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ORIGINAL PAPER

Free groups of ideals

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Abstract
We study the freeness of the group $\mathrm{Inv}(D)$ of invertible ideals of an integral domain D, and the freeness of some related groups of (fractional) ideals. We study the relation between $\mathrm{Inv}(D)$ and $\mathrm{Inv}(D_P)$, in particular in the locally finite case, and we analyze in more detail the case where D is Noetherian (obtaining a characterization of when $\mathrm{Inv}(D)$ is free for one-dimensional analytically unramified Noetherian domains) and where D is Prüfer.

Keywords Invertible ideals · Star operations · Free groups · Prüfer domains · Locally finite domains

Mathematics Subject Classification 13A15 · 13A18 · 13F05 · 13F20 · 13G05 · 20K99

1 Introduction

Let D be a Dedekind domain. The unique factorization of the ideals of D into product of prime ideals can be expressed by saying that the group $\mathrm{Inv}(D)$ of its invertible ideals is the free abelian group on the set $\mathrm{Max}(D)$ of maximal ideals. Recently, it has been shown that the group $\mathrm{Inv}(D)$ is free also when D is only *locally* Dedekind (or, equivalently, when it is locally a discrete valuation ring), that is, when D is a so-called *almost Dedekind domain*. This has been proved first by considering radical factorization (and the class of SP-domains) [15], and then extending these results with the help of a derived set-like sequence [23, 25].

In this paper, we start a more general study of conditions under which the group $\mathrm{Inv}(D)$ of the nonzero invertible ideals of D is free; we also extend our reach to closely related groups, such as the group $\mathrm{Princ}(D)$ of nonzero principal ideals, the group $\mathrm{Inv}^t(D)$ of t-invertible ideals and the group $\mathrm{Div}(D)$ of v-invertible ideals. In Sect. 3 and 5, we relate the group $\mathrm{Inv}(D)$ with the group of invertible ideals on the localizations of D; in particular, we show that if D is a locally finite intersection of domains whose groups of invertible ideals are free then also $\mathrm{Inv}(D)$ is free (Proposition 3.6), and that the same holds in the case of one-dimensional domains with scattered maximal space (Proposition 5.2). In some cases, we work in the more general case of groups of $\star$-invertible $\star$-ideals, where $\star$ is a star operation (see below for the definition); in this context, some problems of freeness of the positive semigroup of this group have been investigated in [5].

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In Sect. 4 we study the case of Noetherian domains: we show that the freeness of $\mathrm{Princ}(D)$ is related to the group of units of D and of its integral closure $\overline{D}$ (Proposition 4.2) and in the one-dimensional analytically unramified case we characterize when $\mathrm{Inv}(D)$ is free and show that this is a rather strong condition in the non-integrally closed case (Theorem 4.15). In Sect. 6 we study in more detail the case of Prüfer domains, and we show that under some finiteness hypothesis it is possible to characterize when $\mathrm{Inv}(D)$ and $\mathrm{Div}(D)$ are free (Propositions 6.10 and 6.13). We also advance the conjecture that $\mathrm{Inv}(D)$ is free whenever D is a strongly discrete Prüfer domain.

In the final Sect. 7 we relate the group $\mathrm{Inv}(R)$ of invertible ideals of a D-algebra R with the group $\mathrm{Inv}(D)$, and prove a sufficient condition for $\mathrm{Inv}(R)$ to be free, based on the properties of the extension $D \subseteq R$ (Theorem 7.1). As a consequence, we show that the group $\mathrm{Inv}(\mathrm{Int}(D))$ (where $\mathrm{Int}(D)$ is the ring of integer-valued polynomials over D) is free for every Dedekind domain D.

2 Preliminaries

Throughout the paper, D is an integral domain, and K will denote its quotient field. We also suppose $D \neq K$, i.e., that D is not a field. We use $\mathcal{U}(D)$ to denote the group of units of D and $\mathrm{Jac}(D)$ to denote the Jacobson radical of D.

2.1 Fractional and invertible ideals

A *fractional ideal* of D is a D-submodule I of K such that $dI \subseteq D$ for some $d \in K, d \neq 0$. We denote by $\mathcal{F}(D)$ the set of nonzero fractional ideals of D. We shall use the term "ideal" to refer to fractional ideals; to refer to ideals in the usual sense (i.e., contained in D) we shall use the expression "integral ideal" or "proper ideal".

A fractional ideal I is *invertible* if there is a fractional ideal J such that $IJ = D$; in this case, $J = (D : I) := \{x \in K \mid xI \subseteq J\}$. Every invertible ideal is finitely generated; moreover, a nonzero ideal I is invertible if and only if I is finitely generated and locally principal (i.e., ID_M is principal for every $M \in \mathrm{Max}(D)$). In particular, if D is local or semilocal (i.e., if $\mathrm{Max}(D)$ is finite) then every invertible ideal is principal.

The set $\mathrm{Inv}(D)$ of invertible ideals is a group under the product of ideals; $\mathrm{Inv}(D)$ contains as a subgroup the set $\mathrm{Princ}(D)$ of nonzero principal ideals of D. The quotient $\mathrm{Inv}(D)/\mathrm{Princ}(D)$ is called the *Picard group* of D, and is denoted by $\mathrm{Pic}(D)$. The natural map $\mathcal{U}(K) \longrightarrow \mathrm{Princ}(D)$, $x \mapsto xD$, induces an exact sequence

$$0 \longrightarrow \mathcal{U}(D) \longrightarrow \mathcal{U}(K) \longrightarrow \mathrm{Princ}(D) \longrightarrow 0.$$

2.2 Star operations

A *star operation* is a map $\star : \mathcal{F}(D) \longrightarrow \mathcal{F}(D)$, $I \mapsto I^\star$, such that the following conditions hold for every $I, J \in \mathcal{F}(D)$ and every nonzero $x \in K$:

- $I \subseteq I^\star$;
- if $I \subseteq J$, then $I^\star \subseteq J^\star$;
- $(I^\star)^\star = I^\star$;
- $D^\star = D$;
- $x \cdot I^\star = (xI)^\star$.

The ideal $I^\star$ is said to be the *$\star$-closure* of I, and I is said to be *$\star$-closed* (or a *$\star$-ideal*) if $I = I^\star$. A star operation $\star$ is *of finite type* if $I^\star = \bigcup\{J^\star \mid J \subseteq I, J \neq (0)$ finitely generated$\}$.

A *$\star$-maximal* ideal is an ideal that is maximal in the set of the $\star$-ideals contained in D; we denote their set with $\mathrm{Max}^\star(D)$. Every $\star$-maximal ideal is prime. If $\star$ is of finite type, then every proper $\star$-ideal is contained in a $\star$-maximal ideal.

The set of star operations has an order, where $\star_1 \leq \star_2$ if and only if $I^{\star_1} \subseteq I^{\star_2}$ for every nonzero ideal I. Under this order, the smallest star operation is the identity (usually denoted by d), while the largest is the *divisorial closure* v, where $I^v := (D : (D : I))$; a v-closed ideal is called a *divisorial ideal*. The *t-operation* is the star operation defined by $I^t := \{x \in K \mid x \in J^v$ for some nonzero finitely generated $J \subseteq I\}$, and is the largest star operation of finite type.

An ideal I is *$\star$-invertible* if there is a fractional ideal J such that $(IJ)^\star = D$; every $\star$-invertible $\star$-ideal is divisorial, and thus is closed by every star operation. The set $\mathrm{Inv}^\star(D)$ of $\star$-closed $\star$-invertible ideals is a group under the "$\star$-product" $I \times_\star J := (IJ)^\star$. If $I_1, \ldots, I_n \in \mathrm{Inv}^\star(D)$, we have $I_1 \times_\star \cdots \times_\star I_n = (I_1 \cdots I_n)^\star$ [12, Proposition 32.2(c)].

By definition, $\mathrm{Inv}(D) = \mathrm{Inv}^d(D)$; we set $\mathrm{Div}(D) := \mathrm{Inv}^v(D)$. If $\star_1 \leq \star_2$, then every $\star_1$-invertible $\star_1$-ideal is a $\star_2$-invertible $\star_2$-ideal; hence, there is a containment $\mathrm{Inv}^{\star_1}(D) \subseteq \mathrm{Inv}^{\star_2}(D)$. In particular, $\mathrm{Inv}(D)$ is contained in $\mathrm{Inv}^\star(D)$ for every star operation $\star$ on D, while $\mathrm{Div}(D)$ contains every $\mathrm{Inv}^\star(D)$.

See [12, §32], [17] or [27] for general results about star operations and $\star$-invertibility.

2.3 Jaffard families

An *overring* of D is a ring contained between D and K; we denote by $\mathrm{Over}(D)$ the set of overrings of D. A *flat overring* is an overring that is flat as a D-module. We say that a family Θ of overrings is:

- *complete* if $I = \bigcap\{IT \mid T \in \Theta\}$ for every ideal I of D;
- *independent* if, for every $T \neq S$ in Θ, there are no nonzero prime ideals P of D such that $PT \neq T$ and $PS \neq S$; if the elements of Θ are flat over D, this is equivalent to asking that $TS = K$ for all $T \neq S$ in Θ (cfr. [6, Section 6.2] and [22, Lemma 3.4 and Definition 3.5]);
- *locally finite* if, for every $x \in D$, we have $xT = T$ for all but finitely many $T \in \Theta$.

A domain is *locally finite* if the set $\Theta := \{D_M \mid M \in \mathrm{Max}(D)\}$ is locally finite.

A *Jaffard family* of D is a family Θ of flat overrings that is complete, independent and locally finite, and such that $K \notin \Theta$ (see [6, Section 6.3] and [21]). An overring T of D is a *Jaffard overring* if it belongs to a Jaffard family of D [22, Definition 3.7].

If D is one-dimensional, the set $\Theta := \{D_M \mid M \in \mathrm{Max}(D)\}$ is always complete and independent, and thus it is a Jaffard family if and only if Θ (or, equivalently, D) is locally finite.

If $\star$ is a star operation on D and T is a flat overring of D, then $\star$ is said to be *extendable* to T if the map

$$\begin{aligned} \star_T : \mathcal{F}(T) &\longrightarrow \mathcal{F}(T), \\ IT &\longmapsto I^\star T \end{aligned}$$

is well-defined, that is, if $IT = JT$ implies $I^\star T = J^\star T$. (Note that, since T is flat, every fractional ideal is extended from D [16, Discussion 2.1(1)].) Every star operation of finite type is extendable to every flat overring. Moreover, if T is a Jaffard overring then every star operation is extendable to T, and every star operation on T is the extension of a star operation on D [21, Theorem 5.4]; in particular, the d-, t- and v-operations on D extend, respectively, to the d-, t- and v-operation on T (use [21, Theorem 5.6]).

