that $\eta \circ id_F = id_F \circ \eta$.

The natural transformation $next\colon Id_{\mathbf{Sh}(A)} \to \blacktriangleright$ induces a $\mathbf{Sh}(A)$-enriched functor $next\colon \mathbf{C} \to {}_{\blacktriangleright}\mathbf{C}$ acting as the identity on objects and as the components of $next$ on hom-sheaves.

DEFINITION 2.11. *A* locally contractive *functor is any* $\mathbf{Sh}(A)$*-enriched functor* $F\colon \mathbf{C} \to \mathbf{D}$ *that factors as a composition of functors enriched over* $\mathbf{Sh}(\alpha)$*:*

$$\mathbf{C} \xrightarrow{next} {}_{\blacktriangleright}\mathbf{C} \longrightarrow \mathbf{D}.$$

The later endofunctor is always locally contractive and so is any composite $F \circ \blacktriangleright$ where F is enriched over $\mathbf{Sh}(A)$. In general, local contractiveness is preserved by composition, dualisation, and products.

LEMMA 2.11. *For A a complete Heyting algebra, the following statements are true.*

1. *The endofunctor* $\blacktriangleright\colon \mathbf{Sh_C}(A) \to \mathbf{Sh_C}(A)$ *is locally contractive.*
2. *For F and G componible and* $\mathbf{Sh}(A)$*-enriched, if F or G is locally contractive then $F \circ G$ is locally contractive.*
3. *For F and G locally contractive, $F \times G$ is locally contractive.*
4. *For F locally contractive, F^{op} is locally contractive.*

For locally contractive functors, (co)algebraic (co)completeness and algebraic compactness coincide for these functors admit at most one invariant, up to isomorphism. In particular, for F locally contractive and X such that $X \cong F(X)$, the isomorphism identifies an initial F-algebra and a final F-coalgebra. Furthermore, $X \cong Y$ whenever $Y \cong F(Y)$.

LEMMA 2.12 ([26]). *Let F be a locally contractive endofunctor over a category* $\mathbf{C}$*. If F has an invariant object, then it is unique up to isomorphism.*

DEFINITION 2.12. *A sheaf enriched category* $\mathbf{C}$ *is called* contractively compact *if it is algebraically compact with respect to locally contractive functors.*

In [26] Birkedal et al. identify conditions on the underlying category and on the Heyting algebra that are sufficient for algebraic compactness: completeness and well-foundedness.

PROPOSITION 2.13 ([26]). *For A a complete Heyting algebra with a well-founded base and* $\mathbf{C}$ *a category enriched over* $\mathbf{Sh}(A)$*, if (the externalisation of)* $\mathbf{C}$ *is complete then it is contractively complete.*

Examples of contractively compact categories are categories of sheaves on Alexandrov topologies induced by ordinal numbers and that take values in a complete category such as **Set**, **Cpo**, $\mathbf{Cpo}_\bot$, **Top**, and **Meas**.

REMARK 2.3. Results and constructions described in this section were presented in [25, 26] in the more general setting of categories enriched over models of guarded terms. A concrete instance of these model categories are sheaves over well-founded complete Heyting algebras and we preferred to focus our exposition to these models because they can be presented without formally introducing models of guarded terms and because they provide a setting that is sufficiently general for the aims of this work.

3 INFINITE TRACE SEMANTICS

Since the seminal paper [117], Kleisli categories have been recognised as the context where to model the linear semantics of several types of transition systems as shown by a plethora of works such as [43, 63, 64, 70, 80, 145, 146]. The key idea behind this approach falls under the motto "change the category not the definition" and can be traced back to Moggi's modelling of side effects in Kleisli categories of monads [23, 109–111]. Roughly speaking, systems are modelled as TF-coalgebras where T is a monad describing the "branching type" (*e.g.* partiality, non-determinism, probabilistic) and F is an endofunctor describing the "linear type" (*e.g.* labelled transitions). Coalgebras of this type form a wide subcategory of coalgebras for certain endofunctors obtained as suitable extensions of F to the Kleisli category of T and called *Kleisli liftings*. In this setting, objects modelling systems are the same of TF-coalgebras whereas coalgebra homomorphisms abstract from branching (the computational effect associated to the monad T) and hence capture the linear behaviour of systems under scrutiny. In general, final semantics for coalgebras of Kleisli liftings may not coincide with any established notion of trace semantics: there are instances where this results in finite, possibly infinite, infinite only traces, or none at all (see *e.g.* [80]). In [64] Hasuo, Jacobs and Sokolova presented general and sufficient conditions that ensure finite trace semantics is captured by the final semantics of coalgebras of certain Kleisli liftings. Although there are works recovering (possibly) infinite trace semantics via canonical maps to weakly final coalgebras (see *e.g.* [43, 64, 70, 145]), a general account on par with those of finite trace semantics is still missing. In this chapter we propose a general approach to infinite trace semantics via final semantics.

Our proposal combines three main ingredients: Kleisli liftings, sheaves on ordinals, and guarded (co)recursion. Although each of them is widely studied, their combination is the key novelty that allows us to systematically capture infinite traces by finality. Clearly, the rôle of Kleisli lifting is to abstract branching. The rôle of sheaves and guarded (co)recursion becomes clear when one notes that an infinite object (*e.g.* a stream) is equivalently described by an infinite family of coherent approximations (*e.g.* the countable family of its prefixes). This is exactly how a global section is characterised from local sections via amalgamation. If we replace infinite traces and finite traces for streams and words, respectively, then infinite traces are *global observations*, finite traces are partial or *local observations*, and *amalgamation* is the mechanism for obtaining the former from coherent families of the latter. From this perspective, observations are naturally organised in sheaves over the Alexandrov topology of an ordinal number—ω, in the case of the example of streams and words considered above.

We introduce the notion of guarded behavioural functor and *guarded coalgebras* as a way to capture local and global observations at once. Guarded behavioural endofunctors are systematically derived from any behavioural endofunctor while preserving the associated final semantics: there is an inclusion functor that exhibits the category of coalgebras as a coreflective subcategory of that of guarded coalgebras. We prove that the final semantics of guarded coalgebras in Kleisli categories always capture infinite trace semantics of the systems under scrutiny.

This result, together with existing accounts of finite trace semantics, allows us to use trace semantics to drive the construction of self-referential endofunctors hence to model self-referential behaviours from the trace semantics perspective.

The chapter is organised as follows. In Section 3.1 we shortly describe the modelling of linear and trace semantics via Kleisli liftings and propose a notion of morphisms for relating such models. In Section 3.2 we consider category-valued sheaves over ordinals equipped with the Alexandrov topology and study extensions of behavioural endofunctors and Kleisli liftings to this setting while preserving the original semantics. In Section 3.3 we consider locally contractive endofunctors over Kleisli categories of monads obtained by pointwise extension as the technical foundation for guarded coalgebras in Kleisli categories. In Section 3.4 we combine all these techniques into the notion of guarded coalgebras and guarded Kleisli liftings and prove that this combination is suitable for capturing infinite trace semantics. Concluding remarks are in Section 3.5.

3.1 LINEAR AND TRACE SEMANTICS VIA KLEISLI LIFTINGS

In this section we recall the approach to modelling linear-time semantics (or linear semantics, for short) introduced in [117] and its application to trace semantics [64, 70]. The section is organised as follows: in Section 3.1.1 we describe endofunctor extensions to Kleisli categories known as Kleisli liftings; in Section 3.1.2 and Section 3.1.3 we study coalgebras for Kleisli liftings and their associated notions of final semantics and behavioural equivalences; in Section 3.1.4 we propose a notion of morphism between models of linear semantics with the property that they preserve linear bisimulations—hence trace equivalences.

3.1.1 Kleisli liftings

Let (T, μ, η) be a monad over a category **C** and write $(K \dashv L)\colon \mathbf{Kl}(T) \to \mathbf{C}$ for the canonical adjunction presenting **C** as a subcategory of $\mathbf{Kl}(T)$. We are interested in extending an endofunctor F from **C** to $\mathbf{Kl}(T)$ in a way that preserves its action on (the image of) **C**. This intuition is formalised by the following definition:

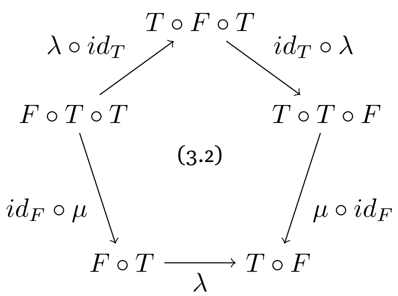
DEFINITION 3.1 ([113]). *Let (T, μ, η) be a monad and F an endofunctor, both over some category* **C**. *A* Kleisli lifting *of F to $\mathbf{Kl}(T)$ is any endofunctor $\overline{F}$ over $\mathbf{Kl}(T)$ such that the diagram below commutes.*

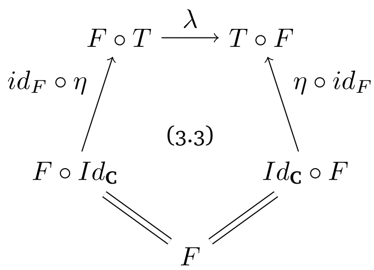
$$\begin{array}{ccc} \mathbf{Kl}(T) & \xrightarrow{\overline{F}} & \mathbf{Kl}(T) \\ {\scriptstyle K}\uparrow & & \uparrow{\scriptstyle K} \\ \mathbf{C} & \xrightarrow[F]{} & \mathbf{C} \end{array} \tag{3.1}$$

REMARK 3.1 (Kleisli extensions). A lifting of $f\colon X \to Y$ along an (epi)morphism $e\colon A \to Y$ is a morphism $h\colon X \to A$ such that $f = g \circ e$. Dually, an extension of $f\colon X \to Y$ along a (mono)morphism $m\colon X \to B$ is a morphism $h\colon B \to Y$ such that $f = h \circ m$. Therefore, Kleisli liftings are actually *extensions* along the inclusion functor $K\colon \mathbf{C} \to \mathbf{Kl}(T)$: $\overline{F}$ is *an* extension of $K \circ F$ along K. For historical reasons we will honour the terminology used in the literature and refer to such extensions as Kleisli liftings.

As noted in [113], Kleisli liftings of an endofunctor F along $K\colon \mathbf{C} \to \mathbf{Kl}(T)$ are uniquely characterised by suitable natural transformations that distribute the monad T over the endofunctor F. Formally:

DEFINITION 3.2. *Let (T, μ, η) be a monad and F an endofunctor, both over some category* **C**. *A law distributing (T, μ, η) over F is a natural transformation $\lambda\colon FT \to TF$ compatible with the monad structure of T as stated by the commuting diagrams below.*





A precise notation would require to write distributive laws as triples such as $((T, \mu, \eta), F, \lambda)$ in order to keep the monad structure explicit; as common practice, we will often write $\lambda\colon FT \to TF$ or just $\lambda^{T,F}$ akin to how we write T for a monad and denote its multiplication and unit by μ^T and η^T, respectively.

The following fact is stated in [113, Theorem 2.2]; see also [96, 97] for further details and generalisations.

PROPOSITION 3.1. *For (T, μ, η) and F a monad and an endofunctor both over a category* **C***, Kleisli liftings of F to* **Kl**(T) *are in bijective correspondence with laws distributing (T, μ, η) over F.*

PROOF. Let λ be a distributive law of (T, μ, η) over F. For X an object and $f\colon X \to Y$ a morphism of **Kl**(T), the assignments

$$X \mapsto FX \qquad f \mapsto \lambda_Y \circ Ff$$

define an endofunctor $\overline{F}$ over **Kl**(T). This functor is a Kleisli lifting since:

$$KFf = \eta_{FY} \circ Ff = \lambda_Y \circ F\eta_Y \circ Ff = \overline{F}Kf$$

for any $f\colon X \to Y$ in **C**.

For the converse assume $\overline{F}$ Kleisli lifting of F and let ε denote the counit of $(K \dashv L)$. Since $\overline{F}K = KF$, define $\lambda'\colon FL \to L\overline{F}$ as the transpose of $\overline{F}\varepsilon\colon \overline{F}KL \to \overline{F}$. Then, λ' is $(id_{L\overline{F}} \circ \varepsilon) \bullet (\eta \circ id_{FL})$ where $\circ$ and $\bullet$ denote horizontal and vertical composition of natural transformations in **Cat**. Since $LK = T$ and $\overline{F}K = KF$, $\lambda' K$ is a natural transformation of type $FLK \to LKF$ as needed. Compatibility with (T, μ, η) and F follows by diagram chasing. $\square$

Canonical liftings via tensorial strength Existence of distributive laws of (T, μ, η) over F is not unusual. In fact, there are several classes of monads and functors of interest, w.r.t. linear and trace semantics, for which Kleisli liftings can be constructed in a canonical way. In particular we mention strong (commutative) monads and polynomial functors [35, 43, 64, 113, 145, 146].

Assume $(\mathbf{C}, \otimes, I)$ to be a monoidal category and let a, l, and r denote its associator, left unitor, and right unitor. A monad (T, μ, η) on **C** is called *strong* if it is equipped with a family of morphisms:

$$\{str_{X,Y}\colon X \otimes TY \to T(X \otimes Y)\}_{X,Y \in \mathbf{C}},$$

called *(tensorial) strength*, which is natural in both components and is coherent with the structure of monads and monoidal categories, *i.e.*:

$$\mu_{X\otimes Y} \circ T(str_{X,Y}) \circ str_{X,TY} = str_{X,Y} \circ (id_X \otimes \mu_Y)$$
$$\eta_{X\otimes Y} = str_{X,Y} \circ (id_X \otimes \eta_Y) \qquad \lambda_{TX} = T(\lambda_X) \circ str_{I,X}$$
$$T(\alpha_{X,Y,Z}) \circ str_{X\otimes Y,Z} = str_{X,Y\otimes Z} \circ id_X \otimes str_{Y,Z} \circ \alpha_{X,Y,TZ}.$$

Dually, a *costrength* for a monad is family:

$$\{cstr_{X,Y} \colon TX \otimes Y \to T(X \otimes Y)\}_{X,Y \in \mathbf{C}},$$

which is natural in both X and Y and coherent with respect to the structure of T and $\mathbf{C}$. Every strong monad on a symmetric monoidal category has a costrength given on each component as $cstr_{X,Y} = T\phi_{Y,X} \circ str_{Y,X} \circ \phi_{TX,Y}$ where $\phi = \{\phi_{X,Y} \colon X \otimes Y \cong Y \otimes X\}_{X,Y \in \mathbf{C}}$ is the braiding natural isomorphism for the symmetric monoidal category $(\mathbf{C}, \otimes, I)$. A strong monad on a symmetric monoidal category is called *commutative* whenever:

$$\mu_{X \otimes Y} \circ T(str_{X,Y}) \circ cstr_{X,TY} = \mu_{X \otimes Y} \circ T(cstr_{X,Y}) \circ str_{TX,Y}.$$

Every strong commutative monad is a symmetric monoidal monad (and *vice versa*). In fact, its *double strength*:

$$\{dstr_{X,Y} \colon TX \otimes TY \to T(X \otimes Y)\}_{X,Y \in \mathbf{C}},$$

can be defined in terms of its (co)strength as follows:

$$dstr_{X,Y} = \mu_{X \otimes Y} \circ T(str_{X,Y}) \circ cstr_{X,TY} = \mu_{X \otimes Y} \circ T(cstr_{X,Y}) \circ str_{TX,Y}.$$

Conversely, $str_{X,Y} = dstr_{X,Y} \circ (\eta_X \otimes id_{TY})$ and $cstr_{X,Y} = dstr_{X,Y} \circ (id_{TX} \otimes \eta_Y)$.

Kleisli categories of (symmetric) monoidal monads have a canonical (symmetric) monoidal structure, induced by the monoidal structure of the monad and such that the canonical adjunction is a monoidal adjunction with respect to this structure. For $(T, \mu, \eta, dstr)$ a strong commutative monad over $(\mathbf{C}, \otimes, I)$, define $(-\overline{\otimes}-) \colon \mathbf{Kl}(T) \times \mathbf{Kl}(T) \to \mathbf{Kl}(T)$ as $(X \otimes X')$, on each pair of objects X and X', and as $f \mathbin{\overline{\otimes}} f' = dstr_{Y,Y'} \circ (f \times f')$, on each pair of morphisms $f \colon X \to Y$ and $f' \colon X' \to Y'$. This functor is a lifting of $(- \otimes -)$ along $K \colon \mathbf{C} \to \mathbf{Kl}(T)$ and forms, together with the unit I of $\otimes$, a monoidal structure on the Kleisli category of T. We refer the reader to [86, 88] for further details on strong and monoidal monads.

EXAMPLE 3.2. *The powerset functor $\mathcal{P}$ assigns to any set the set $\mathcal{P}X$ of all its subsets and to any function $f \colon X \to Y$ the function $\mathcal{P}(f)(X') = \{f(x) \mid x \in X'\}$; it admits a monad structure $(\mathcal{P}, \mu, \eta)$ whose multiplication and unit are given on each component X as $\mu_X(Y) = \bigcup Y$ and $\eta_X(x) = \{x\}$. This monad is equipped with strength, costrength and double strength given, on each component, as:*

$$\begin{aligned} str_{X,Y}\left(x, Y'\right) &= \{x\} \times Y' \\ cstr_{X,Y}\left(X', y\right) &= X' \times \{y\} \\ dstr_{X,Y}\left(X', Y'\right) &= X' \times Y' \end{aligned}$$

and hence is strong and commutative (see e.g. *[64, 86, 88]).*

EXAMPLE 3.3. *The probability distribution functor $\mathcal{D}$ assigns to any set X the set $\mathcal{D}X = \{\phi\colon X \to [0,1] \mid \sum_{x\in X}\phi(x) = 1\}$ of discrete measures and to any function $f\colon X \to Y$ the function $\mathcal{D}f(\phi)(y) = \sum_{f(x)=y}\phi(x)$; it admits a monad structure $(\mathcal{D},\mu,\eta)$ whose multiplication and unit are given on each component X as $\mu_X(\phi)(x) = \sum_{\psi}\psi(x)\cdot\phi(\psi)$ and $\eta_X(x) = \delta_x$ where $\delta_x\colon X \to [0,1]$ is the Dirac's delta function. This monad is equipped strength, costrength and double strength given, on each component, as: (see* e.g. *[64]):*

$$str_{X,Y}(x,\psi)(x',y') = \delta_x(x')\cdot\psi(y')$$
$$cstr_{X,Y}(\phi,y)(x',y') = \phi(x')\cdot\delta_y(y')$$
$$dstr_{X,Y}(\phi,\psi)(x',y') = \phi(x')\cdot\psi(y')$$

and hence is strong and commutative (see e.g. *[35, 64]).*

An endofunctor over a category with products and coproducts is called *polynomial* whenever it is formed by constants, products, and coproducts. Assume **C** has coproducts of cardinality κ, a polynomial endofunctor over **C** is any endofunctor F generated by the grammar:

$$F ::= Id_{\mathbf{C}} \mid A \mid \coprod_{i\in I} F_i \mid F_0 \times F_1$$

where A ranges over $\mathrm{obj}(\mathbf{C})$ and I has cardinality at most κ. Kleisli liftings for polynomial functors can be constructed by structural recursion: all cases are trivial except for products which require the additional assumption that T is a symmetric monoidal monad with respect to the structure $(\mathbf{C},\times,1)$.

- If $F = Id_{\mathbf{C}}$ or $F = A$ then, define $\overline{F}$ as $Id_{\mathbf{Kl}(T)}$ and A, respectively.
- If $F = \coprod_{i\in I} F_i$ then, define its Kleisli lifting as the coproduct $\coprod_{i\in I}\overline{F_i}$ where each $\overline{F_i}$ is the lifting of F_i obtained via this recursive procedure—this yields a Kleisli lifting by construction of each $\overline{F_i}$ and by $K\colon \mathbf{C} \to \mathbf{Kl}(T)$ preserving coproducts.
- If $F = F_0 \times F_1$, define $\overline{F}$ as $\overline{F_0}\ \overline{\times}\ \overline{F_1}$ where $\overline{\times}$ is the tensor product induced by $\times$ and the monoidal structure of T, $\overline{F_0}$ and $\overline{F_1}$ are obtained via this recursive procedure.

EXAMPLE 3.4. *LTSs with labels in a given set A (see e.g [125]) can be viewed as coalgebras for the endofunctor $\mathcal{P}(A \times Id)\colon$ **Set** $\to$ **Set** [121]. Since $A \times Id$ is polynomial and $\mathcal{P}$ is strong and commutative, it is possible to apply the above procedure and construct $\overline{A \times Id}$ canonical Kleisli lifting of $A \times Id$; this endofunctor over* **Kl**$(\mathcal{P})$ *acts as $A \times Id$ on objects and as $dstr(\eta_A \times Id)$ on morphisms. In particular, for any object X we have:*

$$(\overline{A \times Id})X = A \times X$$

and for any morphism $f\colon X \to Y$ *in* $\mathbf{Kl}(\mathcal{P})$ *we have:*

$$(\overline{A \times Id})X = A \times X \quad \textit{and} \quad (\overline{A \times Id})(f)(a,x) = \{(a,y) \mid y \in f(x)\}. \quad \square$$

EXAMPLE 3.5. *Fully-probabilistic systems [56] are modelled as coalgebras for the endofunctor* $\mathcal{D}(A \times Id)$ *on* **Set** *[136]. Since* $A \times Id$ *is polynomial and* $\mathcal{D}$ *is strong and commutative, it is possible to construct* $\overline{A \times Id}$ *canonical Kleisli lifting of* $A \times Id$*; this endofunctor over* $\mathbf{Kl}(\mathcal{D})$ *acts as* $A \times Id$ *on objects and as* $dstr(\eta_A \times Id)$ *on morphisms. Since the probability distribution monad* $\mathcal{D}$ *is strong and commutative, the endofunctor* $(A \times Id)$ *has a canonical Kleisli lifting* $\overline{A \times Id}$ *to* $\mathbf{Kl}(\mathcal{D})$ *acting as* $A \times Id$ *on objects and as* $dstr(\eta_A \times Id)$ *on morphisms. In particular, for any object* X *we have:*

$$(\overline{A \times Id})X = A \times X$$

and for any morphism $f\colon X \to Y$ *in* $\mathbf{Kl}(\mathcal{D})$ *we have:*

$$(\overline{A \times Id})(f)(a,x)(b,y) = \delta_a(b) \cdot f(x)(y). \quad \square$$

3.1.2 Kleisli coinduction, linear and trace semantics

For $\overline{F}$ a Kleisli lifting of F along $K\colon \mathbf{C} \to \mathbf{Kl}(T)$, the category $\mathbf{Coalg}(TF)$ is a wide subcategory of $\mathbf{Coalg}(\overline{F})$: the inclusion functor K lifts along the forgetful functors for $\mathbf{Coalg}(TF)$ and $\mathbf{Coalg}(\overline{F})$, as shown in the diagram below, to a functor that acts as the identity on coalgebras and as K on morphisms:

$$\begin{array}{ccc} \mathbf{Coalg}(TF) & \longrightarrow & \mathbf{Coalg}(\overline{F}) \\ \big\downarrow & & \big\downarrow \\ \mathbf{C} & \xrightarrow[K]{} & \mathbf{Kl}(T) \end{array} \tag{3.4}$$

Although TF-coalgebras are precisely $\overline{F}$-coalgebras, their morphisms capture a different kind of relations between the systems under scrutiny: TF-coalgebra homomorphisms are *functional* bisimulations [121, Theorem 2.5] whereas $\overline{F}$-coalgebra homomorphisms are functional *linear* bisimulations [117, Proposition 2.8]. Here the term linear is intended in a broad sense generalising from non-determinism ([117] considers LTSs only) to effects modelled by an arbitrary monad T.

This difference becomes clear when the definition of $\overline{F}$-coalgebra homomorphisms is expressed as a diagram in $\mathbf{C}$, the category underlying $\mathbf{Kl}(T)$. To this end, let $f\colon (X,h) \to (Y,k)$ be a coalgebra homomorphism with underlying morphisms $f\colon X \to Y$ and consider its associated diagram in $\mathbf{Kl}(T)$:

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle k} \\ \overline{F}X & \xrightarrow[\overline{F}f]{\quad\quad} & \overline{F}Y \end{array}$$

This diagram corresponds to the following diagram in **C**:

$$\begin{array}{ccccccc} X & & & \xrightarrow{\qquad\qquad f \qquad\qquad} & & & TY \\ & & & & & & \big\downarrow{\scriptstyle Tk} \\ {\scriptstyle h}\big\downarrow & & & & & & TTFY \\ & & & & & & \big\downarrow{\scriptstyle \mu_{FY}} \\ TFX & \xrightarrow[TFf]{\quad} & TFTY & \xrightarrow[T\lambda_Y]{\quad} & TTFY & \xrightarrow[\mu_{FY}]{\quad} & TFY \end{array}$$

For instance, consider non-deterministic transition systems by taking T and F to be $\mathcal{P}$ and $A \times Id$, respectively. The diagram above commutes if, and only if, for any label $a \in A$ and states $x, x' \in X$ it holds that:

$$(a, x') \in h(x) \iff \forall y' \in f(x') \exists y \in f(x)((a, y') \in k(y)).$$

Intuitively, $\lambda\colon TF \to FT$ distributes the "branching" part of the behaviour (*i.e.* the computational effects modelled by T) over the observable "linear" part of it (characterised by the endofunctor F) while μ collects and combines effects thus forgetting when and how branching occurred.

From this perspective, final TF-coalgebras, and their associated coinduction principle, capture branching semantics whereas final $\overline{F}$-coalgebras, and their coinduction principle, capture linear semantics.

DEFINITION 3.3. *Assume $\overline{F}$ Kleisli lifting of F to* $\mathbf{Kl}(T)$*. For $h\colon X \to TFX$, its linear semantics is the unique morphisms $lbeh_h$:*

$$\begin{array}{ccc} X & \overset{lbeh_h}{\dashrightarrow} & Z \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle \nu\overline{F}} \\ \overline{F}X & \xrightarrow[\overline{F}lbeh_h]{\quad\quad} & \overline{F}Z \end{array}$$

In general, the linear semantics described by final $\overline{F}$-coalgebra homomorphisms may not capture a known notion of trace semantics for systems modelled as

TF-coalgebras. For instance, if T is powerset monad $\mathcal{P}$ and F is the labelling functor $A \times Id$ then the final $\overline{A \times Id}$-coalgebra coincides with the initial one and hence has the emptyset as its carrier [64, 117]. Even when final $\overline{F}$-coalgebras capture some notion of trace semantics, this is not unique across the range of choices for T and F. In fact, there are several examples in the literature where finality characterises finite, possibly infinite, infinite only traces or none at all. For instance, in [80], Kerstan and König investigate trace semantics for continuous probabilistic transition systems. To this end they consider different combinations of monads and endofunctors over **Meas**, the category of continuous functions between measurable spaces. In particular, they take T to be either the probability measure monad $\mathcal{G}$ (a.k.a. Giry monad [44, 46, 115]) or the sub-probability measure monad $\mathcal{G}_{\leq}$, and F as either $A \times Id + 1$ or $A \times Id$ *i.e.* labelling endofunctors with or without explicit termination. For each combination they compute the final $\overline{F}$-coalgebra and determine whether it captures some established notion of trace semantics. The results are summarised by the table below.

T	F	$\nu\,\overline{F}$	trace semantics
$\mathcal{G}_{\leq}$	$A \times Id$	$K\,\mu\,F$	none
$\mathcal{G}_{\leq}$	$A \times Id + 1$	$K\,\mu\,F$	finite
$\mathcal{G}$	$A \times Id$	$K\,\nu\,F$	infinite
$\mathcal{G}$	$A \times Id + 1$	$K\,\nu\,F$	possibly infinite

In the wake of the examples above, finite and (possibly) infinite trace semantics are abstractly defined by lifting initial F-algebras and final F-coalgebras.

DEFINITION 3.4. *The final $\overline{F}$-coalgebra, whenever it exists, is said to capture:*

- *finite trace semantics if $\nu\,\overline{F} \cong K(\mu\,F)^{-1}$,*
- *(possibly) infinite trace semantics if $\nu\,\overline{F} \cong K\,\nu\,F$.*

In [64] Hasuo, Jacobs and Sokolova present general and sufficient conditions for capturing finite traces via Kleisli coinduction based on suitable order-enrichment. Below we propose a modest generalisation of this seminal result by rephrasing it in terms of algebraic compactness.

PROPOSITION 3.2. *Let $\overline{F}$ be a Kleisli lifting of an endofunctor F to* **Kl**(T)*. Assume F algebraically complete and* **Kl**(T) *algebraically compact with respect to* **E** *such that $\overline{F} \in$* **E**. *The initial F-algebra lifts (along the inclusion K) to the final $\overline{F}$-coalgebra:*

$$\nu\,\overline{F} \cong K(\mu\,F)^{-1}.$$

PROOF. Recall from [66, Theorem. 2.14] that a law distributing (T, μ, η) over F induces a lifting (along the obvious forgetful functors) of the canonical adjunction

$(K \dashv L) \colon \mathbf{Kl}(T) \to \mathbf{C}$ to an adjunction between the categories of algebras for F and $\overline{F}$, receptively, as shown in the following diagram:

$$\begin{array}{ccc} \mathbf{Alg}(F) & \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} & \mathbf{Alg}(\overline{F}) \\ \downarrow & & \downarrow \\ \mathbf{C} & \underset{L}{\overset{K}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} & \mathbf{Kl}(T) \end{array}$$

In particular, the lifting of K maps an F-algebra $g \colon FX \to X$ to the $\overline{F}$-algebra $(\eta_{FX} \circ g) \colon \overline{F}X \to X$ and an F-algebra homomorphism $f \colon (X, g) \to (Y, h)$ to $(\eta_Y \circ f) \colon (X, \eta_{FX} \circ g) \to (Y, \eta_{FY} \circ h)$. Because of the above adjoint situation, $K\,\mu F = \eta_{|\mu F|} \circ \mu F$ is an initial $\overline{F}$-algebra. By algebraic compactness any initial $\overline{F}$-algebra is canonically isomorphic to a final $\overline{F}$-coalgebra hence $K(\mu F)^{-1} = (K\,\mu F)^{-1}$ is (up to isomorphism) the required final $\overline{F}$-coalgebra. □

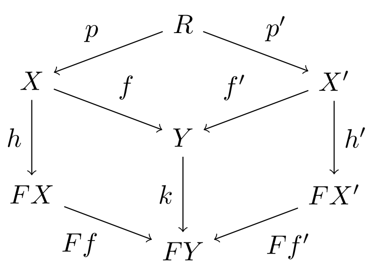
The result presented in [64] is readily recovered as an instance of the above: assume an initial F-algebra can be computed via an initial sequence indexed by ω, that both $\mathbf{Kl}(T)$ and $\overline{F}$ are enriched over the category of continuous maps between ω-CPOs with bottom elements **Cppo** (*i.e.* the full image of the inclusion $\mathbf{Cpo}_\bot \hookrightarrow \mathbf{Cpo}$), and that composition in $\mathbf{Kl}(T)$ is left-strict. In fact, under these assumptions, $\overline{F}$ is algebraically compact and hence Proposition 3.2 applies. These hypothesis are met by polynomial functors and monads modelling several computational effects of interests and encompassing non-deterministic transition systems, weighted transition systems, discrete and continuous probabilistic transition systems among others (*cf.* [35, 64] and Section 3.1.1).

A general account of infinite trace semantics on par with the finite case is currently missing. Several works investigated this issue, see *e.g.* [43, 64, 70, 145, 146], but in all of them liftings of final F-coalgebras are not final $\overline{F}$-coalgebras. They are *weakly final* ones. Although these characterisations are weakly universal in $\mathbf{Coalg}(\overline{F})$, they can be uniquely defined by means of some other properties: [64, 145, 146] assume an order-enriched setting and identify infinite trace semantics as the maximal $\overline{F}$-coalgebras morphism to $K(\nu F)$. Likewise, [43] define infinite trace semantics as the maximal among the mediating maps to $K(\nu F)$ arising from a suitable weak limit in $\mathbf{Kl}(T)$. Finally, we remark that all these works assume F admits a final coalgebra computable via the final sequence construction and in ω steps. It follows that infinite traces are implicitly defined as ω-indexed sequences in contrast to more general definitions such as transfinite traces [45, Chapter 11]. Nonetheless, these constructions capture (countably) infinite traces for labelled non-deterministic systems, discrete and continuous labelled probabilistic systems.

3.1.3 Linear bisimulations and trace equivalences

Notions of strong bisimulations have been captured coalgebraically [2, 121, 139]. Previous works studied behavioural equivalences for coalgebras in Kleisli categories and identified the notion of *kernel bisimulation* as the more suitable for the rôle especially because the resulting notion of bisimilarity agrees with final semantics.

Intuitively, a kernel bisimulation is "a relation which is the kernel of a common compatible refinement of the two systems" [139]. To be more precise, a relation R, *i.e.* a jointly monic span in **C**, is a *kernel bisimulation* between two F-coalgebras (h, X) and (h', X') if there are a third one, say (Y, k), and a cospan of coalgebra homomorphisms $h \to k \leftarrow h'$ for which R is the pullback in **C** of the cospan $X \to Y \leftarrow X'$ underlying $h \to k \leftarrow h'$ as illustrated by the following diagram:



From this perspective, final coalgebras can be thought as "maximal" compatible refinement systems and hence (kernels of) final semantics arrows capture bisimilarity.

In Section 3.1.2 we discussed how Kleisli liftings and their coalgebra homomorphisms capture linear and trace semantics; these results lead us to define coalgebraic linear bisimulations in terms of $\overline{F}$-coalgebra homomorphisms. However, coalgebraic notions of bisimulations for $\overline{F}$-coalgebras, like Aczel and Mendler's bisimulation or kernel bisimulation, consider relations as spans in the Kleisli category of T as opposed to their counterparts for TF-coalgebras which are spans in the underlying category **C**. Consider for instance the case of **C** being **Set**, (jointly monoic) spans in Kleisli categories are subsets of $TX \times TY$: if T is the powerset monad $\mathcal{P}$ or the probability distribution monad $\mathcal{D}$ then spans in their Kleisli category are relations between subsets like $R \subseteq \mathcal{P}X \times \mathcal{P}Y$ and relations between probability distributions like $R \subseteq \mathcal{D}X \times \mathcal{D}Y$, respectively. In [117], Power and Turi argue that spans in Kleisli categories are the right notion of relation to use since they extend the effect of the branching type T to carriers (consider *e.g.* $\mathbf{Kl}(\mathcal{P})$) akin to automata determinisation [74, 131] or transition systems linearisation [117].

Finally, note that the notion of kernel bisimulation as described above assumes relations to be jointly monic spans that are pullbacks for the given cospan. These spans can be thought as compact representations of relations: being pullbacks means being the final span in the category of spans for a given cospan (span homomorphisms are morphisms between their cusps making the obvious triangles commute). We maintained this restriction while describing kernel bisimulations to adhere to the main definitions found in the literature (*cf.* [90, 138, 139]) but the definition can be safely relaxed to accommodate arbitrary spans. In the setting of Kleisli categories this is a necessity since Kleisli categories usually lack pullbacks regardless of their underlying category having such limits.