2.4 Topologies

Let $\mathrm{Spec}(D)$ be the prime spectrum of D. We denote by $V(I)$ and $D(I)$, respectively, the basic closed and the basic open subset of $\mathrm{Spec}(D)$ in the Zariski topology induced by the ideal I. The *inverse topology* on $\mathrm{Spec}(D)$ (and on its subsets) is the topology generated (as a subbasis of open sets) by the $V(I)$, as I ranges among the finitely generated ideals. We denote by $\mathrm{Spec}(D)^{\mathrm{inv}}$ this topological space.

The *Zariski topology* on $\mathrm{Over}(D)$ is the topology generated by the sets $\mathcal{B}(x) := \{T \in \mathrm{Over}(D) \mid x \in T\}$, as x ranges in K, while the *inverse topology* is generated by the complements of the $\mathcal{B}(x)$, i.e., by $\mathrm{Over}(D) \setminus \mathcal{B}(x)$ as x ranges in K.

A point x of a topological space X is said to be *isolated* if $\{x\}$ is an open set, and it is said to be a *limit point* if it is not isolated. The set of all limit points of X is the *derived set* of X, and is denoted by $\mathcal{D}(X)$. More generally, for every ordinal number α, the *α-th derived set* is

$$\mathcal{D}^\alpha(X) := \begin{cases} X & \text{if } \alpha = 0; \\ \mathcal{D}(\mathcal{D}^\beta(X)) & \text{if } \alpha = \beta + 1; \\ \bigcap_{\beta<\alpha} \mathcal{D}^\beta(X) & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

The space X is said to be *scattered* if $\mathcal{D}^\alpha(X) = \emptyset$ for some α.

2.5 Valuation and Prüfer domains

A *valuation domain* is an integral domain whose ideals are linearly ordered; equivalently, it is a domain V with quotient field K such that there is a surjective map $v : K \setminus \{0\} \longrightarrow \Gamma$ (where Γ is a linearly ordered group) such that $v(ab) = v(a) + v(b)$ and $v(a+b) \geq \min\{v(a), v(b)\}$ for every nonzero $a, b \in K$, and $V = \{x \in K \setminus \{0\} \mid v(x) \geq 0\} \cup \{0\}$. The group Γ is called the *value group* of V; we denote it by $\Gamma(V)$. If $\Gamma(V) \simeq \mathbb{Z}$, then V is said to be a *discrete valuation domain* (DVR).

A prime ideal P of V is *branched* if there is a prime ideal $Q \subsetneq P$ such that there are no prime ideals properly contained between Q and P; otherwise P is said to be *unbranched*. Equivalently, P is unbranched if and only if it is the union of the prime ideals properly contained in P.

A *Prüfer domain* is an integral domain that is locally a valuation domain, or equivalently such that every nonzero finitely generated ideal is invertible. A Prüfer domain D is *strongly discrete* if no prime ideal of D is idempotent. See e.g. [12] or [8] for properties of valuation and Prüfer domains.

2.6 Free groups and homology

Let A be an abelian group. Then, A is a *free abelian group* if it is the direct sum of infinite cyclic groups, i.e., copies of $\mathbb{Z}$; equivalently, A is free if and only if it has a basis, i.e., if there is a subset $S \subset A$ such that every element of A can be written uniquely as a sum of elements of S. Equivalently, a free abelian group is a free $\mathbb{Z}$-module. Throughout the paper, whenever we speak of "free" groups we always mean free *abelian* groups.

A free group is always torsionfree, and a subgroup of a free group is always free. A free group A is *projective*, i.e., every exact sequence

$$0 \longrightarrow B \longrightarrow C \longrightarrow A \longrightarrow 0$$

splits, and in particular $C \simeq B \oplus A$.

We shall sometimes use the so-called snake lemma: given a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & H_1 \arrow[r] \arrow[d, "f"] & G_1 \arrow[r] \arrow[d, "g"] & L_1 \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & H_2 \arrow[r] & G_2 \arrow[r] & L_2 \arrow[r] & 0.
\end{tikzcd}$$

there is an exact sequence

$$0 \longrightarrow \ker(f) \longrightarrow \ker(g) \longrightarrow \ker(h) \longrightarrow \operatorname{coker}(f) \longrightarrow \operatorname{coker}(g) \longrightarrow \operatorname{coker}(h) \longrightarrow 0.$$

3 Local finiteness

One of the main tools of the paper is the possibility to extend invertible ideals. If $D \subseteq R$ is an extension of integral domains, we have a natural map

$$\begin{aligned} \phi\colon \operatorname{Inv}(D) &\longrightarrow \operatorname{Inv}(R), \\ I &\longmapsto IR. \end{aligned}$$

Indeed, if I is an invertible ideal, then $IJ = D$ for some fractional ideal J, and

$$\phi(I)\phi(J) = IRJR = IJR = DR = R,$$

so that $\phi(I)$ is an invertible ideal of R. It is straightforward to see that ϕ is a group homomorphism, and that it restricts to a map $\operatorname{Princ}(D) \longrightarrow \operatorname{Princ}(R)$. In general, we cannot say much more about ϕ, as the next examples show.

Example 3.1 (a) If $R = D[X]$ is the polynomial ring over D, then ϕ is injective but not surjective; for example, the principal ideal $fD[X]$ is not in the image of ϕ for every non-constant polynomial f.

(b) If $R = D_P$ is the localization of R at a prime P, then ϕ is not injective, as any principal ideal xD with $x \in D \setminus P$ extends to the whole R. However, it is surjective: indeed, every invertible ideal of D_P is principal (since D_P is local) and thus it is an extension of a principal ideal of D.

(c) If D is Prüfer and $R \in \operatorname{Over}(D)$, then ϕ is surjective: indeed, given $I \in \operatorname{Inv}(R)$, let I_0 be an ideal of D generated by a finite generating set of I: then, I_0 is invertible (since it is a finitely generated ideal of a Prüfer domain) and its extension is I by construction.

(d) In general, ϕ may not be surjective even if R is a localization of D. For example, let D be a local Krull domain of dimension 2, and suppose that there is a nonzero non-unit $f \in D$ such that $R := D[1/f]$ is not a unique factorization domain. Then, R is a Dedekind domain and so $\operatorname{Inv}(R)$ contains non-principal ideals, but every invertible ideal of D is principal (since D is local). Hence there are invertible ideals of R that are not extensions of invertible ideals of D.

More generally, if $\star$ is a star operation on D and T is a flat overring of D such that $\star$ is extendable to T, then we have a map

$$\begin{aligned}\phi_\star\colon \operatorname{Inv}^\star(D) &\longrightarrow \operatorname{Inv}^{\star_T}(T),\\ I &\longmapsto IT,\end{aligned}$$

that is a group homomorphism [21, proof of Proposition 7.1]. In particular, this map exists when $\star$ is of finite type (and thus for the t-operation), but not in general: for example, it cannot be always extended to the v-operation.

Example 3.2 Let D be a one-dimensional Prüfer domain such that every maximal ideal of D is principal except for one, say Q; suppose also that Q is not the radical of any principal ideal and that D_Q is not discrete, and that the Jacobson radical J of D is nonzero. If $x \in \bigcap\{P \mid P \in \operatorname{Max}(D), P \neq Q\}\setminus\{Q\}$, then $(x, Q) = D$ and thus $1 = xy + q$ for some $y \in D$ and $q \in Q$. Thus $q = 1 - xy \in Q$; however, $1 - xy \notin P$ for every $P \in \operatorname{Max}(D)$, $P \neq Q$ and thus Q would be the radical of qD, a contradiction.

Hence $J = \bigcap\{P \mid P \in \operatorname{Max}(D), P \neq Q\}$. Therefore, J is divisorial, since each prime ideal $P \neq Q$ is finitely generated and thus invertible (and so divisorial). Moreover, since D is completely integrally closed, every ideal is v-invertible [12, Theorem 3.4], and in particular J is v-invertible.

The extension JD_Q is equal to QD_Q since J is radical; however, QD_Q is not divisorial in D_Q since QD_Q is not principal [12, §34, Exercise 12]. Hence, the extension map $\operatorname{Div}(D) \longrightarrow \operatorname{Div}(D_Q)$ is not well-defined.

The map ϕ becomes better-behaved when dealing with Jaffard families.

Proposition 3.3 *Let D be an integral domain, $\star$ a star operation on D and Θ be a Jaffard family on D. Then, there is a group isomorphism*

$$\begin{aligned}\Phi\colon \operatorname{Inv}^\star(D) &\longrightarrow \bigoplus_{T\in\Theta} \operatorname{Inv}^{\star_T}(T),\\ I &\longmapsto (IT)_{T\in\Theta}.\end{aligned}$$

In particular,

(a) $\operatorname{Inv}(D) \simeq \bigoplus_{T\in\Theta} \operatorname{Inv}(T)$*;*
(b) $\operatorname{Inv}^t(D) \simeq \bigoplus_{T\in\Theta} \operatorname{Inv}^t(T)$*;*
(c) $\operatorname{Div}(D) \simeq \bigoplus_{T\in\Theta} \operatorname{Div}(T)$.

Thus $\operatorname{Inv}(D)$ *(respectively,* $\operatorname{Inv}^t(D)$, $\operatorname{Div}(D)$*) is free if and only if* $\operatorname{Inv}(T)$ *(respectively,* $\operatorname{Inv}^t(T)$, $\operatorname{Div}(T)$*) is free for every $T \in \Theta$.*

Proof The first part of the theorem is exactly [21, Proposition 7.1], and the second part follows by taking $\star = d$, $\star = t$ and $\star = v$, respectively. The last part follows trivially. □

Corollary 3.4 *Let D be a locally finite one-dimensional domain. Then,* $\operatorname{Inv}(D) \simeq \bigoplus\{\operatorname{Inv}(D_M) \mid M \in \operatorname{Max}(D)\}$*; in particular,* $\operatorname{Inv}(D)$ *is free if and only if* $\operatorname{Inv}(D_M)$ *is free for every maximal ideal M.*

Proof It is enough to note that $\Theta := \{D_M \mid M \in \operatorname{Max}(D)\}$ is a Jaffard family and apply Proposition 3.3. □

Usually, we do not have at our disposal a Jaffard family. However, if we are interested only in the freeness of $\mathrm{Inv}(D)$, we can work with much weaker hypothesis.

Lemma 3.5 *Let D be an integral domain, and let Θ be a family of overrings of D such that $D = \bigcap\{T \mid T \in \Theta\}$. Then, there is an injective group homomorphism*

$$\begin{aligned}\Phi\colon \mathrm{Inv}(D) &\longrightarrow \prod_{T\in\Theta} \mathrm{Inv}(T).\\ I &\longmapsto (IT)_{T\in\Theta}.\end{aligned}$$

Moreover, if each $T \in \Theta$ is flat, then there is an injective group homomorphism

$$\begin{aligned}\Phi_t\colon \mathrm{Inv}^t(D) &\longrightarrow \prod_{T\in\Theta} \mathrm{Inv}^t(T).\\ I &\longmapsto (IT)_{T\in\Theta}.\end{aligned}$$

Proof The discussion in the previous part of the section shows that Φ is a well-defined group homomorphism, since each component $\mathrm{Inv}(D) \longrightarrow \mathrm{Inv}(T)$ is a homomorphism. Likewise, we have a homomorphism $\mathrm{Inv}^t(D) \longrightarrow \mathrm{Inv}^{t_T}(T)$ (where t_T is the extension of the t-operation), and $\mathrm{Inv}^{t_T}(T)$ is contained in $\mathrm{Inv}^t(T)$ since t_T is of finite type and the t-operation is the largest star operation of finite type.

To show that these maps are injective, we note that the map $\wedge_\Theta : J \mapsto \bigcap\{JT \mid T \in \Theta\}$ is a star operation on D; since every t-invertible t-ideal (and, in particular, every invertible ideal) is divisorial, it is also $\wedge_\Theta$-closed, and thus for every $I \in \mathrm{Inv}^t(D)$ we have $I = \bigcap\{IT \mid T \in \Theta\}$, from which the injectivity of Φ and Φ_t follows. The claim is proved. □

Proposition 3.6 *Let D be an integral domain. Let Θ be a family of overrings of D such that:*

- *Θ is locally finite;*
- *$D = \bigcap\{T \mid T \in \Theta\}$*
- *$\mathrm{Inv}(T)$ is free for every $T \in \Theta$.*

Then, $\mathrm{Inv}(D)$ is free.

Proof Consider the map Φ of Lemma 3.5. Since Θ is locally finite, given an $I \in \mathrm{Inv}(D)$ we have $IT = T$ for all but finitely many $T \in \Theta$; thus the range of Φ is contained in the direct sum $\bigoplus \mathrm{Inv}(T)$, which is free since each $\mathrm{Inv}(T)$ is free. Hence $\mathrm{Inv}(D) \simeq \Phi(\mathrm{Inv}(D))$ is (isomorphic to) a subgroup of a free group, and thus it is itself free. The claim is proved. □

Corollary 3.7 *Let D be an integral domain. If D is locally finite and $\mathrm{Inv}(D_M)$ is free for every $M \in \mathrm{Max}(D)$, then $\mathrm{Inv}(D)$ is free.*

Proof Saying that D is locally finite is equivalent to saying that $\Theta := \{D_M \mid M \in \mathrm{Max}(D)\}$ is locally finite. The claim now follows from Proposition 3.6. □

We can also obtain a similar result for t-invertible ideals. Recall that a domain has the *t-finite character* if the set $\{D_M \mid M \in \mathrm{Max}^t(D)\}$ is locally finite.

Proposition 3.8 *Let D be an integral domain with the t-finite character. If $\mathrm{Inv}^t(D_M)$ is free for every $M \in \mathrm{Max}^t(D)$, then $\mathrm{Inv}^t(D)$ is free.*

Proof It is enough to apply Lemma 3.5 on $\Theta := \{D_M \mid M \in \mathrm{Max}^t(D)\}$ and use the same method of the proof of Proposition 3.6 with the t-finite character to guarantee the local finiteness of Θ. □

4 The Noetherian case

In this section, we study the case in which D is a Noetherian domain.

The integrally closed case can be done in greater generality; recall that a *Krull domain D* is an integral domain that is the intersection of a locally finite family of discrete valuation rings, each of which is a localization of D. Every Noetherian integrally closed domain is a Krull domain and, indeed, the integral closure of every Noetherian domain (in its quotient field) is a Krull domain [20].

Let D be a Krull domain. Lemma 3.5 and Proposition 3.8 give a map

$$\begin{aligned}\widetilde{\Phi}\colon \mathrm{Inv}^t(D) &\longrightarrow \bigoplus_{M\in\mathrm{Max}^t(D)} \mathrm{Inv}^t(D_M),\\ I &\longmapsto (ID_M)_{M\in\mathrm{Max}^t(D)}\end{aligned}$$

that becomes an isomorphism, since D_M is a DVR whenever $M \in \mathrm{Max}^t(D)$ and every $M \in \mathrm{Max}^t(D)$ is in $\mathrm{Inv}^t(D)$. Therefore, we obtain back the well-known result (see e.g. [10, Chapter 1, Corollary 3.4]) that, for a Krull domain, the set $X^1(D)$ of prime ideals of height 1 is a basis of $\mathrm{Inv}^t(D) = \mathrm{Div}(D)$ (which, in particular, is a free group). For ease of reference, we state this fact in the following proposition.

Proposition 4.1 *Let D be a Krull domain. Then,* $\mathrm{Div}(D)$, $\mathrm{Inv}(D)$ *and* $\mathrm{Princ}(D)$ *are free groups.*

Proof By the previous reasoning, $\mathrm{Inv}^t(D) = \mathrm{Div}(D)$ is free. Thus, also its subgroups $\mathrm{Inv}(D)$ and $\mathrm{Princ}(D)$ are free. □

In the non-integrally closed case, the freeness of $\mathrm{Princ}(D)$ and $\mathrm{Inv}(D)$ is strongly connected to the units of D.

Proposition 4.2 *Let D be an integral domain such that its integral closure $\overline{D}$ is a Krull domain. Then,* $\mathrm{Princ}(D)$ *is free if and only if* $\mathcal{U}(\overline{D})/\mathcal{U}(D)$ *is free.*

Proof The natural maps $\mathcal{U}(D) \longrightarrow \mathcal{U}(\overline{D})$ and $\mathrm{Princ}(D) \longrightarrow \mathrm{Princ}(\overline{D})$, $xD \mapsto x\overline{D}$, give rise to a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{U}(D) \arrow[r] \arrow[d] & \mathcal{U}(K) \arrow[r] \arrow[d, equal] & \mathrm{Princ}(D) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{U}(\overline{D}) \arrow[r] & \mathcal{U}(K) \arrow[r] & \mathrm{Princ}(\overline{D}) \arrow[r] & 0.
\end{tikzcd}$$

The leftmost vertical map is injective, the middle one is the identity and the rightmost one is surjective; by the snake lemma, the kernel of $\mathrm{Princ}(D) \longrightarrow \mathrm{Princ}(\overline{D})$ is isomorphic to the cokernel of $\mathcal{U}(D) \longrightarrow \mathcal{U}(\overline{D})$, i.e., there is an exact sequence

$$0 \longrightarrow \frac{\mathcal{U}(\overline{D})}{\mathcal{U}(D)} \longrightarrow \mathrm{Princ}(D) \longrightarrow \mathrm{Princ}(\overline{D}) \longrightarrow 0.$$

Since $\overline{D}$ is Krull, the group $\mathrm{Princ}(\overline{D})$ is free by Proposition 4.1; thus, the sequence splits and $\mathrm{Princ}(D) \simeq \mathrm{Princ}(\overline{D}) \oplus \frac{\mathcal{U}(\overline{D})}{\mathcal{U}(D)}$. The claim follows. □

There are two immediate specializations of the previous proposition that are of interest.

Corollary 4.3 *Let D be a local integral domain such that its integral closure $\overline{D}$ is a Krull domain. Then,* $\mathrm{Inv}(D)$ *is free if and only if* $\mathcal{U}(\overline{D})/\mathcal{U}(D)$ *is free.*

Proof If D is local, then $\mathrm{Inv}(D)=\mathrm{Princ}(D)$. The claim follows from Proposition 4.2. □

Corollary 4.4 *Let D be a Noetherian domain and $\overline{D}$ its integral closure. Then,* $\mathrm{Princ}(D)$ *is free if and only if $\mathcal{U}(\overline{D})/\mathcal{U}(D)$ is free.*

Proof If D is Noetherian, then $\overline{D}$ is a Krull domain [20]. The claim follows from Proposition 4.2. □

Putting the previous corollaries together, we have:

Corollary 4.5 *Let D be a local Noetherian domain and $\overline{D}$ its integral closure. Then,* $\mathrm{Inv}(D)$ *is free if and only if $\mathcal{U}(\overline{D})/\mathcal{U}(D)$ is free.*

We now study in more detail the one-dimensional case.

Let $\mathfrak{c}:=(D:\overline{D})$ be the conductor of the extension $D\subseteq\overline{D}$. Then, since D is one-dimensional, $\overline{D}$ is finitely generated (and thus $\mathfrak{c}\neq(0)$) if and only if D is analytically unramified, i.e., if and only if the completion of D is reduced [18]. In this case, we can study the quotient $\mathcal{U}(\overline{D})/\mathcal{U}(D)$ through quotienting by $\mathfrak{c}$.

Lemma 4.6 *Let A be an integral domain, and $I\subseteq$* $\mathrm{Jac}(A)$ *be an ideal. Then, the natural map $\phi:\mathcal{U}(A)\longrightarrow\mathcal{U}(A/I)$ is surjective.*

Proof Let $u=a+I\in\mathcal{U}(A/I)$. Then, there is a $b\in A$ such that $(a+I)(b+I)=1+I$, i.e., $ab-1\in I\subseteq\mathrm{Jac}(A)$. Hence, $ab\in 1+\mathrm{Jac}(A)\subseteq\mathcal{U}(A)$. Thus $a\in\mathcal{U}(A)$ and $u=\phi(a)$, so that ϕ is surjective. □

Lemma 4.7 *Let $A\subseteq B$ be an extension of integral domains, and let I be a common ideal of A and B that is contained in their Jacobson radical. Then, $\dfrac{\mathcal{U}(B)}{\mathcal{U}(A)}\simeq\dfrac{\mathcal{U}(B/I)}{\mathcal{U}(A/I)}$.*

Proof Let $\phi_A:\mathcal{U}(A)\longrightarrow\mathcal{U}(A/I)$ and $\phi_B:\mathcal{U}(B)\longrightarrow\mathcal{U}(B/I)$ be the natural maps; by Lemma 4.6, they are surjective. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{U}(A) \arrow[r] \arrow[d, "\phi_A"] & \mathcal{U}(B) \arrow[r] \arrow[d, "\phi_B"] & \mathcal{U}(B)/\mathcal{U}(A) \arrow[r] \arrow[d, "\psi"] & 0 \\
0 \arrow[r] & \mathcal{U}(A/I) \arrow[r] & \mathcal{U}(B/I) \arrow[r] & \mathcal{U}(B/I)/\mathcal{U}(A/I) \arrow[r] & 0.
\end{tikzcd}$$

By the snake lemma, the surjectivity of ϕ_A and ϕ_B implies that ψ is surjective, and that we have an exact sequence

$$0\longrightarrow\ker\phi_A\longrightarrow\ker\phi_B\longrightarrow\ker\psi\longrightarrow 0.$$

The kernels of ϕ_A and ϕ_B coincide, since they are both equal to $1+I$; thus, we must also have $\ker\psi=0$. Hence, ψ is both injective and surjective, thus an isomorphism. □

Lemma 4.8 *Let K be a field and F be a free group; let G be the additive group of K. Then, for every integer n,* $\mathrm{Hom}(G^n,F)=0$. *In particular, no quotient of G^n is free.*

Proof As subgroups of free groups are free, it is enough to consider surjective homomorphisms; moreover, composing with a nonzero homomorphism $F\longrightarrow\mathbb{Z}$, we can suppose that $F=\mathbb{Z}$ is cyclic. Furthermore, since G^n is the direct sum of n copies of G, by the properties of the hom-sets we can suppose that $n=1$.