DEFINITION 3.5. *A* linear (kernel) bisimulation *for $\overline{F}$-coalgebras (X, h) and (X', h') is a span $(p\colon R \to X, p'\colon R \to X')$ in* **Kl**(T) *such that there exists a cospan of $\overline{F}$-coalgebra homomorphisms making the diagram below commute in* **Kl**(T)*:*

$$\begin{array}{ccccc} & & R & & \\ & \swarrow^{p} & & \searrow^{p'} & \\ X & & & & X' \\ & \searrow^{f} & & \swarrow^{f'} & \\ \big\downarrow h & & Y & & \big\downarrow h' \\ & & \big\downarrow k & & \\ \overline{F}X & & & & \overline{F}X' \\ & \searrow_{\overline{F}f} & & \swarrow_{\overline{F}f'} & \\ & & \overline{F}Y & & \end{array} \tag{3.5}$$

Like in the case of linear semantics, Definition 3.5 may not capture any known notion of trace equivalence for systems modelled as TF-coalgebras. Again, coalgebras for the functor $\mathcal{P}(A \times Id)$ provide us an example of this situation: final $\overline{A \times Id}$-coalgebras are carried by the empty-set which is final in **Kl**$(\mathcal{P})$. The *criterium* behind Definition 3.4 applies to this situation as well.

3.1.4 Distributive law morphisms

Distributive laws can be organised into categories by means of several notions of distributive law morphisms depending on the kind of structures being distributed as discussed in [96, 97, 118, 147]. In [96], these notions are introduced as part of different 2-categorical contexts where to analyse distributive laws arising in operational and denotational semantics. This effort, and especially works such as [83, 147], resulted in the proposal of (suitable formulations of) distributive law morphisms as the abstract understanding of translations between SOS specifications. As a consequence, this result extends the theory of abstract GSOS [82, 144] with morphisms able to relate models while preserving structures of interest.

The study of Kleisli coinduction faces a situation similar to that of abstract GSOS prior to the aforementioned works: this theory lacks morphisms between models. To this end, we propose the use of following notion of distributive law morphisms because they induce functors between categories of coalgebras and transformations between Kleisli liftings that are coherent with respect to linear bisimulation and the relevant structures of Kleisli categories such as the canonical inclusion of their underlying category.

DEFINITION 3.6. *Let $\lambda\colon FT \to TF$ and $\lambda'\colon F'T' \to T'F'$ be distributive laws of monads over endofunctors over* **C**. *A distributive law morphism from λ to λ' is a pair (θ, υ) composed by a monad morphism $\theta\colon (T, \mu^T, \eta^T) \to (T', \mu^{T'}, \eta^{T'}) \in$* **Mnd**(**C**) *and an endofunctor morphism $\upsilon\colon F \to F' \in$* **End**(**C**) *subject to the following coherence condition:*

$$\begin{array}{ccc} F \circ T & \xrightarrow{\lambda} & T \circ F \\ {\scriptstyle \upsilon \circ \theta}\downarrow & & \downarrow{\scriptstyle \theta \circ \upsilon} \\ F' \circ T' & \xrightarrow[\lambda']{} & T' \circ F' \end{array} \tag{3.6}$$

Distributive laws of monads over endofunctors on **C** together with their morphisms for the category **MndEnd**(**C**). Forgetting either component of distributive law morphisms gives rise to two functors from **MndEnd**(**C**) to **Mnd**(**C**) and **End**(**C**), respectively. Both these forgetful functors have sections since any monad distributes over the identity functor and the identity monad distributes over any endofunctor.

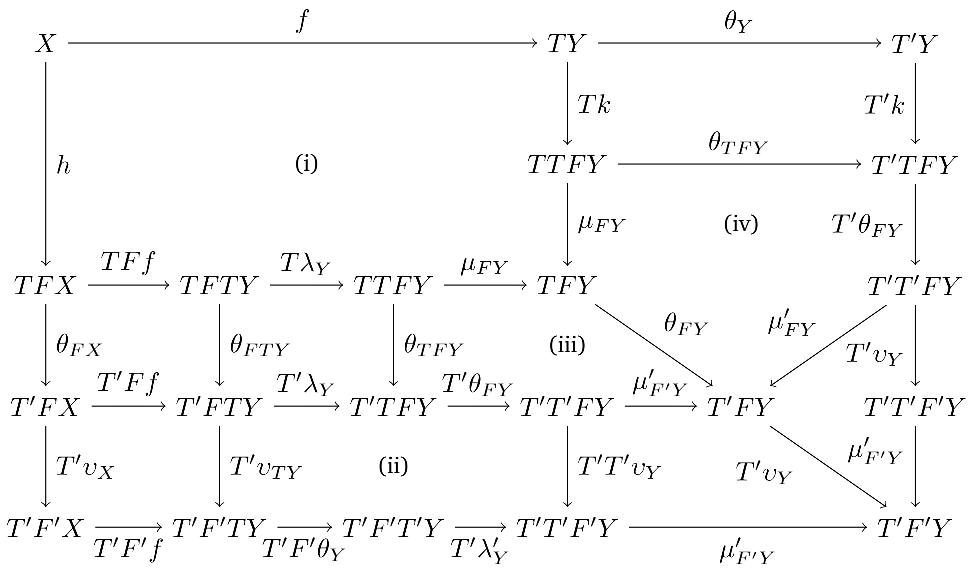
An interesting property of distributive law morphisms (as defined above) is that they induce functors between categories of coalgebras modelling branching and linear semantics that are coherent with respect to the canonical inclusion into Kleisli categories of their underlying category (*cf.* Proposition 3.3) and to linear semantics (*cf.* Proposition 3.4). Let (θ, υ) be a distributive law morphism from $\lambda\colon FT \to TF$ to $\lambda'\colon F'T' \to T'F'$. Consider the assignments mapping each TF-coalgebra (X, g) and homomorphism $f\colon (X, g) \to (Y, h)$ as follows:

$$(X, g) \mapsto (X, (\theta \circ \upsilon)_X \circ g) \qquad f \mapsto f \tag{3.7}$$

It follows from naturality of θ and υ that these assignments define the functor **Coalg**$(\theta \circ \upsilon)$: **Coalg**$(TF) \to$ **Coalg**$(T'F')$. For a $\overline{F}$-coalgebra $g\colon X \to \overline{F}X$ and a $\overline{F}$-coalgebra homomorphism $f\colon (X, g) \to (Y, h)$, consider the assignments

$$(X, g) \mapsto (X, (\theta \circ \upsilon)_X \circ g) \qquad f \mapsto \theta_Y \circ f. \tag{3.8}$$

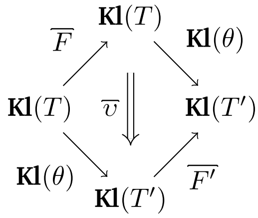
These assignments map $\overline{F}$-coalgebras to $\overline{F'}$-coalgebras and homomorphisms accordingly as the following commuting diagram asserts:



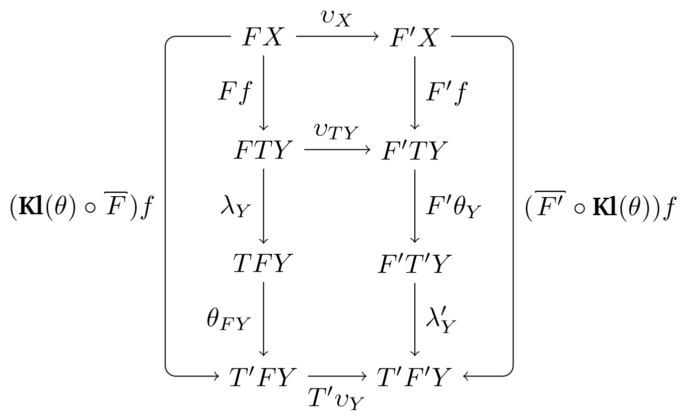
In order to check that the diagram above indeed commutes note that (i) is the expansion in **C** of the diagram asserting that f is a $\overline{F}$-coalgebra homomorphism, that (ii) is a component of (3.6) under T, that (iii) and (iv) follow from the fact that θ is a monad morphism, and that all remaining sub-diagrams are naturality squares. As a consequence of the fact that the above diagram commutes and the assumption of (θ, υ) that is a distributive law morphism, the assignments (3.8) define a functor $\mathbf{Coalg}(\theta, \upsilon)\colon \mathbf{Coalg}(\overline{F}) \to \mathbf{Coalg}(\overline{F})$. Moreover, this functor is a lifting of $\mathbf{Kl}(\theta)\colon \mathbf{Kl}(T) \to \mathbf{Kl}(T')$ along the forgetful functors depicted in the following diagram:

$$\begin{array}{ccc} \mathbf{Coalg}(\overline{F}) & \xrightarrow{\mathbf{Coalg}(\theta,\upsilon)} & \mathbf{Coalg}(\overline{F'}) \\ \downarrow & & \downarrow \\ \mathbf{Kl}(T) & \xrightarrow[\mathbf{Kl}(\theta)]{} & \mathbf{Kl}(T') \end{array}$$

From a more abstract perspective, this situation can be seen to follow from lifting $\upsilon\colon F \to F'$ to a natural transformation exchanging $\mathbf{Kl}(\theta)$ with Kleisli liftings as depicted in the diagram below. In fact, any distributive law morphism (θ, υ) determines a natural transformation $\overline{\upsilon}$ such that:



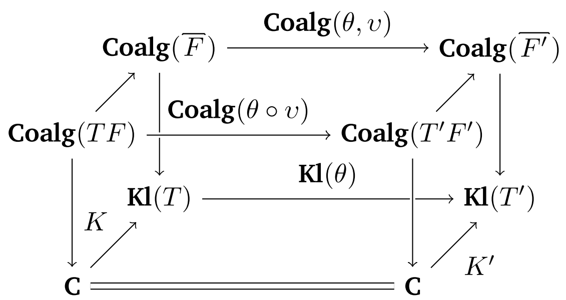
The natural transformation $\overline{\upsilon}\colon \mathbf{Kl}(\theta) \circ \overline{F} \to \overline{F'} \circ \mathbf{Kl}(theta)$ is given on each object X as $K' \circ \upsilon_X$. To see that this is indeed a natural transformation it suffices to note that, for any $f\colon X \to TY$, the corresponding naturality square reduces to the following diagram in the underlying category **C**:



The diagram above commutes since (θ, υ) is a distributive law morphism. Note that the diagram above lies in the lower part of the diagram unfolding (3.8).

Actually, the functor $\mathbf{Coalg}(\theta \circ \upsilon)$ is the restriction of $\mathbf{Coalg}(\theta, \upsilon)$ to the wide subcategory determined by TF-coalgebra homomorphisms.

PROPOSITION 3.3. *For (θ, υ) a distributive law morphism from $\lambda\colon FT \to TF$ to $\lambda'\colon F'T' \to T'F'$, the diagram below commutes:*

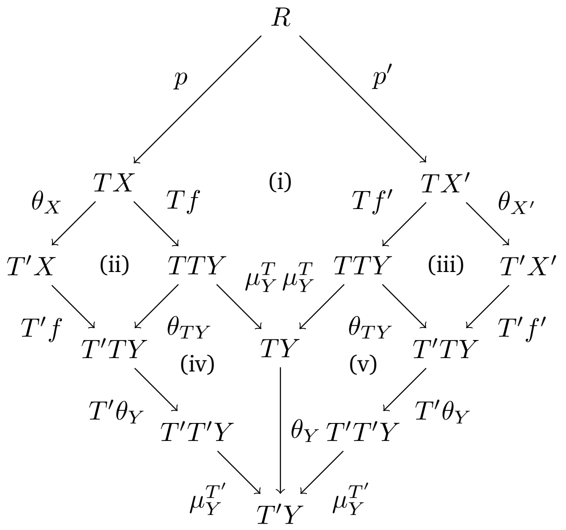


PROOF. Left and right faces assert that the inclusions K and K' lift along the forgetful functors for the coalgebra categories involved and this holds true for any Kleisli lifting as per (3.4). The front face of the diagram commutes by definition of $\mathbf{Coalg}(\theta \circ \upsilon)$ since, as clear from (3.7), this functor acts as the identity on coalgebra homomorphisms. The back face of the diagram commutes by definition of $\mathbf{Coalg}(\theta, \upsilon)$ since, as clear from (3.8), this functor composes coalgebra homomorphisms to the opportune components of θ *i.e.* it acts as $\mathbf{Kl}(\theta)$. The bottom of the diagram commutes since $\theta\colon T \to T'$ is a monad morphism and by definition of the functor $\mathbf{Kl}(\theta)$. To see that the top of the diagram commutes as well note that **Coalg**(TF) and **Coalg**$(T'F')$ are wide subcategories of **Coalg**$(\overline{F})$ and **Coalg**$(\overline{F'})$, respectively and that on coalgebras **Coalg**$(\theta \circ \upsilon)$ and **Coalg**(θ, υ) act in the very same way, whereas on coalgebra homomorphisms the first acts as the identity and the second as **Kl**(θ): **Kl**$(T) \to$ **Kl**(T'). Therefore, the top diagram commutes since **Kl**$(\theta) \circ K = K'$ $(\theta \bullet \eta = \eta')$. Finally, the whole diagram commutes since all of its faces are so. □

PROPOSITION 3.4. *For $(\theta, \upsilon)\colon \lambda \to \lambda'$ a distributive law morphism, the following statements are true.*

- *If (p, p') is a linear bisimulation for the $\overline{F}$-coalgebras h and h', then (p, p') is a linear bisimulation for the $\overline{F'}$-coalgebras* **Coalg**$(\theta, \upsilon)(h)$ *and* **Coalg**$(\theta, \upsilon)(h')$.
- *If θ is componentwise monic, the functor* **Coalg**(θ, υ) *reflects linear bisimulations whose witness lies in the (essential) image of* **Coalg**(θ, υ).

PROOF. Let (p, p') be a linear bisimulation for the $\overline{F}$-coalgebras h and h'. In order to check that (p, p') is also a linear bisimulation for the $\overline{F'}$-coalgebras **Coalg**$(\theta, \upsilon)(h)$ and **Coalg**$(\theta, \upsilon)(h')$ let (f, f') be the cospan of $\overline{F}$-coalgebra homomorphisms rendering (p, p') a linear bisimulation for h and h'. Then, the claim follows from the commuting diagram below:



(vi)

The diagram above indeed commutes since: (i) holds by assumption on (p, p') and (f, f'); (ii) and (iii) are naturality squares for θ; (iv) and (v) follow from θ being a monad morphism. Assume that θ is componentwise monic and let (p, p') be a span in $\mathrm{eimg}($**Coalg**$(\theta, \upsilon))$, let (q, q') be any span taken (p, p') up to isomorphism, and let (f, f') be a cospan of $\overline{F}$-coalgebra homomorphisms from h

and h' to some common refinement coalgebra k such that its image shows (p, p') as a linear bisimulation for $\mathbf{Coalg}(\theta, \upsilon)(h)$ and $\mathbf{Coalg}(\theta, \upsilon)(h')$. The diagram associated to the later assumption can be decomposed in (vi) and, since θ_Y is monic, the diagram (i) commutes. This proves that the span (q, q') is a linear bisimulation for the $\overline{F}$-coalgebras h and h'. □

The situation described by Propositions 3.3 and 3.4 confirms Definition 3.6 as a suitable notion of morphisms since these coherently induce functors between categories of coalgebras capturing the linear and branching semantics of systems.

3.2 POINTWISE EXTENSIONS TO SHEAF CATEGORIES

In this section we consider **C**-valued sheaves over ordinal numbers equipped with the Alexandrov topology and study the pointwise extension of endofunctors, monads, and their distributive laws to this setting. Hereafter, let α be limit ordinal and assume that the constant sheaf adjunction $(\Delta \dashv \Gamma)\colon \mathbf{Sh_C}(\alpha) \to \mathbf{C}$ is defined (*cf.* Section 2.1).

3.2.1 The pointwise extension functor

For F an endofunctor over **C** consider the endofunctor $\mathbf{Fun}(Id, F)$ over $\mathbf{PSh_C}(\alpha)$ defined on any presheaf X, morphism f, and stages $\beta \leq \beta'$ as follows:

$$\mathbf{Fun}(Id,F)X_\beta = FX_\beta \quad \mathbf{Fun}(Id,F)X_{\iota_{\beta,\beta'}} = FX_{\iota_{\beta,\beta'}} \quad \mathbf{Fun}(Id,F)f_\beta = Ff_\beta$$

Because this functor acts on values as F we call $\mathbf{Fun}(Id, F)$ the *pointwise extension (to presheaves) of* F.

This endofunctor need not to preserve sheaves since F may not preserve the necessary limits (which are pointwise in $\mathbf{PSh_C}(\alpha)$). Therefore, to obtain an extension of F to the category of sheaves we need to apply the associated sheaf functor **a** that, together with its right adjoint **i**, yields the endofunctor:

$$\mathbf{a} \circ \mathbf{Fun}(Id, F) \circ \mathbf{i}\colon \mathbf{Sh_C}(\alpha) \to \mathbf{Sh_C}(\alpha).$$

We call this functor the *pointwise extension (to sheaves) of* F and denote it as $\underline{F}$. This functor takes any sheaf X and any morphism $f\colon X \to Y$ to:

$$\begin{aligned} &\underline{F}X_\beta = FX_\beta && \underline{F}X_{\iota_{\beta,\beta+1}} = FX_{\iota_{\beta,\beta+1}} && \underline{F}f_\beta = Ff_\beta \\ &\underline{F}X_\gamma = \lim_{\beta<\gamma} FX_\beta && \underline{F}X_{\iota_{\beta,\gamma}} = \pi_\beta && \underline{F}f_\gamma = \rho_\gamma \end{aligned}$$

where β is a successor ordinal, γ a limit one, $\pi_\beta\colon \lim_{\beta'<\gamma} FX_{\beta'} \to FX_\beta$ is the component at β of the limiting cone, and $\rho_\gamma\colon \lim_{\beta<\gamma} FX_\beta \to \lim_{\beta<\gamma} FY_\beta$ is the mediating map for the cone $\{f_\beta \circ \pi_\beta\}_{\beta<\gamma}$.

The endofunctor $\underline{F}$ is *an* extension of F (actually of $\Delta \circ F$) along the constant sheaf functor Δ since it makes the diagram below commute.

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{\Delta} & \mathbf{Sh_C}(\alpha) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle \underline{F}} \\ \mathbf{C} & \xrightarrow[\Delta]{} & \mathbf{Sh_C}(\alpha) \end{array} \tag{3.9}$$

Intuitively, this diagram abstractly describes the idea that $\underline{F}$ acts as F on the image of the subcategory **C** of $\mathbf{Sh_C}(\alpha)$, especially on the final object 1. As a consequence, the final sequence for the endofunctor F extends along Δ to the final sequence for $\underline{F}$. Formally, the constant sheaf functor $\Delta \colon \mathbf{C} \to \mathbf{Sh_C}(\alpha)$ lifts along the forgetful functors for $\mathbf{Coalg}(F)$ and $\mathbf{Coalg}(\underline{F})$ as in the diagram below.

$$\begin{array}{ccc} \mathbf{Coalg}(F) & \longrightarrow & \mathbf{Coalg}(\underline{F}) \\ \downarrow & & \downarrow \\ \mathbf{C} & \xrightarrow{\Delta} & \mathbf{Sh_C}(\alpha) \end{array}$$

The lifted functor is an inclusion functor and takes final F-coalgebras to final $\underline{F}$-coalgebras.

PROPOSITION 3.5. *For F an endofunctor on* **C***, the constant sheaf functor Δ takes final F-coalgebras to final $\underline{F}$-coalgebras.*

PROOF. First note that Δ lifts along the forgetful functors for $\mathbf{Coalg}(F)$ to $\mathbf{Coalg}(\underline{F})$ to an inclusion functor from $\mathbf{Coalg}(F)$ to $\mathbf{Coalg}(\underline{F})$. For (X, h) a F-coalgebra, Δh has type $\Delta h \to \Delta F X$ and $\Delta F = \underline{F}\Delta$ from which we conclude that Δh is a $\underline{F}$-coalgebra. Likewise, Δ maps F-coalgebra homomorphisms to $\underline{F}$-coalgebra homomorphisms. Assume νF exists, we show its image final. For (Y, k) a $\underline{F}$-coalgebra and β a successor ordinal in α, the component k_β is a F-coalgebra and hence there is a unique F-coalgebra homomorphism $!_{k_\beta} \colon k_\beta \to \nu F$. For any successor ordinal β' such that $\beta \le \beta'$, the restriction morphism $Y_{\iota_{\beta,\beta'}} \colon Y_{\beta'} \to Y_\beta$ carries a F-coalgebra homomorphism from $k_{\beta'}$ to k_β such that $!_{k_{\beta'}} = !_{k_\beta} \circ Y_{\iota_{\beta,\beta'}}$. Reworded, for β and β' successor ordinals, $!_{k_\beta}$ and $!_{k_{\beta'}}$ satisfy naturality. Since successor ordinals form a base for $\mathcal{A}(\alpha)$, the family of F-coalgebra homomorphisms $\{!_{k_\beta}\}$ uniquely extends to a $\underline{F}$-coalgebra homomorphism from k to $\Delta\, \nu F$. This homomorphism is necessarily unique by assumption on νF and hence exhibits $\Delta\, \nu F$ as final in $\mathbf{Coalg}(\underline{F})$. □

REMARK 3.6. Assume that **C** is bicomplete with respect to α-sequences hence that the constant sheaf functor Δ is both left and right adjoint; examples of this situation are the categories **Set** and **Meas**. Since under these assumption **C** is a (co)reflective subcategory of $\mathbf{Sh_C}(\alpha)$, Proposition 3.5 follows from a result known as "Freyd's Reflective Subcategory Lemma" [53]: final coalgebras for endofunctors which restrict to reflective subcategories lie in such categories and coincide with final coalgebras for their restrictions.

Symmetrically to the situation described in (3.9), the endofunctor F is the right extension of $\underline{F}$ (actually of $\Gamma \circ \underline{F}$) along the global section functor Γ since there is a (unique) 2-cell ϱ such that:

$$\begin{array}{ccc} \mathbf{Sh_C}(\alpha) & \xrightarrow{\Gamma} & \mathbf{C} \\ {\scriptstyle \underline{F}}\downarrow & \overset{\varrho}{\swarrow} & \downarrow{\scriptstyle F} \\ \mathbf{Sh_C}(\alpha) & \xrightarrow[\Gamma]{} & \mathbf{C} \end{array} \tag{3.10}$$

In particular, $(F, \varrho) \cong Ran_\Gamma(\Gamma\underline{F})$. The natural transformation ϱ is given on each sheaf X as the mediating map $\varrho_X \colon F \lim X \to \lim FX$ which is clearly unique. This characterisation might appear backward since we obtained F from $\underline{F}$–one might prefer to call $\underline{F}$ a lifting of F along Γ. Nonetheless, the natural transformation ϱ and its sections are be of relevance for lifting the adjunction $(\Delta \dashv \Gamma)$ to categories of coalgebras as stated by Proposition 3.6 below.

PROPOSITION 3.6. *If* $\varrho \colon F\Gamma \to \Gamma\underline{F}$ *from* (3.10) *is a retraction then the constant sheaf adjunction* $(\Delta \dashv \Gamma) \colon \mathbf{Sh_C}(\alpha) \to \mathbf{C}$ *lifts along the forgetful functors for* $\mathbf{Coalg}(F)$ *and* $\mathbf{Coalg}(\underline{F})$ *as shown in the diagram below.*

$$\begin{array}{ccc} \mathbf{Coalg}(F) & \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} & \mathbf{Coalg}(\underline{F}) \\ \downarrow & & \downarrow \\ \mathbf{C} & \underset{\Gamma}{\overset{\Delta}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} & \mathbf{Sh_C}(\alpha) \end{array}$$

PROOF. Let η and ε denote the unit and counit of $(\Delta \dashv \Gamma)$, respectively. It follows from [81, Theorem 2.5] that natural transformations $\vartheta \colon \Delta F \to \underline{F}\Delta$ and $\varsigma \colon \Gamma\underline{F} \to F\Gamma$ define a lifting of $(\Delta \dashv \Gamma)$ along the forgetful functors for $\mathbf{Coalg}(F)$ and $\mathbf{Coalg}(\overline{F})$ whenever the diagrams below commute:

$$\begin{array}{ccc} F & \xrightarrow{id_F \circ \eta} & F \circ \Gamma \circ \Delta \\ {\scriptstyle \eta \circ id_F}\downarrow & \text{(i)} & \uparrow{\scriptstyle \varsigma \circ id_\Delta} \\ \Gamma \circ \Delta \circ F & \xrightarrow[id_\Gamma \circ \vartheta]{} & \Gamma \circ \underline{F} \circ \Delta \end{array} \qquad \begin{array}{ccc} \underline{F} & \xleftarrow{\varepsilon \circ id_{\underline{F}}} & \Delta \circ \Gamma \circ \underline{F} \\ {\scriptstyle id_{\underline{F}} \circ \varepsilon}\uparrow & \text{(ii)} & \downarrow{\scriptstyle id_\Delta \circ \varsigma} \\ \underline{F} \circ \Delta \circ \Gamma & \xleftarrow[\vartheta \circ id_\Gamma]{} & \Delta \circ F \circ \Gamma \end{array}$$

In this setting, the desired lifting for Δ is given, on each F-coalgebra (X, g) and homomorphism f, by the assignments:

$$(X, g) \mapsto (\Delta X, \vartheta_X \circ \Delta g) \qquad f \mapsto \Delta f.$$

Its right adjoint, that is the desired lifting for Γ, is given on each $\underline{F}$-coalgebra (Y, h) and homomorphism f, by the assignments:

$$(Y, h) \mapsto (\Gamma Y, \varsigma_Y \circ \Gamma h) \qquad f \mapsto \Gamma f.$$

We apply [81, Theorem 2.5] by choosing ϑ and ς as the identity and as any section for ϱ, respectively. This choice is justified by the fact that $\underline{F}$ is the pointwise extension of F hence an extension along Δ as per (3.9) and ϱ is a retraction by hypothesis. The natural transformation $\varrho \circ id_\Delta \circ \Gamma \underline{F} \Delta \to F \Gamma \Delta$ is an identity since for any object X of **C** we have that $\Gamma \underline{F} \Delta X = \Gamma \Delta F X$ by (3.9) and that $\Gamma \Delta F X = \lim \Delta F X = F X$. It follows that $\varsigma \circ id_\Delta$ is an identity as well and that diagram (i) is made from identities. Note that for X a sheaf, the component of ε_X at any stage β is the restriction arrow $X_{\iota_{\beta,\alpha}}$, and that the sheaves $\Delta\Gamma\underline{F}X$ and $\Delta F \Gamma X$ take values $\lim_{\beta'<\alpha} F X_{\beta'}$ and $F \lim_{\beta'<\alpha} X_{\beta'}$ at any stage, respectively. Thus, for β a stage, the component at stage β of (ii) is the following square in **C**:

$$\begin{array}{ccc}
\lim_{\beta'<\alpha} F X_{\beta'} & \xrightarrow{\varsigma_X} & F \lim_{\beta'<\alpha} X_{\beta'} \\
{\scriptstyle \underline{F}X_{\iota_{\beta,\alpha}}} \downarrow & & \downarrow {\scriptstyle F X_{\iota_{\beta,\alpha}}} \\
F X_\beta & = & F X_\beta
\end{array}$$

For X a sheaf, the components at X of $\varepsilon \circ id_F$ and $(id_F \circ \varepsilon) \bullet (\vartheta \circ id_\Gamma)$ shown in (ii) describe two cones for the α-sequence $(\underline{F}X_{\iota_{\beta,\beta'}} \colon \underline{F}X_{\beta'} \to \underline{F}X_\beta)_{\beta \leq \beta' < \alpha}$ (since $\underline{F}X_{\iota_{\beta+1,\beta'+1}} = F X_{\iota_{\beta,\beta}}$) and such cones can be safely restricted to the successors base for $\mathcal{A}(\alpha)$ (*cf.* Section 2.1.2). We conclude by noting that the first of these cones is limiting, the associated mediating map is exactly the component at X of ϱ, and $\varsigma_X \varrho_X = id_X$ by hypothesis. $\square$

Note that for ϱ to be a retraction entails for F to weakly preserve limits of α-sequences since any section of ϱ represents a coherent choice of weak mediating morphisms. In particular, ϱ is a natural isomorphism if and only if the endofunctor F (strongly) preserves limits of α-sequences. This is a mild assumption in the context of this chapter: Proposition 3.6 will be applied to choices of F and α such that the final sequence of F is stable at α (*cf.* Section 3.4.2).

Taking an endofunctor to its pointwise extension is a functorial operation. There is a functor:

$$(\underline{\ \ }) \colon \mathbf{End}(\mathbf{C}) \to \mathbf{End}(\mathbf{Sh_C}(\alpha))$$

defined as follows:

$$\underline{(-)} \triangleq \mathbf{a} \circ \mathbf{Fun}(Id, -) \circ \mathbf{i}.$$

We call the *pointwise extension functor* (to sheaves). In the remaining of this section we show that this functor lifts to distributive laws and that it enriches over the category of sheaves.

3.2.2 Pointwise extension of distributive laws

The pointwise extension functor preserves the identity functor over **C** up to isomorphism: the unit of the reflection ($\mathbf{a} \dashv \mathbf{i}$) defines the isomorphism

$$\psi\colon Id_{\mathbf{Sh_C}(\alpha)} \cong \underline{Id_{\mathbf{C}}} \qquad Id_{\mathbf{Sh_C}(\alpha)} \cong \mathbf{a} \circ \mathbf{i} = \underline{Id_{\mathbf{C}}}.$$

Likewise, $\underline{(-)}$ preserves endofunctor composition up to isomorphism: for F and G in $\mathbf{End}(\mathbf{C})$ there is an isomorphism

$$\phi_{F,G}\colon \underline{F} \circ \underline{G} \cong \underline{F \circ G}$$

natural in F and G. For β and γ a successor and a limit ordinal, the component β of $\phi_{F,G}$ is defined as:

$$(\underline{F} \circ \underline{G})X_\beta = F(\underline{G}X)_\beta = (F \circ G)X_\beta = (\underline{F \circ G})X_\beta$$

and the component γ as:

$$(\underline{F} \circ \underline{G})X_\gamma = \lim_{\beta<\gamma} F\underline{G}X_\beta \overset{\ddagger}{\cong} \lim_{\beta<\gamma}(F \circ G)X_\beta = (\underline{F \circ G})X_\gamma$$

where (‡) easily follows by restriction to the successors base of $\mathcal{A}(\alpha)$. As any category of endofunctors, $\mathbf{End}(\mathbf{C})$ and $\mathbf{End}(\mathbf{Sh_C}(\alpha))$ are (strict) monoidal categories whose tensor product and unit are endofunctor composition and the identity functor, respectively. The isomorphisms ψ and ϕ from above render the functor $\underline{(-)}$ a monoidal functor.

THEOREM 3.7. *The following data defines a strong monoidal functor going from* $(\mathbf{End}(\mathbf{C}), \circ, Id_{\mathbf{C}})$ *to* $(\mathbf{End}(\mathbf{Sh_C}(\alpha)), \circ, Id_{\mathbf{Sh_C}(\alpha)})$*:*

- *the pointwise extension functor* $\underline{(-)}\colon \mathbf{End}(\mathbf{C}) \to \mathbf{End}(\mathbf{Sh_C}(\alpha))$*,*
- *the natural isomorphism* $\phi_{F,G}\colon \underline{F} \circ \underline{G} \cong \underline{F \circ G}$*, and*
- *the isomorphism* $\psi\colon Id_{\mathbf{Sh_C}(\alpha)} \cong \underline{Id_{\mathbf{C}}}$*.*

PROOF. The monoidal structure of an endofunctor category under composition is strict since its associator, left unitor, and right unitor are all identities—in fact $(F \circ G) \circ H = F \circ (G \circ H)$, $F \circ Id = F$, and $Id \circ F = F$ for any F, G, and H. Therefore, the coherence diagrams stating the compatibility of $(\underline{-})$, ϕ, and ψ with respect to the associator, left and right unitor (*cf.* [100, Section X.2]) instantiate as follows:

$$\begin{array}{ccccc}
 & \underline{F} \circ (\underline{G} \circ \underline{H}) & = & (\underline{F} \circ \underline{G}) \circ \underline{H} & \\
 {\scriptstyle id_{\underline{F}} \circ \phi_{G,H}} \swarrow & & & & \searrow {\scriptstyle \phi_{F,G} \circ id_{\underline{H}}} \\
 \underline{F} \circ (\underline{G \circ H}) & & & & (\underline{F \circ G}) \circ \underline{H} \\
 {\scriptstyle \phi_{F,G\circ H}} \searrow & & & & \swarrow {\scriptstyle \phi_{F\circ G,H}} \\
 & \underline{F \circ (G \circ H)} & = & \underline{(F \circ G) \circ H} &
\end{array}$$

$$\begin{array}{ccc}
\underline{F} \circ Id_{\mathbf{Sh}_{\mathbf{C}}(\alpha)} & = & \underline{F} \\
{\scriptstyle id_{\underline{F}} \circ \psi} \downarrow & & \Vert \\
\underline{F} \circ \underline{Id_{\mathbf{C}}} & \xrightarrow[\phi_{G,Id_{\mathbf{C}}}]{} & \underline{G \circ Id_{\mathbf{C}}}
\end{array}
\qquad
\begin{array}{ccc}
Id_{\mathbf{Sh}_{\mathbf{C}}(\alpha)} \circ \underline{F} & = & \underline{F} \\
{\scriptstyle \psi \circ id_{\underline{F}}} \downarrow & & \Vert \\
\underline{Id_{\mathbf{C}}} \circ \underline{F} & \xrightarrow[\phi_{Id_{\mathbf{C}},G}]{} & \underline{Id_{\mathbf{C}} \circ G}
\end{array}$$

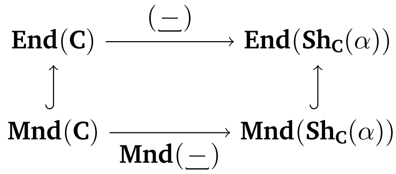
Let β a successor ordinal. At stage β any component of the isomorphisms ϕ and ψ is an identity and thus each corresponding component of the diagrams above has only identities as arrows. Stages associated to limit ordinals follow by universality and the coherent choice of limits inherent in fixing $\mathbf{a}$. □

A defining property of monoidal functors is that they send monoids to monoids. In the case of $(\underline{-})$ this means that if T carries a monad structure, then its extension $\underline{T}$ carries a monad structure derived from the extensions of μ^T and η^T. Note that $\underline{\mu^T}$ and $\underline{\eta^T}$ have types $\underline{T \circ T} \to \underline{T}$ and $\underline{Id_{\mathbf{C}}} \to \underline{T}$ instead of $\underline{T} \circ \underline{T} \to \underline{T}$ and $Id_{\mathbf{Sh}_{\mathbf{C}}(\alpha)} \to \underline{T}$ (expected from any multiplication and unit for $\underline{T}$). The necessary gluing is provided by the isomorphisms ϕ and ψ which allows us to derive a multiplication and unit for $\underline{T}$ from those of T as follows:

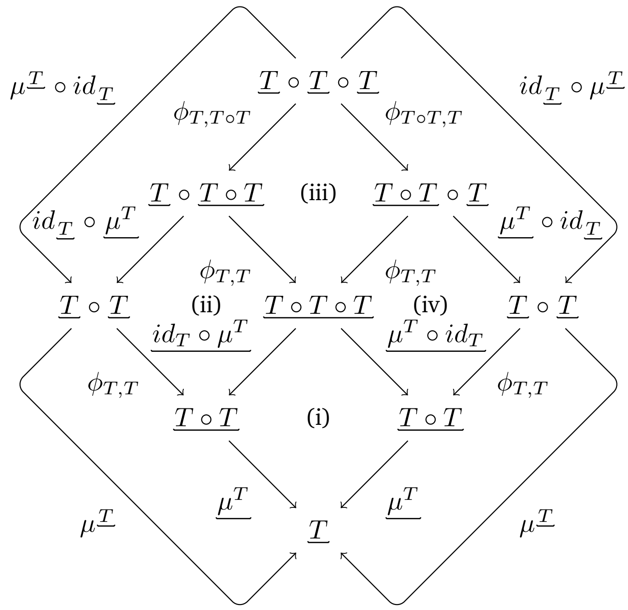
$$\mu^{\underline{T}} \triangleq \underline{\mu^T} \bullet \phi_{T,T} \qquad \eta^{\underline{T}} \triangleq \underline{\eta^T} \bullet \phi_{T,T}. \tag{3.11}$$

It follows from simple diagram chasing that $\mu^{\underline{T}}$ and $\eta^{\underline{T}}$ satisfy the usual diagrams of associativity and unit (*cf.* Corollary 3.8 below) and hence define a monad structure on $\underline{T}$. We call the monad $(\underline{T}, \mu^{\underline{T}}, \eta\underline{T})$ the pointwise extension of $(T, \mu^T, \eta T)$. This structure is uniquely defined and hence by assigning to each monad its extension we obtain a functor $\mathbf{Mnd}(\underline{-})$ between monad categories that is the restriction to monads of the pointwise extension functor $\underline{-}$ for endofunctors.