Let thus $\phi:G\longrightarrow\mathbb{Z}$ be such a homomorphism.

Let $n > 1$ be a positive integer that is not a multiple of the characteristic of K. Since ϕ is surjective, there is an $x \in G$ such that $\phi(x) = 1$. By the choice of n, there is an $y \in G$ such that $x = ny$: thus $1 = \phi(x) = \phi(ny) = n\phi(y)$, which is impossible in $\mathbb{Z}$. Thus ϕ must be the zero homomorphism, as claimed.

The "in particular" statement follows from the fact that a quotient $G^n \longrightarrow H$ is a nonzero homomorphism. □

We now distinguish three cases according to the properties of $\mathfrak{c}$ as an ideal of $\overline{D}$.

Proposition 4.9 *Let* $(D, \mathfrak{m})$ *be a local Noetherian domain of dimension* 1*, and let* $\overline{D}$ *be its integral closure; suppose that* $D \neq \overline{D}$ *and that* $\mathfrak{c} := (D : \overline{D}) \neq (0)$. *If* $\mathfrak{c}$ *is not a radical ideal of* $\overline{D}$*, then* $\mathrm{Inv}(D)$ *is not free.*

Proof Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ be the maximal ideals of $\overline{D}$ (there are only finitely many of them since $\overline{D}$ is the integral closure of the one-dimensional local Noetherian domain); note that $\mathfrak{c} \subseteq \mathfrak{m}_i$ for every i and thus $\mathfrak{c} \subseteq \mathrm{Jac}(\overline{D})$. Since $\mathfrak{c} \neq (0)$, we can write $\mathfrak{c} = \mathfrak{m}_1^{e_1} \cdots \mathfrak{m}_n^{e_n}$ for some natural numbers $e_i > 0$. As $\mathfrak{c}$ is not radical, we have $e_i > 1$ for some i; without loss of generality we can suppose that $e_1 > 1$.

Let $A := D/\mathfrak{c}$ and $B := \overline{D}/\mathfrak{c}$: then, A is a local Artinian ring while B is an Artinian ring with n maximal ideals, say $\mathfrak{n}_i := \mathfrak{m}_i/\mathfrak{c}$. By Lemma 4.7, $\mathcal{U}(B)/\mathcal{U}(A) \simeq \mathcal{U}(\overline{D})/\mathcal{U}(D)$; by Corollary 4.5, we only need to prove that $\mathcal{U}(B)/\mathcal{U}(A)$ is not free.

Let $I := \mathfrak{n}_1^{e_1-1} \cdots \mathfrak{n}_n^{e_n}$; then, $I \subseteq \mathrm{Jac}(B)$ and $(0) = I\mathfrak{n}_1$. Moreover, $I \nsubseteq A$ since I is the image of $I_0 := \mathfrak{m}_1^{e_1-1} \cdots \mathfrak{m}_n^{e_n}$ and I_0 is larger than $\mathfrak{c} = (D : \overline{D})$, that by definition is the largest ideal that is common to D and $\overline{D}$.

Consider $H := 1 + I$. Since $I \subseteq \mathrm{Jac}(B)$, $H \subseteq \mathcal{U}(B)$; moreover, if $1 + t, 1 + t' \in H$, then

$$(1 + t)(1 + t') = 1 + t + t' + tt' = 1 + t + t'$$

since $tt' \in I^2 = \mathfrak{n}_1^{2(e_1-1)} \cdots \mathfrak{n}_n^{2e_n} = (0)$ (using $e_1 > 1$). Therefore, H is a subgroup of $\mathcal{U}(B)$, and the map

$$\begin{aligned} H &\longrightarrow (I, +),\\ 1 + t &\longmapsto t \end{aligned}$$

is a group isomorphism. By construction, $I\mathfrak{n}_1 = (0)$; therefore, I is a $k := B/\mathfrak{n}_1$-vector space, that is of finite dimension since B is Noetherian.

The quotient $H' := H/(H \cap \mathcal{U}(A))$ is a subgroup of $\mathcal{U}(B)/\mathcal{U}(A)$; if the latter were free, then also H' would be free. Moreover, since $I \nsubseteq A$ we have $H' \neq (0)$. However, H' is isomorphic to a quotient of $(I, +)$, which is isomorphic to the direct sum of finitely many copies of $(k, +)$; by Lemma 4.8, no such quotient can be free, and thus H' is not free. Hence $\mathcal{U}(B)/\mathcal{U}(A)$ is not free, and thus neither are $\mathcal{U}(\overline{D})/\mathcal{U}(D)$ and $\mathrm{Inv}(D)$. □

We now turn to the case where $\mathfrak{c}$ is a nonzero radical ideal. One case can only be solved almost tautologically.

Proposition 4.10 *Let* $(D, \mathfrak{m})$ *be a local Noetherian domain of dimension* 1*, and let* $\overline{D}$ *be its integral closure; suppose that* $\mathfrak{c} := (D : \overline{D})$ *is the unique maximal ideal of* $\overline{D}$. *Then,* $\mathrm{Inv}(D)$ *is free if and only if* $\mathcal{U}(\overline{D}/\mathfrak{c})/\mathcal{U}(D/\mathfrak{m})$ *is free.*

Proof By Lemma 4.7 and Corollary 4.5, $\mathrm{Inv}(D)$ is free if and only if $\mathcal{U}(\overline{D})/\mathcal{U}(D) \simeq \mathcal{U}(\overline{D}/\mathfrak{c})/\mathcal{U}(D/\mathfrak{m})$ is free. □

When $\overline{D}$ has more than one maximal ideal, the condition equivalent to the freeness of $\mathrm{Inv}(D)$ is more complicated. We premise a group-theoretic lemma.

Lemma 4.11 *Let $G, A_1, \dots, A_n$ be groups. For each i, let $\phi_i : G \longrightarrow A_i$ be an injective homomorphism and suppose that $A_i = B_i \oplus \phi_i(G)$ for some subgroup B_i. Let $\phi : G \longrightarrow A_1 \oplus \cdots \oplus A_n$ be the composite of all the ϕ_i. Then, $(A_1 \oplus \cdots \oplus A_n)/\phi(G) \simeq B_1 \oplus \cdots \oplus B_n \oplus G^{n-1}$.*

Proof For each i, let $\pi_i : A_i \longrightarrow B_i$ be the canonical quotient, and let $\theta_i : A_i \longrightarrow G$ be the left inverse of ϕ_i. Consider the following two maps:

$$\begin{aligned} \psi_1 \colon A_1 \oplus \cdots \oplus A_n &\longrightarrow B_1 \oplus \cdots \oplus B_n, \\ (a_1, \dots, a_n) &\longmapsto (\pi_1(a_1), \dots, \pi_n(a_n)), \end{aligned}$$

and

$$\begin{aligned} \psi_2 \colon A_1 \oplus \cdots \oplus A_n &\longrightarrow G^{n-1}, \\ (a_1, \dots, a_n) &\longmapsto (\theta_1(a_1) - \theta_n(a_n), \theta_2(a_2) - \theta_n(a_n), \dots, \theta_{n-1}(a_{n-1}) - \theta_n(a_n)). \end{aligned}$$

Let $\psi : A_1 \oplus \cdots \oplus A_n \longrightarrow B_1 \oplus \cdots \oplus B_n \oplus G^{n-1}$ be the composite of ψ_1 and ψ_2. We claim that ψ is surjective with kernel $\phi(G)$.

Indeed, if $(a_1, \dots, a_n) \in \ker \psi$ then $a_i \in \phi_i(G)$, and thus $a_i = \phi_i(g_i)$ for some $g_i \in G$; by definition, $\theta_i(a_i) = (\theta_i \circ \phi_i)(g_i) = g_i$, and thus the condition $\theta_i(a_i) = \theta_n(a_n)$ implies $b_i = b_n$ for each i, that is, all the g_i are equal to the same g. Therefore, $a_i = \phi_i(g)$, and $(a_1, \dots, a_n) = \phi(g) \in \phi(G)$. The inclusion $\phi(G) \subseteq \ker \psi$ follows trivially.

Let now $(b_1, \dots, b_n, g_1, \dots, g_{n-1}) \in B_1 \oplus \cdots \oplus B_n \oplus G^{n-1}$, and set $a_i = (b_i, g_i)$ for $i < n$ and $a_n = (b_n, 0)$. Then, $b_i = \pi_i(a_i)$ for each i, while $\theta_i(a_i) - \theta_n(a_n) = g_i$; therefore, $\psi(a_1, \dots, a_n) = (b_1, \dots, b_n, g_1, \dots, g_{n-1})$. Thus ψ is surjective, and the claim is proved. □

Proposition 4.12 *Let $(D, \mathfrak{m})$ be a local Noetherian domain of dimension 1, and let $\overline{D}$ be its integral closure; suppose that $\overline{D}$ is not local. Let $\mathfrak{c} := (D : \overline{D})$; suppose that $\mathfrak{c} \neq (0)$ is a radical ideal of $\overline{D}$. Let k be the residue field of D and $L_1, \dots, L_n$ be the residue fields of $\overline{D}$. Then, $\mathrm{Inv}(D)$ is free if and only if, for each i, $\mathcal{U}(L_i)$ is free and the natural map $\phi_i : \mathcal{U}(k) \longrightarrow \mathcal{U}(L_i)$ induced by the inclusion $k \hookrightarrow L_i$ makes $\phi_i(\mathcal{U}(k))$ a direct summand of $\mathcal{U}(L_i)$.*

Proof By Lemma 4.7 and Corollary 4.4, we need to show that $\mathcal{U}(B)/\mathcal{U}(A)$ is free, where $A := D/\mathfrak{c}$ and $B := \overline{D}/\mathfrak{c}$. By hypothesis, $B \simeq L_1 \oplus \cdots \oplus L_n$, and thus $\mathcal{U}(B) \simeq \mathcal{U}(L_1) \oplus \cdots \oplus \mathcal{U}(L_n)$; on the other hand, $\mathcal{U}(A) \simeq \mathcal{U}(k)$, and the inclusion $\mathcal{U}(A) \hookrightarrow \mathcal{U}(B)$ is the diagonal embedding of $\mathcal{U}(k)$ into the product. Note that $n > 1$ since $\overline{D}$ is not local.