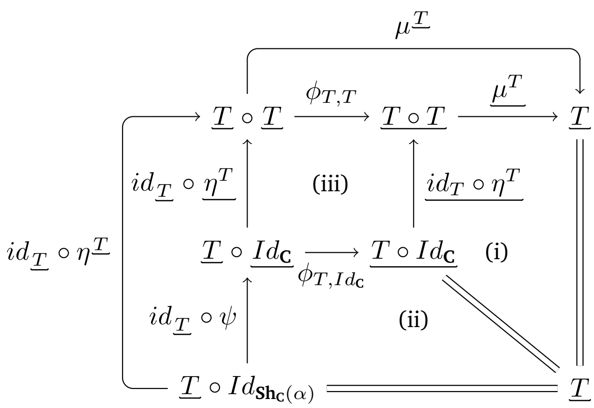
COROLLARY 3.8. *The pointwise extension functor restricts to a functor between the categories of monads over* **C** *and over* $\mathbf{Sh_C}(\alpha)$ *as illustrated by the diagram below.*



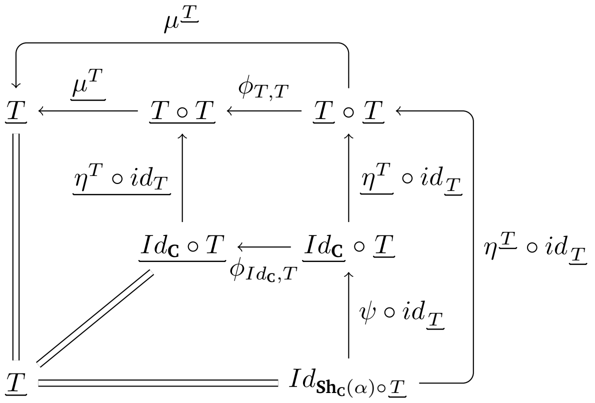
PROOF. Let (T, μ^T, η^T) be a monad over **C** and consider the following diagram asserting that $\mu^{\underline{T}} = \underline{\mu^T} \bullet \phi_{T,T}$ is associative:



The diagram above commutes since: (i) commutes by hypothesis on (T, μ, η), squares (ii-iv) follow from naturality of ϕ and μ^T, and all remaining diagrams commute by definition of $\mu^{\underline{T}}$. Consider the following decomposition of the diagram asserting that $\eta^{\underline{T}}$ is a right unit for $\mu^{\underline{T}}$:

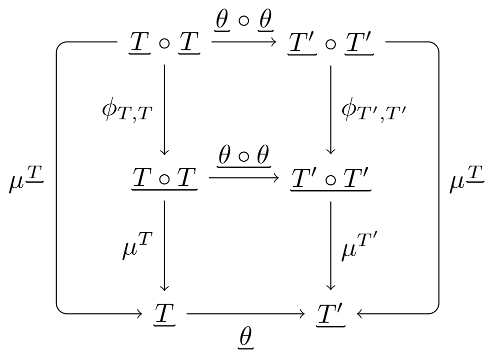


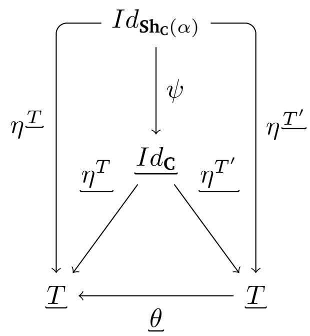
This diagram commutes for: (i) asserts that η^T is a right unit of μ^T, (ii) is states the compatibility of ψ with the right unitor and ϕ, and (iii) is a naturality square for ϕ. The diagram describing $\eta^{\underline{T}}$ as a left unit for $\mu^{\underline{T}}$ can be shown to commute by a similar decomposition:



Thus, $\underline{T}$ equipped with $\mu^{\underline{T}}$ and $\eta^{\underline{T}}$ as defined in (3.11) is a monad over $\mathbf{Sh_C}(\alpha)$.

For $\theta\colon T \to T'$ a monad morphism, the natural transformation $\underline{\theta}\colon \underline{T} \to \underline{T'}$ is a morphism between the extensions for T and T' as the following commuting diagrams assert:





We conclude by noting that the functorial assignments above act on monad morphisms and on the functorial part of monads as the pointwise extension functor $(\underline{-})\colon \mathbf{End}(\mathbf{C}) \to \mathbf{End}(\mathbf{Sh_C}(\alpha))$. □

For notational convenience, we will often write just $(\underline{-})$ instead of $\mathbf{Mnd}(\underline{-})$.

REMARK 3.7 (Generalised writer monad). Let (T, μ, η) be a monad over $\mathbf{C}$. The endofunctor $\mathbf{Fun}(Id_{\mathbf{D}}, T)$ over $\mathbf{Fun}(\mathbf{D}, \mathbf{C})$ carries a monad structure whose multiplication and unit are derived from μ and η as follows:

$$(\mu \circ -)\colon (T \circ T \circ -) \to (T \circ -) \qquad (\eta \circ -)\colon (Id \circ -) \to (T \circ -)$$

The usual coherence diagrams can be directly checked by simple diagram chasing. This construction is an instance of the *writer monad transformer* [78] where the monoid of writes is (T, μ, η) and the transformed monad is the identity on $\mathbf{Fun}(\mathbf{D}, \mathbf{C})$. The pointwise extension of T is the "sheafified" version of the above.

Let (T, μ^T, η^T) be a monad, F an endofunctor over $\mathbf{C}$ and $\lambda^{T,F}$ be a distributive law for them. The pointwise extension of the natural transformation $\lambda^{T,F}$ is not a distributive law of the extensions of (T, μ^T, η^T) over F, yet. In fact, the type of $\underline{\lambda^{T,F}}$ is $\underline{F \circ T} \to \underline{T \circ F}$ instead of $\underline{F} \circ \underline{T} \to \underline{T} \circ \underline{F}$ required by distributive laws of $\underline{T}$ over $\underline{F}$. Similarly to (3.11), the isomorphisms ϕ and ψ that render $(\underline{-})$ a monoidal functor, provide us with the required gluing: for $\lambda^{T,F}$ a distributive law, define its pointwise extension as the natural transformation $\lambda^{\underline{T},\underline{F}} \colon \underline{F} \circ \underline{T} \Rightarrow \underline{T} \circ \underline{F}$ defined as follows:

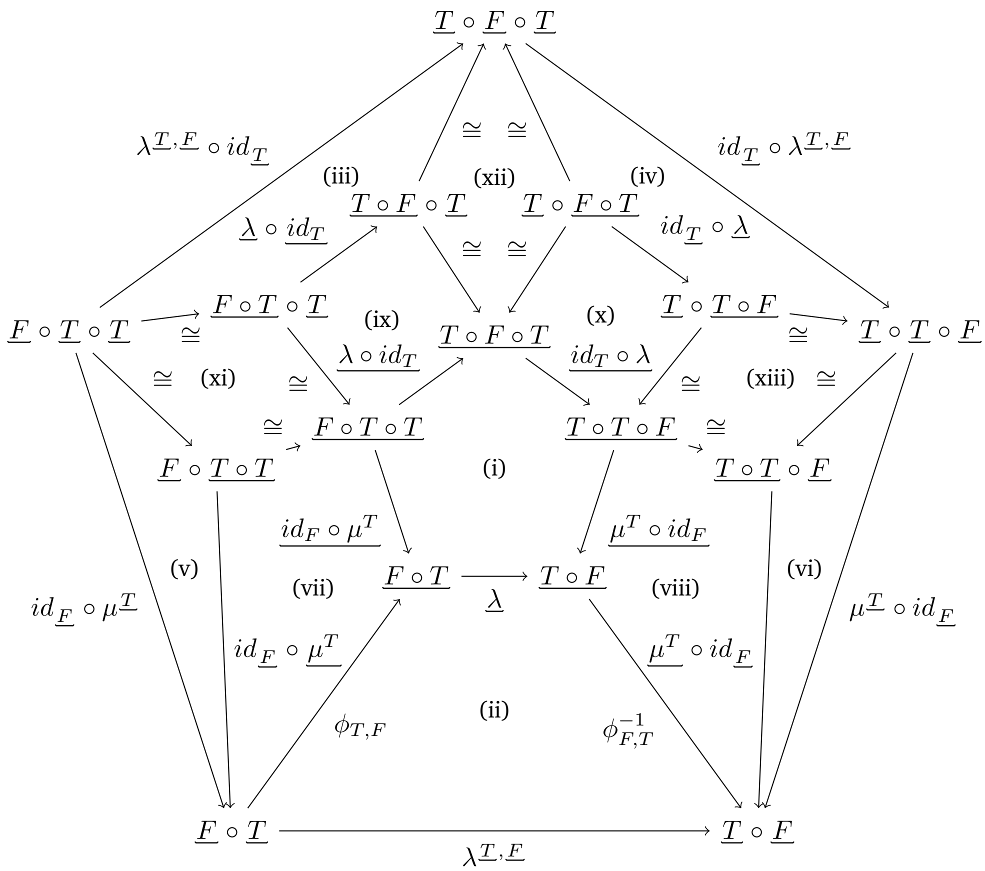
$$\lambda^{\underline{T},\underline{F}} \triangleq \phi^{-1}_{T,F} \bullet \underline{\lambda^{T,F}} \bullet \phi_{F,T}.$$

Recall that by Corollary 3.9 the application of $(\underline{-})$ to natural transformations underlying monad morphisms yields monad morphisms for the extended monads. Therefore, the component-wise application of $(\underline{-})$ to a distributive law morphism (θ, υ) yields the pair $(\underline{\theta}, \underline{\upsilon})$ whose components are those of a distributive law morphisms. It follows from simple diagram chasing that the pair $(\underline{\theta}, \underline{\upsilon})$ makes the necessary diagram commute and hence is a distributive law morphism. By assigning to each distributive law its extension and to each morphism the pair formed by the extensions of its components we obtain a functor between categories of distributive laws $\mathbf{MndEnd}(\underline{-}) \colon \mathbf{MndEnd}(\mathbf{C}) \to \mathbf{MndEnd}(\mathbf{Sh}_{\mathbf{C}}(\alpha))$ that projects on the categories of monads and endofunctors as $\mathbf{Mnd}(\underline{-})$ and $(\underline{-})$.

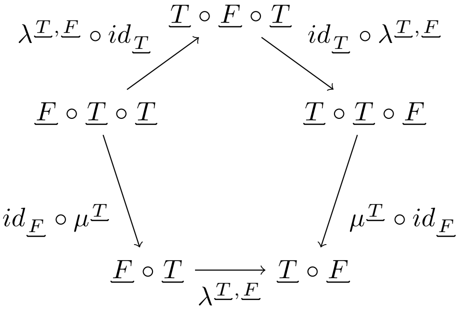
COROLLARY 3.9. *The pointwise extension of distributive laws is functorial and commutes with the projections to the categories of monads and endofunctors as shown by the diagram below.*

$$\begin{array}{ccc}
\mathbf{End}(\mathbf{C}) & \xrightarrow{\;(\underline{-})\;} & \mathbf{End}(\mathbf{Sh}_{\mathbf{C}}(\alpha)) \\
\uparrow & & \uparrow \\
\mathbf{MndEnd}(\mathbf{C}) & \xrightarrow[\;\mathbf{MndEnd}(\underline{-})\;]{} & \mathbf{MndEnd}(\mathbf{Sh}_{\mathbf{C}}(\alpha)) \\
\downarrow & & \downarrow \\
\mathbf{Mnd}(\mathbf{C}) & \xrightarrow[\;\mathbf{Mnd}(\underline{-})\;]{} & \mathbf{Mnd}(\mathbf{Sh}_{\mathbf{C}}(\alpha))
\end{array}$$

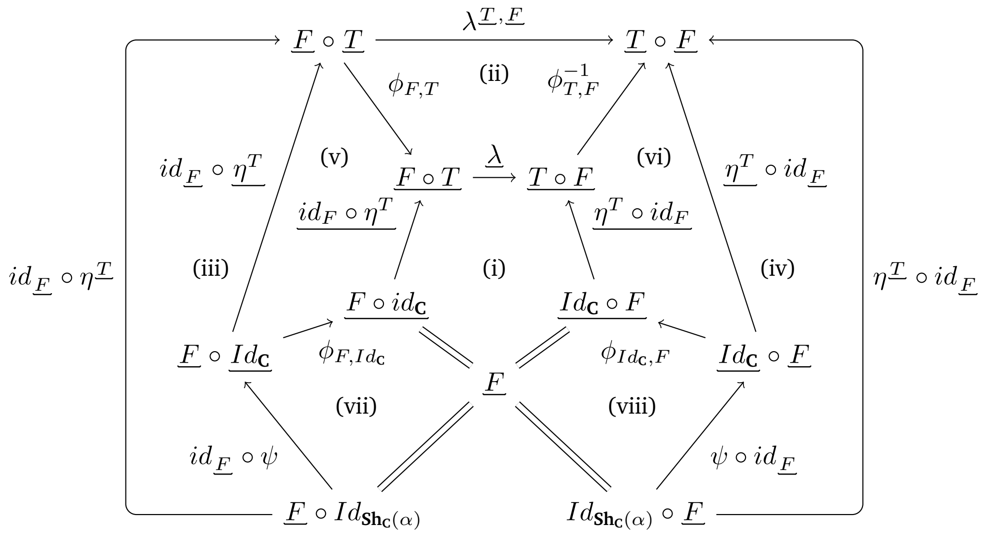
PROOF. First we prove that, for $\lambda \colon FT \to TF$ a distributive law, the natural transformation $\lambda^{\underline{T},\underline{F}} \colon \underline{F}\,\underline{T} \to \underline{T}\,\underline{F}$ is compatible with the pointwise extensions of (T, μ, η) and F. Consider the following decomposition of the compatibility diagram for $\lambda^{\underline{T},\underline{F}}\colon \underline{F}\,\underline{T} \to \underline{T}\,\underline{F}$ and $\mu^{\underline{T}}\colon \underline{T}\,\underline{T} \to \underline{T}$:



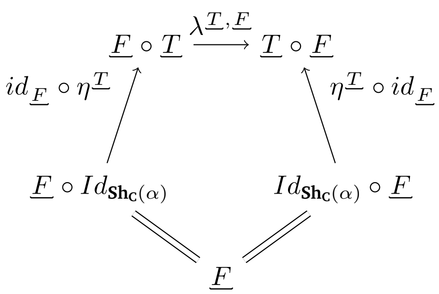
The diagram (i) is (3.2) and commutes since $\lambda\colon TF \to FT$ is assumed compatible with the structure of the monad T. Diagrams (ii–iv) and (v, vi) commute by distributivity of horizontal and vertical composition of natural transformations and by definition of $\lambda^{\underline{T},\underline{F}}\colon \underline{F}\,\underline{T} \to \underline{T}\,\underline{F}$ and $\mu^{\underline{T}}\colon \underline{T}\,\underline{T} \to \underline{T}$, respectively. Diagrams (vii) and (ix) are naturality squares for ϕ. Diagrams (xi) and (xiii) follow by coherence of ϕ with the monoidal associator. Thus the whole diagram commutes and in particular the outer pentagon *i.e.* the compatibility diagram for $\lambda^{\underline{T},\underline{F}}$ and $\mu^{\underline{T}}$:



Consider the following decomposition of the compatibility diagram for $\lambda^{\underline{T},\underline{F}} \colon \underline{F}\,\underline{T} \to \underline{T}\,\underline{F}$ and $\eta^{\underline{T}} \colon Id_{\mathbf{Sh}_{\mathbf{C}}(\alpha)} \to \underline{T}$:

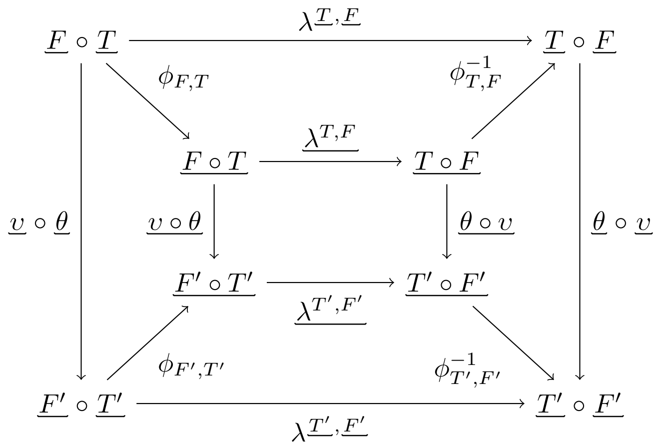


The pentagon (i) is (3.3) and commutes by hypothesis on $\lambda \colon FT \to TF$. The square (ii) follows by definition of $\lambda^{\underline{T},\underline{F}} \colon \underline{F}\,\underline{T} \to \underline{T}\,\underline{F}$. Triangles (iii) and (iv) commute by vertical-horizontal composition of natural transformations and by definition of $\eta^{\underline{T}}$. Diagrams (v) and (v) are naturality squares of ϕ hence commute. Squares (vii) and (viii) commute by Theorem 3.7 for they assert ϕ and ψ coherent with right and left suitors, respectively. By pasting, the outer pentagon



commutes. Therefore, the natural transformation $\lambda^{\underline{T},\underline{F}}$ is a distributive law of the extensions of (T, μ, η) over F.

For the second part of the proof assume that (θ, υ) is a distributive law morphism from $\lambda^{T,F}$ to $\lambda^{T',F'}$. The following diagram commutes by hypothesis on (θ, υ):



The outer part of this commuting diagram exhibits the pair $(\underline{\theta}, \underline{\upsilon})$ as a distributive law morphism from the extension of $\lambda^{T,F}$ to that of $\lambda^{T',F'}$. Functoriality conditions follow from $(\underline{-})$ being a functor and by diagram chasing. Finally, the show that the diagram in the claim commutes note that distributive law morphisms are extended as pairs of extensions. $\square$

3.2.3 Sheaf enrichment

The pointwise extension functor enriches over $\mathbf{Sh}(\alpha)$ in the sense that endofunctors in its essential image are $\mathbf{Sh}(\alpha)$-enriched (*cf.* Section 2.1.4) as stated by Theorem 3.10 below. From this result it follows that Kleisli categories and Kleisli liftings for pointwise extension of monads and endofunctors are sheaf enriched and hence admit a notion of local contractiveness.

THEOREM 3.10. *The essential image of* $(\underline{-})$ *lies in the* $\mathbf{Sh}(\alpha)$**-End**$(\mathbf{Sh}_{\mathbf{C}}(\alpha))$.

PROOF. In order to prove the thesis it suffices to show that the pointwise extension of any endofunctor F over $\mathbf{C}$ is $\mathbf{Sh}(\alpha)$-enriched *i.e.* that each assignment $\underline{F}_{X,Y} \colon \mathbf{Sh}_{\mathbf{C}}(\alpha)(X,Y) \to \mathbf{Sh}_{\mathbf{C}}(\alpha)(\underline{F}X, \underline{F}Y)$ lies in $\mathbf{Sh}(\alpha)$ when hom-objects of $\mathbf{Sh}_{\mathbf{C}}(\alpha)$ are seen as sheaves of sets. Recall from Section 2.1.4 that for sheaves $X, Y \in \mathbf{Sh}_{\mathbf{C}}(\alpha)$, the hom-sheaf $\mathbf{Sh}_{\mathbf{C}}(\alpha)(X,Y)$ takes each stage β to the value:

$$\mathbf{Sh}_{\mathbf{C}}(\alpha)(X,Y)_\beta = \{(f, f_\beta) \mid f \in \mathbf{Sh}_{\mathbf{C}}(\alpha)(X,Y)\}$$

and each morphism $\beta \to \beta'$ to the restriction:

$$\mathbf{Sh}_{\mathbf{C}}(\alpha)(X,Y)_{\iota_{\beta,\beta'}}(f, f_\beta) = (f, f_{\beta'}).$$

Then, the component of $\underline{F}_{X,Y}$ at stage β, for β a successor ordinal, is:

$$\underline{F}_{X,Y,\beta}(f, f_\beta) = (\underline{F}f, \underline{F}f_\beta) = (\underline{F}f, Ff_\beta)$$

and the component at stage γ, for γ a limit ordinal, is

$$\underline{F}_{X,Y,\gamma}(f, f_\gamma) = (\underline{F}f, \underline{F}f_\gamma) = (\underline{F}f, \lim_{\beta<\gamma} Ff_\beta)$$

where $\lim_{\beta<\gamma} Ff_\beta \colon \lim_{\beta<\gamma} FX_\beta \to \lim_{\beta<\gamma} FY_\beta$ is the mediating map for the cone $\{f_\beta \circ \pi_\beta\}_{\beta<\gamma}$. It follows from functoriality of $\underline{F}$ that these components are well-defined and satisfy the required naturality conditions as illustrated by the diagrams below:

$$\begin{array}{ccc}
\mathbf{Sh_C}(\alpha)(X,Y)_\gamma & \xrightarrow{\underline{F}_{X,Y,\gamma}} & \mathbf{Sh_C}(\alpha)(\underline{F}X,\underline{F}Y)_\gamma \\
\downarrow & & \downarrow \mathbf{Sh_C}(\alpha)(\underline{F}X,\underline{F}Y)_{\iota_{\beta',\gamma}} \\
\mathbf{Sh_C}(\alpha)(X,Y)_{\beta'} & \xrightarrow{\underline{F}_{X,Y,\beta}} & \mathbf{Sh_C}(\alpha)(\underline{F}X,\underline{F}Y)_{\beta'} \\
\downarrow & & \downarrow \mathbf{Sh_C}(\alpha)(\underline{F}X,\underline{F}Y)_{\iota_{\beta,\beta'}} \\
\mathbf{Sh_C}(\alpha)(X,Y)_\beta & \xrightarrow[\underline{F}_{X,Y,\beta}]{} & \mathbf{Sh_C}(\alpha)(\underline{F}X,\underline{F}Y)_\beta
\end{array}
\qquad
\begin{array}{ccc}
f_\gamma & \longmapsto & \lim_{\beta<\gamma} Ff_\beta \\
\downarrow & & \downarrow \\
f_{\beta'} & \longmapsto & Ff_{\beta'} \\
\downarrow & & \downarrow \\
f_\beta & \longmapsto & Ff_\beta
\end{array}$$

where γ is a limit ordinal and $\beta < \beta'$ are successor ordinals in γ. □

It follows from the above result that pointwise extensions of endofunctors are sheaf enriched, especially if they carry a monad structure. Kleisli categories of such monads share the same enrichment of their underlying category $\mathbf{Sh_C}(\alpha)$ as stated by the following corollary.

COROLLARY 3.11. *For (T, μ, η) a monad, the category $\mathbf{Kl}(\underline{T})$ is enriched over $\mathbf{Sh}(\alpha)$.*

PROOF. Hom-objects of $\mathbf{Kl}(\underline{T})$ are objects of $\mathbf{Sh}(\alpha)$ since the underlying category $\mathbf{Sh_C}(\alpha)$ is enriched over $\mathbf{Sh}(\alpha)$. For X, Y, and Z, Kleisli composition is given as:

$$\begin{array}{ccc}
\mathbf{Kl}(\underline{T})(Y,Z) \times \mathbf{Kl}(\underline{T})(X,Y) & = & \mathbf{Sh_C}(\alpha)(Y,\underline{T}Z) \times \mathbf{Sh_C}(\alpha)(X,TY) \\
 & & \downarrow \underline{T}_{Y,\underline{T}Z} \times id \\
 & & \mathbf{Sh_C}(\alpha)(\underline{T}Y,\underline{T}\,\underline{T}Z) \times \mathbf{Sh_C}(\alpha)(X,TY) \\
\downarrow (- \circ_{X,Y,Z} -) & & \downarrow (- \circ_{X,\underline{T}Y,\underline{T}\,\underline{T}Z} -) \\
 & & \mathbf{Sh_C}(\alpha)(X,\underline{T}\,\underline{T}Z) \\
 & & \downarrow (- \circ_{X,\underline{T}\,\underline{T}Z,\underline{T}Z}\, \mu_Z) \\
\mathbf{Kl}(\underline{T})(X,Z) & = & \mathbf{Sh_C}(\alpha)(X,\underline{T}Z)
\end{array}$$

It follows from Theorem 3.10 and Lemma 2.7 that the morphism $(- \circ_{X,Y,Z} -) \colon \mathbf{Kl}(\underline{T})(Y,Z) \times \mathbf{Kl}(\underline{T})(X,Y) \to \mathbf{Kl}(\underline{T})(X,Z)$ lies in $\mathbf{Sh}(\alpha)$ for it is given as the composition of morphisms in $\mathbf{Sh}(\alpha)$. Finally, associativity and existence of identities follow from definition of Kleisli category. □

A prerequisite of locally contractive endofunctors over $\mathbf{Kl}(\underline{T})$ is to share its sheaf enrichment. This holds for Kleisli lifting of pointwise extensions (or, equivalently, for pointwise extensions of Kleisli lifting).

COROLLARY 3.12. *Any lifting $\overline{\underline{F}}$ to $\mathbf{Kl}(\underline{T})$ is enriched over $\mathbf{Sh}(\alpha)$.*

PROOF. Let $\lambda^{\underline{T},\underline{F}} \colon \underline{F}\,\underline{T} \to \underline{T}\,\underline{F}$ be the distributive law induced by the Kleisli lifting $\overline{\underline{F}}$ as per Proposition 3.1. For objects X and Y, the functorial assignment $\overline{\underline{F}}_{X,Y} \colon \mathbf{Kl}(\underline{T})(X,Y) \to \mathbf{Kl}(\underline{T})(\overline{\underline{F}}X, \overline{\underline{F}}Y)$ is given as

$$\begin{array}{ccc}
\mathbf{Kl}(\underline{T})(X,Y) & = & \mathbf{Sh}_{\mathbf{C}}(\alpha)(X, \underline{T}Y) \\
\Big\downarrow \overline{\underline{F}}_{X,Y} & & \Big\downarrow \underline{F}_{X,\underline{T}Y} \\
 & & \mathbf{Sh}_{\mathbf{C}}(\alpha)(\underline{F}X, \underline{F}\,\underline{T}Y) \\
 & & \Big\downarrow (- \circ_{X,\underline{F}\,\underline{T}Y,\underline{T}\,\underline{F}Y} \lambda^{\underline{T},\underline{F}}_{Y}) \\
\mathbf{Kl}(\underline{T})(\overline{\underline{F}}X, \overline{\underline{F}}Y) & = & \mathbf{Sh}_{\mathbf{C}}(\alpha)(\underline{F}X, \underline{T}\,\underline{F}Y)
\end{array}$$

and it is a morphism of sheaves in $\mathbf{Sh}(\alpha)$ by Theorem 3.10 and Lemma 2.7. □

3.3 TOWARDS GUARDED KLEISLI (CO)RECURSION

In this section we consider locally contractive endofunctors over Kleisli categories of monads obtained by pointwise extension. In particular, we are interested in Kleisli liftings of its "guarded pointwise extension" $\underline{F}\blacktriangleright$ and in the systematic derivation of these liftings from liftings of F since these will be required by the constructions we introduce in Section 3.4 in order to capture infinite trace semantics by finality. In Section 3.3.2 we show that there are settings of interest for modelling infinite traces that support this systematic derivation. In particular, we consider sheaves over $\mathcal{A}(\omega)$. In this setting, we identify a class of Kleisli liftings of F that always extend to liftings of $\underline{F}\blacktriangleright$ and we prove that this class covers all Kleisli liftings if, and only if, the given monad is affine.

3.3.1 Locally contractive Kleisli liftings

For this section let $\mathbf{D}$ be a category enriched over $\mathbf{Sh}(\alpha)$ and let (T,μ,η) be a monad over $\mathbf{D}$ such that its Kleisli category is enriched over $\mathbf{Sh}(\alpha)$—baring in mind that our prototypical example are pointwise extensions of monads to $\mathbf{Sh}_{\mathbf{C}}(\alpha)$. We are interested in locally contractive endofunctors over these Kleisli categories, especially in those that are Kleisli liftings of locally contractive endofunctors.

Recall from Section 2.2.2 that a locally contractive endofunctor over $\mathbf{Kl}(T)$ is any functor $F\colon \mathbf{Kl}(T) \to \mathbf{Kl}(T)$ that factors as a composition of functors enriched over $\mathbf{Sh}(\alpha)$:

$$\mathbf{Kl}(T) \xrightarrow{next^{\mathbf{Kl}(T)}} {}_{\blacktriangleright}\mathbf{Kl}(T) \longrightarrow \mathbf{Kl}(T)$$

where $next^{\mathbf{Kl}(T)}$ is the functor induced by the point $next\colon Id_{\mathbf{Sh}(\alpha)} \to \blacktriangleright$. Likewise, an endofunctor over $\mathbf{D}$ is locally contractive if it factors as a composition of functors enriched over $\mathbf{Sh}(\alpha)$:

$$\mathbf{D} \xrightarrow{next^{\mathbf{D}}} {}_{\blacktriangleright}\mathbf{D} \longrightarrow \mathbf{D}.$$

Local contractiveness is only inherited by Kleisli liftings, not *vice versa*.

PROPOSITION 3.13. *For F an endofunctor over* $\mathbf{D}$*, if F is locally contractive, then any of its liftings to* $\mathbf{Kl}(T)$ *is locally contractive but not* vice versa.

PROOF. Let λ be the distributive law induced by the Kleisli lifting $\overline{F}$ let F factor as $F' \circ next^{\mathbf{D}}$ as per hypothesis. It follows from definition of Kleisli category that $next^{\mathbf{Kl}(T)}$ is given on each hom-sheaf $\mathbf{Kl}(T)(X,Y) = \mathbf{D}(X,TY)$, as the sheaf morphism $next^{\mathbf{D}}_{X,TY}\colon \mathbf{D}(X,TY) \to {}_{\blacktriangleright}\mathbf{D}(X,TY)$.

For any X and Y in $\mathbf{Kl}(T)$, the functorial assignment $\overline{F}_{X,Y}$ factors as follows:

$$\begin{array}{ccccccc}
\mathbf{Kl}(T)(X,Y) & = & \mathbf{D}(X,TY) & = & \mathbf{D}(X,TY) & = & \mathbf{Kl}(T)(X,Y) \\
 & & & & \downarrow{\scriptstyle next^{\mathbf{D}}_{X,TY}} & & \downarrow{\scriptstyle next^{\mathbf{Kl}(T)}_{X,Y}} \\
\downarrow{\scriptstyle \overline{F}_{X,Y}} & & \downarrow{\scriptstyle F_{X,TY}} & & {}_{\blacktriangleright}\mathbf{D}(X,TY) & = & {}_{\blacktriangleright}\mathbf{Kl}(T)(X,Y) \\
 & & & & \downarrow{\scriptstyle F'_{X,Y}} & & \downarrow{\scriptstyle \overline{F}'_{X,Y}} \\
 & & \mathbf{D}(FX,FTY) & = & \mathbf{D}(FX,FTY) & & \\
 & & \downarrow & {\scriptstyle (-\circ_{X,FTY,TFY}\lambda_Y)} & \downarrow & & \\
\mathbf{Kl}(T)(\overline{F}X,\overline{F}Y) & = & \mathbf{D}(FX,TFY) & = & \mathbf{D}(FX,TFY) & = & \mathbf{Kl}(T)(\overline{F}X,\overline{F}Y)
\end{array}$$

The factorisation $\overline{F} = \overline{F}' \circ next^{\mathbf{Kl}(T)}$ proves that the lifting $\overline{F}$ is locally contractive. Finally, note that it is not possible to identify $F'_{X,Y}$ from only $\underline{F}'_{X,Y}$. We conclude that the derivation in the other direction is not be possible. □

If the category $\mathbf{D}$ comes equipped with its own instance of the modality $\blacktriangleright$ (*i.e.* ${}_{\blacktriangleright}\mathbf{D}(X,Y) \rightarrowtail \mathbf{D}(\blacktriangleright X, \blacktriangleright Y)$) then this endofunctor provides us with a convenient way to turn any endofunctor over $\mathbf{D}$ into a locally contractive one: simply compose it with $\blacktriangleright$. In fact, any instance of $\blacktriangleright$ is locally contractive and composition with locally contractive functors preserves this property. This is indeed the case in our setting of interest: pointwise extensions to $\mathbf{Sh}_{\mathbf{C}}(\alpha)$.

COROLLARY 3.14. *For F an endofunctor over* **C**, *any Kleisli lifting to* $\mathbf{Kl}(\underline{T})$ *for* $\underline{F}\blacktriangleright$ *is locally contractive.*

PROOF. It follows from Corollary 3.11 and Proposition 3.13 that $\mathbf{Kl}(\underline{T})$ and $\underline{F}$ are enriched over $\mathbf{Sh}(\alpha)$ and hence from Corollary 3.12 that $\underline{F}\blacktriangleright$ is locally contractive. □

3.3.2 Kleisli liftings for guarded pointwise extensions

For the remaining of the section we return to our original setting: Kleisli categories of monads obtained by pointwise extension and Kleisli liftings of endofunctors obtained by "guarding" pointwise extensions. We investigate the derivation of liftings for $\underline{F}\blacktriangleright$ from liftings for F and *vice versa*.

The opposite derivation is always possible: the desired distributive laws are constructed from components in the image of Δ and stages associated to the so called "double successor" ordinals (*i.e.* any $\beta + 2$). The derivation and its correctness are stated in Lemma 3.15 below; we will refer to distributive laws (resp. Kleisli lifting) obtained in this way as *induced* laws (resp. lifting).

LEMMA 3.15. *Let $\alpha > 1$ be an ordinal, (T, μ, η) a monad, F an endofunctor, and ξ a distributive law of $(\underline{T}, \mu^{\underline{T}}, \eta^{\underline{T}})$ over $\underline{F}\blacktriangleright$ on $\mathbf{Sh}_{\mathbf{C}}(\alpha)$. There is a distributive law λ of (T, μ, η) over F such that $\lambda_X = \xi_{\Delta X,\beta+2}$ for any $X \in \mathbf{C}$ and $\beta + 1 < \alpha$.*

PROOF. Fix an ordinal β such that $\beta+1 < \alpha$. For any object X and any morphism $f\colon X \to Y$ in **C**, the following equalities hold:

$$(\underline{F}\blacktriangleright\underline{T}\Delta X)_{\beta+2} = FTX \qquad (\underline{F}\blacktriangleright\underline{T}\Delta f)_{\beta+2} = FTf$$
$$(\underline{T}\,\underline{F}\blacktriangleright\Delta X)_{\beta+2} = TFX \qquad (\underline{T}\,\underline{F}\blacktriangleright\Delta f)_{\beta+2} = TFf$$

It follows from these equalities that the naturality square for $\xi_{\Delta X}$, $\xi_{\Delta Y}$, and $\Delta f\colon \Delta X \to \Delta Y$ at stage $\beta + 2$ is exactly that for λ_X, λ_Y, and $f\colon X \to Y$. Therefore, the family $\{\lambda_X\}_{X\in\mathbf{C}}$ is a natural transformation of type $FT \to TT$.