Suppose first that the conditions in the statement hold. Then, we are in the setting of Lemma 4.11: $\mathcal{U}(L_i) \simeq B_i \oplus \phi_i(\mathcal{U}(k))$ and thus $\mathcal{U}(B)/\mathcal{U}(A)$ is isomorphic to $B_1 \oplus \cdots \oplus B_n \oplus \mathcal{U}(k)^{n-1}$. Since each $\mathcal{U}(L_i)$ is free, so are the B_i and $\mathcal{U}(k)$ (as they are isomorphic to subgroups of a free group), and thus $\mathcal{U}(B)/\mathcal{U}(A)$ is free, as claimed.

Conversely, suppose that $\mathcal{U}(B)/\mathcal{U}(A)$ is free. Let $G_1 \simeq \mathcal{U}(L_1) \oplus (0) \oplus \cdots \oplus (0) \simeq \mathcal{U}(L_1)$. Consider the subgroup H_1 of $\mathcal{U}(B)$ generated by G_1 and by $(1, \dots, 1)\mathbb{Z}$, where $(1, \dots, 1)$ is the element whose i-th component is the unit of L_i. Then, $G_1 \cap (1, \dots, 1)\mathbb{Z} = (0)$, and thus $H_1 = G_1 \oplus (1, \dots, 1)\mathbb{Z}$. Then, $\mathcal{U}(A) \cap H_1 = \phi_1(\mathcal{U}(k))$, and so $H_1/(\mathcal{U}(A) \cap H_1) \simeq \mathcal{U}(L_1)/\phi_i(\mathcal{U}(k))$. Since $H_1/(\mathcal{U}(A) \cap H_1)$ is a subgroup of $\mathcal{U}(B)/\mathcal{U}(A)$, it must be free; hence, the exact sequence

$$0 \longrightarrow \mathcal{U}(k) \longrightarrow \mathcal{U}(L_1) \longrightarrow \mathcal{U}(L_1)/\phi_1(\mathcal{U}(k)) \longrightarrow 0$$

splits and so $\mathcal{U}(L_1) \simeq B_1 \oplus \phi_1(\mathcal{U}(k))$ for some free subgroup B_1. The same holds for each $\mathcal{U}(L_i)$. An application of Lemma 4.11 now implies that $\mathcal{U}(B)/\mathcal{U}(A)$ is isomorphic to

$B_1 \oplus \cdots \oplus B_n \oplus \mathcal{U}(k)^{n-1}$; therefore, also $\mathcal{U}(k)$ must be free, and so $\mathcal{U}(L_1)$ is free, being the direct sum of two free groups. The same happens for every $\mathcal{U}(L_i)$. Therefore, the conditions of the statement are fulfilled. □

In the proposition above, the condition that the unit group of a field is free is very rare.

Lemma 4.13 *Let K be a field. If $\mathcal{U}(K)$ is free, then K has characteristic 2 and $\mathbb{F}_2$ is algebraically closed in K.*

Proof If the characteristic of K is not 2 then $\mathcal{U}(K)$ contains the torsion element $-1 \neq 1$, and thus it is not free. If K has characteristic 2, then any $x \in K$ that is algebraic over $\mathbb{F}_2$ is torsion; thus if $\mathcal{U}(K)$ is free then $\mathbb{F}_2$ must be algebraically closed in K. □

Corollary 4.14 *Let $(D, \mathfrak{m})$ be a local Noetherian domain of dimension 1, and let $\overline{D}$ be its integral closure; suppose that $\overline{D}$ is not local. Let $\mathfrak{c} := (D : \overline{D})$; suppose that $\mathfrak{c} \neq (0)$ is a radical ideal of $\overline{D}$. If $\mathrm{Inv}(D)$ is free, then the characteristic of $D/\mathfrak{m}$ is 2.*

Proof If $\mathrm{Inv}(D)$ is free, by Proposition 4.12 the multiplicative group of the residue fields of $\overline{D}$ must be free, and thus also $\mathcal{U}(D/\mathfrak{m})$ must be free. By Lemma 4.13, it follows that $D/\mathfrak{m}$ has characteristic 2. □

We collect the previous results in the following theorem.

Theorem 4.15 *Let $(D, \mathfrak{m})$ be a local Noetherian domain of dimension 1, and suppose that D is analytically unramified and not integrally closed. Let $\overline{D}$ be its integral closure and $\mathfrak{c} := (D : \overline{D})$. Let k be the residue field of D and $L_1, \ldots, L_n$ be the residue fields of $\overline{D}$, and consider k as a subfield of each L_i. Then, the following hold.*

(a) If $\mathfrak{c}$ is not radical in $\overline{D}$, then $\mathrm{Inv}(D)$ is not free.
(b) If $\overline{D}$ is local and $\mathfrak{c}$ is radical in $\overline{D}$, then $\mathrm{Inv}(D)$ is free if and only if $\mathcal{U}(L_1)/\mathcal{U}(k)$ is free.
(c) If $\overline{D}$ is not local and $\mathfrak{c}$ is radical in $\overline{D}$, then $\mathrm{Inv}(D)$ is free if and only if, for every i, $\mathcal{U}(L_i)$ is free and $\mathcal{U}(k)$ is a direct summand of $\mathcal{U}(L_i)$.

Proof Since D is analytically unramified, $\overline{D}$ is finite over D [18] and thus $\mathfrak{c} := (D : \overline{D}) \neq (0)$. The claim now follows putting together Proposition 4.9, Proposition 4.10 and Proposition 4.12. □

Remark 4.16

(1) In Proposition 4.10, it is possible for $\mathcal{U}(\overline{D})/\mathcal{U}(D)$ to be free even if $\mathcal{U}(D) \neq \mathcal{U}(\overline{D})$. For example, let $K \subseteq L$ be a field extension, and let $D := K + XL[X]_{(X)}$; i.e., D is the pullback of K inside $\overline{D} = L[X]_{(X)}$. Then, $\mathcal{U}(D) = \mathcal{U}(K) + XL[X]_{(X)}$ while $\mathcal{U}(\overline{D}) = \mathcal{U}(L) + XL[X]_{(X)}$, and thus $\mathcal{U}(\overline{D})/\mathcal{U}(D) \simeq \mathcal{U}(L)/\mathcal{U}(K)$. Suppose now that $K = \mathbb{Q}$ and $K \subseteq L$ is finite. By [19, Proposition 1] $\mathcal{U}(L)/\mathcal{U}(K) \simeq F \oplus A$, where F is free and A is finite. In particular, if A is trivial then $\mathcal{U}(\overline{D})/\mathcal{U}(D)$ is free.
For an explicit example, let $\alpha := \zeta_7 + \zeta_7^{-1}$ (where $\zeta_7 \neq 1$ and $\zeta_7^7 = 1$) and $L := \mathbb{Q}(\alpha)$: then, L is a Galois extension of $\mathbb{Q}$ of degree 3, and if A is not trivial there would be an $x \in L \setminus \mathbb{Q}$ such that $y := x^k \in \mathbb{Q}$ for some k. Thus L should contain the Galois closure of $\mathbb{Q}(\sqrt[k]{y})$, which however cannot have degree 1 or 3 over $\mathbb{Q}$, and thus in particular cannot be contained in L.

(2) Corollary 4.5 does not hold when D is not local. For example, suppose that K be a field and let $D = K[X^2, X^3]$. Then, $\overline{D} = K[X]$ and thus $\mathcal{U}(D) = K \setminus \{0\} = \mathcal{U}(\overline{D})$. However, if $\mathrm{Inv}(D)$ is free then so is $\mathrm{Inv}(D_M)$, for every maximal ideal M (by Proposition 3.3, since

D is a locally finite domain): choosing $M = (X^2, X^3)$, we obtain that $\overline{D_M} = K[X]_{(X)}$ and $(D_M : \overline{D_M}) = X^2\overline{D_M}$ is not radical, and thus $\mathrm{Inv}(D_M)$ is not free by Proposition 4.9.

(3) If K is a field of characteristic 2, then it is possible for $\mathcal{U}(K)$ to be free even if $K \neq \mathbb{F}_2$. For example, if $\mathbf{X}$ is a set of indeterminates and $K = \mathbb{F}_2(\mathbf{X})$, then $\mathcal{U}(K)$ is isomorphic to $\mathrm{Princ}(\mathbb{F}_2[\mathbf{X}])$, which is free since $\mathbb{F}_2[\mathbf{X}]$ is a unique factorization domain.

(4) Likewise, the fact that $\mathcal{U}(K)$ and $\mathcal{U}(K')$ are free (where $K \subseteq K'$ are fields) does not guarantee that $\mathcal{U}(K)$ is a direct summand of $\mathcal{U}(K')$. Indeed, suppose that X is an indeterminate over $\mathbb{F}_2$, and let $K = \mathbb{F}_2(X^2)$ and $K' = \mathbb{F}_2(X)$. Then, $\mathcal{U}(K) \simeq \mathcal{U}(K')$ are free, but $X + \mathcal{U}(K')$ has degree 2 in the quotient, so that $\mathcal{U}(K')/\mathcal{U}(K)$ is not torsionfree and, henceforth, not free.

5 Beyond local finiteness

In this section, we generalize Proposition 3.3 from Jaffard to pre-Jaffard families. Recall that a *pre-Jaffard family* on D is a set Θ of flat overrings of D that is complete, independent and compact, with respect to the Zariski topology of $\mathrm{Over}(D)$, and such that $K \notin \Theta$.

For every ordinal number α, we associate to Θ a subset $\mathcal{N}^\alpha(\Theta)$ and an overring T_α of D in the following way:

- $\mathcal{N}^0(\Theta) := \Theta$, $T_0 := D$;
- if $\alpha = \gamma + 1$ is a successor ordinal, then $\mathcal{N}^\alpha(\Theta)$ is the set of members of $\mathcal{N}^\gamma(\Theta)$ that are not Jaffard overrings of T_γ;
- if α is a limit ordinal, then $\mathcal{N}^\alpha(\Theta) := \bigcap\{\mathcal{N}^\gamma(\Theta) \mid \gamma < \alpha\}$;
- $T_\alpha := \bigcap\{T \mid T \in \mathcal{N}^\alpha(\Theta)\}$.