For any object X and any morphism $f\colon X \to Y$ in **C**, the following hold:

$$(\underline{F}\blacktriangleright\Delta X)_{\beta+2} = FX \qquad (\underline{F}\blacktriangleright\Delta f)_{\beta+2} = Ff$$
$$(\underline{F}\blacktriangleright\underline{T}\,\underline{T}\Delta X)_{\beta+2} = FTTX \qquad (\underline{F}\blacktriangleright\underline{T}\,\underline{T}\Delta f)_{\beta+2} = FTTf$$
$$(\underline{T}\,\underline{F}\blacktriangleright\underline{T}\Delta X)_{\beta+2} = TFTX \qquad (\underline{T}\,\underline{F}\blacktriangleright\underline{T}\Delta f)_{\beta+2} = TFTf$$
$$(\underline{T}\,\underline{T}\,\underline{F}\blacktriangleright\Delta X)_{\beta+2} = TTFX \qquad (\underline{T}\,\underline{T}\,\underline{F}\blacktriangleright\Delta f)_{\beta+2} = TTFf$$
$$(\xi \circ id_{\underline{T}})_{\Delta X,\beta+2} = (\lambda \circ id_T)_X \qquad (id_{\underline{T}} \circ \xi)_{\Delta X,\beta+2} = (id_T \circ \lambda)_X$$
$$(id_{\underline{F}\circ\blacktriangleright} \circ \mu^{\underline{T}})_{\Delta X,\beta+2} = id_F \circ \mu_X \qquad (\mu^{\underline{T}} \circ id_{\underline{F}\circ\blacktriangleright})_{\Delta X,\beta+2} = \mu_X \circ id_F$$
$$(id_{\underline{F}\circ\blacktriangleright} \circ \eta^{\underline{T}})_{\Delta X,\beta+2} = id_F \circ \eta_X \qquad (\eta^{\underline{T}} \circ id_{\underline{F}\circ\blacktriangleright})_{\Delta X,\beta+2} = \eta_X \circ id_F$$

Consider the diagrams asserting that ξ is compatible with the structure of $(\underline{T}, \mu^{\underline{T}}, \eta^{\underline{T}})$ and, in particular, their components at stage $\beta + 2$: it follows from the equalities above that these are exactly the compatibility diagrams for λ and the structure of (T, μ, η).

It follows from definition of constant sheaves that the choice of the ordinal β made at the beginning of this proof is irrelevant: indeed any restriction map $\Delta X_{\iota_{\beta+2,\beta'+2}}$ is an identity. □

In general, the converse of Lemma 3.15 does not hold: it is not true that a distributive law of (T, μ, η) over F extends to one of $(\underline{T}, \mu^{\underline{T}}, \eta^{\underline{T}})$ and $\underline{F}\blacktriangleright$. In order to illustrate why this is not the case, assume a distributive law ξ of $(\underline{T}, \mu^{\underline{T}}, \eta^{\underline{T}})$ over $\underline{F}\blacktriangleright$ and write λ for the distributive law induced by ξ. Stages for double successor ordinals contains the data used in the derivation of λ and stages for limit ordinals are covered by definition of $\mathbf{Sh}_{\mathbf{C}}(\alpha)$: as a consequence, the issue at hand must arise from (first) successors of limits ordinals. In fact, for γ a limit ordinal and X a sheaf, the component $\xi_{X,\gamma+1} \colon \underline{F}\blacktriangleright\underline{T}X_{\gamma+1} \to \underline{F}\blacktriangleright X_{\gamma+1}$ is a morphism:

$$\xi_{X,\gamma+1} \colon F \lim_{\beta<\gamma} TX_\beta \to FT \lim_{\beta<\gamma} X_\beta.$$

It follows from the type of these components that their naturality, their compatibility with the structure of $\underline{T}$, and even their existence are non-trivial and can not be derived from $\lambda \colon FT \to TF$ without any additional information. For instance, when F is the identity and λ an automorphism for T, the naturality and compatibility conditions for the induced ξ result to be even stronger than imposing that T weakly preserve limits of γ-chains.

3.3.3 Affiness and extension of Kleisli liftings

In this section we focus on pointwise extensions to sheaves over $\mathcal{A}(\omega)$. In this setting, we identify a class of Kleisli liftings we call *ω-suitable* and such that the converse of Lemma 3.15 always holds. Remarkably, a monad has the property that all liftings to its Kleisli category are ω-suitable if and only if it has the affiness property (*cf.* Theorem 3.18).

DEFINITION 3.7. *A law λ distributing a monad (T, μ, η) over an endofunctor F is called* ω-suitable *whenever the diagram below commutes for any object X in* **C***:*

$$\begin{array}{ccc} FTX & \xrightarrow{\lambda_X} & TFX \\ {\scriptstyle F!_{TX}}\downarrow & & \downarrow{\scriptstyle TF!_X} \\ F1 & \xrightarrow[\eta_{F1}]{} & TF1 \end{array} \qquad (3.12)$$

Kleisli liftings are called ω-suitable whenever their associated distributive laws are ω-suitable.

The ω-suitability property provides us with the necessary information for extending distributive laws of T over F to distributive laws of $\underline{T}$ over $\underline{F}\blacktriangleright$.

THEOREM 3.16. *Let (T, μ, η) be a monad and F an endofunctor, both over* $\mathbf{C}$*. The following statements, where pointwise extensions target $\mathbf{Sh}_{\mathbf{C}}(\omega)$, are true.*

- *For $\xi\colon \underline{F}\blacktriangleright\underline{T} \to \underline{T}\,\underline{F}\blacktriangleright$ a distributive law, the distributive law $\lambda\colon FT \to TF$ induced by ξ is ω-suitable.*
- *For $\lambda\colon FT \to TF$ an ω-suitable distributive law, there is $\xi\colon \underline{F}\blacktriangleright\underline{T} \to \underline{T}\,\underline{F}\blacktriangleright$ a distributive law of $\underline{T}$ over $\underline{F}\blacktriangleright$ given on each sheaf Y as:*

  $$\xi_{Y,0} = id_1 \qquad \xi_{Y,1} = \eta_{F1} \qquad \xi_{Y,n+2} = \lambda_{Y_{n+2}} \qquad \xi_{Y,\omega} = \rho_Y$$

  where $n \in \omega$ and ρ_Y is the mediating map for the cone $\{\xi_{Y,n} \circ \underline{F}\blacktriangleright\underline{T}Y_{\iota_{n,\omega}}\}_{n<\omega}$.

PROOF. Let $\xi\colon \underline{F}\blacktriangleright\underline{T} \to \underline{T}\,\underline{F}\blacktriangleright$ be a distributive law and write λ for the distributive law induced by ξ as per Lemma 3.15. We proceed by showing that for each $X \in \mathbf{C}$ and finite $n > 1$, diagram (3.12) corresponds to the naturality square:

$$\begin{array}{ccc} FTX & \xrightarrow{\xi_{\Delta X,n}} & TFX \\ {\scriptstyle F!_{TX}}\downarrow & & \downarrow{\scriptstyle TF!_X} \\ F1 & \xrightarrow[\xi_{\Delta X,1}]{} & TF1 \end{array}$$

It follows from definition of λ and from compatibility of ξ with the unit of $\underline{T}$ that $\xi_{\Delta X,1}$ is η_{F1}. In fact, for any sheaf Y, the component at stage 1 of the compatibility diagram for ξ_Y and $\eta^{\underline{T}}_Y$ corresponds to the diagram below:

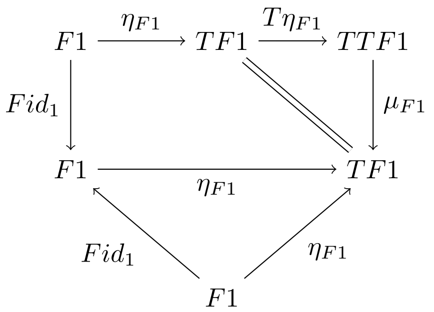
$$\begin{array}{ccc} 1 & \xrightarrow{\xi_{Y,1}} & T1 \\ {\scriptstyle id_1}\nwarrow & & \nearrow{\scriptstyle \eta_{F1}} \\ & 1 & \end{array}$$

Finally, $\xi_{\Delta X,n}$ is λ_X by definition of λ and by assumption on n. Thus, λ is ω-suitable.

For the converse assume $\lambda\colon FT \to TF$ an ω-suitable distributive law and let ξ be the family $\{\xi_{Y,\beta}\colon \underline{F}\blacktriangleright\underline{T}Y_\beta \to \underline{T}\,\underline{F}\blacktriangleright Y_\beta\}_{Y \in \mathbf{Sh}_{\mathbf{C}}(\omega),\, \beta \in \mathcal{A}(\omega)}$ as defined above. First we prove that the family ξ is natural in both Y and β and then that it is compatible with the structure of $\underline{T}$. For sheaves $Y, Y' \in \mathbf{Sh}_{\mathbf{C}}(\omega)$ and finite ordinals $n, n' > 1$, the components $\xi_{Y,n}$ and $\xi_{Y',n'}$ of ξ are, by construction, λ_{Y_n}

and $\lambda_{Y'_{n'}}$, respectively, and satisfy the naturality condition for them because λ is a natural transformation. For all sheaves $Y, Y' \in \mathbf{Sh_C}(\omega)$ and any finite ordinal $n > 1$, the components $\xi_{Y,n}$ and $\xi_{Y',1}$ of ξ are λ_{Y_n} and η_1, respectively, and satisfy the naturality condition since the associated naturality square corresponds to diagram (3.12) which commutes by ω-suitability of λ. All the remaining components are at stage 0 or ω and, by definition of morphisms in $\mathbf{Sh_C}(\omega)$, are mediating maps. Therefore, ξ is a natural transformation.

For Y a sheaf consider the compatibility diagrams associated with the component ξ_Y. At stage 1 these instantiate to the diagrams in **C**:

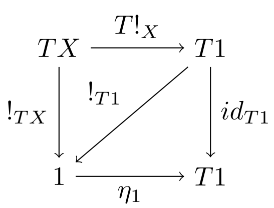


which commute by basic properties of η and μ. At stage n, for any finite $n > 1$, the compatibility diagrams for ξ_Y are those for $\lambda_{Y_{n-1}}$ and commute by hypothesis on λ. Finally, diagrams at stages 0 and ω follow from definition of pointwise extension and of morphisms in $\mathbf{Sh_C}(\omega)$. $\square$

A monad (T, μ, η) on a category with a final object 1 is called *affine* whenever its unit exhibits the isomorphism $T1 \cong 1$ [69, 71, 72, 87, 98].[1] Examples of affine monads are the non-empty powerset monad $\mathcal{P}^+$, the probability distribution monad $\mathcal{D}$, or the Giry monad $\mathcal{G}$.

LEMMA 3.17. *Let λ be distributive law of (T, μ, η) over F. If T is affine, then λ is ω-suitable.*

PROOF. It follows from finality of 1 in **C** and the affiness of T that the diagram below commutes for any object X of **C**:



(i)

[1] Older formulations for strong commutative monads require that components of double strengths are sections to $\langle T\pi_1, T\pi_2 \rangle$ [69, 87].

For X an object of **C**, consider the following decomposition of diagram (3.12):

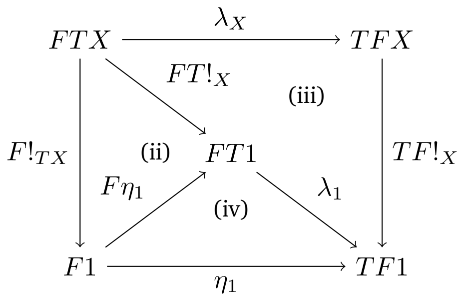


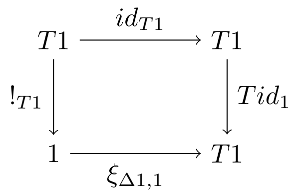
Diagram (ii) commutes since it is the image of (i). Diagrams (iii) and (iv) follow from naturality and compatibility of λ with η, respectively. □

It follows from Theorem 3.16 and Lemma 3.17 that if (T, μ, η) if affine, then any distributive law of T over F induces a distributive law of $\underline{T}$ over $\underline{F}\blacktriangleright$ (where pointwise extensions target $\mathbf{Sh}_{\mathbf{C}}(\omega)$). Remarkably, this property of distributive laws and affiness of monads are equivalent:

THEOREM 3.18. *For (T, μ, η) a monad on* **C***, the following statements are equivalent:*

- *The monad (T, μ, η) is affine.*
- *For any endofunctor F and any distributive law $\lambda\colon FT \to TF$, there is $\xi\colon \blacktriangleright\underline{T} \to \underline{T}\blacktriangleright$ such that it is compatible with the pointwise extension of (T, μ, η) to $\mathbf{Sh}_{\mathbf{C}}(\omega)$ and it induces λ.*

PROOF. Assume that (T, μ, η) is an affine monad, then the implication follows from Theorem 3.16 and Lemma 3.17. For the converse assume that for any endofunctor F, all distributive laws of T over F are induced by laws for $\underline{T}$ and $\underline{F}\blacktriangleright$. Note that laws for $\underline{T}$ and $\blacktriangleright$ induce laws for T and Id and these are exactly endomorphisms for the monad T. Thus, by assumption there is a distributive law $\xi\colon \blacktriangleright\underline{T} \to \underline{T}\blacktriangleright$ such that its induced law for T and Id is the identity on T. In particular, consider its component for final sheaf $\Delta 1$. By construction, $\xi_{\Delta 1, n} = id_{T1}$ for any finite successor ordinal n. It follows from naturality of ξ that $\xi_{\Delta 1, 1}$ is an isomorphism since naturality of components 1 and $n > 1$ corresponds to the following diagram:



It follows from compatibility of ξ with $\eta^{\underline{T}}$ that the component $\xi_{\Delta 1,1}$ is η_1 since the associated diagram is the following:

$$\begin{array}{ccc} 1 & \xrightarrow{\xi_{\Delta 1,1}} & T1 \\ {\scriptstyle id_1} \nwarrow & & \nearrow {\scriptstyle \eta_1} \\ & 1 & \end{array}$$

Therefore, η_1 is an isomorphism and (T, μ, η) is affine. □

Recall from [69, 98] that the *affine part* of a monad (T, μ, η) over **C** is the greatest affine submonad T^{a} of T and that, assuming **C** has enough finite limits, T^{a} is determined on each object X by pulling back η_1 along $T!_X$:

$$\begin{array}{ccc} T^{\mathrm{a}}X & \longrightarrow & TX_\alpha \\ \downarrow & \lrcorner & \downarrow {\scriptstyle T!_X} \\ 1 & \xrightarrow[\eta_1]{} & T1 \end{array}$$

(*cf.* [69, Definition 4.5].) For instance, Id, $\mathcal{P}^+$, and $\mathcal{D}$ are the affine part of the writer monad $M \times Id$, the powerset monad $\mathcal{P}$, and the generalised multiset monad $\mathcal{M}_{[0,\infty]}$, respectively. As noted in [43, Proposition 3.2] any law distributing a monad over an endofunctor restricts to a law distributing its affine part over the same endofunctor.

On alternatives to the pointwise extension The pointwise extension is not the only way to extend an endofunctor from **C** to $\mathbf{Sh_C}(\alpha)$. We discuss two possible alternatives detailing why they are unsatisfactory for the aims of this chapter. The first approach relies on the constant sheaf adjunction $(\Delta \dashv \Gamma)\colon \mathbf{Sh_C}(\alpha) \to \mathbf{C}$ and defines the extension of an endofunctor F as the composite $\Delta F \Gamma\colon \mathbf{C} \to \mathbf{C}$. This definition extends to a functor but not a monoidal functor since in this situation a natural isomorphism $Id_{\mathbf{Sh_C}(\alpha)} \cong \Delta\Gamma$ exhibits an equivalence of categories for $\mathbf{Sh_C}(\alpha)$ and **C** (for a counterexample consider sheaves of sets). However, this is not an issue: the fact that pointwise extension defines a monoidal functor allows us to prove Corollaries 3.8 and 3.9 from general properties of monoidal functors but these can also proven directly. In fact, the extension functor defined from $(\Delta \dashv \Gamma)$ preserves monads and lifts to the category of distributive laws. Moreover, its essential image is enriched similarly to the pointwise extension functor $(\underline{\ })$. Then, why is the pointwise extension preferable to this alternative notion? It turns out that this kind of extensions pose very stringent constraints on Kleisli liftings of locally contractive endofunctors to the point of impacting their suitability with respect to the aims of this chapter. To understand the severity of this limitation consider a natural transformation $\xi\colon \blacktriangleright\Delta T\Gamma \to \Delta T\Gamma\blacktriangleright$ (the type of ξ is exactly that of transformations associated to Kleisli liftings for $\blacktriangleright$). Naturality requires each component at any stage above 1 to factor through the morphism to the final object of **C** as illustrated by the naturality square below.

$$\begin{array}{ccc} TX_\alpha & \xrightarrow{id_{T1}} & TX_\alpha \\ {\scriptstyle !_{TX_\alpha}}\downarrow & & \downarrow{\scriptstyle Tid_{X_\alpha}} \\ 1 & \xrightarrow[\xi_{X,\beta}]{} & TX_\alpha \end{array}$$

Compatibility with the monad structure imposes similar constraints also on the unit and multiplication of the monad. The second approach we discuss is usually known as *right extension* and (assuming enough limits exists) associates F to (the functorial part of) the right Kan extension along Δ of ΔF *i.e.* $Ran_\Delta(\Delta F)$. Assume, for the sake of the argument, that Δ has also a left adjoint (*e.g.* when **C** is **Set**), then right Kan extensions are preserved by Δ and $Ran_\Delta(\Delta F) \cong \Delta\circ Ran_\Delta F$. In particular, the right extension of $Id_{\mathbf{C}}$ is $\Delta \circ Ran_\Delta Id_{\mathbf{C}} \cong \Delta\Gamma$ and at this point the argument detailed above applies.

3.4 INFINITE TRACE SEMANTICS VIA GUARDED KLEISLI (CO)RECURSION

In this section we introduce a construction for capturing infinite trace semantics of systems modelled as TF-coalgebras via finality in a suitable category of coalgebras.

The key observation supporting our construction is that infinite traces can be characterised by amalgamation of certain families of coherent approximations akin to how a stream is described by the infinite family of its prefixes. In general, these approximations can be thought of as observations obtained from monitoring executions for a given number of steps (the prefix length) and such observations are associated to intermediate steps of final sequences [3, 17, 18].

In Section 3.4.1 we present the final sequence for an endofunctor F as the unique invariant object of its guarded pointwise extension $\underline{F}\blacktriangleright$. In Section 3.4.2 we study coalgebras of type $\underline{F}\blacktriangleright$ and show that associated notion of bisimulation generalises known behavioural pseudo-ultrametrics induced by final sequences [3, 18]. Finally, in Section 3.4 we consider the categories of coalgebras for Kleisli liftings of $\underline{F}\blacktriangleright$, we characterise their final objects, and provide embeddings from the category of TF-coalgebras.

3.4.1 Final sequences as invariant objects

Final sequences were introduced by [17] in order to compute final coalgebras and, together with their dual structures (*i.e.* initial sequences and initial algebras), can be thought as generalisations of Kleene's chains. These constructions have been successfully used to provide sufficient conditions for a functor to admit final (resp. initial) invariant objects (see for example Barr [18], Smyth and Plotkin [133], Adámek [4–7], Worrell [148–150], Bacci [12, 13]). In this section we characterise final sequences for endofunctors as unique invariants for suitable endofunctors over categories of sheaves. These objects are proxy to all the information usually found in final coalgebras together with the sequences of observations approximating them. For instance, if final coalgebras for the given endofunctor describe infinite streams, then we obtain a sheaf that represents them by means of their prefixes.

Recall from [17] that the final sequence for an endofunctor F over $\mathbf{C}$ is the ordinal-indexed sequence of objects $(F^{\beta})_{\beta\in\mathbf{Ord}}$ and arrows $(f_{\beta}^{\beta'})_{\beta\leq\beta'\in\mathbf{Ord}}$ such that:

$$F^{\beta+1} = F(F^{\beta}) \qquad F^{\gamma} = \lim_{\beta<\gamma} F^{\beta} \qquad f_{\beta+1}^{\beta'+1} = F f_{\beta}^{\beta'} \qquad f_{\beta}^{\gamma} = \pi_{\beta}$$

where γ is a limit ordinal (note that 0 is considered a limit ordinal as well) and the projection $\pi_{\beta}\colon F^{\gamma} \to F^{\beta}$ is the β-component of the limiting cone. The final sequence for F corresponds to a $\mathbf{C}$-valued sheaf over the category of ordinals $\mathbf{Ord}$.

DEFINITION 3.8. *For F an endofunctor over* $\mathbf{C}$*, the* final sequence of F *is any limit-preserving functor* $\mathrm{fin}(F)\colon \mathbf{Ord} \to \mathbf{C}$ *such that, for all ordinals $\beta \leq \beta'$:*

- $\mathrm{fin}(F)(\beta+1) = F(\mathrm{fin}(F)(\beta)$*;*
- $\mathrm{fin}(F)(\iota_{\beta+1,\beta+1}) = F(\mathrm{fin}(F)(\iota_{\beta,\beta'}))$.

In particular, the functor $\mathrm{fin}(F)$ is given on any ordinal β and on any inclusion $\iota_{\beta,\beta'}\colon \beta \to \beta'$ as follows:

$$\mathrm{fin}(F)_{\beta} = F^{\beta} \qquad \mathrm{fin}(F)_{\iota_{\beta,\beta'}} = f_{\beta}^{\beta'}.$$

In the following we will be interested in the first α steps of the final sequence (*e.g.* when the sequence is stable after α steps) and hence will restrict $\mathrm{fin}(F)$ to $\mathcal{A}(\alpha)^{op}$. Formally, this restriction yields an object in $\mathbf{Sh}_{\mathbf{C}}(\alpha)$ and corresponds to the action on $\mathrm{fin}(F)$ of the inverse image i^*, where i is given by the inclusion of $\mathcal{A}(\alpha)$ into $\mathbf{Ord}$. The final sequence is said to be *stable* at some ordinal α provided that $f_{\alpha}^{\alpha+1}$ is an isomorphism. For notational convenience, we will write $\mathrm{fin}(F)$ instead of $i^*(\mathrm{fin}(F))$ when i is clear from the context.

EXAMPLE 3.8. *Consider the endofunctor $A \times Id$ where A is a (non-empty) set of labels and let α be ω—the final sequence for $A \times Id$ is stable after ω. The sheaf* $\mathrm{fin}(A \times Id)$ *on $\mathcal{A}(\omega)$ is given as follows:*

$$\begin{aligned} \mathrm{fin}(A \times Id)_0 &= 1 & \mathrm{fin}(A \times Id)_{\iota_{0,n}} &= !_{A^n} \\ \mathrm{fin}(A \times Id)_n &= A^n & \mathrm{fin}(A \times Id)_{\iota_{n,m}} &= A^n!_{A^{m-n}} \\ \mathrm{fin}(A \times Id)_\omega &= \lim_{n<\omega} A^n \cong A^\omega & \mathrm{fin}(A \times Id)_{\iota_{n,\omega}} &= A^n!_{A^\omega} = \pi_n \end{aligned}$$

Finite words over the alphabet A are the observations characterising streams i.e. *the abstract behaviours for $A \times Id$-coalgebras.*

By considering final sequences as sheaves we are able to "internalise" their information about how final coalgebras are identified via sequences of approximations. Besides the above direct construction, these sheaves are characterised as unique invariants (*i.e.* final coalgebras) of guarded pointwise extensions.

LEMMA 3.19. *There is a unique $\underline{F}\blacktriangleright$-invariant given (up to isomorphism by) the identity on* $\mathrm{fin}(F)$.

PROOF. Let α be an ordinal. It follows from Theorem 3.10 and Lemma 2.11 that $\underline{F}$ is enriched over $\mathbf{Sh}(\alpha)$ and that $\underline{F}\blacktriangleright$ is locally contractive. The category $\mathbf{Sh}_{\mathbf{C}}(\alpha)$ has limits of α-sequences since, by hypothesis, we have $(\Delta \dashv \Gamma)$ and limits of α-sequences in $\mathbf{C}$. It follows from Proposition 2.13 that the endofunctor $\underline{F}\blacktriangleright$ has a unique (up to isomorphism) invariant. Therefore, to prove the claim it suffices to show that $id_{\mathrm{fin}(F)}$ is an $\underline{F}\blacktriangleright$-(co)algebra. On successors ordinals we have that:

$$\mathrm{fin}(F)_{\beta+1} = F^{\beta+1} = FF^\beta = (\underline{F}\blacktriangleright \mathrm{fin}(F))_{\beta+1}$$

and on limit ones that:

$$\mathrm{fin}(F)_\gamma = \lim_{\beta<\gamma} \mathrm{fin}(F)_\beta \overset{\ddagger}{=} \lim_{\beta+1<\gamma} \mathrm{fin}(F)_{\beta+1} = \lim_{\beta<\gamma} (\underline{F}\blacktriangleright \mathrm{fin}(F))_\beta = (\underline{F}\blacktriangleright \mathrm{fin}(F))_\gamma$$

where (‡) follows by restriction to a family of successor ordinals covering γ. □

3.4.2 Guarded coalgebras

For an endofunctor F, we refer to coalgebras of type $\underline{F}\blacktriangleright$ as *guarded*. Intuitively, the modality $\blacktriangleright$ guarding $\underline{F}$ forces transitions at any successor stage $\beta + 1$ to have targets at their predecessor stage β:

$$X_{\beta+1} \xrightarrow{h_{\beta+1}} FX_\beta$$

whereas transitions at stages that are limit ordinals have targets at the same stage and are obtained as mediating maps.

For instance, take F as the endofunctor $A \times Id$ and α as ω since, as discussed in Example 3.8, the final sequence for $A \times Id$ is stable at ω. Then, $\underline{F}\blacktriangleright$ is the endofunctor $\Delta A \times \blacktriangleright$ over **Sh**(ω). Let $h\colon X \to \Delta A \times \blacktriangleright X$ be a guarded coalgebra. The component at stage 0 of h is determined by the structure of sheaves and is the identity on the singleton 1; hence the only element inhabiting this stage can be seen as a sink state $\bot$. This interpretation for h_0 and $\bot$ is fostered by looking at the other components of h. At stage 1, $\blacktriangleright X$ takes value $X_0 = 1$ and hence all transitions described by $h_1\colon X_1 \to A \times X_0$ necessarily end in the sink $\bot$ which essentially means they terminate producing a label ($A \times X_0 \cong A$). In general, transitions described by $h_{n+2}\colon X_{n+2} \to A \times X_{n+1}$ start at stage $n+2$ and end at stage $n+1$, those described by $h_{n+1}\colon X_{n+1} \to A \times X_n$ go from $n+1$ to n, and so on until $\bot$ is reached after $n+2$ steps. At stage ω, X and h are defined by amalgamation from the underlying stages:

$$h_\omega(x) = (a, x') \iff \forall n < \omega\, (h_{n+1} \circ X_{\iota_{n+1,\omega}})(x) = (a, X_{\iota_{n,\omega}}(x')).$$

Computations described by this component of h never leave stage ω and each of their countably many steps projects, coherently with restriction maps, to a step at stage n for any $n < \omega$. It follows that h outputs streams and words forming their prefixes.

This example fosters the intuition of stages as describing the "number of available steps" or the "observations length". Form this perspective, the component at stage β of the final semantics map describes behaviours distinguishable by means of observations at stage β that is, baring with the above intuition, "by considering executions up to β-many[2] transition steps". This perspective generalises ideas from [3, 18] where Barr and Adámek observed how final sequences for ω-continuous endofunctors over **Set** determine a pseudo-ultrametric on their final coalgebra carrier (and hence on each coalgebra carrier). In particular, for (X, h) an $(A \times Id)$-coalgebra, the distance of two states x and x' in X is defined as 2^{-n} where n is the length of the longest prefix shared by the streams generated from x and x':

$$d(x, x') = \inf \left\{ 2^{-n} \;\middle|\; \begin{array}{l} \exists (x_i \in X)_{i<n},\ (x'_i \in X)_{i<n},\ (a_i \in A)_{i<n} \\ \text{such that } x_0 = x,\ x'_0 = x',\ \forall i < n-1 \\ h(x_i) = (a_i, x_{i+1}) \text{ and } h(x'_i) = (a_i, x'_{i+1}) \end{array} \right\} \tag{3.13}$$

Thus, $d(x, x') = 0$ if and only if x and x' are behaviourally equivalent.

Coalgebras of type $\underline{F}\blacktriangleright$ and their bisimulations rephrase the above situation in the language of sheaves: these structures localise the information contained in

[2] In general β is the index of a step in the final sequence and not an actual length, however the two coincide for sequences stable at ω like those arising from the examples considered in this section.

the pseudo-ultrametric (3.13) by restriction to the values associated to each n or, equivalently, to each step of the final sequence for $F = A \times Id$. In this setting, a bisimulation is a span $X \leftarrow R \rightarrow X'$ of sheaves making the usual diagram commute and can be understood (without loss of generality) as a decreasing ω-indexed sequence of relations $R_0 \supseteq R_1 \supseteq \dots$ such that:

$$x\ R_n\ x' \implies d(x, x') \leq 2^{-n}.$$

From this perspective, a bisimulation at stage β captures observational equivalence where observations are restricted to those described by the β-step of the final sequence. Therefore, guarded coalgebras and their bisimulations are a (conservative) generalisation of Barr's ideas to arbitrary endofunctors (albeit a metric cannot be defined in general, *e.g.* when $\mathcal{A}(\alpha)$ is not metrizable).

To conclude this section, we show that all coalgebras are guarded in the sense that F-coalgebras form a subcategory of **Coalg**$(\underline{F}\blacktriangleright)$. Recall from Section 2.2.2 that $\blacktriangleright$ is a well-pointed endofunctor and its point is $next\colon Id \to \blacktriangleright$. This natural transformation induces the functor between coalgebra categories:

$$\mathbf{Coalg}(id_{\underline{F}} \circ next)\colon \mathbf{Coalg}(\underline{F}) \to \mathbf{Coalg}(\underline{F}\blacktriangleright)$$

given, on each coalgebra (X, h) and homomorphism f by the assignments:

$$(X, h) \mapsto (X, \underline{F}(next_X) \circ h) \qquad f \mapsto f.$$

Intuitively, this functor uses restriction morphisms to "guard transition targets" as clear from unfolding the definition of $(\underline{F}(next_X) \circ h)_{\beta+1}$:

$$X_{\beta+1} \xrightarrow{h_{\beta+1}} FX_{\beta+1} \xrightarrow{FX_{\iota_{\beta,\beta+1}}} FX_{\beta}.$$

We remark that **Coalg**$(id_F \circ next)$ is a lifting of the identity on $\mathbf{Sh}_{\mathbf{C}}(\alpha)$ along the forgetful functors for **Coalg**$(\underline{F})$ and **Coalg**$(\underline{F}\blacktriangleright)$ since it acts as the identity coalgebra homomorphisms. We write $(-)^{\blacktriangleright}$ for **Coalg**$(id_{\underline{F}} \circ next)$.

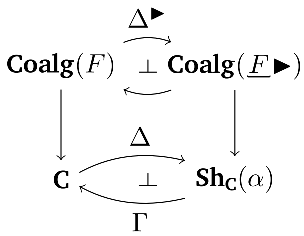
Recall from Section 3.2 that the constant sheaf functor Δ lifts to categories of coalgebras and hence, by composition with $(-)^{\blacktriangleright}$, we have a functor $\Delta^{\blacktriangleright}$ turning every F-coalgebra into a guarded coalgebra while acting as Δ on their carrier.

$$\begin{array}{ccc} \mathbf{Coalg}(F) & \xrightarrow{\Delta^{\blacktriangleright}} & \mathbf{Coalg}(\underline{F}\blacktriangleright) \\ \downarrow & & \downarrow \\ \mathbf{C} & \xrightarrow[\Delta]{} & \mathbf{Sh}_{\mathbf{C}}(\alpha) \end{array} \tag{3.14}$$

Moreover, this functor is an inclusion whenever α is greater than 1 since $\Delta X_{\iota_{\beta,\beta'}}$ is id_X for any $\beta' \geq \beta > 0$. It follows from Proposition 3.5 and Lemma 3.19 that

$\Delta^{\blacktriangleright}$ takes final coalgebras to final (guarded) coalgebras. The functor $\Delta^{\blacktriangleright}$ exhibits $\mathbf{Coalg}(F)$ as a subcategory of $\mathbf{Coalg}(\underline{F}\blacktriangleright)$ and, under mild assumptions akin to Proposition 3.6, as a coreflective subcategory.

PROPOSITION 3.20. *For α an infinite limit ordinal, if $\varrho\colon F\Gamma \to \Gamma\underline{F}$ from* (3.10) *is a retraction then the constant sheaf adjunction $(\Delta \dashv \Gamma)\colon \mathbf{Sh}_{\mathbf{C}}(\alpha) \to \mathbf{C}$ lifts along the forgetful functors for $\mathbf{Coalg}(F)$ and $\mathbf{Coalg}(\underline{F}\blacktriangleright)$. Moreover, the lifted adjunction is a coreflection.*



PROOF. Assume $\varrho\colon F\Gamma \to \Gamma\underline{F}$ is a retraction and let $\varsigma\colon \Gamma\underline{F} \to F\Gamma$ be any of its sections. Akin to Proposition 3.6, the statement can be shown to follow from [81, Theorem 2.5] by providing $\vartheta^{\blacktriangleright}\colon \Delta F \to \underline{F}\blacktriangleright\Delta$ and $\varsigma^{\blacktriangleright}\colon \Gamma\underline{F}\blacktriangleright \to F\Gamma$ such that the necessary diagrams commute.

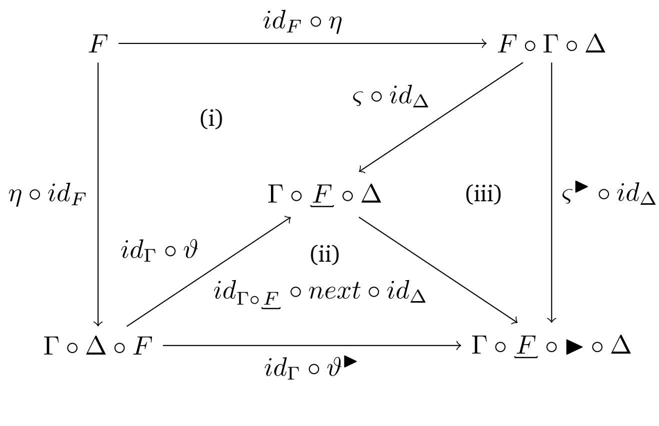
Note that the natural transformation $\Gamma\underline{F}next\colon \Gamma\underline{F} \to \Gamma\underline{F}\blacktriangleright$ has an inverse for it is given, on each sheaf X, by the equality

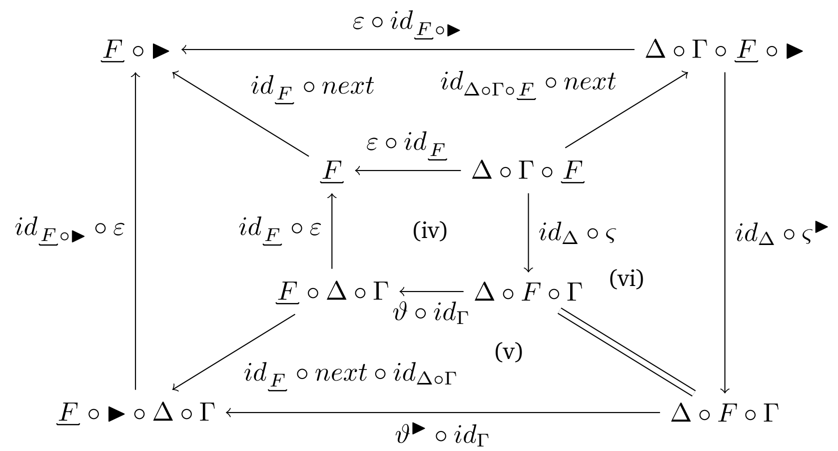
$$\Gamma\underline{F}\blacktriangleright X = \underline{F}\blacktriangleright X_\alpha = \lim_{\beta+1<\alpha} \underline{F}\blacktriangleright X_{\beta+1} = \lim_{\beta+1<\alpha} FX_\beta = \underline{F}X_\alpha = \Gamma\underline{F}X$$

where the limits are restricted to the base of successor ordinals. Define the natural transformations $\vartheta^{\blacktriangleright}\colon \Delta F \to \underline{F}\blacktriangleright\Delta$ and $\varsigma^{\blacktriangleright}\colon \Gamma\underline{F}\blacktriangleright \to F\Gamma$ as

$$\Delta F \xrightarrow{\vartheta} \underline{F}\Delta \xrightarrow{\underline{F}next_\Delta} \underline{F}\blacktriangleright\Delta \qquad \text{and} \qquad \Gamma\underline{F}\blacktriangleright \xrightarrow{(\Gamma\underline{F}next)^{-1}} \Gamma\underline{F} \xrightarrow{\varsigma} F\Gamma$$

where $\vartheta\colon \Delta F \to \underline{F}\Delta$ and $\varsigma\colon \Gamma\underline{F} \to F\Gamma$ are the equality (3.9) and a section for ϱ as per (3.10). In order to prove the necessary diagrams commute decompose them as follows:





Both diagrams commute: (i) and (iv) are shown to commute in the proof of Proposition 3.6; (ii) and (v) define $\vartheta^{\blacktriangleright}$; (iii) and (vi) define $\varsigma^{\blacktriangleright}$; the remaining squares follow by naturality of ε e $next$. It follows from [81, Theorem 2.5] that the desired lifting exists. In particular, the lifting of Γ, is given on each $\underline{F}$-coalgebra (X, h) and homomorphism f, by the assignments

$$(X, h) \mapsto (\Gamma X, \varsigma^{\blacktriangleright}_X \circ \Gamma h) \qquad f \mapsto \Gamma f$$

and the lifting of Δ given on each F-coalgebra (Y, k) and homomorphism g, by the assignments

$$(Y, k) \mapsto (\Delta Y, \vartheta^{\blacktriangleright}_Y \circ \Delta k) \qquad g \mapsto \Delta g.$$

The latter is exactly the inclusion functor $\Delta^{\blacktriangleright} = \textbf{Coalg}(id_{\underline{F}} \circ next)$ since $\vartheta^{\blacktriangleright}$ is defined as $(id_{\underline{F}} \circ next) \bullet \vartheta$ and ϑ as the equality $\Delta F = \underline{F}\Delta$. □

3.4.3 Infinite trace semantics

In this section we combine guarded coalgebras with extensions to Kleisli categories in order to capture infinite trace semantics via final semantics. Intuitively, the former provides us with the tools for collecting observations into coherent families whereas the latter offers us the setting where to abstract the effects modelled by the branching type *i.e.* ensure observations come from the linear semantics of systems under scrutiny. In practice, for systems modelled as TF-coalgebras, we consider coalgebras for Kleisli liftings of $\underline{F}\blacktriangleright$ where the pointwise extension targets sheaves on an ordinal α large enough for the final sequence of F to stabilise.

Before we discuss $\overline{\underline{F}\blacktriangleright}$-coalgebras in general let us illustrate the construction in the case of non-deterministic labelled transition systems. To this end, take T, F, λ, and α as follows:

- the affine and commutative monad $\mathcal{P}^+$ (the double strength of $\mathcal{P}$ readily restricts to its affine part);

- the polynomial functor $A \times Id$ (for A non-empty);
- the distributive law $\lambda\colon A \times \mathcal{P}^+ \to \mathcal{P}^+(A \times Id)$ associated to the canonical Kleisli lifting of $A \times Id$ to $\mathbf{Kl}(\mathcal{P}^+)$;
- the first infinite ordinal ω (the final sequence for $A \times Id$ stabilises at ω).

Since $\mathcal{P}^+$ is affine, we can apply Theorems 3.16 and 3.18 to λ obtaining the distributive law of $\underline{\mathcal{P}^+}$ over $\underline{F}\blacktriangleright$:

$$\underline{A \times Id} \circ \blacktriangleright \circ \mathcal{P}^+ \xrightarrow{id_{\underline{A\times Id}} \circ \theta} \underline{A \times Id} \circ \mathcal{P}^+ \circ \blacktriangleright \xrightarrow{\underline{\lambda} \circ id_{\blacktriangleright}} \mathcal{P}^+ \circ \underline{A \times Id} \circ \blacktriangleright. \quad (3.15)$$

The fact that this distributive law factors through $\underline{A \times Id} \circ \mathcal{P}^+ \circ \blacktriangleright$ corresponds to its associated Kleisli lifting of $\underline{F}\blacktriangleright$ being the composition of Kleisli liftings of $\underline{F}$ and $\blacktriangleright$ given by the extension of λ and the distributive law ξ constructed in the proof of Theorem 3.18, respectively. We remark that this strategy is the equivalent for Kleisli liftings of the constructions for distributive laws presented in Section 3.3. The distributive law (3.15) acts essentially as λ since its component for a sheaf X is the arrow given at stage 1 as the function

$$\begin{array}{ccc}
(\Delta A \times \blacktriangleright \underline{\mathcal{P}^+} X)_1 & = & A \\
{\scriptstyle (id_{\underline{A\times Id}} \circ \theta)_{X,1}} \downarrow & & \downarrow {\scriptstyle id_A} \\
(\Delta A \times \blacktriangleright \underline{\mathcal{P}^+} X)_1 & = & A \\
{\scriptstyle (\underline{\lambda} \circ id_{\blacktriangleright})_{X,1}} \downarrow & & \downarrow {\scriptstyle \eta_A} \\
\underline{\mathcal{P}^+}(\Delta A \times \blacktriangleright X)_1 & = & \mathcal{P}^+ A
\end{array}$$

and at stage $n+2$ as the function

$$\begin{array}{ccc}
(\Delta A \times \blacktriangleright \underline{\mathcal{P}^+} X)_{n+2} & = & A \times \mathcal{P}^+ X_{n+1} \\
{\scriptstyle (id_{\underline{A\times Id}} \circ \theta)_{X,n+2}} \downarrow & & \downarrow {\scriptstyle id_{A \times \mathcal{P}^+ X_{n+1}}} \\
(\Delta A \times \blacktriangleright \underline{\mathcal{P}^+} X)_{n+2} & = & A \times \mathcal{P}^+ X_{n+1} \\
{\scriptstyle (\underline{\lambda} \circ id_{\blacktriangleright})_{X,n+2}} \downarrow & & \downarrow {\scriptstyle \lambda_{X_{n+1}}} \\
\underline{\mathcal{P}^+}(\Delta A \times \blacktriangleright X)_{n+2} & = & \mathcal{P}^+(A \times X_{n+1})
\end{array}$$

Coalgebras for endofunctors like these are guarded coalgebras: this means that the executions they describe are intertwined with stages whose ordinal number represents the number of steps available to the computation. Akin to

Section 3.4.2, consider the components of a coalgebra $h\colon X \to \underline{\mathcal{P}^+}(\Delta A \times \blacktriangleright X)$: h_0 has type $1 \to 1$ but since $\mathcal{P}^+$ is affine this is an isomorphism which means the computation it describes cannot evolve in any meaningful way; h_1 has type $X_1 \to \mathcal{P}^+\Delta A$ and the computations it describes non-deterministically output a label before reaching the sink at stage 0; h_n for $n > 1$ has type $X_n \to \mathcal{P}^+(\Delta A \times X_{n-1})$ and the computations it describes non-deterministically output a label before reaching a state at stage $n-1$. The fundamental difference with respect to the situation discussed in Section 3.4.2 is that steps are now concatenated by means of Kleisli composition hence abstracting from non-deterministic branching. In order to illustrate this difference and elucidate the key rôle played by the Kleisli category let us consider sequences of steps. Sequences of two steps in the system modelled by h are described by the composite

$$h' \triangleq \overline{\Delta A \times \blacktriangleright}\,(h) \circ h$$

i.e. the arrow $h'\colon X \to \underline{\mathcal{P}^+}\,(\Delta A \times \blacktriangleright(\Delta A \times \blacktriangleright X))$ in $\mathbf{Sh}(\omega)$ defined as:

$$h' \triangleq \underline{\mu}_X \circ \underline{\mathcal{P}^+}\left(\underline{\lambda}_{\blacktriangleright(\Delta A \times \blacktriangleright X)}\right) \circ \underline{\mathcal{P}^+}\,(\Delta A \times \blacktriangleright\theta_{\Delta A \times X}) \circ \underline{\mathcal{P}^+}\,(\Delta A \times \blacktriangleright h) \circ h.$$

At stage 1, h' equals to h:

$$a \in h'_1(x) \iff a \in h_1(x)$$

since the outermost occurrence of $\blacktriangleright$ in the behavioural functor takes value 1 at this stage. We encounter the first difference at stage 2:

$$(a, a') \in h'_2(x) \iff \exists x' \in X_1 \text{ s.t. } (a, x') \in h_2(x) \land a' \in h_1(x').$$

At this stage the outermost occurrence of $\blacktriangleright$ takes the value of its argument at stage 1 and hence the innermost occurrence takes value at stage 0 meaning that sequences always end in the sink $\bot$. Because of Kleisli composition, intermediate states (x' above) are stripped from the outcome. Components at greater stages behave similarly except for the ending state not being the sink. In particular, at any stage $n+3$ we have that:

$$(a, a', x'') \in h'_{n+3}(x) \iff \exists x' \in X_{n+2}((a, x') \in h_{n+3}(x) \land (a', x'') \in h_{n+2}(x')).$$

The same considerations apply to sequences of arbitrary length: observations at stage n are partial traces of length[3] n and partial traces observed at different

[3] In presence of explicit termination, as in the case for $F = A \times id + 1$, length of executions at stage n is *at most* n.

stages abide restriction maps as illustrated by the schema:

$$\begin{array}{ccccccccccccccc}
n+1 \vdash & x_1 & \xrightarrow{a_1} & x_2 & \xrightarrow{a_2} & \cdots & \xrightarrow{a_{n-1}} & x_n & \xrightarrow{a_n} & x_{n+1} & \xrightarrow{a_{n+1}} & \bot & \longrightarrow & \cdots \\
 & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
n \vdash & x_1|_n & \xrightarrow{a_1} & x_2|_n & \xrightarrow{a_2} & \cdots & \xrightarrow{a_{n-1}} & x_n|_n & \xrightarrow{a_n} & \bot & \longrightarrow & \bot & \longrightarrow & \cdots
\end{array}$$

where vertical arrows are mappings induced by the restriction function $X_{\iota_{n,n+1}}$ and horizontal arrows are transitions described by h at stages n and $n+1$.

Stage ω is defined by amalgamation and the associated observations are ω-sequences. It follows from definition of sheaves and their morphisms that observations made at stage ω restrict, for each finite ordinal n, to observations at stage n. Symmetrically, a family with an observation for each stage $n < \omega$ that is coherent with respect to restriction maps induces an observation at stage ω. We conclude that since observations at stage $n < \omega$ are (partial) trace of length n observations at stage ω are necessarily infinite traces.

Consider the diagram asserting that $f\colon X \to Y$ extends to some morphism of $\overline{\Delta A \times \blacktriangleright}$-coalgebras $f\colon (X,h) \to (Y,k)$ and in particular its unfolding in **Sh**(ω):

$$\begin{array}{ccccc}
X & & \xrightarrow{\quad h \quad} & & \underline{\mathcal{P}^+}(\Delta A \times \blacktriangleright X) \\
 & & & & \downarrow{\scriptstyle \underline{\mathcal{P}^+}(\Delta A \times \blacktriangleright f))} \\
 & & & & \underline{\mathcal{P}^+}(\Delta A \times \blacktriangleright \underline{\mathcal{P}^+} Y) \\
 & & & & \downarrow{\scriptstyle \underline{\mathcal{P}^+}(\Delta A \times \theta_Y)} \\
{\scriptstyle f}\downarrow & & & & \underline{\mathcal{P}^+}(\Delta A \times \underline{\mathcal{P}^+} \blacktriangleright Y) \\
 & & & & \downarrow{\scriptstyle \underline{\mathcal{P}^+}\lambda_{\blacktriangleright Y}} \\
 & & & & \underline{\mathcal{P}^+}\underline{\mathcal{P}^+}(\Delta A \times \blacktriangleright Y) \\
 & & & & \downarrow{\scriptstyle \underline{\mu}_{\Delta A \times \blacktriangleright Y}} \\
\underline{\mathcal{P}^+} Y & \xrightarrow[\underline{\mathcal{P}^+} k]{} & \underline{\mathcal{P}^+}\underline{\mathcal{P}^+}(\Delta A \times \blacktriangleright Y) & \xrightarrow[\underline{\mu}_{\Delta A \times \blacktriangleright Y}]{} & \underline{\mathcal{P}^+}(\Delta A \times \blacktriangleright Y)
\end{array}$$

In order to show that the above diagram commutes, it suffices to show that it commutes when restricted to successor ordinals in ω and hence only two cases need to be checked; the first corresponds to stage 1 and the second to all the remaining $1 < n < \omega$. The component at stage 1 commutes if, and only if, for any label $a \in A$ and any state $x \in X_1$ it holds that:

$$a \in h_1(x) \iff \exists y \in f_1(x)(a \in k_1(y)).$$

A component at stage $n > 1$ commutes if, and only if, for any $a \in A$, $x \in X_n$, and $x' \in X_{n-1}$, it holds that:

$$(a, x') \in h_n(x) \iff \forall y' \in f_{n-1}(x') \exists y \in f_n(x)((a, y') \in k_n(y)).$$

Therefore, $\overline{\underline{F}\blacktriangleright}$-coalgebra homomorphisms, like $\underline{F}\blacktriangleright$-coalgebra homomorphisms, tie execution steps to stages and, like $\underline{F}$-coalgebra homomorphisms, they abstract from branching.

The example above suggests that final $\overline{\Delta A \times \blacktriangleright}$-coalgebras capture infinite trace semantics for labelled transition systems (without implicit termination). Indeed, this is the case and the same result holds for arbitrary systems modelled by TF-coalgebras—provided that the sequence for F stabilises and that $\underline{F}\blacktriangleright$ has a Kleisli lifting. Under these assumptions, final $\overline{\underline{F}\blacktriangleright}$-coalgebras are images through the canonical inclusion $K\colon \mathbf{C} \to \mathbf{Kl}(T)$ of final $\underline{F}\blacktriangleright$-coalgebras and the latter characterise, by construction, final F-coalgebras (*cf.* Lemma 3.19 and Proposition 3.20). Therefore, final semantics for $\overline{\underline{F}\blacktriangleright}$-coalgebras captures (possibly) infinite trace semantics (*cf.* Definition 3.4) for systems modelled by TF-coalgebras. Formally:

THEOREM 3.21. *Let (T, μ, η) be a monad and F an endofunctor, both over some category* **C**. *Let α be an ordinal such that the adjunction $(\Delta \dashv \Gamma)\colon \mathbf{Sh}_{\mathbf{C}}(\alpha) \to \mathbf{C}$ is defined and the final sequence for F is stable at α. For $\overline{\underline{F}\blacktriangleright}$ Kleisli lifting of $\underline{F}\blacktriangleright$, there is a unique $\overline{\underline{F}\blacktriangleright}$-invariant and it is the identity on $\operatorname{fin}(F)$.*

PROOF. It follows from Lemma 3.19 that the identity on $\operatorname{fin}(F) \in \mathbf{Sh}_{\mathbf{C}}(\alpha)$ exhibits the unique $\underline{F}\blacktriangleright$-invariant. Because the canonical inclusion $K\colon \mathbf{Sh}_{\mathbf{C}}(\alpha) \to \mathbf{Kl}(\underline{T})$ transports initial invariants to initial invariants of Kleisli liftings (see Proposition 3.2), $K\left(id_{\operatorname{fin}(F)}\right) = \eta_{\operatorname{fin}(F)}$ is the initial invariant of $\overline{\underline{F}\blacktriangleright}$. We conclude by noting that by Proposition 3.13 $\overline{\underline{F}\blacktriangleright}$ is locally contractive and thus it follows from Lemma 2.12 that $\overline{\underline{F}\blacktriangleright}$ has a unique invariant *i.e.* $id_{\operatorname{fin}(F)} \in \mathbf{Kl}(\underline{T})$. □

COROLLARY 3.22. *Let (T, μ, η) be a monad and F an endofunctor, both over some category* **C**. *Let α be an ordinal such that the adjunction $(\Delta \dashv \Gamma)\colon \mathbf{Sh}_{\mathbf{C}}(\alpha) \to \mathbf{C}$ is defined and the final sequence for F is stable at α. The final $\overline{\underline{F}\blacktriangleright}$-coalgebra captures infinite trace semantics.*

We conclude the section by noting that there is a functor associating TF-coalgebras to $\overline{\underline{F}\blacktriangleright}$-coalgebras while acting as the constant sheaf functor Δ on carriers and as K on homomorphisms. In fact, there is a lifting of $K\Delta$ given by composition of (3.14) and (3.4) as the commuting diagram below illustrates.

$$
\begin{array}{ccccc}
\mathbf{Coalg}(TF) & \longrightarrow & \mathbf{Coalg}(\underline{T}\,\underline{F}\blacktriangleright) & \longrightarrow & \mathbf{Coalg}(\overline{\underline{F}\blacktriangleright}) \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathbf{C} & \xrightarrow[\Delta]{} & \mathbf{Sh_C}(\alpha) & \xrightarrow[K]{} & \mathbf{Kl}(\underline{T})
\end{array}
$$

If $\alpha > 1$, then this functor is an inclusion for the category of TF-coalgebras into that of $\overline{\underline{F}\blacktriangleright}$-coalgebras. In particular, it acts on any F-coalgebra (X, h) as $\Delta^{\blacktriangleright}$ and on any F-coalgebra homomorphism $f\colon (X,h) \to (Y,k)$ as $K\Delta$:

$$
(X,h) \mapsto (\Delta X, \underline{TF}(next_{\Delta X}) \circ \Delta h) \qquad f \mapsto \eta^{\underline{T}}_{\underline{F}\Delta Y} \circ \Delta f.
$$

Thanks to this inclusion we are able to define the infinite trace semantics of a TF-coalgebra (X, h) as the unique coalgebra homomorphism from $(\Delta X, \Delta^{\blacktriangleright} h)$ to the final $\overline{\underline{F}\blacktriangleright}$-coalgebra. By definition unfolding, this morphism is the unique arrow $!_{\Delta^{\blacktriangleright} h}$ that makes the following diagram in $\mathbf{Sh_C}(\alpha)$ commute:

$$
\begin{array}{ccccc}
\Delta X & \xrightarrow{\Delta^{\blacktriangleright} h} & \underline{T}\,\underline{F}\Delta X & \xrightarrow{\underline{T}\,\underline{F} next_{\Delta X}} & \underline{T}\,\underline{F}\blacktriangleright\Delta X \\
 & & & & \big\downarrow{\scriptstyle \underline{T}\,\underline{F}\blacktriangleright !_{\Delta^{\blacktriangleright} h}} \\
 & & & & \underline{T}\,\underline{F}\blacktriangleright\underline{T}\,\mathrm{fin}(F) \\
 & & & & \big\downarrow{\scriptstyle \underline{T}\,\underline{F}\theta_{\mathrm{fin}(F)}} \\
{\scriptstyle !_{\Delta^{\blacktriangleright} h}}\big\downarrow & & & & \underline{T}\,\underline{F}\,\underline{T}\blacktriangleright\mathrm{fin}(F) \\
 & & & & \big\downarrow{\scriptstyle \underline{T}\lambda_{\blacktriangleright\,\mathrm{fin}(F)}} \\
 & & & & \underline{T}\,\underline{T}\,\underline{F}\blacktriangleright\mathrm{fin}(F) \\
 & & & & \big\downarrow{\scriptstyle \underline{\mu}_{\underline{F}\blacktriangleright\,\mathrm{fin}(F)}} \\
\underline{T}\,\mathrm{fin}(F) & = & \underset{\underline{T}\,\nu\,\underline{F}\blacktriangleright}{=\!=\!=} & = & \underline{T}\,\underline{F}\blacktriangleright\mathrm{fin}(F)
\end{array}
$$

Let T be an affine monad, F an endofunctor whose final sequence is stable at ω, and λ a distributive law for them. Given a TF-coalgebra (X, h), $!_{\Delta^{\blacktriangleright} h}$ is the unique morphism of sheaves such that:

$$
!_{\Delta^{\blacktriangleright} h,1} = TF!_X \circ h \qquad !_{\Delta^{\blacktriangleright} h,n} = \mu_{F^n} \circ T\lambda_{F^{n-1}} \circ F!_{\Delta^{\blacktriangleright} h,n-1} \circ h
$$

where $1 < n < \omega$. With reference to our initial example on non-deterministic labelled transition systems, when (X, h) is a $\mathcal{P}^+(A \times Id)$-coalgebra, $!_{\Delta^{\blacktriangleright} h}$ is the unique morphism of sheaves such that:

$$!_{\Delta^\blacktriangleright h,1} = \mathcal{P}^+(A\times !_X) \circ h \qquad !_{\Delta^\blacktriangleright h,n} = \mu_{A^n} \circ \mathcal{P}^+\lambda_{A^{n-1}} \circ \mathcal{P}^+!_{\Delta^\blacktriangleright h,n-1} \circ h$$

where $1 < n < \omega$. The first equation corresponds to the double implication

$$a \in !_{\Delta^\blacktriangleright h,1}(x) \iff \exists x' \in X \text{ s.t. } (a, x') \in h(x)$$

whereas the second to:

$$(a_1, a_2, \ldots, a_{n+1}) \in !_{\Delta^\blacktriangleright h,n+1}(x) \iff \exists x' \in X \text{ s.t. } (a_1, x') \in h(x) \\ \wedge (a_2, \ldots, a_n) \in !_{\Delta^\blacktriangleright h,n}(x').$$

In other words, a state x is assigned by $!_{\Delta^\blacktriangleright h,n}$ to the set of its (partial) traces of length n. Restriction from stage $n+1$ to n corresponds to the implication:

$$(a_1, \ldots, a_n, a_{n+1}) \in !_{\Delta^\blacktriangleright h,n+1}(x) \implies (a_1, \ldots, a_n) \in !_{\Delta^\blacktriangleright h,n}(x)$$

and amalgamation to the double implication:

$$(a_1, a_2, \ldots) \in !_{\Delta^\blacktriangleright h,\omega}(x) \iff \forall n < \omega (a_1, a_2, \ldots, a_n) \in !_{\Delta^\blacktriangleright h,n}(x)$$

In other words, $!_{\Delta^\blacktriangleright h,\omega}$ captures infinite trace semantics.

3.5 CONCLUDING REMARKS

In this chapter we presented a general coalgebraic account of infinite trace semantics covering several systems such as non-deterministic, discrete and continuous probabilistic labelled transition systems. Many authors and works have investigated infinite trace semantics under the lens the theory of coalgebras; we mention [63, 64, 70, 80, 145, 146] and [43] which is perhaps the closest to ours. The main improvements with respect to related works introduced in this thesis are summarised below:

- infinite trace semantics coincides with final semantics in a suitable category of coalgebras;
- monads modelling the branching type considered need not to induce enriched Kleisli categories;
- the final sequence for the functor modelling linear behaviours can stabilise after ω.

The first contribution is of relevance for the remaining of this thesis since the definition of self-referential endofunctors is based on final coalgebras. Therefore, the construction proposed in this chapter enables the modelling of self-referential behaviours based on the notion of (infinite) trace semantics. In retrospective, the main motivation behind this chapter is that related works capture infinite trace semantics by means of certain maps into weakly final coalgebras [43, 64, 145, 146].

As a further contribution, we proved in Section 3.4.2 that final semantics for guarded coalgebras provides a conservative generalisations of certain behavioural pseudo-metrics due to Barr and Adámek [3, 18]. In comparison to the rich theory of behavioural metrics [14, 15, 37–39] this result is preliminary, only simple behavioural metrics were considered, nonetheless it calls for future investigations.

The only non-trivial piece of information required in order to apply our construction are Kleisli liftings for guarded pointwise extensions of behavioural endofunctors. A strategy for obtaining this data is to extend Kleisli liftings for "unguarded" behavioural functors (in practice, to start from $\lambda\colon FT \to TF$) and then compose them with liftings for $\blacktriangleright$. Although this path is not always available, we identified mild assumptions that are sufficient for the strategy to succeed: if the construction is done in the context of sheaves on $\mathcal{A}(\omega)$, then existence of these liftings is equivalent to affiness of T. Affine monad where considered also in [43] where Cîrstea identified in this property a sufficient condition for her construction of canonical maps to the weakly final coalgebra of infinite traces. We remark that our is an equivalence result and holds whenever the final sequence for F stabilises at ω *i.e.* the same assumption made in *loc. cit.* Affiness and stabilisation at ω are met by monads and functors used in the modelling of several systems of interest, especially those considered in [43, 64, 145, 146]: non-deterministic, discrete and continuous probabilistic labelled transition systems.

There are other approaches to coalgebraic trace semantics besides the use of Kleisli liftings. In particular we mention *forgetful logics* [84] and *coalgebraic determinisation* [31, 33, 74, 131]. The main reason behind our choice to base this work on Kleisli liftings is that these are compatible with the notion self-referential behaviours and the constructions we introduce in the other chapters. Nonetheless, the investigation of alternatives to our construction is of interest, especially in the wake of the remarkable results achieved thanks to coalgebraic determinisation [29, 30, 52].

4 SELF-REFERENTIAL ENDOFUNCTORS

In this chapter we consider generic characterisations of behaviour types that are parameterised by objects representing input and output values. Formally, these are functors from the category of values into that of behavioural endofunctors (*e.g.* functors of type $\mathbf{C}^{op} \times \mathbf{C} \to \mathbf{End}(\mathbf{C})$). We refer to these functors as *behavioural schemata*. Examples of these parametrically defined behaviours are stream systems, labelled transition systems, automata, and any other class of computational devices with inputs or outputs. In the self-referential case, input and output objects are equipped with dynamics of the same type and, to an external observer, they are indistinguishable whenever they exhibit the same behaviour. We are interested in the coalgebraic modelling of these systems. As we discussed in Chapter 1, we achieve this result by means of *self-referential endofunctors i.e.* behavioural endofunctors that are instances of behavioural schemata determined by their final coalgebras. This circularity is the gist of self-referential systems and renders the construction of these models non-trivial.

The chapter is organised as follows. In Section 4.1 we consider systems that can only perform outputs and abstract behavioural endofunctors modelling them by means of covariant behavioural schemata. This initial simplification allows us to focus on the challenge of modelling self-referential behaviours without the additional issues due to contravariant occurrences of systems as values *i.e.* inputs. Systems with inputs are covered in Section 4.2 where we consider mixed-variance behavioural schemata. Finally, in Section 4.3 we study conservative generalisations of self-referential endofunctors that support situations where values and systems need to be modelled in distinct categories.

4.1 COVARIANT BEHAVIOURAL SCHEMATA

In this section we consider covariant behavioural schemata *i.e.* functors of type:

$$\mathbf{C} \to \mathbf{End}(\mathbf{C}).$$

These functors provide an abstract definition of behaviours generic in their outputs such as stream systems which we covered in Chapter 1. Self-referential endofunctors for covariant behavioural schemata are instances defined by their own final coalgebra. Formally:

DEFINITION 4.1. *For a covariant behavioural schema* $S\colon \mathbf{C} \to \mathbf{End}(\mathbf{C})$*, an endofunctor* F *over* **C** *is called* self-referential *whenever:*

$$\begin{cases} F \cong S_Z \\ Z \cong |\,\nu\, F| \end{cases} \tag{4.1}$$

Self-referential endofunctors for a behavioural schema S on **C** form a subcategory of $\mathbf{End}(\mathbf{C})$ where morphisms are natural transformations that are also coherent with final coalgebras and the schema S. Formally, a natural transformation $\phi\colon F \to G$ is a morphism of solutions to (4.1) provided that there is an isomorphism $\phi \cong S_f$ in the arrow category $\mathbf{End}(\mathbf{C})^{\to}$ where f denotes the final semantics for the G-coalgebra $\phi_{|\,\nu\, F|} \circ \nu\, F$ depicted below:

$$\begin{array}{ccc} |\,\nu\, F| & \overset{f}{\dashrightarrow} & |\,\nu\, G| \\ {\scriptstyle \nu\, F}\downarrow & & \\ F|\,\nu\, F| & & \downarrow{\scriptstyle \nu\, G} \\ {\scriptstyle \phi_{|\,\nu\, F|}}\downarrow & & \\ G|\,\nu\, F| & \xrightarrow[G(f)]{} & G|\,\nu\, G| \end{array} \tag{4.2}$$

This notion of morphisms between solutions is motivated by the fact that the equation system (4.1) admits equivalent formulations that use either the unknown Z or F. Reworded, self-referential endofunctors are identified by their final coalgebra and *vice versa*.

Like all natural transformations, solution morphisms induce functors between categories of coalgebras: for $\phi\colon F \to G$ a solution morphism, the functor $\mathbf{Coalg}(\phi)\colon \mathbf{Coalg}(F) \to \mathbf{Coalg}(G)$ assigns every F-coalgebra (X, h) to the G-coalgebra $(X, \phi_X \circ h)$ and acts as the identity on homomorphisms:

$$(X, h) \mapsto (X, \phi_X \circ h) \qquad f \mapsto f.$$

This functor is a lifting of the identity on the underlying **C** along the forgetful functors:

$$\begin{array}{ccc} \mathbf{Coalg}(F) & \xrightarrow{\mathbf{Coalg}(\phi)} & \mathbf{Coalg}(G) \\ \downarrow & & \downarrow \\ \mathbf{C} & = & \mathbf{C} \end{array}$$

The functor, $\mathbf{Coalg}(\phi)$ preserves behavioural equivalences and reflects them whenever components of ϕ are monomorphisms in **C**. As a consequence,

- any solution morphism $\phi\colon F \to G$ exhibits F as *sound* with respect to G and, *vice versa* G as *complete* with respect to F;
- any component-wise solution morphism $\phi\colon F \to G$ exhibits F as *adequate* with respect to G.

These notions extend from solutions to their associated models of self-referential systems: we say that models of type F are sound with respect to models of type G whenever there is a morphism of solutions $\phi\colon F \to G$ and that are adequate whenever ϕ is also componentwise monic. Complete models are defined likewise. Solution morphisms characterise two distinguished kinds of solutions and their associated models: final and initial ones.

- Final solutions identify model types that are complete with respect to any other model type.
- Dually, initial solutions identify model types that are sound with respect to any other model type.

We call solutions and models *canonical* whenever they are initial or final. Depending on the application, canonical solutions support reasoning about self-referential systems (for the same schema) even if they are described by semantic models of different type. Solution morphism into the final solution induce functors into a shared category of coalgebras such that they preserve behavioural equivalences. Dually, solution morphisms from the initial solution induce functors from a shared category of coalgebras and such that they preserve behavioural equivalences.

4.1.1 Self-referential endofunctors as invariants

Solutions to recursive equations like those considered in this section can be understood as fixed points of certain endofunctors derived from the equation under scrutiny. In the case of (4.1) the first step to reformulate the system of equations in one unknown and as one clause *i.e.* either as:

$$F \cong S_{|\nu F|} \tag{4.3}$$

or as:

$$Z \cong |\nu S_Z|. \tag{4.4}$$

Both formulations are very close to describing an endofunctor over $\mathbf{End}(\mathbf{C})$ (taking F to $S_{|\nu F|}$) or one over $\mathbf{C}$ (taking Z to $|\nu S_Z|$), except for two main issues that prevent these to be actual endofunctors: the first is due to the assumption that enough final coalgebras exist and the second is due to the presence of isomorphisms in (4.1). To this end, we need to be able to choose final coalgebras for all endofunctors described by the given schema. We remark that all choices are equivalent since final coalgebras are defined only up to isomorphism.

LEMMA 4.1. *Let* **E** *be a subcategory of* **End**(**C**) *and assume that* **C** *is coalgebraically cocomplete with respect to* **E**. *Any family of assignments* $\{F \mapsto |\,\nu\, F|\}_{F \in \mathbf{E}}$ *induces a functor* $N\colon \mathbf{E} \to \mathbf{C}$ *coherent with the assignments.*

PROOF. On objects, the functor is defined by the given assignments. Each transformation $\phi \in \mathbf{End}(\mathbf{C})(F,G)$ defines a G-coalgebra on the carrier of the final F-coalgebra:

$$|\,\nu\, F| \xrightarrow{\nu F} F|\,\nu\, F| \xrightarrow{\phi_{|\,\nu\, F|}} G|\,\nu\, F|$$

The associated coinductive extension into the final G-coalgebra (which exists by coalgebraic cocompleteness) defines the action $N(\phi)$. It follows from basic properties of natural transformations that the assignment is functorial. □

For the sake of exposition, let us assume to be given an assignment for final coalgebras and thus fix $N\colon \mathbf{E} \to \mathbf{C}$. We remark that this mild assumption can be avoided by carrying out all constructions in the (more technically involved) setting of anafunctors [103, 114, 119].

For a covariant behavioural schema $S\colon \mathbf{C} \to \mathbf{End}(\mathbf{C})$, assume **C** is coalgebraically cocomplete with respect to the essential image $\mathrm{eimg}(S)$, fix a choice of coalgebras $N\colon \mathrm{eimg}(S) \to \mathbf{C}$ and replace S with its restriction to $\mathrm{eimg}(S)$ (*i.e.* the functor $S'\colon \mathbf{C} \to \mathrm{eimg}(S)$ such that composition with the inclusion $\mathrm{eimg}(S) \hookrightarrow \mathbf{End}(\mathbf{C})$ yields S). For the sake of exposition, we will refer to S as a functor into **End**(**C**) and hence regard N as if it were defined on **End**(**C**) while baring in mind that we actually mean their restrictions to $\mathrm{eimg}(S)$. The similarity between the definition of N and that of solution morphisms is not by chance: self-referential endofunctors for S are precisely fixed points of the endofunctor

$$\mathbf{End}(\mathbf{C}) \xrightarrow{N} \mathbf{C} \xrightarrow{S} \mathbf{End}(\mathbf{C}).$$

In fact, under the mild assumption of having chosen final coalgebras, equation (4.3), the reformulation of (4.1) in the unknown F, is precisely:

$$F \cong SN(F).$$

Solution morphisms are homomorphisms between SN-coalgebras that are also SN-invariants *i.e.* solutions to (4.1). From this perspective, initial and final solutions are initial and final SN-invariants, respectively *i.e.* initial SN-algebras and final NS-coalgebras. Symmetrically, the endofunctor NS over **C** corresponds to (4.4), the reformulation of (4.1) in the unknown Z that represents the instantiation parameter of the schema S. The two formulations lead to the same canonical solutions: initial (resp. final) SN-invariants induce initial (resp. final) NS-invariants and *vice versa*. We formalise this correspondence in terms of (co)algebraic (co)completeness.