Then, $\{\mathcal{N}^\alpha(\Theta)\}_\alpha$ is a decreasing sequence of subsets of Θ and $\{T_\alpha\}_\alpha$ is an increasing sequence of overrings of D; moreover, $\mathcal{N}^\alpha(\Theta)$ is always a pre-Jaffard family of T_α. There are always ordinal numbers α such that $\mathcal{N}^\alpha(\Theta) = \mathcal{N}^{\alpha+1}(\Theta)$ (and thus $T_\alpha = T_{\alpha+1}$); the smallest of such ordinal numbers is the *Jaffard degree* of Θ. If $T_\alpha = K$ for some α, we say that Θ is *sharp*. See [22] for properties of pre-Jaffard families and of this sequence.

Lemma 5.1 *Let D be an integral domain, Θ a pre-Jaffard family of D, and let $I \neq D$ be a finitely generated fractional ideal of D such that $IT_\alpha = T_\alpha$ for some ordinal number α. Then, there is an ordinal β such that $IT_\beta \neq T_\beta$ and $IT_{\beta+1} = T_{\beta+1}$.*

Proof Let Γ be the set of all ordinal numbers λ such that $IT_\lambda = T_\lambda$; then, $\alpha \in \Gamma$, and thus Γ has a minimum γ. If $\gamma = \beta + 1$ is a successor ordinal, then β is the ordinal we were looking for; we claim that γ cannot be a limit ordinal.

Indeed, let $I = (x_1, \ldots, x_m)$. Since $IT_\gamma = T_\gamma$, we have $x_1, \ldots, x_m \in T_\gamma$, and there are $t_1, \ldots, t_m \in T_\gamma$ such that $1 = x_1t_1 + \cdots + x_mt_m$. If γ is a limit ordinal, then by [24, Lemma 7.1] we have $T_\gamma = \bigcup_{\lambda<\gamma} T_\lambda$, and thus there is a $\overline{\lambda} < \gamma$ such that $x_1, \ldots, x_m, t_1, \ldots, t_m \in T_{\overline{\lambda}}$. However, this implies that $IT_{\overline{\lambda}} = T_{\overline{\lambda}}$, against the definition of γ; hence γ cannot be a limit ordinal, as claimed. □

The proof of the following proposition follows the ideas laid out in [23, Section 5] and [24, Section 7].

Proposition 5.2 *Let D be an integral domain, Θ a pre-Jaffard family on D, and let $\{T_\alpha\}_\alpha$ be the derived sequence. Fix an ordinal number α. If $\mathrm{Inv}(T)$ is free for every $T \in \Theta\setminus\mathcal{N}^\alpha(\Theta)$,* *then there is an exact sequence*

$$0 \longrightarrow \bigoplus_{T\in\Theta\setminus\mathcal{N}^\alpha(\Theta)} \mathrm{Inv}(T) \longrightarrow \mathrm{Inv}(D) \longrightarrow \mathrm{Inv}(T_\alpha) \longrightarrow 0. \tag{1}$$

In particular, if Θ *is sharp with Jaffard degree at most* α*, then* $\mathrm{Inv}(D) \simeq \bigoplus_{T\in\Theta} \mathrm{Inv}(T)$*, and* $\mathrm{Inv}(D)$ *is free.*

Proof By [24, Theorem 7.2(a)], the extension map $\mathrm{Inv}(D) \longrightarrow \mathrm{Inv}(T_\alpha)$ is always surjective; we denote by K_α its kernel. We proceed by induction on α.

If $\alpha = 0$ the claim is obvious. If $\alpha = 1$, the kernel of $\mathrm{Inv}(D) \longrightarrow \mathrm{Inv}(T_1)$ is the set of all ideals I such that $IT_1 = T_1$; by [26, Lemma 8.2] (and see [22, Proposition 5.6]), for every such I there are only finitely many $T \in \Theta$ (all of them out of $\mathcal{N}^1(\Theta)$) such that $IT \neq T$. Thus, the kernel K_1 is contained in the direct sum $H := \bigoplus\{\mathrm{Inv}(T) \mid T \in \Theta\setminus\mathcal{N}^1(\Theta)\}$. If now $I_1, \dots, I_n$ are invertible ideals of $A_1, \dots, A_n$, respectively (for some $A_1, \dots, A_n \in \Theta\setminus\mathcal{N}^1(\Theta)$), then $I_1 \cap \cdots \cap I_n \cap S$ is an invertible ideal of D with inverse $J_1 \cap \cdots \cap J_n \cap S$ (where J_i is the inverse of I_i in A_i and $S = \bigcap\{T \mid T \in \Theta, T \neq T_1, \dots, T_n\}$). Thus H is contained in the kernel, and so K_1 has the required decomposition.

Suppose now that the claim holds for all $\beta < \alpha$. If $\alpha = \gamma + 1$ is a successor ordinal, then the map $\mathrm{Inv}(D) \longrightarrow \mathrm{Inv}(T_\alpha)$ factors through $\mathrm{Inv}(T_\gamma)$; therefore, there is an exact sequence

$$0 \longrightarrow K_\gamma \longrightarrow K_\alpha \longrightarrow K_{\alpha,\gamma} \longrightarrow 0, \tag{2}$$

where $K_{\alpha,\gamma}$ is the kernel of $\mathrm{Inv}(T_\gamma) \longrightarrow \mathrm{Inv}(T_\alpha)$. By the case $\alpha = 1$, $K_{\alpha,\gamma}$ is free and isomorphic to $\bigoplus\{\mathrm{Inv}(T) \mid T \in \mathcal{N}^\gamma(\Theta)\setminus\mathcal{N}^\alpha(\Theta)\}$; thus the exact sequence (2) splits, and $K_\alpha \simeq K_\gamma \oplus K_{\alpha,\gamma}$. By induction, K_α is free and has the required decomposition.

Suppose that α is a limit ordinal. Then, the sequence $\{K_\beta\}_{\beta<\alpha}$ is an ascending chain of subgroups of K_α, and by the previous reasoning $K_{\beta+1} \simeq K_\beta \oplus H_\beta$, where $H_\beta := \bigoplus\{\mathrm{Inv}(T) \mid T \in \mathcal{N}^\beta(\Theta)\setminus\mathcal{N}^{\beta+1}(\Theta)\}$. Let $I \in K_\alpha$. Then, $IT_\alpha = T_\alpha$, and thus by Lemma 5.1 there is a $\beta < \alpha$ such that $IT_\beta \neq T_\beta$ and $IT_{\beta+1} = T_{\beta+1}$. Since γ is a limit ordinal, $\beta + 1 < \gamma$ and $I \in K_{\beta+1}$. Thus $\bigcup_{\beta<\alpha} K_\beta = K_\alpha$, and by [23, Lemma 5.6] (see also [11, Chapter 3, Lemma 7.3]) we have that K_α has the required decomposition. By induction, the sequence (1) is always exact.

For the last statement, the hypothesis implies that $\mathcal{N}^\alpha(\Theta) = \emptyset$ and $T_\alpha = K$, and thus (1) becomes

$$0 \longrightarrow \bigoplus_{T\in\Theta} \mathrm{Inv}(T) \longrightarrow \mathrm{Inv}(D) \longrightarrow 0 \longrightarrow 0.$$

The claim follows. □

In the context of one-dimensional domains, we obtain the following.

Corollary 5.3 *Let* D *be a one-dimensional domain. If* $\mathrm{Max}(D)^{\mathrm{inv}}$ *is scattered and* $\mathrm{Inv}(D_M)$ *is free for every* $M \in \mathrm{Max}(D)$*, then*

$$\mathrm{Inv}(D) \simeq \bigoplus\{\mathrm{Inv}(D_M) \mid M \in \mathrm{Max}(D)\}.$$

In particular, $\mathrm{Inv}(D)$ *is free.*

Proof The family $\Theta := \{D_M \mid M \in \mathrm{Max}(D)\}$ is a pre-Jaffard family, and $\mathcal{N}^\alpha(\Theta) = \mathcal{D}^\alpha(\Theta^{\mathrm{inv}}) \simeq \mathcal{D}^\alpha(\mathrm{Max}(D)^{\mathrm{inv}})$ [22, Theorem 8.4]. We can now apply Proposition 5.2. □

Remark 5.4 If D is one-dimensional and $\mathrm{Max}(D)^{\mathrm{inv}}$ is scattered, but the groups $\mathrm{Inv}(D_M)$ are not free, in general we do not have a decomposition $\mathrm{Inv}(D) \simeq \bigoplus_M \mathrm{Inv}(D_M)$. For example, suppose that D is a one-dimensional Prüfer domain such that $\mathrm{Max}(D)^{\mathrm{inv}}$ has a unique limit point, say Q; suppose that D_P is a DVR for every $P \neq Q$ and that $\Gamma(D_Q) \simeq \mathbb{Q}$. The set $\Theta = \{D_M \mid M \in \mathrm{Max}(D)\}$ is a pre-Jaffard family with $\mathcal{N}^1(\Theta) = \{D_Q\}$, and thus from the extension map $\mathrm{Inv}(D) \longrightarrow \mathrm{Inv}(D_Q)$ we get an exact sequence

$$0 \longrightarrow F \longrightarrow \mathrm{Inv}(D) \longrightarrow \mathbb{Q} \longrightarrow 0,$$

where F is a free group. We claim that $\mathrm{Inv}(D)$ does not contain any subgroup isomorphic to $\mathbb{Q}$.

Indeed, say that an element g of a group G is *divisible* if for every positive integer n there is an $h \in G$ such that $h^n = g$. Then, every element of $\mathbb{Q}$ is divisible, and thus every group containing $\mathbb{Q}$ contains nonzero divisible elements.

Suppose that $I \in \mathrm{Inv}(D)$ is divisible. Since Q is a limit point of $\mathrm{Max}(D)^{\mathrm{inv}}$, there is a $P \neq Q$ such that $ID_P \neq D_P$. If $n > |v_P(I)| \neq 0$, then $|v_P(J^n)| = n|v_P(J)|$ is either 0 or at least n, and thus cannot be equal to $v_P(I)$. Thus $J^n \neq I$ for every J, and I is not divisible. Hence, $\mathrm{Inv}(D)$ cannot contain a group isomorphic to $\mathbb{Q}$, as claimed.

6 Prüfer domains

In this section, we analyze the case of Prüfer domains. We start with a very straightforward lemma.

Lemma 6.1 *Let V be a valuation domain. Then,* $\mathrm{Inv}(V) \simeq \Gamma(V)$.

Proof Every invertible ideal of a valuation domain is principal; hence, we have a map

$$\begin{aligned} \gamma\colon \mathrm{Inv}(V) &\longrightarrow \Gamma(V),\\ xV &\longmapsto v(x). \end{aligned}$$

It is easy to see that γ is well-defined, injective and surjective. Thus $\mathrm{Inv}(V) \simeq \Gamma(V)$, as claimed. □

Thus, Corollary 3.7 immediately becomes:

Corollary 6.2 *Let D be a locally finite Prüfer domain. If $\Gamma(D_M)$ is free for every $M \in \mathrm{Max}(D)$, then $\mathrm{Inv}(D)$ is free.*

Likewise, the results of the previous sections can be specialized to the case of one-dimensional Prüfer domains.