LEMMA 4.2. *For* $G\colon \mathbf{C} \to \mathbf{D}$ *and* $H\colon \mathbf{D} \to \mathbf{C}$*, the following statements are true:*

- *the endofunctor* GH*, is coalgebraically cocomplete if and only if so is* HG*;*
- *the endofunctor* GH*, is algebraically complete if and only if so is* HG*.*

PROOF. Assume HG admits a final coalgebra $h\colon X \to HGX$. The image through G of h provides us with an invariant for GH, namely $G(h)\colon GX \to GHG(X)$. First we prove that the GH-coalgebra $G(h)$ is weakly final. Let $g\colon Y \to GH(Y)$ be any GH-coalgebra and consider its image $H(g)\colon HY \to HGH(Y)$. Write $f\colon HY \to X$ for the coinductive extension of $H(g)$ into the final HG-coalgebra h. The morphism $G(f) \circ g\colon Y \to G(X)$ in $\mathbf{D}$ induces a GH-coalgebra homomorphism from g into $G(h)$. This proves the claim that $G(h)$ is a weakly final GH-coalgebra. Secondly, we prove that homomorphisms into $G(h)$ are indeed unique hence final semantics of their source coalgebras. It follows from finality of $h\colon X \to HG(X)$ that the image through H of any GH-coalgebra homomorphism into $G(h)$ factors through h. In particular, for any $f'\colon Y \to G(X)$ that induces a GH-coalgebra homomorphism for g into $G(h)$, we have that $H(f') = h \circ f$. As a consequence:

$$GH(f') = G(h \circ f) = G(HG(f) \circ H(g)) = G(f \circ H(g)).$$

Since f' and $G(f) \circ g$ are GH-coalgebra homomorphisms, the:

$$G(f \circ h) \circ g = GH(G(f) \circ g) = GH(f') \circ g = G(h) \circ f'$$

We conclude that $G(h) \circ G(f) \circ g$ is unique up to the isomorphism $G(h)$ *i.e.* the chosen final GH-coalgebra. The second statement follows by duality. □

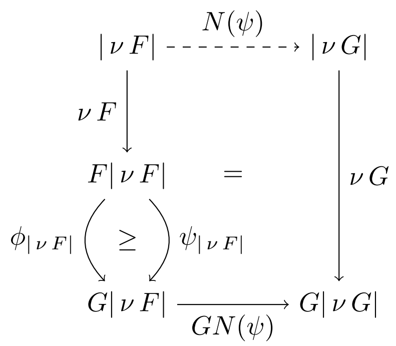
Proving the existence of initial (resp. final) solutions corresponds to proving algebraic completeness (resp. coalgebraic cocompleteness) of the endofunctors $SN\colon \mathbf{End}(\mathbf{C}) \to \mathbf{End}(\mathbf{C})$ and $NS\colon \mathbf{C} \to \mathbf{C}$ induced by (4.1). This task is not trivial and can benefit from the ability to reduce the issue to checking certain properties of S and N separately, especially since N is fixed. In the remaining of this section, we assume $\mathbf{C}$ is either $\mathbf{Cpo}$-(co)algebraically (co)complete or contractively compact[1] and demonstrate that existence of initial and final solutions follows from local continuity or local contractiveness of the behavioural schema S.

[1]Recall from Section 2.2.2 that coalgebraic cocompleteness, algebraic completeness, and compactness are all equivalent for locally contractive endofunctors since, as stated in Lemma 2.12, these endofunctors admit at most one invariant object (up to isomorphism).

In order to present the system (4.1) as a suitably enriched endofunctors akin to SN and NS above, we need to demonstrate that any choice for final coalgebras induce enriched functors akin to Lemma 4.1.

PROPOSITION 4.3. *Assume* **C** *that is* **Cpo**-*coalgebraically cocomplete. Any family of assignments* $\{F \mapsto |\,\nu\, F|\}_{F \in \mathbf{Cpo\text{-}End}(\mathbf{C})}$ *extends to a functor* $N\colon \mathbf{Cpo\text{-}End}(\mathbf{C}) \to \mathbf{C}$ *enriched over* **Cpo**.

PROOF. It follows from Lemma 4.1 that the provided family of assignments induce a functor N between the categories underlying $\mathbf{Cpo\text{-}End}(\mathbf{C})$ and $\mathbf{C}$. Therefore, to prove the thesis it suffices to show that N is locally monotonous and continuous. Let $\phi \geq \psi$ be a 2-cell in $\mathbf{Cpo\text{-}End}(\mathbf{C})(F, G)$. The morphism $N(\psi)\colon |\,\nu\, F| \to |\,\nu\, G|$ carries a lax-homomorphism from $\left(\phi_{|\,\nu\, F|} \circ \nu\, F\right)$ to the (chosen) final G-coalgebra:



Consider the continuous endomorphism Ω over $\mathbf{C}(|\,\nu\, F|, |\,\nu\, G|)$ defined as follows:

$$\Omega(f) = (\nu\, G)^{-1} \circ G(f) \circ \phi_{|\,\nu\, F|} \circ \nu\, F.$$

The sequence $\left(\Omega^{i+1}(N(\psi))\right)_{i<\omega}$ is an ω-chain and its supremum is the unique G-coalgebra homomorphism from $\left(\phi_{|\,\nu\, F|} \circ \nu\, F\right)$ into $\nu\, G$ since:

$$\begin{aligned}
\bigsqcup_{i<\omega} \Omega^{i+1}(N(\psi)) &= \bigsqcup_{i<\omega} \left((\nu\, G)^{-1} \circ G\left(\Omega^{i}(N(\psi))\right) \circ \left(\phi_{|\,\nu\, F|} \circ \nu\, F\right)\right)\\
&= (\nu\, G)^{-1} \circ \bigsqcup_{i<\omega} G\left(\Omega^{i}(N(\psi))\right) \circ \left(\phi_{|\,\nu\, F|} \circ \nu\, F\right)\\
&= (\nu\, G)^{-1} \circ G\Big(\bigsqcup_{i<\omega} \Omega^{i}(N(\psi))\Big) \circ \left(\phi_{|\,\nu\, F|} \circ \nu\, F\right).
\end{aligned}$$

We conclude that the functor N is locally monotonous since:

$$N(\phi) = \bigsqcup_{i<\omega} \Omega^{i+1}(N(\psi)) \geq N(\psi).$$

Let $(\phi_i)_{i<\omega}$ be an ω-chain in **Cpo-End**(**C**)(F, G) and consider the ω-chain $(N(\phi_i))_{i<\omega}$. The supremum of the latter is the unique G-coalgebra homomorphism from $\left(\bigsqcup_{i<\omega} \phi_{i,|\nu F|} \circ \nu F\right)$ into νG since:

$$\begin{aligned}\bigsqcup_{i<\omega} N(\phi_i) &= \bigsqcup_{i<\omega} \left((\nu G)^{-1} \circ GN(\phi_i) \circ \left(\phi_{i,|\nu F|} \circ \nu F\right)\right) \\ &= (\nu G)^{-1} \circ \bigsqcup_{i<\omega} GN(\phi_i) \circ \left(\bigsqcup_{i<\omega} \phi_{i,|\nu F|} \circ \nu F\right) \\ &= (\nu G)^{-1} \circ G\left(\bigsqcup_{i<\omega} N(\phi_i)\right) \circ \left(\bigsqcup_{i<\omega} \phi_{i,|\nu F|} \circ \nu F\right).\end{aligned}$$

We conclude that the functor N is locally continuous since:

$$\bigsqcup_{i<\omega} N(\phi_i) = N\left(\bigsqcup_{i<\omega} \phi_i\right).$$

□

PROPOSITION 4.4. *Assume* **C** *that is contractively complete and enriched over sheaves of sets on a complete Heyting algebra A. Let* **E** *be a* **Sh**(A)*-enriched category of locally contractive endofunctors over* **C**. *Any family of assignments* $\{F \mapsto |\nu F|\}_{F \in \mathbf{E}}$ *extends to a functor* $N\colon \mathbf{E} \to \mathbf{C}$ *enriched over* **Sh**(A).

PROOF. For F an object of **E**, define $N(F)$ as the object $|\nu F|$ associated to F by the provided assignments. For F and G objects from **E**, define $N_{F,G}\colon \mathbf{E}(F, G) \to \mathbf{C}(|\nu F|, |\nu G|)$ as the unique fixed point of the morphism

$$\Omega\colon \mathbf{E}(F, G) \times \mathbf{C}(|\nu F|, |\nu G|) \to \mathbf{C}(|\nu F|, |\nu G|)$$

in **Sh**(A) defined on each $\phi \in \mathbf{E}(F, G)$ and $f \in \mathbf{C}(|\nu F|, |\nu G|)$ as follows:

$$\Omega(\phi, f) = (\nu G)^{-1} \circ G_{|\nu F|,|\nu G|}(f) \circ \phi_{|\nu F|} \circ \nu F.$$

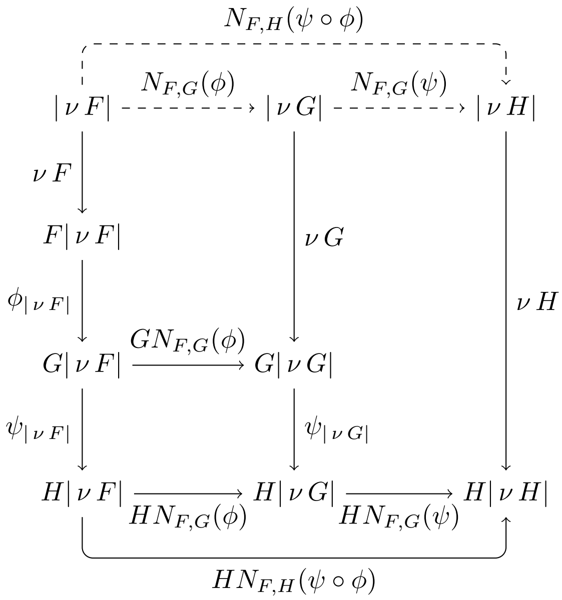
In order to prove that $N_{F,G}\colon \mathbf{E}(F, G) \to \mathbf{C}(|\nu F|, |\nu G|)$ is well-defined and that it describes the desired action of N we need to prove that said fixed-point uniquely exists and that it associates each G-coalgebra $\phi\colon F \to G$ with the coinductive extension of $\phi_{|\nu F|} \circ \nu F$. Recall from [26, 55] that a morphism $f\colon Y \times X \to X$ has a unique fixed point $\text{fix}(f)\colon Y \to X$ provided it is contractive in the component X *i.e.* it factors through $(id \times next_X)$. The morphism Ω is contractive by construction since $G_{|\nu F|,|\nu G|}$ factors through $next_{\mathbf{C}(|\nu F|,|\nu G|)}$ by hypothesis on **E**. Therefore $N_{F,G}$ is always defined and takes any $\phi_{|\nu F|} \circ \nu F$ to the coinductive extension of the G-coalgebra $\phi_{|\nu F|} \circ \nu F$:

$$N_{F,G}(\phi) = (\nu G)^{-1} \circ G(N_{F,G}(\phi)) \circ (\phi_{|\nu F|} \circ \nu F).$$

Functoriality conditions for N follow from basic properties of final coalgebras and natural transformations. For each F, the morphism $N_{F,F}$ takes identities to identities since the coinductive extension of νF is the identity. For F, G, and H endofunctors over **C**, the associativity condition requires that:

$$N_{F,Z} = (- \circ_{|\nu F|,|\nu G|,|\nu H|} -)(N_{G,H} \times N_{F,G})$$

and this is indeed true since the diagram below commutes for any $\phi\colon F \to G$ and any $\psi\colon G \to H$ in **E**.



□

We are now able to state the main result of this section, namely that existence of canonical self-referential endofunctors reduces to assessing local continuity or local contractiveness of behavioural schemata.

THEOREM 4.5. *Assume that the covariant behavioural schema S is* **Cpo**-*enriched. The following statements are true:*

- *if* **C** *is* **Cpo**-*coalgebraically cocomplete, then there is a final solution to* (4.1)*;*
- *if* **C** *is* **Cpo**-*coalgebraically cocomplete and* **Cpo**-*algebraically complete, then there is an initial solution to* (4.1).

PROOF. It follows from Proposition 4.3 that there is a **Cpo**-enriched functor $N\colon$ **Cpo-End**(**C**) $\to$ **C** associating each endofunctor with the carrier of a chosen final coalgebra for it. Composition with S yields a **Cpo**-enriched functor that describes (4.1) under the choice of final coalgebras described by the functor N. We conclude from **Cpo**-coalgebraic cocompleteness of **C** that SN and NS admit final invariants. These invariants identify final solutions to (4.1). Assume that **C** is also **Cpo**-algebraically complete. It follows that SN and NS admit initial invariants. These invariants identify initial solutions to (4.1). □

THEOREM 4.6. *Let* **E** *be a sheaf enriched category of locally contractive endofunctors on a contractively compact category* **C**. *If S is locally contractive, then there is a unique (up to isomorphism) solution to* (4.1).

PROOF. Recall from Section 2.2.2 that local contractiveness (as per Definition 2.11) implies enrichment over sheaves of sets on a complete Heyting algebra. It follows from Proposition 4.4 that there is a suitably enriched functor $N\colon \mathbf{E} \to \mathbf{C}$ associating each endofunctor in **E** with the carrier of a chosen final coalgebra for it. Composition with the locally contractive functor S yields a locally contractive endofunctor by Lemma 2.11. We conclude from contractive completeness of **C** that this functor admits an invariant and from Lemma 2.12 that the invariant is unique up to isomorphism. This invariant is the requested solution. □

4.1.2 Final self-referential endofunctors via diagonalisation

Final solutions can be obtained from final coalgebras for endofunctors that are "diagonalisations" of covariant behavioural schemata akin to Bekič's Lemma [21, 22]. For S a behavioural schema, define $\widehat{S}\colon \mathbf{C} \to \mathbf{C}$ as the composite of the diagonal functor for **C** with the transpose of $S\colon \mathbf{C} \to \mathbf{End}(\mathbf{C})$. This functor is given on each object V and morphism f of **C** as:

$$\widehat{S}(V) = S_V(V) \qquad \widehat{S}(f) = S_f(f).$$

where $S_f(f)\colon S_V(V) \to S_U(U)$ is given by the naturality square:

$$S_f(f) = S_f(U) \circ S_V(f) = S_U(f) \circ S_f(V).$$

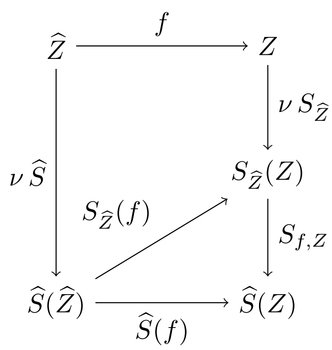
Solutions to (4.1) and their morphisms are equivalent to coalgebras of type $\widehat{S}$ and their homomorphisms. If an endofunctor F solves (4.1), then its final coalgebra is also a $\widehat{S}$-coalgebra since it holds that:

$$|\nu F| \cong S_{|\nu F|}|\nu F| = \widehat{S}|\nu F|.$$

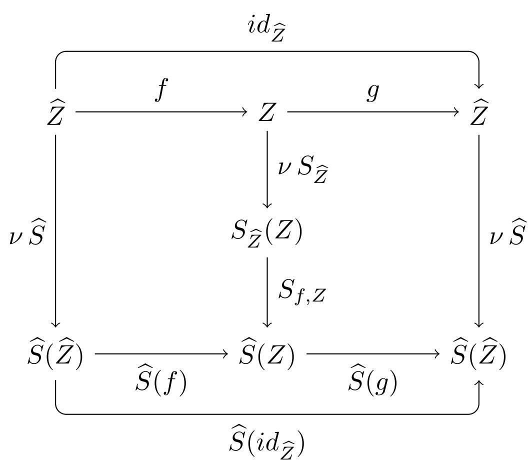
If a natural transformation $\phi\colon F \to G$ is a solution morphism, then the corresponding morphism into the final G-coalgebra $f\colon |\nu F| \to |\nu G|$ is also an homomorphism between νF and νG seen as $\widehat{S}$-coalgebras: the claim follows from (4.2) and definition of $\widehat{S}$ on morphisms.

THEOREM 4.7. *For S a covariant behavioural schema, assume that $\widehat{S}$ admits final coalgebras. If $S_{|\nu \widehat{S}|}$ admits final coalgebras, then final $\widehat{S}$-coalgebras coincide with final solutions to* (4.1).

PROOF. We restrict to endofunctors that are instance of S *i.e.* endofunctors of the form S_V for $V \in \mathbf{C}$. This restriction does not introduce any loss of generality: Definition 4.1 already requires self-referential endofunctors to be isomorphic to instances of S. Assume that there is a final $\widehat{S}$-coalgebra and write $\widehat{Z}$ for its carrier. Assume that there is a final $S_{\widehat{Z}}$-coalgebra and write Z for its carrier. We claim that $\nu\,\widehat{S} \colon \widehat{Z} \to \widehat{S}\widehat{Z}$ regarded as a $S_{\widehat{Z}}$-coalgebra $\nu\,\widehat{S} \colon \widehat{Z} \to S_{\widehat{Z}}(\widehat{Z})$ is final ($\widehat{Z} \cong |\,\nu\,S_{\widehat{Z}}|$). As a consequence of our claim, $S_{\widehat{Z}}$ (or equivalently $\widehat{Z}$) is a solution to (4.1). Regard $\nu\,\widehat{S}$ as a coalgebra for $S_{\widehat{Z}}$ and let $f \colon \widehat{Z} \to Z$ denote its coinductive extension. This morphism induce a homomorphisms of $\widehat{S}$-coalgebras from $\nu\,\widehat{S}$ into $S_{f,Z} \circ \nu\,S_{\widehat{Z}}$ as illustrated by the diagram below.

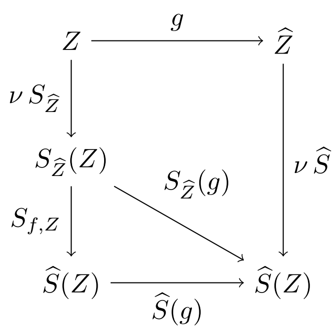


In particular, the upper part of the diagram commutes by assumption on f and the lower part by definition of $\widehat{S}$. Let $g \colon Z \to \widehat{Z}$ be the coinductive extension of $S_{f,Z} \circ \nu\,S_{\widehat{Z}}$. It follows from finality of $\nu\,\widehat{S}$ that $g \circ f = id_{\widehat{Z}}$ as shown by the commuting diagram below.



As a consequence, if $\nu\,\widehat{S}$ is regarded as a $S_{\widehat{Z}}$-coalgebra, then the arrow $g \colon Z \to \widehat{Z}$

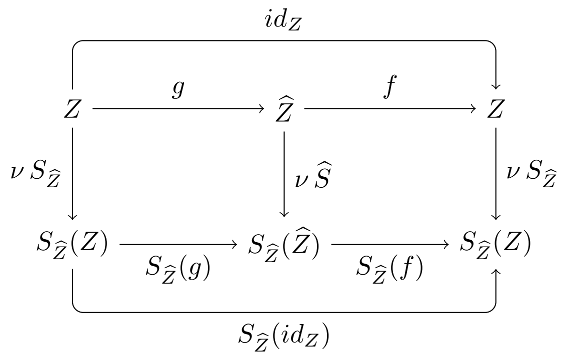
induces a $S_{\widehat{Z}}$-coalgebra homomorphism into $\nu\,\widehat{S}$. Consider the following diagram:



The diagram commutes since its contour commutes by construction of g and the lower triangle commutes by definition of $\widehat{S}$ and $g \circ f$:

$$\widehat{S}(g) \circ S_{f,Z} = S_{\widehat{Z}}(g) \circ S_{g\circ f,Z} = S_{\widehat{Z}}(g) \circ S_{id_{\widehat{Z}},Z} = S_{\widehat{Z}}(g).$$

Therefore, the arrows f and g extend to homomorphisms of $S_{\widehat{Z}}$-coalgebras that split the identity on $\nu\,S_{\widehat{Z}}$:



The pair of arrows f and g exhibit Z and $\widehat{Z}$ as isomorphic which proves our initial claim *i.e.* that final $\widehat{S}$-coalgebras are solutions. Recall that solutions and their morphisms can be equivalently regarded as $\widehat{S}$-coalgebras and their homomorphisms. We conclude that final $\widehat{S}$-coalgebras are final among all solutions. □

Note that coalgebras used to describe the dynamics associated to the parameter used for schema instantiation are always coalgebras for the diagonalised endofunctor since $\widehat{S}(V) = S_V(V)$. It follows from Theorem 4.7 that for any concrete representation of value dynamics $d\colon V \to S_V(V)$ there is a unique structure preserving morphism into the dynamics associated to the final self-referential endofunctor *i.e.* the $\widehat{S}$-coalgebra homomorphism into the final $\widehat{S}$-coalgebra. This morphism induce a functor between categories of coalgebras

$\mathbf{Coalg}(S_{!_d})\colon \mathbf{Coalg}(S_V) \to \mathbf{Coalg}(S_{|\nu S_V|})$ that maps concrete representations based on d into semantic models for the final self-referential endofunctor by replacing values with their semantics.

COROLLARY 4.8. *There is functorial assignment associating concrete representations of self-referential systems with their canonical semantic models.*

As a consequence of Theorem 4.7, the problem of obtaining final self-referential endofunctors can be reduced to the existence and computation of certain final coalgebras. This is of relevance because it allows the application of several powerful results about final coalgebras. We mention Barr [18], Smyth and Plotkin [133], Adámek [4–7], Worrell [148–150], and Bacci [12, 13].

EXAMPLES OF FINAL SOLUTIONS

Self-referential stream systems Recall from Section 1.3 that stream systems are characterised by the behavioural schemata

$$S(A) = A \times Id$$

where A is the set of output symbols. Models of self-referential stream systems are coalgebra whose behavioural endofunctor satisfies (1.3) and this equation is precisely what we get by instantiating Definition 4.1 on the schema of stream systems. In Chapter 1 we claimed that the initial and final solutions are identified by the alphabets 0 and 1, these corresponds to the constant functor $S_0 = 0$ and the identity functor $S_1 = Id_{\mathbf{Set}}$. There is a unique morphism between these two solutions rendering S_1 final. As a sanity check, let us apply Theorem 4.7 to this simple example. The first step is to check whether there is a final coalgebra for

$$\widehat{S} = Id \times Id.$$

The pair $(1, id_1)$ is said coalgebra. The second is to instantiate the schema on the carrier of this coalgebra and check that the resulting behavioural endofunctor admits a final coalgebra: the pair $(1, id_1)$ is a final coalgebra for $S_1 = Id$, trivially. Then, it follows from Theorem 4.7 that $S_1 = Id$ is final among all solutions. Interestingly, the initial $\widehat{S}$-algebra coincides with the initial solution S_0. We remark that this correspondence does not hold for general schemata.

REMARK 4.1. The approach behind Theorem 4.7 cannot be adapted to the problem of finding initial solutions because of mixed occurrences of initial and final invariants in the resulting definition. In fact, the dual of Theorem 4.7 characterises initial solutions to the equation system dual of (4.1), namely:

$$\begin{cases} F \cong S_A \\ A \cong |\,\mu F| \end{cases}$$

Self-referential partial stream systems As discussed in Section 1.2, self-referential stream systems admit only degenerate canonical models. In order to apply Theorem 4.7 to a more interesting example let us consider partial stream systems. Recall from Section 1.3 that these systems are characterised by the behavioural schema

$$S(A) = A \times Id + A$$

where A is the set of output symbols. Consider the endofunctor

$$\widehat{S} = Id \times Id + Id$$

diagonalisation of S. This endofunctor admits a final coalgebra and its carrier is the set T of all infinite trees whose nodes have either one or two children. Note that the set T is exactly the alphabet described in the introduction as a solution to (1.2). The functor S_T admits a final coalgebra since it is finitary [150]. It follows from Theorem 4.7 that S_T is final among all types of models for self-referential partial stream systems. Partial stream systems are another example of schemata with the property that initial algebras for its diagonalisation determine initial model types. In fact, the initial solution is $S_0 = 0$ and the initial $\widehat{S}$-algebra is the pair $(0, id_0)$. The cause of this fact is not the degeneracy of this solution but the constant nature of S_0: any category is algebraically compact with respect to the class of constant endofunctors, trivially.

Self-referential non-deterministic systems Non-deterministic labelled transition systems are characterised by the behavioural schema

$$S(A) = \mathcal{P}_f(A \times Id)$$

where the parameter A is the set of labels. Final coalgebras for S_A determine the set of all finitely-branching possibly infinite trees with labels in A modulo children ordering. In the self-referential case, the set of labels carries its own transition system. Intuitively, semantic models of self-referential LTSs use as labels trees labelled with the same type of trees. In order to produce a concrete instance of these set of recursive labels we apply Theorem 4.7 and construct the final type of semantic models for these systems. The diagonalisation of S is

$$\widehat{S} = \mathcal{P}_f(Id \times Id)$$

which admits final coalgebras since it is finitary. Up to isomorphism, these are carried by the set T of all possibly infinite trees that alternate the following two types of branching: nodes at even depth (the root has depth 0) have an arbitrary but finite set of children whereas nodes at odd depth have exactly two children. Since S_T admits final coalgebras, we conclude that S_T is the final self-referential endofunctor for the given schema. The final S_T-coalgebra is carried by T and coincide with final $\widehat{S}$-coalgebras as per Theorem 4.7. In particular, arbitrary and binary branches correspond to the non-deterministic and the output part of the self-referential behaviour, respectively. The latter can be thought of as forks and provide the characteristic covariant self-referentiality. Indeed, a tree whose branches are labelled with trees of the same type is essentially a tree whose branches end in two children: the label and the actual child.

4.1.3 Self-referential endofunctors via observations refinement

In the first part of this section we presented general results about existence and computation of canonical self-referential endofunctors. In particular, we proved that, under mild conditions, any concrete representation of self-referential systems admits a canonical semantic model whose type is the final self-referential endofunctor. It may be the case that one is interested in models whose type is not final or that such type does not exist at all. To this end, we here present an alternative approach based on observations and dynamics refinement. Intuitively, the idea is to regard value dynamics as ordinary coalgebras and use the associated semantics to derive new ones. This step produces values, dynamics, and observations that are complete with respect to the original ones (when regarded as ordinary coalgebras) and sound with respect to self-referential ones. The procedure reaches a fixed point as soon as it encounters a solution to (4.1).

From the perspective of coalgebras, sequences of observations refinements are certain functors from the category of ordinals into that of $\widehat{S}$-coalgebras.

DEFINITION 4.2. *For S a covariant schema, the* observations refinement sequence *for $(V, d\colon V \to \widehat{S}(V))$ is the functor $\mathrm{ref}(d)\colon$ **Ord** $\to$ **Coalg**$(\widehat{S})$ such that:*

- $\mathrm{ref}(d)(0) = (V, d)$*;*
- $\mathrm{ref}(d)(\beta+1) = \big(|\,\nu\, S_{V_\beta}|, S_{!_{d_\beta}, V_{\beta+1}} \circ \nu\, S_{V_\beta}\big)$ *where* $(V_\beta, d_\beta) = \mathrm{ref}(d)(\beta)$*;*
- $\mathrm{ref}(d)(\iota_{\beta,\beta+1}) = {!_{d_\beta}}$.
- $\mathrm{ref}(d)(\varinjlim_\gamma \beta) \cong \varinjlim_\gamma \mathrm{ref}(d)(\beta)$ *for any transfinite limit ordinal γ;*

The observations refinement sequence for d is said to stabilise at some ordinal α whenever $\mathrm{ref}(d)(\iota_{\alpha,\alpha+1})$ is an isomorphism.

PROPOSITION 4.9. *If $\mathrm{ref}(d)$ stabilises at α, then $S_{|\,\mathrm{ref}(d)(\alpha)|}$ is a self-referential endofunctor.*

PROOF. Write (V_α, d_α) for $\mathrm{ref}(d)(\alpha)$ and $(V_{\alpha+1}, d_{\alpha+1})$ for $\mathrm{ref}(d)(\alpha+1)$. Recall from above that $V_{\alpha+1}$ carries a final S_{V_α}-coalgebra by construction. We conclude that the morphism in **C** that carries the isomorphism of $\widehat{S}$-coalgebras

$\mathrm{ref}(d)(\iota_{\alpha,\alpha+1})\colon d_\alpha \to d_{\alpha+1}$ is an isomorphisms $V_\alpha \cong |S_{V_\alpha}|$ and exhibits S_{V_α} as a self-referential endofunctor. $\square$

Observations refinement sequences induce functors assigning concrete representations of self-referential systems to models. Assume the observations refinement sequence for $\mathrm{ref}(d)$ stabilises at some ordinal α and write Z for the carrier of $\mathrm{ref}(d)(\alpha)$. Note that once the representation of value dynamics (V, d) is fixed, concrete representations of self-referential systems operating on (V, d) coincide with S_V-coalgebras. Then, semantic models for these systems are S_Z-coalgebras. The morphism underlying $\mathrm{ref}(d)(\iota_{0,\alpha})\colon d \to d_\alpha$ identifies the functor

$$\mathbf{Coalg}\big(S_{\mathrm{ref}(d)(\iota_{0,\alpha})}\big)\colon \mathbf{Coalg}(S_V) \to \mathbf{Coalg}(S_Z)$$

that associates concrete representations with semantic models such that final semantics capture self-referential semantics.

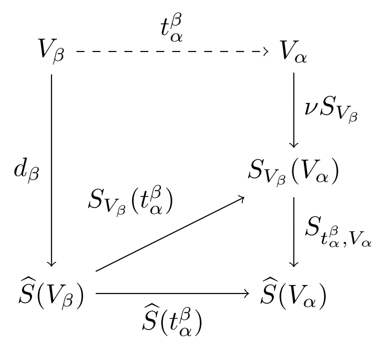
The value taken by $\mathrm{ref}(d)$ at some ordinal α is uniquely defined from values at ordinals smaller than α. As a consequence, observations refinement sequences can be constructed by transfinite induction. For every object of values V equipped with dynamics $d\colon V \to \widehat{S}(V)$, define the ordinal-indexed sequence of $\widehat{S}$-coalgebras $\big(V_\beta,\, d_\beta\colon V_\beta \to \widehat{S}(V_\beta)\big)_{\beta\in\mathbf{Ord}}$ together with homomorphisms $\big(t^\gamma_\beta\colon (V_\gamma, d_\gamma) \to (V_\beta, d_\beta)\big)_{\gamma\leq\beta}$ by transfinite induction on α as described below.

First step. Let α be 0. Define (V_0, d_0) as (V, d) and t^0_0 as id_V.

Isolated step. Let α be $\beta+1$ and assume S_{V_β} admits a final coalgebra. Define the carrier V_α as the carrier of the final S_{V_β}-coalgebra regarded as a $\widehat{S}$-coalgebra and the structure $d_\alpha\colon V_\alpha \to \widehat{S}(V_\alpha)$ as the S_{V_α}-coalgebra

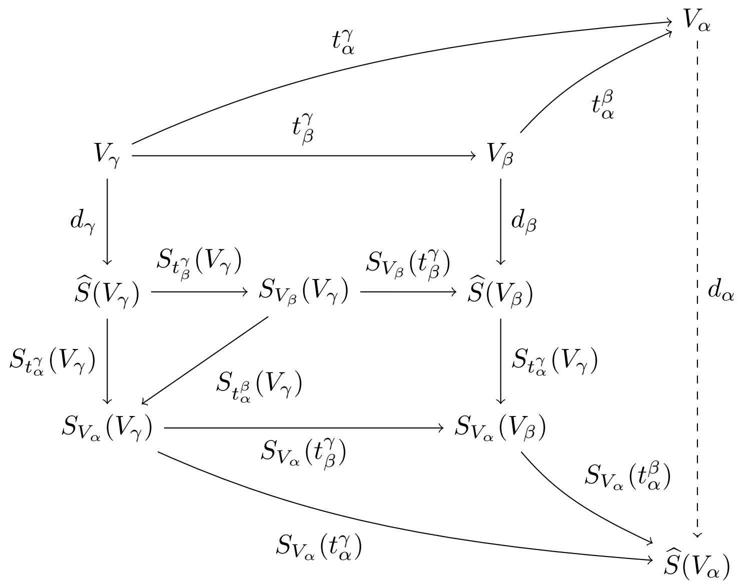
$$V_\alpha \xrightarrow{\nu\, S_{V_\beta}} S_{V_\beta}(V_\alpha) \xrightarrow{S_{t^\beta_\alpha, V_\alpha}} S_{V_\alpha}(V_\alpha)$$

regarded as a $\widehat{S}$-coalgebra. Define $t^\beta_\alpha\colon V_\beta \to V_\alpha$ as the coinductive extension of d_β and note that this morphism carries a $\widehat{S}$-coalgebra homomorphism from d_β to d_α as illustrated by the commuting diagram below.



Define t_α^α as the identity on V_α and, for any $\gamma \leq \beta$, define t_α^γ as the composite $t_\alpha^\beta \circ t_\beta^\gamma$.

Limit step. Let α be a transfinite limit ordinal. Define the carrier V_α as the colimit $\varinjlim_\alpha V_\beta$ for the diagram $(t_\beta^\gamma \colon V_\gamma \to V_\beta)_{\gamma \leq \beta \leq \alpha}$ and the structure $d_\alpha \colon V_\alpha \to \widehat{S}(V_\alpha)$ as the mediating morphism for the cocone $\big(\widehat{S}(t_\alpha^\beta) \circ d_\beta \colon V_\beta \to \widehat{S}(V_\alpha)\big)_{\beta < \alpha}$ on the α-indexed sequence $\big(t_\beta^\gamma \colon V_\gamma \to V_\beta\big)_{\gamma \leq \beta < \alpha}$. Define t_α^α as the identity on V_α and for every $\beta < \alpha$, define t_α^β as the injection of the component β into the colimit V_α. These morphisms carry the required $\widehat{S}$-coalgebra homomorphisms as illustrated by the commuting diagram below.



From the above inductive construction it is clear that observations refinement sequences as per Definition 4.2 implement the intuitive procedure outlined in the opening of this section.

Under the mild assumption of that a choice for final coalgebras can be made (*cf*. Lemma 4.1), the isolated step from the observations refinement construction induces an endofunctor over the category of $\widehat{S}$-coalgebras. As a consequence, the whole construction is functorial and observations refinement sequences are initial sequences. Before we formalise and prove this claim let us introduce some auxiliary results.

LEMMA 4.10. *Assume a choice of final coalgebras* $N \colon \operatorname{eimg}(S) \to \mathbf{C}$. *There is an endofunctor* $R \colon \mathbf{Coalg}(\widehat{S}) \to \mathbf{Coalg}(\widehat{S})$ *given on every* $\widehat{S}$*-coalgebra by the mapping:*

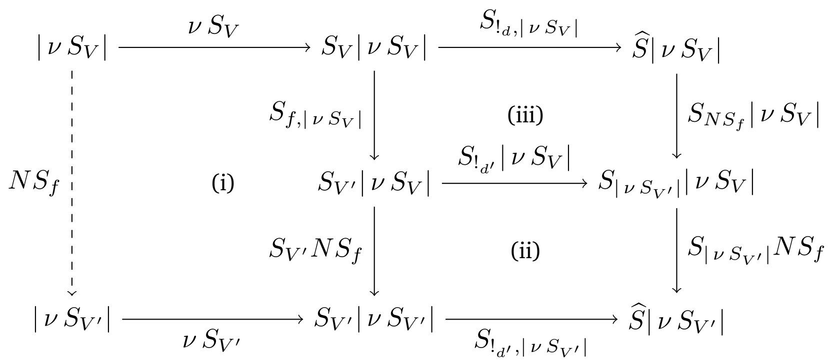
$$V \xrightarrow{d} \widehat{S}V \quad \mapsto \quad |\nu\, S_V| \xrightarrow{\nu\, S_V} S_V|\nu\, S_V| \xrightarrow{S_{!_d, |\nu\, S_V|}} \widehat{S}|\nu\, S_V|$$

and on every $\widehat{S}$-coalgebra homomorphism by the mapping:

$$d \xrightarrow{f} d' \quad \mapsto \quad \nu\, S_V \xrightarrow{NS_f} \nu\, S_{V'}.$$

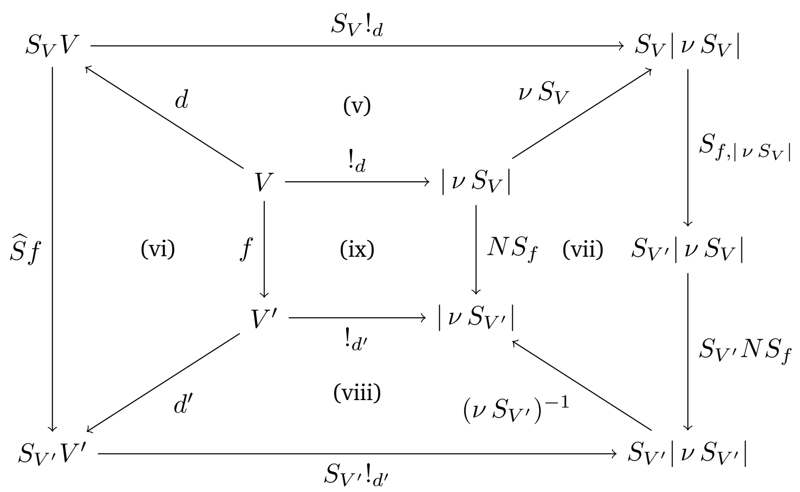
The family of morphisms $\{!_d \colon V \to |\,\nu\, S_V|\}_{d \in \mathbf{Coalg}(\widehat{S})}$ from **C** *defines a natural transformation $\rho\colon Id_{\mathbf{Coalg}(\widehat{S})} \to R$.*

PROOF. For the first part of the proof it suffices to prove that for every homomorphism $f\colon d \to d'$, NS_f carries a $\widehat{S}$-homomorphism from $R(d)$ to $R(d')$. Then, preservation of identities and associativity would follow by simple diagram chasing. To this end consider the following decomposition of the diagram asserting that the morphism $NS_f\colon |\,\nu\, S_V| \to |\,\nu\, S_{V'}|$ carries the desired homomorphism:



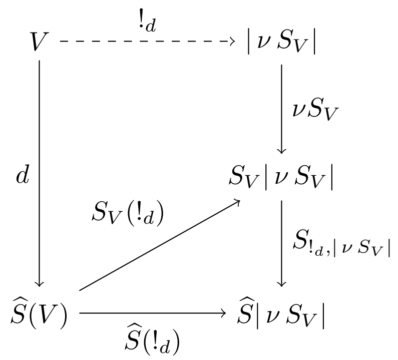
(iv)

Diagram (i) commutes by definition since NS_f is the unique homomorphism from $(S_{f,|\,\nu\, S_V|} \circ \nu\, S_V)$ into the final $S_{V'}$-coalgebra. Diagram (ii) is a naturality square for $S_{!_{d'}}\colon S_{V'} \to S_{|\,\nu\, S_{V'}|}$ and commutes by hypothesis on S. This leaves only (iii) to be checked. To this end consider the following diagram:



We claim that this diagram commutes. Diagrams (v) and (viii) are homomorphisms into final coalgebras and commute by construction. Diagram (vi) commutes since f carries an homomorphism for the $\widehat{S}$-coalgebras d and d' by hypothesis. Diagram (vii) is (i) except for the inverse of the isomorphism $\nu\, S_{V'}$. We conclude that (ix) commutes. Note that (iii) is the component at $|\,\nu\, S_V|$ of the square of natural transformations of (ix) under S and hence commutes. We conclude that (iv) commutes and R is indeed a functor.

For the second part of the proof, note that (ix) is the naturality square for the components at d and d' of ρ. As a consequence, to prove that the family $\{!_d \colon V \to |\,\nu\, S_V|\}_{d \in \mathbf{Coalg}(\widehat{S})}$ defines a natural transformation it suffices to prove that each morphism $!_d \colon V \to |\,\nu\, S_V|$ carries an homomorphism from d to $R(d)$. To this end consider the commuting diagram below.



□

Initial sequences for endofunctors are dual to final sequences and are usually (and historically) presented by transfinite induction as the iterative application of endofunctors (see [17]). The following is an equivalent but more compact formulation.

DEFINITION 4.3. *For F an endofunctor over* **C***, the* initial sequence of F *is any colimit-preserving functor* $\mathrm{ini}(F) \colon$ **Ord** $\to$ **C** *such that, for all ordinals $\beta \leq \alpha$:*

- $\mathrm{ini}(F)(\alpha + 1) = F(\mathrm{ini}(F)(\alpha)$*;*
- $\mathrm{ini}(F)(\iota_{\beta+1,\alpha+1}) = F(\mathrm{ini}(F)(\iota_{\beta,\alpha}))$.

Note that initial sequences take by definition the ordinal 0 (*i.e.* the initial object of **Ord**) to the initial object of the underlying category **C** whereas the observations refining sequence for d takes 0 to coalgebra d. In order to recover $\mathrm{ref}(d)$ as an initial sequence we need to change the underlying category and move to the coslice category $(d \downarrow \mathbf{Coalg}(\widehat{S}))$: there d is initial *per definitionem*. For every coalgebra d, the pointed functor (R, ρ) induces an endofunctor R_d over the under category $(d \downarrow \mathbf{Coalg}(\widehat{S}))$ given on objects by the mapping:

$$d \xrightarrow{f} d' \quad \mapsto \quad d \xrightarrow{\rho_d} R(d) \xrightarrow{R(f)} R(d')$$

and on morphisms by the mapping:

$$(d \xrightarrow{f} d') \xrightarrow{g} (d \xrightarrow{f'} d'') \quad \mapsto \quad (d \xrightarrow{R(f)\circ\rho_d} R(d')) \xrightarrow{R(g)} (d \xrightarrow{R(f')\circ\rho_d} R(d'')).$$

Preservation of identities and associativity readily follow once we observe that R_d factors through the fibre $(R(d) \downarrow \mathbf{Coalg}(\widehat{S}))$ as:

$$(d \downarrow \mathbf{Coalg}(\widehat{S})) \xrightarrow{R^{\to}} (R(d) \downarrow \mathbf{Coalg}(\widehat{S})) \xrightarrow{(\rho_d \downarrow \mathbf{Coalg}(\widehat{S}))} (d \downarrow \mathbf{Coalg}(\widehat{S}))$$

The first component is the lifting of R to the arrow category $\big(\mathbf{Coalg}(\widehat{S})\big)^{\to}$ (*i.e.* the comma category $(\mathbf{Coalg}(\widehat{S}) \downarrow \mathbf{Coalg}(\widehat{S}))$) and is given on objects by the mapping:

$$d \xrightarrow{f} d' \quad \mapsto \quad R(d) \xrightarrow{R(f)} R(d')$$

and on morphisms by the mapping:

$$(d \xrightarrow{f} d') \xrightarrow{g} (d \xrightarrow{f'} d'') \quad \mapsto \quad (R(d) \xrightarrow{f} R(d')) \xrightarrow{R(g)} (d \xrightarrow{R(f')} R(d'')).$$

The second component is the functor over $\rho_d \colon d \to R(d)$ going from the fibre $(R(d) \downarrow \mathbf{Coalg}(\widehat{S}))$ to $(d \downarrow \mathbf{Coalg}(\widehat{S}))$ in the domain fibration for $\mathbf{Coalg}(\widehat{S})$ *i.e.* $\mathrm{dom}\colon \big(\mathbf{Coalg}(\widehat{S})\big)^{\to} \to \mathbf{Coalg}(\widehat{S})$. This functor precomposes ρ_d to object arrows and acts as the identity on morphisms

$$R(d) \xrightarrow{f} d' \quad \mapsto \quad d \xrightarrow{\rho_d} R(d) \xrightarrow{f} d'$$
$$(R(d) \xrightarrow{f} d') \xrightarrow{g} (R(d) \xrightarrow{f'} d'') \quad \mapsto \quad (d \xrightarrow{f\circ\rho_d} d') \xrightarrow{g} (d \xrightarrow{f'\circ\rho_d} d'').$$

Like any coslice category, $(d \downarrow \mathbf{Coalg}(\widehat{S}))$ comes equipped with a projection $\mathrm{cod}\colon (d \downarrow \mathbf{Coalg}(\widehat{S})) \to \mathbf{Coalg}(\widehat{S})$ that takes object arrows to their codomain and morphisms accordingly. This last piece of information allows us to formalise the claim that observations refinement sequences are initial sequences.

THEOREM 4.11. *Assume that* **C** *is coalgebraically cocomplete with respect to the essential image of the schema* S. *For* d *a* $\widehat{S}$*-coalgebra, the diagram below commutes.*

$$\begin{array}{ccc} & \mathbf{Ord} & \\ {\scriptstyle \mathrm{ini}(R_d)} \swarrow & & \searrow {\scriptstyle \mathrm{ref}(d)} \\ (d \downarrow \mathbf{Coalg}(\widehat{S})) & \xrightarrow[\mathrm{cod}]{} & \mathbf{Coalg}(\widehat{S}) \end{array}$$

PROOF. To prove the thesis it suffices to show that $\mathrm{ini}(R_d)(\iota_{\beta,\alpha}) = \mathrm{ref}(d)(\iota_{\beta,\alpha})$ for all ordinals $\alpha \geq \beta$. From this property it follows that $\mathrm{ini}(R_d)(\alpha) = \mathrm{ref}(d)(\iota_{0,\alpha})$ and $\mathrm{cod}(\mathrm{ini}(R_d)(\alpha)) = \mathrm{ref}(d)(\alpha)$ for any ordinal α. The proof proceeds by transfinite induction on $\alpha \in \mathbf{Ord}$.

First step. Let $\alpha = 0$. The observations refinement sequence and initial sequences begin with (V, d) and $id_{(V,d)}$, respectively. We conclude that $\mathrm{ref}(d)(\iota_{0,0})$ and $\mathrm{cod}(\mathrm{ini}(R_d)(\iota_{0,0}))$ are both equal to $t_0^0 = id_{(V,d)}$.

Isolated step. Let $\alpha = \beta+1$ and assume by inductive hypothesis that $\mathrm{ini}(R_d)(\iota_{\gamma,\beta})$ is equal to $\mathrm{ref}(d)(\iota_{\gamma,\beta}) = t_\beta^\gamma$ for any ordinal $\gamma \leq \beta$. It follows that

$$\begin{aligned}\mathrm{ini}(R_d)(\iota_{\gamma+1,\alpha}) &\overset{(i)}{=} R_d(\mathrm{ini}(R_d)(\iota_{\gamma,\beta})) \overset{(ii)}{=} R(t_\beta^\gamma) \overset{(iii)}{=} N(t_\beta^\gamma)\\ &\overset{(iv)}{=} !_{S_{t_\beta^\gamma, |\nu S_{V_\gamma}|}} \circ \nu S_{V_\gamma} \overset{(v)}{=} \mathrm{ref}(d)(\iota_{\gamma+1,\alpha})\end{aligned}$$

where (i) holds by definition of initial sequence, (ii) by inductive hypothesis, (iii) by definition of R, (iv) by definition of N, and (v) by definition of observations refinement sequence on successor ordinals. This proves that $\mathrm{ini}(R_d)(\iota_{\gamma+1,\alpha})$ is equal to $\mathrm{ref}(d)(\iota_{\gamma+1,\alpha})$ for any ordinal $\gamma < \alpha$. From this fact, functoriality, and inductive hypothesis we conclude that:

$$\begin{aligned}\mathrm{ini}(R_d)(\iota_{0,\alpha}) &= \mathrm{ini}(R_d)(\iota_{0,1}) \circ \mathrm{ini}(R_d)(\iota_{1,\alpha})\\ &= \mathrm{ref}(d)(\iota_{0,1}) \circ \mathrm{ref}(d)(\iota_{1,\alpha}) = \mathrm{ref}(d)(\iota_{0,\alpha})\end{aligned}$$

Finally, $\mathrm{ini}(R_d)(\iota_{\alpha,\alpha}) = id = \mathrm{ref}(d)(\iota_{\alpha,\alpha})$ holds by functoriality.

Limit step. Let α be a transfinite limit ordinal and assume that $\mathrm{ini}(R_d)(\iota_{\gamma,\beta})$ is equal to $\mathrm{ref}(d)(\iota_{\gamma,\beta}) = t_\beta^\gamma$ for any ordinal $\gamma \leq \beta < \alpha$. The thesis follows by noting that the diagrams for the two colimits coincide since $\mathrm{ini}(R_d)(\beta)$ and $\mathrm{ini}(R_d)(\iota_{0,\beta})$ are the same coalgebra homomorphism for any ordinal β, especially 0. □

4.2 MIXED-VARIANCE BEHAVIOURAL SCHEMATA

In this section we consider mixed-variance behavioural schemata *i.e.* functors of type

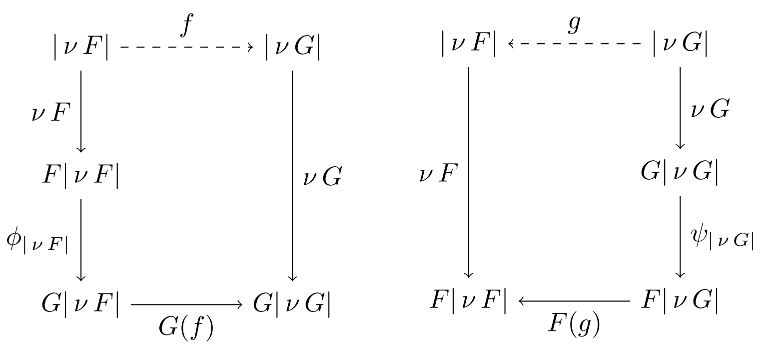
$$\mathbf{C}^{op} \times \mathbf{C} \to \mathbf{End}(\mathbf{C}).$$

These functors provide an abstract definition of behaviours generic in their inputs and outputs. Self-referential endofunctors for behavioural schemata are instances defined by their own final coalgebra. As a consequence, objects representing inputs and output are always isomorphic. Formally:

DEFINITION 4.4. *For a mixed-variance behavioural schema $S\colon \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{End}(\mathbf{C})$, an endofunctor F over $\mathbf{C}$ is called* self-referential *whenever it solves:*

$$\begin{cases} F \cong S_{Z,Z} \\ Z \cong |\nu F| \end{cases} \tag{4.5}$$

Like the covariant case discussed in Section 4.1, solutions are organised in a category. Because of mixed variance, morphisms are given by pairs of opposing natural transformations that are coherent with the behavioural schema and with final coalgebras. For F and G solutions to (4.5), a solution morphisms from the former to the latter is formally defined as a pair (ϕ, ψ) where ϕ and ψ are natural transformations of type $F \to G$ and $G \to F$, respectively, with the property that there are isomorphisms $\phi \cong S_{g,f}$ and $\psi \cong S_{f,g}$ where f and g are the unique coalgebra homomorphisms depicted in the commuting diagrams below:



Every solution morphism $(\phi, \psi) \colon F \to G$ defines by symmetry a morphism $(\psi, \phi) \colon G \to F$ in the opposite direction (note that this is not necessarily an inverse). As a consequence, the category of solutions is self-dual with the swapping and dualising functors providing an involution. We observe that initial solutions identify final ones by symmetry and *vice versa*. From this observation we conclude that either they both exist or they both do not exist and that they are necessarily isomorphic. This isomorphism is expected because of the symmetry between inputs and outputs for self-referential behaviours.

4.2.1 Self-referential endofunctors as invariants

Along the lines of Section 4.1.1, in this section we characterise self-referential endofunctors as fixed points of certain endofunctors derived from the equation system (4.5) and provide results concerning existence and construction of canonical solutions. Before we present the endofunctor modelling (4.5) let us recall some auxiliary definitions.

Recall from [1, 40, 49, 92] that a category with involution, or dagger category, $\big(\mathbf{C}, (-)^\dagger\big)$ is a category $\mathbf{C}$ equipped with an involution[2] $(-)^\dagger \colon \mathbf{C} \to \mathbf{C}^{op}$ *i.e.* a functor that is its own inverse $(-)^{\dagger\dagger} \cong Id$. An object in an involutive category is called *symmetric* whenever it is equal to its involution ($X = X^\dagger$). A morphism

[2] We adopt the definition from [49] for it offers a cleaner definition of universal involutive categories.

from an involutive category $\big(\mathbf{C}, (-)^{\dagger}\big)$ into $\big(\mathbf{D}, (-)^{\ddagger}\big)$, is any functor $F\colon \mathbf{C} \to \mathbf{D}$ that respects involutions as formalised by the diagram below.

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{\quad F \quad} & \mathbf{D} \\ {\scriptstyle (-)^{\dagger}}\big\downarrow & & \big\downarrow{\scriptstyle (-)^{\ddagger}} \\ \mathbf{C}^{op} & \xrightarrow[\quad F^{op} \quad]{} & \mathbf{D}^{op} \end{array}$$

Functors that are also morphisms of involutive categories are often called *symmetric functors*. A category $\mathbf{C}$ induces an involutive category $\big(\mathbf{Inv}(\mathbf{C}), (-)^{\S}\big)$ where the category $\mathbf{Inv}(\mathbf{C})$ is defined as $\mathbf{C}^{op} \times \mathbf{C}$ and the involution $(-)^{\S}\colon \mathbf{Inv}(\mathbf{C}) \to \mathbf{Inv}(\mathbf{C})^{op}$ is given by the isomorphism of categories

$$(-)^{\S}\colon \mathbf{C}^{op} \times \mathbf{C} \cong \mathbf{C} \times \mathbf{C}^{op}.$$

This structure is called *universal involutive category* of $\mathbf{C}$. Universal involutive categories are self-dual via their involution and are universal in the sense that there is the following bijective correspondence

$$\frac{\mathbf{C} \xrightarrow{F} \mathbf{D}}{(\mathbf{C}, (-)^{\dagger}) \xrightarrow{\widehat{F}} (\mathbf{Inv}(\mathbf{D}), (-)^{\S})} \tag{4.6}$$

In particular, the symmetric functor $\widehat{F}\colon \mathbf{C} \to \mathbf{D}^{op} \times \mathbf{D}$ is universally defined as follows:

$$\widehat{F} = \langle (F^{op})^{\dagger}, F \rangle.$$

In the sequel, we will omit the involution $(-)^{\dagger}$ when clear from the context and thus write just $\langle (F^{op}), F \rangle$. The mapping $\mathbf{C} \to \big(\mathbf{Inv}(\mathbf{C}), (-)^{\S}\big)$ associating a category with its universal involutive categories extends to a functor $\mathbf{Inv}(-)$ from $\mathbf{Cat}$ into $\mathbf{InvCat}$ the category of involutive categories. This functor takes each functor $F\colon \mathbf{C} \to \mathbf{D}$ into $\mathbf{Inv}(F)\colon \mathbf{Inv}(\mathbf{C}) \to \mathbf{Inv}(\mathbf{D})$ defined as the product $F^{op} \times F$. It follows from basic properties of products that the functor $\mathbf{Inv}(-)$ left and right distributes over $(\widehat{-})$ in the sense that for any $F\colon \mathbf{B} \to \mathbf{C}$, $G\colon \mathbf{Inv}(\mathbf{C}) \to \mathbf{D}$, $H\colon \mathbf{A} \to \mathbf{B}$, and $\big(\mathbf{A}, (-)^{\dagger}\big)$ involutive, in $\mathbf{InvCat}$ there are the following isomorphisms:

$$\widehat{G \circ \mathbf{Inv}(F)} \cong \widehat{G} \circ \mathbf{Inv}(F) \qquad \widehat{F \circ H} \cong \mathbf{Inv}(F) \circ \widehat{H}. \tag{4.7}$$

Universal involutive categories offer us the language necessary to formalise the idea of "symmetric solutions" discussed in the opening of this section and tools for turning mixed-variance functors into covariant symmetric functors.

As in the case of Section 4.1.1, the first step towards characterising self-referential endofunctors as invariant objects is to put (4.5) in a form that uses unknowns ranging either over object of values (*i.e.* **C**) or over behavioural functors (*i.e.* **End**(**C**)). A first attempt could be to consider the following formulations of (4.5):

$$Z \cong |\,\nu\, S(Z,Z)| \qquad F \cong S(|\,\nu\, F|, |\,\nu\, F|)$$

In both cases, the unknown occurs in covariant and contravariant position and this prevents us from characterising solutions in terms of fixed points. To this end, we need to replace Z and F with pairs of unknowns: one for occurrences in covariant positions and one for its occurrences in contravariant position. In the first case we obtain the following system of equations:

$$\begin{cases} Z \cong |\,\nu\, S(Z',Z)| \\ Z' \cong |\,\nu\, S^{op}(Z,Z')| \\ Z' \cong Z \end{cases} \tag{4.8}$$

where Z and Z' range over **C** and $\mathbf{C}^{op}$, respectively. The last clause is required to ensure symmetry for solutions *i.e.* $(Z,Z') \cong (Z,Z')^{\S}$. In the second case, we obtain the following system of equations:

$$\begin{cases} F \cong S(|\,\nu\, F'|, |\,\nu\, F|) \\ F' \cong S^{op}(|\,\nu\, F|, |\,\nu\, F'|) \\ F' \cong F \end{cases} \tag{4.9}$$

where F and F' range over **End**(**C**) and $\mathbf{End}(\mathbf{C})^{op}$, respectively.

As discussed in Section 4.1.1, we assume that every endofunctor described by the given schema S comes equipped with a chosen final coalgebras. In other words, we assume to be given functor $N\colon \operatorname{eimg}(S) \to \mathbf{C}$ taking endofunctors to their chosen final coalgebra. For the sake of exposition, we will refer to S as a functor into **End**(**C**) and regard N as if it is defined over the entire category **End**(**C**).

Let us ignore for a few lines the equations concerning solution symmetry. Under the mild assumption of chosen final coalgebras, the first equation of (4.8) corresponds to the functor:

$$\mathbf{Inv}(\mathbf{C}) \xrightarrow{S} \mathbf{End}(\mathbf{C}) \xrightarrow{N} \mathbf{C}$$

and the second equation to its opposite:

$$\mathbf{Inv}(\mathbf{C})^{op} \xrightarrow{S^{op}} \mathbf{End}(\mathbf{C})^{op} \xrightarrow{N^{op}} \mathbf{C}^{op}.$$

From this observation we conclude that the first two equations of (4.8) correspond to the symmetric endofunctor over the universal involutive category for **C**:

$$\widehat{NS} = \langle ((NS)^{op})^{\S}, NS \rangle.$$

Then, by (4.7), we have the following correspondence:

$$\frac{\mathbf{Inv}(\mathbf{C}) \xrightarrow{S} \mathbf{End}(\mathbf{C}) \xrightarrow{N} \mathbf{C}}{\mathbf{Inv}(\mathbf{C}) \xrightarrow{\widehat{S}} \mathbf{Inv}(\mathbf{End}(\mathbf{C})) \xrightarrow{\mathbf{Inv}(N)} \mathbf{Inv}(\mathbf{C})}$$

The very same reasoning applies to the system (4.9). The first equation of the system corresponds to the functor

$$\mathbf{Inv}(\mathbf{End}(\mathbf{C})) \xrightarrow{\mathbf{Inv}(N)} \mathbf{Inv}(\mathbf{C}) \xrightarrow{S} \mathbf{End}(\mathbf{C}) \tag{4.10}$$

and the second to its opposite. From (4.10) and the correspondence (4.6) we obtain a symmetric endofunctor over the universal involutive category for **End**(**C**):

$$\frac{\mathbf{Inv}(\mathbf{End}(\mathbf{C})) \xrightarrow{\mathbf{Inv}(N)} \mathbf{Inv}(\mathbf{C}) \xrightarrow{S} \mathbf{End}(\mathbf{C})}{\mathbf{Inv}(\mathbf{End}(\mathbf{C})) \xrightarrow{\mathbf{Inv}(N)} \mathbf{Inv}(\mathbf{C}) \xrightarrow{\widehat{S}} \mathbf{Inv}(\mathbf{End}(\mathbf{C}))}$$

Algebras for symmetric endofunctors like $\mathbf{Inv}(N) \circ \widehat{S}$ and $\widehat{S} \circ \mathbf{Inv}(N)$ from above are suitable pairs of algebras and coalgebras called dialgebras [57]. Whenever they exists, initial algebras of symmetric endofunctor determine final coalgebras via the involution $(-)^{\S}$ and *vice versa*. For instance, consider a functor $F\colon \mathbf{Inv}(\mathbf{C}) \to \mathbf{C}$ and the corresponding symmetric endofunctor $\widehat{F} = \langle (F^{op})^{\S}, F \rangle$ over $\mathbf{Inv}(\mathbf{C})$. An algebra for $\widehat{F}$ is a pair

$$(g, h)\colon (F^{op}(Y, X), F(X, Y)) \to (X, Y)$$

where (X, g) is a coalgebra for $F(Y, -)$ and (Y, h) is an algebra for $F(X, -)$. It follows that the pair (h, g) is a coalgebra for $\widehat{F}$ and that the involution $(-)^{\S}$ lifts to a functor from $\mathbf{Alg}(\widehat{F})$ to $\mathbf{Alg}(\widehat{F}^{op}) \cong \mathbf{Coalg}(\widehat{F})$ as illustrated in the commuting diagram below.

$$\begin{array}{ccc}
\mathbf{Alg}(\widehat{F}) & \longrightarrow & \mathbf{Alg}(\widehat{F}^{op}) \\
\downarrow & & \downarrow \\
\mathbf{C} & \xrightarrow[(-)^{\S}]{} & \mathbf{C}^{op}
\end{array}$$

In particular, the involution $(-)^{\S}$ takes the inductive extension of an $\widehat{F}$-algebra (g, h) to the coinductive extension of (h, g). Reworded, $(-)^{\S}$ takes initial algebras to final coalgebras. However, these are not guaranteed to be symmetric. For an example of non-symmetric initial invariants consider as F the projection $\pi_r \colon \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{C}$. The symmetric endofunctor $\widehat{\pi_r} \colon \mathbf{Inv}(\mathbf{C}) \to \mathbf{Inv}(\mathbf{C})$ is the identity since $((\pi_r)^{op})^{\S}$ is the projection $\pi_l \colon \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{C}^{op}$. It follows that the initial algebra is the identity on $(1,0)$ and that the final coalgebra is the identity on $(0,1)$—provided that **C** has initial and final objects.

We observe that symmetry of initial or final invariants of symmetric endofunctors corresponds to algebraic compactness.

PROPOSITION 4.12. *Let F be an endomorphism over an involutive category $\big(\mathbf{C}, (-)^{\dagger}\big)$. The following statements are equivalent:*

- *there is an initial F-algebra and its carrier is symmetric;*
- *the endofunctor F is algebraically compact.*

PROOF. Assume an initial F-algebra $\mu\, F$. It follows that $(\mu\, F)^{\dagger}$ is an initial F^{op}-algebra *i.e.* a final F-coalgebra. Symmetry imposes $|\, \mu\, F| = |(\mu\, F)^{\dagger}| \cong |(\nu\, F)|$. The converse follows likewise. □

As a consequence, when it comes to canonical self-referential endofunctors, the conditions $Z \cong Z'$ and $F \cong F'$ imposing symmetry on solutions to systems (4.8) and (4.9) coincide with algebraic compactness of the associated endofunctors $\mathbf{Inv}(N) \circ \widehat{S}$ and $\widehat{S} \circ \mathbf{Inv}(N)$ over $\mathbf{Inv}(\mathbf{C})$ and $\mathbf{Inv}(\mathbf{End}(\mathbf{C}))$, respectively.

In this work we consider the two main classes of algebraically compact functors: locally continuous and locally compact ones. Recall from Section 2.2 that if **C** is **Cpo**-algebraically compact or contractively compact, then so is its universal involutive category $\mathbf{C}^{op} \times \mathbf{C}$. However, it is not known whether the category of **Cpo**-algebraically compact categories is closed under exponentiation [48]; it remains an open question even if we restrict to the case of "self-exponentials" such as $\mathbf{End}(\mathbf{C})$. We remark that algebraic compactness is preserved when the exponential base is $\mathbf{Cpo}_{\perp}$ or, in general, any *algebraically super-compact* category and refer the interested reader to [48] for further details. Likewise, it is not known whether the class of contractively compact categories are closed under exponentiation or self-exponentiation [26].

Although we cannot state that if **C** is **Cpo**-algebraically compact (resp. contractively compact), so is the category of all locally continuous (resp. locally contractive) endofunctors on it, a weaker result will suffice for our aims. In fact, unlike arbitrary endofunctors over $\mathbf{Inv}(\mathbf{End}(\mathbf{C}))$, we are interested in endofunctors that factor through $\mathbf{Inv}(\mathbf{C})$, by construction. Furthermore, it is reasonable to assume that **C** is algebraically compact with respect to the class of functors of interest *e.g.* locally continuous or locally contractive functors.

LEMMA 4.13. *For $G\colon \mathbf{C} \to \mathbf{D}$ and $H\colon \mathbf{D} \to \mathbf{C}$, the endofunctor GH, is algebraically compact if and only if so is HG.*

PROOF. Assume GH is algebraically compact. It follows from Lemma 4.2 that HG admits an initial algebra and a final coalgebra: $H(\mu GH)$ and $H(\nu GH)$. We conclude that the image through H of the canonical isomorphism between μGH and νGH is an isomorphism for $H(\mu GH)$ and $H(\nu GH)$ and hence HG is algebraically compact. □

We are now able to state the main result of this section: existence of canonical self-referential endofunctors reduces to assessing local continuity or local contractiveness of behavioural schemata.

THEOREM 4.14. *Assume the covariant behavioural schema S is* **Cpo**-*enriched. If* **C** *is* **Cpo**-*algebraically compact, then there is a canonical solution to* (4.5).

PROOF. It follows from Proposition 4.3 and hypothesis on **C** that there is a **Cpo**-enriched functor $N\colon$ **Cpo-End**$(\mathbf{C}) \to \mathbf{C}$ associating each endofunctor with the carrier of a chosen final coalgebra for it. Both functors $\mathbf{Inv}(N) \circ \widehat{S}$ and $\widehat{S} \circ \mathbf{Inv}(N)$ are locally continuous since **Cpo-Cat** has products. It follows from Corollary 2.10 and hypothesis on **C** that $\mathbf{Inv}(\mathbf{C})$ is **Cpo**-algebraically compact. We conclude from Lemma 4.13 that $\mathbf{Inv}(N) \circ \widehat{S}$ and $\widehat{S} \circ \mathbf{Inv}(N)$ are algebraically compact endofunctors. Initial and final invariants for these endofunctors are canonically isomorphic hence symmetric objects. These invariants identify canonical solutions to (4.8) and (4.9), respectively. Thus, there is a canonical solution to (4.5). □

COROLLARY 4.15. *Assume the covariant behavioural schema S is* **Cpo**-*enriched. If* **C** *is* **Cpo**-*algebraically compact, canonical self-referential endofunctors are both sound and complete with respect to any other self-referential endofunctor.*

PROOF. It follows from algebraic compactness that any non-canonical solution factors the isomorphism between the initial and final solutions. In a **Cpo**-enriched setting this situation defines a coreflection formed by an embedding from the initial solution together with a projection into the final one. □

THEOREM 4.16. *Let* **E** *be a sheaf enriched category of locally contractive endofunctors on a contractively compact category* **C**. *If S is locally contractive, then there is a unique (up to isomorphism) solution to* (4.5).

PROOF. Recall from Section 2.2.2 that local contractiveness (as per Definition 2.11) implies enrichment over sheaves of sets on a complete Heyting algebra. By Proposition 4.4 there is a suitably enriched functor $N\colon \mathbf{E} \to \mathbf{C}$ associating each endofunctor in **E** with the carrier of a chosen final coalgebra for it. By Lemma 2.11

and hypothesis on S, the endofunctors $\mathbf{Inv}(N) \circ \widehat{S}$ and $\widehat{S} \circ \mathbf{Inv}(N)$ are locally contractive. It follows from Proposition 2.13 and hypothesis on **C**, that the category **Inv**(**C**) is **Cpo**-algebraically compact and from Lemma 4.13 that the endofunctors $\mathbf{Inv}(N) \circ \widehat{S}$ and $\widehat{S} \circ \mathbf{Inv}(N)$ are algebraically compact. We conclude that initial and final invariants for these endofunctors are canonically isomorphic hence symmetric objects. These invariants identify canonical solutions to (4.8) and (4.9), respectively. Therefore, there is a canonical solution to (4.5). □

4.2.2 Examples of canonical self-referential endofunctors

Deterministic self-referential behaviours Consider the mixed-variance schema:

$$S_{V,U} = Id^V + U$$

where the underlying category is **Set**. Instances of this schema describe behaviours of systems that can deterministically input a value from V or terminate producing a value from U. This behaviours schema does not admit self-referential functors. In fact, the cardinality of the set carrying the final $S_{V,U}$-coalgebra always exceeds that of V and of U. Behaviours characterised by this functor are closely related to the domain equation $D \cong (D \to D) + D$: this equation cannot be solved in **Set** but admits a unique dominating solution in $\mathbf{Cpo}_\perp$. This observation prompted us to study $Id^V + U$ as a schema on $\mathbf{Cpo}_\perp$.

Let $(X \to_\perp Y)$ denote the space of continuous bottom-strict functions equipped with the pointwise ordering and consider the behavioural schema:

$$S_{V,U} = (V \to_\perp Id) + U.$$

For any V and U, the final $S_{V,U}$-coalgebra exists and describes all trees whose leaves are in U and whose branches are indexed by continuous bottom-strict functions from V. Strictness renders bottom elements sink states which can be interpreted as modelling divergent behaviours. From this perspective, the function space $(V \to_\perp Id)$ characterises eager deterministic inputs since divergent inputs results in divergence. Strictness of coproducts (which equate bottom elements) means that divergence on inputs or outputs coincide and all diverging computations are assigned the same abstract behaviour. In other words, the schema S models eager deterministic computations. The behavioural schema S is **Cpo**-enriched and the category $\mathbf{Cpo}_\perp$ is **Cpo**-algebraically compact. It follows from Theorem 4.14 that there exists a canonical self-referential endofunctor for the behavioural schemata S. This solution can be obtained as an initial/final sequence for $\mathbf{Inv}(N) \circ \widehat{S}$. The sequence stabilises after the first iteration since $1 \cong |\,\nu\, S_{1,1}|$. Indeed, 1 is solution to the domain equation $D \cong (D \to_\perp D) + D$ in $\mathbf{Cpo}_\perp$.