Proposition 6.3 *Let D be a one-dimensional Prüfer domain.*

(a) [3, Theorem 5] If D is locally finite, then

$$\mathrm{Inv}(D) \simeq \bigoplus_{M \in \mathrm{Max}(D)} \Gamma(D_M).$$

In particular, $\mathrm{Inv}(D)$ is free if and only if $\Gamma(D_M)$ is free for every $M \in \mathrm{Max}(D)$.

(b) If $\mathrm{Max}(D)^{\mathrm{inv}}$ is scattered and $\Gamma(D_M)$ is free for every $M \in \mathrm{Max}(D)$, then $\mathrm{Inv}(D)$ is free.

Proof Let $\Theta := \{D_M \mid M \in \mathrm{Max}(D)\}$. If D is locally finite, then Θ is a Jaffard family, and the claim follows from Proposition 3.3. In general, Θ is a pre-Jaffard family, and if $\mathrm{Max}(D)^{\mathrm{inv}}$ is scattered then Θ is sharp [22, Corollary 8.6]; the second claim now follows from Proposition 5.2. □

We can say something more about valuation domains.

Proposition 6.4 *Let V be a valuation domain without unbranched prime ideals. Suppose that, for every prime ideal $P \neq (0)$, the group $\Gamma(V_P/Q)$ is free, where Q is the prime ideal directly below P. Then, $\Gamma(V)$ is free.*

Proof Note first that, if W is a valuation domain and $\mathrm{Inv}(W) \simeq \Gamma(W)$ is free, we can always find a basis of integral invertible ideal by passing, if needed, from xW to $x^{-1}W$.

For every prime ideal $P \neq (0)$, let $\mathcal{B}(P)$ be a subset of $\mathrm{Inv}(V)$ such that the image of $\mathcal{B}(P)$ into V_P/Q is a basis for $\mathrm{Inv}(V_P/Q)$; in particular, each element of $\mathcal{B}(P)$ is generated by an element of $P\setminus Q$. Then, $\mathcal{B}(P)$ is an independent subset of $\mathrm{Inv}(V)$, and thus it generates a free group, say $H(P)$. Let H be the group generated by all the $H(P)$.

We claim that $H = \bigoplus_P H(P)$. Indeed, suppose that there are $y_1, \dots, y_k \in K$ such that $y_iV \in H(P_i)$ and $y_1 \cdots y_kV = V$, with $P_i \neq P_j$ if $i \neq j$. We can suppose without loss of generality that $P_1 \supsetneq \cdots \supsetneq P_k$. Then, $y_1 \cdots y_kV_{P_k} = V_{P_k}$. If $i < k$ and $xV \in \mathcal{B}(P_i)$, then $x \in P_i\setminus P_k$, and thus $xV_{P_k} = V_{P_k}$: it follows that $y_iV_{P_k} = V_{P_k}$ for all $i < k$, and thus also $y_kV_{P_k} = V_{P_k}$. Since the quotient map $V \longrightarrow V_P/Q$ induces an isomorphism between $H(P)$ and $\mathrm{Inv}(V_P/Q)$, it follows that y_kV is the identity in $\mathrm{Inv}(V)$, i.e., $y_kV = V$. Repeating the process we obtain that $y_iV = V$ for every i. Thus the $H(P)$ are independent subgroups, and H is their direct sum.

We now claim that $H = \mathrm{Inv}(V)$. Let $xV \in \mathrm{Inv}(V)$, and let P be the smallest prime ideal containing x (we can suppose $x \in V$). Then, $xV_P = y_0V_P$ for some $y_0 \in H(P)$: indeed, xV_P/Q can be written as a product $w_1^{n_1} \cdots w_t^{n_t}$, where $w_i = \pi(z_i)$ for some $z_i \in \mathcal{B}(P)$ (and $\pi : V \longrightarrow V_P/Q$ is the canonical map) and we can take $y_0 := z_1^{n_1} \cdots z_t^{n_t}$. Let x_1 be either xy_0^{-1} or $x^{-1}y_0$, according to which is in V: if $x_1V \neq V$, then $x_1V_P = V_P$, and thus the minimal prime P_1 over x_1V properly contains P. Continuing in this way, we obtain an ascending chain $P \subsetneq P_1 \subsetneq P_2 \subsetneq \cdots$ of prime ideals; however, since there are no unbranched prime ideals, such a chain must stop. (The union of a strictly ascending chain of prime ideals is an unbranched prime.) Thus, at one point we must have $x_nV = V$, and thus $x_nV \in H$; by construction, also $xV \in H$. Hence $H = \mathrm{Inv}(V)$, as claimed.

In particular, as a direct sum of free groups, $\mathrm{Inv}(V) \simeq \Gamma(V)$ is free. □

Corollary 6.5 *Let D be a locally finite strongly discrete Prüfer domain. Then,* $\mathrm{Inv}(D)$ *is free.*

Proof Suppose first that $D = V$ is a valuation domain: then, no prime ideal of V is unbranched. Moreover, for every prime P, since $P \neq P^2$ then $\Gamma(V_P/Q) \simeq \mathbb{Z}$ (where Q is the prime ideal directly below P) and thus it is free. The claim follows from Proposition 6.4.

If D is locally finite, the claim now follows from the previous part of the proof and Proposition 3.6. □

We now want to use the properties of Prüfer domains to study more deeply the conditions under which $\mathrm{Inv}(D)$ is free, and to extend as far as possible those results to the group $\mathrm{Div}(D)$ of v-invertible v-ideals. The idea is to use quotients by divided primes. Recall that a prime ideal P is *divided* if $P = PD_P$.

Let D be a Prüfer domain, $\star$ a star operation on D, and let P be a non-maximal divided prime ideal. Let $\pi : D \longrightarrow D/P$ be the quotient. Then, $\star$ induces a star operation $\sharp$ on D/P, defined by

$$I^\sharp := \pi(\pi^{-1}(I)^\star)$$

for every fractional ideal I of D/P (see [9] and [21, Section 6]). The sets of $\star$-invertible and $\sharp$-invertible ideals are closely related.

Proposition 6.6 *Let D be a Prüfer domain and let P be a non-maximal divided prime ideal. Let $\star$ be a star operation on D and let $\sharp$ be the corresponding star operation on $R := D/P$. Then, there is an exact sequence*

$$0 \longrightarrow \mathrm{Inv}^\sharp(R) \longrightarrow \mathrm{Inv}^\star(D) \longrightarrow \Gamma(D_P) \longrightarrow 0.$$

In particular, if $\Gamma(D_P)$ is free, then $\mathrm{Inv}^\star(D)$ is free if and only if $\mathrm{Inv}^\sharp(R)$ is free.

Proof Let I be a $\star$-invertible $\star$-ideal of D: we claim that $v_P(I)$ has a minimum in $\Gamma(D_P)$. Indeed, if I is $\star$-invertible then it is also v-invertible, and thus $(I : I) = D$. If $x \in I$ and $p \in P$ we have $xp \in I$; therefore, since $PD_P = P \subseteq D$, if $\gamma \in v_P(I)$ then I contains all y such that $v_P(y) > \gamma$. Suppose $v_P(I)$ has not a minimum, and let $x \in I$. Then, there is an $x' \in I$ such that $v_P(x') < v_P(x)$, and thus I contains all elements y such that $v_P(y) = v_P(x)$; in particular, $xD_P \subseteq I$, and since x was arbitrary we would have $ID_P \subseteq I$, against $(I : I) = D$. Thus $v_P(I)$ has a minimum.

In particular, the map

$$\begin{aligned} \pi : \mathrm{Inv}^\star(D) &\longrightarrow \Gamma(D_P), \\ I &\longmapsto \min v_P(I) \end{aligned}$$

is well-defined and a surjective group homomorphism. Its kernel is $\ker\pi = \{I \in \mathrm{Inv}^\star(D) \mid P \subsetneq I \subseteq D_P\}$; by the proof of [21, Proposition 7.3], the quotient $D \longrightarrow R$ induces an isomorphism between this set and $\mathrm{Inv}^\sharp(R)$. The existence of the exact sequence follows.

For the "in particular" statement, if $\Gamma(D_P)$ is free then the sequence splits and $\mathrm{Inv}^\star(D) \simeq \mathrm{Inv}^\sharp(R) \oplus \Gamma(D_P)$. In particular, $\mathrm{Inv}^\star(D)$ is free if and only if $\mathrm{Inv}^\sharp(R)$ is free. □

Corollary 6.7 *Let D be a Prüfer domain and let P be a non-maximal divided prime ideal. Suppose that $\Gamma(D_P)$ is free. Then:*

(a) $\mathrm{Inv}(D)$ *is free if and only if* $\mathrm{Inv}(D/P)$ *is free;*
(b) $\mathrm{Div}(D)$ *is free if and only if* $\mathrm{Div}(D/P)$ *is free.*

Proof Both statements follow from Proposition 6.6: the first one using $\star = d$ (so $\sharp = d$), the second one by using $\star = v$ (and thus also $\sharp = v$). □

Let D be a Prüfer domain. Following [21], we say that a prime P of D is a *branching point* for $\mathrm{Spec}(D)$ if there is a family $\Delta \subseteq \mathrm{Spec}(D)$ of pairwise incomparable primes, each one strictly larger than P, such that $P = \inf\Delta$ (in the containment order). We denote by $\mathrm{Spec}_{\mathrm{hi}}(D)$ the union of (0), $\mathrm{Max}(D)$ and the branching points of $\mathrm{Spec}(D)$, and we call it the *homeomorphically irreducible tree* associated to $\mathrm{Spec}(D)$.

Lemma 6.8 *Let D be a Prüfer domain and let P be a non-maximal prime ideal. Then, P is a branching point for* $\mathrm{Spec}(D)$ *if and only if* $P = \inf(V(P) \cap \mathrm{Max}(D))$.