Inspired by the intuitive correlation with domain equations, consider the behavioural schema:

$$S_{V,U} = (V \to_\perp Id) + U + A.$$

Computations are strict as before but can now terminate returning an atom from A. Theorem 4.14 applies to this schema and the sequence leading to the canonical solution stabilises at ω (provided $|A| > 1$). Intuitively, abstract behaviours modelled by the canonical self-referential endofunctor for S are infinite trees with atoms from A as leaves and branching indexed by abstract behaviours (subject to continuity and strictness).

Behavioural schemata considered above are examples of polynomial functors parameterised in (V, U). These are generated by the following grammar:

$$S_{V,U}, S'_{V,U} ::= (A \to_\perp Id) \mid A \mid (V \to_\perp Id) \mid U \mid S_{V,U} + S'_{V,U} \mid S_{V,U} \times S'_{V,U}$$

All functors generated from this grammar meet the hypotheses of Theorem 4.14 and hence admit canonical self-referential endofunctors.

Self-referential non-deterministic behaviours Bounded non-determinism is modelled in the context of **Set** by means of the finite powerset $\mathcal{P}_f$ (and variations thereof). Mixed-variance behavioural schemata based on $\mathcal{P}_f$ rarely admit self-referential endofunctors. For instance, consider the schema $S_{V,U} = \mathcal{P}_f(U \times Id)^V$. Although all instances of the schema admit final coalgebras, the cardinality of their carrier always exceed that of the parameters U and V. Likewise deterministic computations, we model self-referential non-deterministic ones in the context of $\mathbf{Cpo}_\perp$.

Let $\mathbb{B}$ be the boolean lattice. Functions in $\mathbf{Cpo}(X, \mathbb{B})$ are predicates that describe all upward closed subsets of X: for $\phi\colon X \to \mathbb{B}$ a continuous (non necessarily strict) function, it follows from monotonicity that $\phi^{-1}(\top)$ is upward closed. There is an endofunctor $\mathcal{U}$ over **Cpo** given on each object X and on each continuous map $f\colon X \to Y$ as follows:

$$\mathcal{U}(X) = \mathbf{Cpo}(X, \mathbb{B}) \qquad \mathcal{U}(f)(\phi)(y) = \bigvee_{x \in f^{-1}(y)} \phi(x)$$

(Note that the join is always defined since it takes place in the boolean lattice.) The functor $\mathcal{U}$ restricts along the inclusion $\mathbf{Cpo}_\perp \hookrightarrow \mathbf{Cpo}$ to an endofunctor over $\mathbf{Cpo}_\perp$. Recall that if the order on $X \in \mathbf{Cpo}$ is the anti-chain ordering, then any subset of X is trivially upward closed. As a consequence, the endofunctor $\mathcal{U}$ over **Cpo** yields the powerset $\mathcal{P}$ by composition with the insertion functor $I\colon \mathbf{Set} \to \mathbf{Cpo}$ (which equips each set with the anti-chain ordering) and with the forgetful functor $U\colon \mathbf{Cpo} \to \mathbf{Set}$, its adjoint. These observations suggest that the endofunctor $\mathcal{U}$ is a good candidate for modelling non-determinism in **Cpo** and $\mathbf{Cpo}_\perp$. The endofunctor $\mathcal{U}$ has a "strict" version defined on objects as $\mathbf{Cpo}_\perp(-,\mathbb{B})$, instead of $\mathbf{Cpo}(-,\mathbb{B})$, and on morphisms as $\mathcal{U}$:

$$\mathcal{U}_\perp(X) = \mathbf{Cpo}_\perp(X,\mathbb{B}) \qquad \mathcal{U}_\perp(f)(\phi)(y) = \bigvee_{x\in f^{-1}(y)} \phi(x)$$

The additional constraint imposed by bottom-strictness means that subsets described by $\mathcal{U}_\perp(X)$ cannot contain $\perp_X$. The composite of $\mathcal{U}_\perp$ with the insertion I and the forgetful U, yields the non-empty powerset functor $\mathcal{P}^+$. As a consequence, adding strictness models computation steps that either diverge or progress non-deterministically. Like $\mathcal{U}$, the endofunctor $\mathcal{U}_\perp$ restricts to an endofunctor over $\mathbf{Cpo}_\perp$ along the inclusion $\mathbf{Cpo}_\perp \hookrightarrow \mathbf{Cpo}$.

Both $\mathcal{U}$ and $\mathcal{U}_\perp$ are enriched over **Cpo** and hence their restrictions along $\mathbf{Cpo}_\perp \hookrightarrow \mathbf{Cpo}$ are algebraically compact. Final coalgebras (and initial algebras) for $\mathcal{U}$ and $\mathcal{U}_\perp$ are carried by the ordinals 1 and ω, respectively. This difference reflects the shape of non-deterministic behaviours the two endofunctors models: in the first case computation steps can non-deterministically progress or diverge whereas in the second can either progress non-deterministically or diverge. To draw a parallel between internal and external non-determinism, this situation mirrors that of lazy and eager inputs discussed in the previous paragraph and suggests $\mathcal{U}$ and $\mathcal{U}_\perp$ as the types of "lazy" and "eager" internal non-determinism.

Self-referential CCS The late semantics of the CCS with values [65] has been shown in [51] to be captured by the endofunctor over **Set**:

$$\mathcal{P}(\overbrace{C \times U \times Id}^{\text{output}} + \overbrace{C \times Id^V}^{\text{input}} + \overbrace{Id}^{\tau})$$

where C is the set of channels and V the set of exchanged values. Regard V as the only parameter and let S denote the resulting behaviours schema. Although all instances of S are coalgebraically cocomplete, there is not choice for the set of values V that identifies a self-referential endofunctor for this schema. This fact follows from the same argument discussed in the examples of mixed-variance schemata over **Set** from above. Therefore, we characterise the behaviour of self-referential systems with synchronous exchanges in the context of $\mathbf{Cpo}_\perp$ instead of **Set**.

Fix an object V representing exchanged values and an object C representing communication channels. Deterministic outputs over channels are characterised by the endofunctor $C \ltimes (V \times Id)$ where $\ltimes$ denotes left-coalesced products[3].

[3]The left-coalesced products identify all pairs whose left component is bottom.

The use of left-coalesced products corresponds to the interpretation of $\perp_C$ as an "unspecified" communication channel whose use results in divergence. The use of cartesian products, instead of coalesced ones, for $V \times Id$ means that outputs are regarded as lazy: systems can output and then diverge or output diverging values and progress their computation. Deterministic inputs are modelled by the endofunctor $C \ltimes (V \to Id)$. Inputs are assumed lazy for symmetry with outputs hence the use of non-strict functions. The non-deterministic component of the behaviour is provided by the endofunctor $\mathcal{U}$. By combining these elements we obtain the behavioural schemata:

$$S_{V,U} = \mathcal{U}(\overbrace{C \ltimes U \times Id}^{\text{output}} + \overbrace{C \ltimes (V \to Id)}^{\text{input}} + \overbrace{Id}^{\tau}).$$

To obtain an eager version of this schema it suffices to replace $\mathcal{U}$ and $(V \to Id)$ with their strict equivalent. By construction, the schema S is **Cpo**-enriched and by Theorem 4.14 there is a canonical self-referential endofunctor instance of S.

4.3 GENERALISED BEHAVIOURAL SCHEMATA

In the previous section, we proved that canonical self-referential endofunctors instances of a mixed-variance schema S on **C** are canonical invariants for the symmetric endofunctor $\widehat{S} \circ \mathbf{Inv}(N)$ over $\mathbf{Inv}(\mathbf{End}(\mathbf{C}))$. As a consequence, self-referential endofunctors can be computed via initial/final sequences for $\widehat{S} \circ \mathbf{Inv}(N)$—equivalently, sequences for $\mathbf{Inv}(N) \circ \widehat{S}$. Assume a **Cpo**-enriched setting and bi-chains as per Proposition 2.9, by unfolding the sequence leading to the canonical solutions we obtain the diagram illustrated below.

$$\begin{array}{ccccccccc}
1 & \longleftarrow & 1 & \longleftarrow & 1 & \longleftarrow & \cdots & 1 & \\
\uparrow\downarrow & & \uparrow\downarrow & & \uparrow\downarrow & & & \uparrow\downarrow & \\
1 & \longleftarrow & S_{1,1}(1) & \longleftarrow & S^2_{1,1}(1) & \longleftarrow & \cdots & Z_1 & \cong S_{1,1}(Z_1) \\
\uparrow\downarrow & & \uparrow\downarrow & & \uparrow\downarrow & & & \uparrow\downarrow & \\
1 & \longleftarrow & S_{Z_1,Z_1}(1) & \longleftarrow & S^2_{Z_1,Z_1}(1) & \longleftarrow & \cdots & Z_2 & \cong S_{Z_1,Z_1}(Z_2) \\
\uparrow\downarrow & & \uparrow\downarrow & & \uparrow\downarrow & & & \uparrow\downarrow & \\
1 & \longleftarrow & S_{Z_2,Z_2}(1) & \longleftarrow & S^2_{Z_2,Z_2}(1) & \longleftarrow & \cdots & Z_3 & \cong S_{Z_2,Z_2}(Z_3) \\
\uparrow\downarrow & & \uparrow\downarrow & & \uparrow\downarrow & & & \uparrow\downarrow & \\
\vdots & & \vdots & & \vdots & & \ddots & \vdots & \\
1 & \longleftarrow & S_{Z,Z}(1) & \longleftarrow & S^2_{Z,Z}(1) & \longleftarrow & \cdots & Z & \cong S_{Z,Z}(Z)
\end{array}$$

Horizontal arrows form final sequences for instances of the schema S and each final coalgebra determines the instance of the successive layer. Vertical arrows form chains of embedding-projection pairs and characterise horizontal layers as approximations that converge to the limiting final sequence depicted in the bottom of the diagram. To draw a parallel with higher-order languages, behaviours of type $S_{1,1}$ can be thought of as first-order ones because since their parameters represent static values (recall that the final object in a slice category $(\mathbf{C} \downarrow V)$ is id_V). Then, behaviours of type S_{Z_1,Z_1} are of second order since they operate on first order behaviours and so on. In general, behaviours of type S_{Z_n,Z_n} exchange behaviours of order n end hence belong to the order $n+1$. This ω-sequence is limited by higher-order behaviours or, more precisely, ω-order behaviours like those exhibited by calculi with ω-order abstractions *e.g.* the λ-calculus, the HO π-calculus, CHOCS.

We observe that bi-chains lie in the category of parameters and conclude that algebraic compactness can be limited to this context. This suggests a way to decouple the category of parameters used by behavioural schemata from that where behavioural endofunctors act *i.e.* consider schemata of type $\mathbf{Inv}(\mathbf{D}) \to \mathbf{End}(\mathbf{C})$. This distinction allows us to relax the assumptions on **C**, the category where systems are modelled, and require only that **D**, the category where values are modelled, is algebraically compact. Although this result may appear mainly technical, it enables the modelling of a wider range of self-referential systems. For instance, behavioural endofunctors might be defined on a (suitably enriched) category of spaces whereas parameters are restricted to range over its subcategory of exponentiable ones. Likewise, one might consider the Kleisli category for a monad and its underlying category—along the lines of [132]. Modelling values and systems in different categories poses the additional challenge of how to derive the former from the latter. In the self-referential case, this situation means that although systems are defined on **C** their semantics is modelled in **D**. In Sections 4.3.1 and 4.3.2, we formalise this scenario by means of behavioural endofunctors equipped with extensions/liftings that are well-behaved with respect to the respective final coalgebras. The constructions we introduce in Sections 4.3.1 and 4.3.2 require some basic 2-categorical machinery. As a consequence, we are able to instantiate them on locally continuous functors but not on locally contractive ones.

4.3.1 Behavioural schemata of extensions

In this section we consider behavioural schemata of type:

$$\mathbf{Inv}(\mathbf{D}) \to \mathbf{End}(\mathbf{C})$$

and assume that their instances are extensions along some fixed functor $R\colon \mathbf{D} \to \mathbf{C}$ of endofunctors over **D**. The idea is to have R act as a mediator between behavioural endofunctors and parameters.

We organise extensions along R into a category **Ext**(R). An endofunctor F over **C** is *an extension along* R of some endofunctor G whenever there exists an isomorphism[4]:

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{F} & \mathbf{C} \\ R \uparrow & \cong & \uparrow R \\ \mathbf{D} & \xrightarrow[G]{} & \mathbf{D} \end{array}$$

in **Fun**$(\mathbf{D}, \mathbf{C})$. In order to generalise this condition beyond objects in **End**$(\mathbf{C})$ and **End**$(\mathbf{D})$ consider the following 2-pullback:

$$\begin{array}{ccc} \mathbf{P} & \xrightarrow{p_2} & \mathbf{End}(\mathbf{C}) \\ p_1 \downarrow & \cong & \downarrow (- \circ R) \\ \mathbf{End}(\mathbf{D}) & \xrightarrow[(R \circ -)]{} & \mathbf{Fun}(\mathbf{D}, \mathbf{C}) \end{array}$$

Then, the projection of **P** into **End**$(\mathbf{C})$ identifies all extensions along R. Formally, we define **Ext**(R) as the replete image[5] of p_2:

$$\mathbf{Ext}(R) \cong \mathrm{rimg}(p_2).$$

This definition extends to the order enriched setting as it is. Let **V** stand for either **Cpo** or $\mathbf{Cpo}_\perp$. For R a **V**-functor, **Ext**(R) is (isomorphic to) the sub-**V**-category of **V-End**$(\mathbf{C})$ with the following properties:

- an endofunctor F over **C** is an object of **Ext**(R) provided that there are G in **End**$(\mathbf{D})$ and $R \circ G \cong F \circ R$ in **Fun**$(\mathbf{D}, \mathbf{C})$;
- a natural transformation $f\colon F \to F'$ is a morphism of **Ext**(R) provided there are $g\colon G \to G'$ in **End**$(\mathbf{D})$ and $Rg \cong fR$ in **Fun**$(\mathbf{D}, \mathbf{C})$ (*i.e.* there are isomorphisms $\phi\colon R \circ G \cong F \circ R$ and $\psi\colon R \circ G' \cong F' \circ R$ such that $\psi \circ Rg \circ \phi = fR$);
- it holds that $f \leq f'$ whenever there are $g \leq g'$ in **End**$(\mathbf{D})$ and $Rg \cong fR$ and $Rg' \cong f'R$ in **Fun**$(\mathbf{D}, \mathbf{C})$.

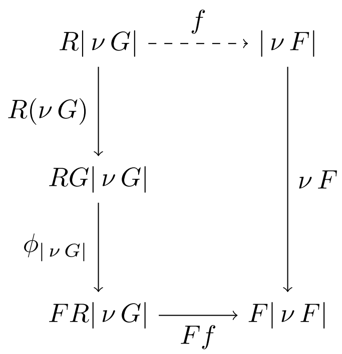
[4]This is the non-evil definition of extensions; the evil one replaces isomorphisms with equalities. In Chapter 3 we restricted to the evil formulation for adherence with the literature on Kleisli liftings.

[5]A subcategory **D** of **C** is *replete* provided that for any $f \in \mathbf{D}$ if $f \cong g$ in the arrow category $\mathbf{C}^{\to}$, then $g \in \mathbf{D}$. Equivalently, a subcategory **D** of **C** is replete if the inclusion $\mathbf{D} \hookrightarrow \mathbf{C}$ is an isofibration. The replete image of a functor $F\colon \mathbf{C} \to \mathbf{D}$ is the *repletion* the image of F *i.e.* the smallest replete category of **C** that has $\mathrm{img}(F)$ as a subcategory.

Functors of type $\mathbf{Inv}(\mathbf{D}) \to \mathbf{Ext}(R)$ are precisely behavioural schemata whose instances are extensions along R. A first attempt at generalising (4.5) to this sort of schemata is given by the following equation system:

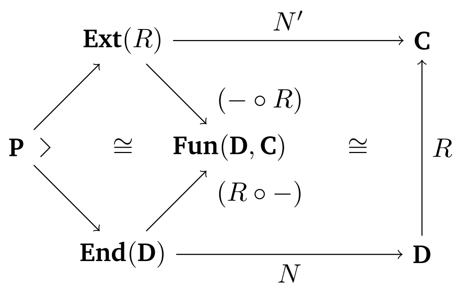
$$\begin{cases} F \cong S_{Z,Z} \\ Z \cong |\nu G| \\ R \circ G \cong F \circ R \end{cases} \tag{4.11}$$

As in the setting of Section 4.2, we are interested in the unknowns F and Z. The new unknown G ranges over functors that admit F as an extension along R and only serves to the purpose of correlating the instantiation parameter Z and behaviours of type F. In order to asses whether this is condition is met, assume G admits a final coalgebra and consider its image through R. Let $\phi\colon R \circ G \cong F \circ R$ be the natural isomorphism that exhibits the extension. The composite $\phi_{|\nu G|} \circ R(\nu G)$ is a coalgebra of type F but not necessarily the final one:



To this end, we need to impose the further constraint that the F-coalgebra $\phi_{|\nu G|} \circ R(\nu G)$ is final. In this setting this translates to the requirement that in $\mathbf{C}$ there is an isomorphism $R|\nu G| \cong |\nu F|$. We call this condition "extension of final invariants". As usual, we assume enough chosen final coalgebras.

DEFINITION 4.5. *Let N and N' chose final coalgebras. For a functor $R\colon \mathbf{D} \to \mathbf{C}$, we say that* final invariants extend along R *whenever the diagram below commutes.*



We remark that the extension of final invariants is a mild assumption since it is met by all inclusions with a reflector as stated by Freyd's "reflective subcategory lemma" [53]. Below we provide a reformulation.

LEMMA 4.17. *Final invariants extend along right adjoints.*

The notion of final invariant extension formalises the intuitive requirement that R translates abstract behaviours into parameters for the schema $S\colon \mathbf{Inv}(\mathbf{D}) \to \mathbf{Ext}(R)$. Then, we generalise Definition 4.4 to this setting as follows:

DEFINITION 4.6. *For a behavioural schema* $S\colon \mathbf{Inv}(\mathbf{D}) \to \mathbf{Ext}(R)$*, an endofunctor* F *over* $\mathbf{C}$ *is called* self-referential *whenever:*

$$\begin{cases} F \cong S_{Z,Z} \\ Z \cong |\,\nu\, G| \\ R \circ G \cong F \circ R \\ RZ \cong |\,\nu\, F| \end{cases} \tag{4.12}$$

Note that the system (4.12) coincides with (4.11) under the assumption that final invariants extend along R. This observation is crucial for characterising canonical self-referential endofunctors as invariants of suitable endofunctors along the lines of Sections 4.1 and 4.2. Henceforth, we assume that final invariants lifts along R.

Assume, for the sake of the argument, that there exists a rule E associating each endofunctor from $\mathbf{Ext}(R)$ to one of its extensions. Under the assumption of chosen extensions, the system (4.12) is equivalent to the following one:

$$\begin{cases} F \cong S_{Z,Z} \\ Z \cong |\,\nu\, E(F)| \end{cases}$$

Additionally, if this rule induces a functor going from $\mathbf{Ext}(R)$ to $\mathbf{End}(\mathbf{D})$, then assuming chosen extensions effectively turns the behavioural schema:

$$S\colon \mathbf{Inv}(D) \to \mathbf{Ext}(R)$$

into a the mixed-variance schema over $\mathbf{D}$:

$$E \circ S\colon \mathbf{Inv}(\mathbf{D}) \to \mathbf{End}(\mathbf{D})$$

and these are covered in Section 4.2. In general, there are no guarantees about existence and uniqueness of functors choosing extensions. Concerning uniqueness, we observe that when there are multiple options as E, these are always equivalent if final invariants extend along the mediating functor R. In fact, the following implication holds true:

$$\begin{array}{ccc} \mathbf{Ext}(R) & \longrightarrow & \mathbf{End}(\mathbf{C}) \\ {\scriptstyle E}\downarrow & \cong & \downarrow{\scriptstyle (-\circ R)} \\ \mathbf{End}(\mathbf{D}) & \xrightarrow[(R\circ -)]{} & \mathbf{Fun}(\mathbf{C},\mathbf{D}) \end{array} \implies \begin{array}{ccc} \mathbf{Ext}(R) & \xrightarrow{N} & \mathbf{C} \\ {\scriptstyle E}\downarrow & \cong & \uparrow{\scriptstyle R} \\ \mathbf{End}(\mathbf{D}) & \xrightarrow[N']{} & \mathbf{D} \end{array}$$

A functor $F\colon \mathbf{C} \to \mathbf{D}$ is 2-monic[6] provided that for every pair of parallel functors G and H with codomain $\mathbf{D}$, the following implication holds true:

$$F \circ G \cong F \circ H \implies G \cong H.$$

Extensions along 2-monic functors are isomorphic whenever they come from isomorphic functors. In other words, if R is 2-monic, then E is universally defined by the construction of $\mathbf{Ext}(R)$.

LEMMA 4.18. *For R 2-monic, there exists E and such that:*

$$\begin{array}{ccc} \mathbf{Ext}(R) & \longrightarrow & \mathbf{End}(\mathbf{C}) \\ {\scriptstyle E}\,\vdots & \cong & \downarrow{\scriptstyle (-\circ R)} \\ \mathbf{End}(\mathbf{D}) & \xrightarrow[(R\circ -)]{} & \mathbf{Fun}(\mathbf{D},\mathbf{C}) \end{array}$$

PROOF. The functor $(R \circ -)$ is 2-monic since, by hypothesis, R is so. By construction, the projection $p_2\colon \mathbf{P} \to \mathbf{End}(\mathbf{C})$ is 2-monic and thus $\mathbf{P}$ coincides with $\mathbf{Ext}(R)$ the replete image of p_2. The projection p_1 provides the desired choice. □

The following result is a direct consequence of Lemma 4.18 and Theorem 4.14.

COROLLARY 4.19. *Assume that the behavioural schemata $S\colon \mathbf{Inv}(\mathbf{C}) \to \mathbf{Ext}(R)$ is enriched over* **Cpo***, that $R\colon \mathbf{D} \to \mathbf{C}$ is 2-monic in* **Cpo-Cat***, and that final invariants extend along R. If* **D** *is* **Cpo***-algebraically compact, then there exists a canonical solution to* (4.12).

4.3.2 Behavioural schemata of liftings

In this section we consider behavioural schemata of type:

$$\mathbf{Inv}(\mathbf{D}) \to \mathbf{End}(\mathbf{C})$$

[6] A morphism f in a 2-category is said to be 2-monic provided that $f \circ g \cong f \circ h \implies g \cong h$.

and assume that their instances are liftings along some fixed mediating functor $R\colon \mathbf{C} \to \mathbf{D}$. This setting is similar to that of Section 4.3.1 except for the direction of R. As a consequence, we have to replace liftings for extensions and in general "symmetrise" all constructions described in Section 4.3.1.

We organise liftings along R into the category $\mathbf{Lift}(R)$. An endofunctor F over $\mathbf{C}$ is *a lifting along* R for some endofunctor G if there is an isomorphism:

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{\;F\;} & \mathbf{C} \\ {\scriptstyle R}\downarrow & \cong & \downarrow{\scriptstyle R} \\ \mathbf{D} & \xrightarrow[\;G\;]{} & \mathbf{D} \end{array}$$

in $\mathbf{Fun}(\mathbf{C},\mathbf{D})$. In order to generalise this condition beyond objects in $\mathbf{End}(\mathbf{C})$ and $\mathbf{End}(\mathbf{D})$ consider the following 2-pullback:

$$\begin{array}{ccc} \mathbf{P} \lrcorner & \xrightarrow{\;p_2\;} & \mathbf{End}(\mathbf{D}) \\ {\scriptstyle p_1}\downarrow & \cong & \downarrow{\scriptstyle (-\circ R)} \\ \mathbf{End}(\mathbf{C}) & \xrightarrow[\;(R\circ -)\;]{} & \mathbf{Fun}(\mathbf{C},\mathbf{D}) \end{array}$$

Then, the projection of $\mathbf{P}$ into $\mathbf{End}(\mathbf{C})$ identifies all liftings along R. Formally, we define $\mathbf{Lift}(R)$ as the replete image of p_1:

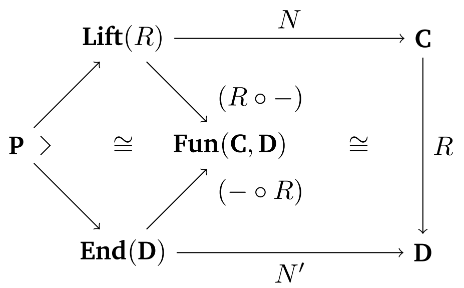
$$\mathbf{Lift}(R) \cong \mathrm{rimg}(p_1).$$

This definition extends to the order enriched setting as it is. Let $\mathbf{V}$ stand for either $\mathbf{Cpo}$ or $\mathbf{Cpo}_\perp$. For R a $\mathbf{V}$-functor, $\mathbf{Lift}(R)$ is (isomorphic to) the sub-$\mathbf{V}$-category of $\mathbf{V}\text{-}\mathbf{End}(\mathbf{C})$ with the following properties:

- an endofunctor F over $\mathbf{C}$ is an object of $\mathbf{Lift}(R)$ provided that there are G in $\mathbf{End}(\mathbf{D})$ and $R \circ F \cong G \circ R$ in $\mathbf{Fun}(\mathbf{C},\mathbf{D})$;
- a natural transformation $f\colon F \to F'$ is a morphism of $\mathbf{Lift}(R)$ provided there are $g\colon G \to G'$ in $\mathbf{End}(\mathbf{D})$ and $Rf \cong gR$ in $\mathbf{Fun}(\mathbf{C},\mathbf{D})$ (*i.e.* there are isomorphisms $\phi\colon R \circ F \cong G \circ R$ and $\psi\colon R \circ F' \cong G' \circ R$ such that $\psi \circ Rf \circ \phi = g_R$);
- it holds that $f \leq f'$ whenever there are $g \leq g'$ in $\mathbf{End}(\mathbf{D})$, $Rf \cong gR$ and $Rf' \cong g'R$ in $\mathbf{Fun}(\mathbf{C},\mathbf{D})$.

The lifting condition alone does not capture the idea that the functor R translates abstract behaviours between liftings and their underlying functors. More precisely, we need to further assume that final invariants lift along R.

DEFINITION 4.7. *Let N and N' chose final coalgebras. For a functor $R\colon \mathbf{C} \to \mathbf{D}$, we say that* final invariants lift along R *whenever the diagram below commutes.*



For instance, let F be a lifting of G and write ϕ for the isomorphism $R \circ F \cong G \circ F$. Definition 4.7 requires the G-coalgebra $\phi_{|\,\nu F|} \circ \nu F$ to be final. We remark that this is a mild assumption since it is met by all right adjoints.

LEMMA 4.20. *Final invariants lift along right adjoints.*

We are now able to generalise Definition 4.4 to this setting.

DEFINITION 4.8. *For a behavioural schema $S\colon \mathbf{Inv}(\mathbf{D}) \to \mathbf{Lift}(R)$, an endofunctor F over $\mathbf{C}$ is called* self-referential *whenever:*

$$\begin{cases} F \cong S_{Z,Z} \\ Z \cong |\,\nu G| \\ R \circ F \cong G \circ R \\ Z \cong R|\,\nu F| \end{cases} \tag{4.13}$$

The assumption that final invariants lift along R renders the auxiliary unknown G of the system (4.13) unnecessary. Under the aforementioned assumption, (4.13) admits the equivalent formulation below:

$$\begin{cases} F \cong S_{Z,Z} \\ Z \cong R|\,\nu F| \end{cases} \tag{4.14}$$

Following the same procedure described in Section 4.2, we derive symmetric endofunctors whose invariants are solutions to this equation. To this end, assume chosen final coalgebras for behavioural endofunctors in the essential image of S. Canonical solutions with respect to the unknown Z are captured by:

$$\frac{\mathbf{Inv}(\mathbf{D}) \xrightarrow{S} \mathbf{Lift}(R) \xrightarrow{N} \mathbf{C} \xrightarrow{R} \mathbf{D}}{\mathbf{Inv}(\mathbf{D}) \xrightarrow{\widehat{S}} \mathbf{Inv}(\mathbf{Lift}(R)) \xrightarrow{\mathbf{Inv}(N)} \mathbf{Inv}(\mathbf{C}) \xrightarrow{\mathbf{Inv}(R)} \mathbf{Inv}(\mathbf{D})}$$

and canonical solutions with respect to the unknown F are captured by:

$$\frac{\mathbf{Inv}(\mathbf{Lift}(R)) \xrightarrow{\mathbf{Inv}(N)} \mathbf{Inv}(\mathbf{C}) \xrightarrow{\mathbf{Inv}(R)} \mathbf{Inv}(\mathbf{D}) \xrightarrow{S} \mathbf{Lift}(R)}{\mathbf{Inv}(\mathbf{Lift}(R)) \xrightarrow{\mathbf{Inv}(N)} \mathbf{Inv}(\mathbf{C}) \xrightarrow{\mathbf{Inv}(R)} \mathbf{Inv}(\mathbf{D}) \xrightarrow{\widehat{S}} \mathbf{Inv}(\mathbf{Lift}(R))}$$

Existence of canonical self-referential endofunctors follows from local continuity of S and R.

THEOREM 4.21. *Assume that the behavioural schema S is* **Cpo***-enriched and that final invariants lift along R. If* **D** *is* **Cpo***-algebraically compact, then there is a canonical solution to* (4.14).

PROOF. We observe that both $\mathbf{Inv}(R) \circ \mathbf{Inv}(N) \circ \widehat{S}$ and $\widehat{S} \circ \mathbf{Inv}(R) \circ \mathbf{Inv}(N)$ are locally continuous by construction. It follows from Corollary 2.10 and hypothesis on **D** that $\mathbf{Inv}(\mathbf{D})$ is **Cpo**-algebraically compact. We conclude from Lemma 4.13 that both endofunctors are algebraically compact. Initial and final invariants for these endofunctors are canonically isomorphic hence symmetric objects. These invariants identify canonical solutions to (4.14). □

4.4 CONCLUDING REMARKS AND FUTURE WORK

In this chapter we presented a general coalgebraic account of self-referential systems: we introduced the notion of self-referential endofunctor and showed that coalgebras for these behavioural endofunctors capture the semantics of self-referential systems. In order to formalise this notion, we considered behavioural schemata and defined self-referential endofunctors as those instances identified by their final coalgebras. We observed that a behavioural schema may admit several self-referential endofunctors and organised them into a category by finding an appropriate notion of morphism. In particular, morphisms of self-referential endofunctors are natural transformations but also morphisms between objects of values with the additional requirement that they are coherent with respect to final semantics. This definition reflects the characteristic interpretation of self-referential systems and offers a notion of soundness: morphisms induce functors between categories of self-referential systems with the property of preserving behavioural equivalence. Initial and final objects of this category describe canonical semantic models of self-referential systems in the sense that:

- initial self-referential endofunctors correspond to semantic models of self-referential systems that are sound with respect to all other models;
- dually, final self-referential endofunctors correspond to semantic models that are complete with respect to all other models.

These canonical characterisations support reasoning about self-referential systems (for the same schema) even if they have semantic models of different type. Morphism into the final self-referential endofunctor induce functors into a shared category of coalgebras such that they preserve bisimulations and carriers. Dually, for morphism from the initial self-referential endofunctor.

We provided general results for determining canonical self-referential endofunctors. In Section 4.1 we considered covariant behavioural schemata which abstract systems that can only perform outputs. This initial simplification allowed us to focus on the challenge of modelling self-referential behaviours without the additional issues due to contravariant occurrences of values *i.e.* inputs. We proposed three main methodologies for determining canonical self-referential endofunctors in this setting:

- the first is a characterisation of initial (resp. final) self-referential endofunctors as initial (resp. final) invariants for certain endofunctors synthesised from behavioural schemata;
- the second considers determines final self-referential endofunctors from final coalgebras for diagonalisations of behavioural schemata;
- the third relies on the iterative refinement of values and their dynamics to construct sequences that stabilise once a self-referential endofunctor is encountered.

Each of these methodologies has its advantages and peculiarities: the first requires chosen final coalgebras for all instances of a schema but characterises all self-referential endofunctor as invariants, not only canonical ones; the second does not assume chosen final coalgebras but covers only final self-referential endofunctors; the third can be used to obtain self-referential endofunctors starting from concrete representations of self-referential systems. As an application, we modelled self-referential (partial) streams and labelled transition systems.

In Section 4.2 we considered mixed-variance behavioural schemata which abstract systems that can perform inputs and outputs and illustrated how to obtain canonical self-referential endofunctors as invariants of certain endofunctors. Compared to Section 4.1, mixed-variance schemata presented us with two additional challenges: the first is, of course, mixed-variance occurrences of values and the second is symmetry between inputs and outputs. We addressed the first using symmetric functors on universal involutory categories and the second using algebraic compactness—two established tools from domain theory. Then, we proved that existence of canonical self-referential endofunctors reduces to assessing local continuity or local contractiveness of behavioural schemata. As an application, we modelled self-referential non-deterministic systems with synchronous exchanges *à la* CCS with values. Of the three methodologies we presented in Section 4.1, only the first extends to mixed-variance schemata: diagonalisation and observations refinement methodologies cannot be applied to this setting. We identified the cause of this fact in the symmetry of inputs and outputs assumed by self-referential systems. We remark that once this condition is dropped, all constructions discussed in Section 4.1 extend to the mixed-variance setting—with minor technicalities. The only modification required is to perform all constructions in the context of dialgebras instead of coalgebras. Finally, in Section 4.3 we explored conservative generalisations of self-referential endofunctors that support situations where values and systems need to be modelled in distinct categories.

We focussed on the definition and construction of self-referential endofunctors: defining behavioural endofunctors is the crucial step of the coalgebraic method, as it corresponds to specify the observable dynamics of systems under scrutiny. Once a behavioural endofunctor is defined, many important properties and general results can be instantiated. Nonetheless, we consider the notion of self-referential endofunctor more as a founding reference for the study of self-referential systems. From a more applicative point of view, working with behavioural endofunctors defined in terms of their own final coalgebras can be challenging. This calls for the development of models and efficient proof techniques that avoid any prior knowledge about the final coalgebra. We consider this line of research as the most pressing continuation of this work.

This is the first work to propose a general coalgebraic model of self-referential systems. Related works can be found in the vast literature about higher-order languages. Perhaps, the closest works to this thesis are [68, 95] where Honsell and Lenisa studied the final semantics of the untyped λ-calculus and of a simple while language with higher-order assignments. We remark that these works take terms as values—not abstract behaviours. For instance, [68] considers coalgebras of type $Id^{\Lambda_0} + 1$, and variations thereof, where Λ_0 is the set of closed λ-terms. In general, coalgebras for behavioural functors like $Id^{\Lambda_0} + 1$ need not to respect the semantics of terms. Indeed this information is not present in the definition of the functor and this forced Honsell and Lenisa to restrict to certain well-behaved coalgebras. These facts are among the motivations that prompted us to consider abstract behaviours as values while modelling the semantics self-referential systems.

One of the most recent developments regarding bisimulations for higher-order languages is the notion of environmental bisimulation proposed by Sangiorgi, Kobayashi and Sumii [127]. As discussed in the opening of Chapter 1, environmental bisimulation *et similia* focus on capturing contextual equivalence and to this end test systems with challenges devised on the guise of contexts. In particular, environmental bisimulation require processes to behave likewise only on inputs that are indistinguishable with respect to contexts forged starting from the current knowledge, not necessarily bisimilar. Thus, the notion of bisimulation obtained from this work does not always coincide with of environmental bisimulation.

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