Proof If $P = \inf(V(P) \cap \mathrm{Max}(D))$, then we can use $V(P) \cap \mathrm{Max}(D)$ as the set in the definition of a branching point. Conversely, suppose that $P \neq \inf(V(P) \cap \mathrm{Max}(D)) =: Q$

(note that Q exists since $\operatorname{Spec}(D)$ is a tree). Then, $P \subseteq Q$, and thus $V(P) = \Delta \cup V(Q)$, where Δ is the set of prime ideals between P and Q; note that Δ is linearly ordered, and each element of Δ is contained in all elements of $V(Q)$. Let $\Lambda \subseteq V(P)\setminus\{P\}$ be a set of pairwise incomparable elements. If $\Delta \cap \Lambda = \emptyset$, then $\Lambda \subseteq V(Q)$ and $\inf \Lambda \supseteq Q$; if $\Delta \cap \Lambda \neq \emptyset$, then by construction Λ must be a singleton $\{L\}$, and so $\inf \Lambda = L$. Therefore, $\inf \Lambda \neq P$ for every such Λ, and thus P is not a branching point. □

Lemma 6.9 *Let D be a semilocal Prüfer domain. Then,* $\operatorname{Spec}_{\mathrm{hi}}(D)$ *is finite.*

Proof By Lemma 6.8, a branching point is uniquely defined by a subset of $\operatorname{Max}(D)$. If D is semilocal, then $\operatorname{Max}(D)$ is finite, and thus $\operatorname{Spec}(D)$ has only finitely many branching points. It follows that $\operatorname{Spec}_{\mathrm{hi}}(D)$ is finite. □

Let D be a Prüfer domain. We say that two maximal ideals P, Q are *dependent* if $P \cap Q$ contains a nonzero prime ideal, or equivalently if $D_P D_Q \neq K$; the fact that $\operatorname{Spec}(D)$ is a tree implies that dependence is an equivalence relation on $\operatorname{Max}(D)$. For an equivalence class Δ, let $T(\Delta) := \bigcap\{D_P \mid P \in \Delta\}$; we call the family $\{T(\Delta)\}$ (as Δ ranges among the equivalence classes) the *standard decomposition* of D. If D is semilocal, or if $\operatorname{Spec}(D)$ is a Noetherian space, the standard decomposition is a Jaffard family [21, Proposition 6.2]; if D has finite dimension, then the members of the standard decomposition are in bijective correspondence with the height-1 primes of D (see [21, Lemma 6.1]), and thus each element of Θ is in the form $T(P) := \bigcap\{D_Q \mid Q \in V(P)\}$ for some height-1 prime P.

Proposition 6.10 *Let D be a semilocal Prüfer domain. Suppose that $\Gamma(D_P)$ is free for every $P \in$* $\operatorname{Spec}_{\mathrm{hi}}(D)\setminus\operatorname{Max}(D)$. *Then,* $\operatorname{Inv}(D)$ *is free if and only if $\Gamma(D_M)$ is free for every $M \in$* $\operatorname{Max}(D)$.

Proof If each $\operatorname{Inv}(D_M)$ is free, the claim follows from Corollary 6.2.

Suppose that $\operatorname{Inv}(D)$ is free: we have to prove that each $\operatorname{Inv}(D_M)$ is free. Since D is semilocal, $\operatorname{Spec}_{\mathrm{hi}}(D)$ is finite by Lemma 6.9. We proceed by induction on its cardinality.

If $|\operatorname{Spec}_{\mathrm{hi}}(D)| = 1$ then D is a field, and $\operatorname{Inv}(D) = \Gamma(D)$ is trivial. If $|\operatorname{Spec}_{\mathrm{hi}}(D)| = 2$ then D is a valuation domain and the claim follows from the fact that $\operatorname{Inv}(D) \simeq \Gamma(D)$.

Suppose now that the claim holds up to $n-1$. Consider the standard decomposition Θ of D; since D is semilocal, Θ is a Jaffard family of D and thus $\operatorname{Inv}(D) \simeq \bigoplus\{\operatorname{Inv}(T) \mid T \in \Theta\}$.

If $|\Theta| > 1$, then $|\operatorname{Spec}_{\mathrm{hi}}(T)| < |\operatorname{Spec}_{\mathrm{hi}}(D)|$ for every $T \in \Theta$, and thus the inductive hypothesis applies to each T. Therefore, each $\operatorname{Inv}(T)$ and thus by induction $\Gamma(T_N)$ is free for every $T \in \Theta$ and every $N \in \operatorname{Max}(T)$. However, the set of these T_N is just the set of all D_M as M ranges in $\operatorname{Max}(D)$; thus each $\operatorname{Inv}(D_M)$ is free.

If $|\Theta| = 1$, then $P := \inf \operatorname{Max}(D) \in \operatorname{Spec}_{\mathrm{hi}}(D)$ is nonzero; since P is contained in every maximal ideal of D, moreover, P is divided. Since $\Gamma(D_P)$ is free by hypothesis, by Corollary 6.7 $\operatorname{Inv}(D/P)$ is free. However, $|\operatorname{Spec}_{\mathrm{hi}}(D/P)| = |\operatorname{Spec}_{\mathrm{hi}}(D)| - 1$, since the elements of $\operatorname{Spec}_{\mathrm{hi}}(D/P)$ are exactly the quotients of the nonzero elements of $\operatorname{Spec}_{\mathrm{hi}}(D)$; hence, $\Gamma((D/P)_N)$ is free for every $N \in \operatorname{Max}(D/P)$. For every maximal ideal M of D, we have $(D/P)_{M/P} \simeq D_M/PD_M$; hence, we have an exact sequence

$$0 \longrightarrow \Gamma((D/P)_{M/P}) \longrightarrow \Gamma(D_M) \longrightarrow \Gamma(D_P) \longrightarrow 0,$$

that splits since $\Gamma(D_P)$ is free. Thus $\Gamma(D_M)$ is free, and the statement follows. By induction, the claim holds for every semilocal Prüfer domain. □

Corollary 6.11 *Let D be Prüfer domain that is finite-dimensional and locally finite. Suppose that $\Gamma(D_P)$ is free for every $P \in$* $\operatorname{Spec}_{\mathrm{hi}}(D)\setminus\operatorname{Max}(D)$. *Then,* $\operatorname{Inv}(D)$ *is free if and only if $\Gamma(D_M)$ is free for every $M \in$* $\operatorname{Max}(D)$.

Proof For every $P \in X^1(D)$, let $T(P) := \bigcap\{D_Q \mid Q \in V(P)\}$. Then, $\Theta := \{T(P) \mid P \in X^1(D)\}$ is the standard decomposition of D, and it is a Jaffard family of D since D is locally finite and finite-dimensional (see the paragraph before Proposition 6.10). Thus $\mathrm{Inv}(D) \simeq \bigoplus\{\mathrm{Inv}(T) \mid T \in \Theta\}$. Moreover, each $T \in \Theta$ is semilocal, and thus by Proposition 6.10 $\mathrm{Inv}(T)$ is free if and only if $\Gamma(T_N)$ is free for every $N \in \mathrm{Max}(T)$. Putting all together, it follows that $\mathrm{Inv}(D)$ is free if and only if each $\Gamma(D_M)$ is free, as claimed. □

We now turn to study the group $\mathrm{Div}(D)$: the first result deals with valuation domains.

Proposition 6.12 *Let V be a valuation domain with maximal ideal M. Suppose that M is branched, and let P be the prime ideal directly below M. Then, the following hold.*

(a) If M is finitely generated, then $\mathrm{Div}(V) \simeq \Gamma(V)$.
(b) If M is not finitely generated, then $\mathrm{Div}(V) \simeq \mathbb{R} \oplus \Gamma(V_P)$. *In particular,* $\mathrm{Div}(V)$ *is not free.*

Proof (a) By [2, Corollary 3.6], if M is principal then every v-invertible v-ideal is principal. Thus $\mathrm{Div}(V) = \mathrm{Inv}(V) \simeq \Gamma(V)$, as claimed.

(b) By Proposition 6.6, we have an exact sequence

$$0 \longrightarrow \mathrm{Div}(V/P) \longrightarrow \mathrm{Div}(V) \longrightarrow \Gamma(V_P) \longrightarrow 0.$$

Since M is not finitely generated, then by the proof of [2, Theorem 2.7] $\mathrm{Div}(V/P) \simeq \mathbb{R}$. In particular, it is a divisible group, and thus the above exact sequence splits; it follows that $\mathrm{Div}(V) \simeq \mathbb{R} \oplus \Gamma(V_P)$. Since $\mathbb{R}$ is not free, neither is $\mathrm{Div}(V)$. □

The proof of the following proposition follows a path similar to the proof of Proposition 6.10.

Proposition 6.13 *Let D be a semilocal Prüfer domain. Suppose that every maximal ideal is branched and that $\Gamma(D_P)$ is free for every $P \in \mathrm{Spec}_{\mathrm{hi}}(D) \setminus \mathrm{Max}(D)$. Then,* $\mathrm{Div}(D)$ *is free if and only if every maximal ideal of D is finitely generated.*

Proof Since D is semilocal, $\mathrm{Spec}_{\mathrm{hi}}(D)$ is finite by Lemma 6.9. We proceed by induction on its cardinality.

If $|\mathrm{Spec}_{\mathrm{hi}}(D)| = 1$ then D is a field, $\mathrm{Div}(D)$ is trivial and its maximal ideal is the zero ideal, which is finitely generated. If $|\mathrm{Spec}_{\mathrm{hi}}(D)| = 2$ then D is a valuation domain; the claim now follows from Proposition 6.12.

Suppose now that the claim holds up to $n-1$. Consider the standard decomposition Θ of D; since D is semilocal, Θ is a Jaffard family of D and thus $\mathrm{Div}(D) \simeq \bigoplus\{\mathrm{Div}(T) \mid T \in \Theta\}$.

If $|\Theta| > 1$, then $|\mathrm{Spec}_{\mathrm{hi}}(T)| < |\mathrm{Spec}_{\mathrm{hi}}(D)|$, and thus the inductive hypothesis applies to each T. If every maximal ideal of D is finitely generated, so is every maximal ideal of T: thus each $\mathrm{Div}(T)$ is free and $\mathrm{Div}(D)$ is free. Conversely, if $\mathrm{Div}(D)$ is free then each $\mathrm{Div}(T)$ is free, and so every maximal ideal of T is finitely generated. If $M \in \mathrm{Max}(D)$, then $M = MT \cap D$ for some $T \in \Theta$; by [21, Lemma 5.9], since T is a Jaffard overring of D and MT is finitely generated also M is finitely generated, as claimed.

If $|\Theta| = 1$, then $P := \inf \mathrm{Max}(D) \in \mathrm{Spec}_{\mathrm{hi}}(D)$ is nonzero; since P is contained in every maximal ideal of D, moreover, P is divided. Since $\Gamma(D_P)$ is free by hypothesis, by Corollary 6.7 $\mathrm{Div}(D)$ is free if and only if $\mathrm{Div}(D/P)$ is free. However, $|\mathrm{Spec}_{\mathrm{hi}}(D/P)| = |\mathrm{Spec}_{\mathrm{hi}}(D)| - 1$, since the elements of $\mathrm{Spec}_{\mathrm{hi}}(D/P)$ are exactly the quotients of the nonzero elements of $\mathrm{Spec}_{\mathrm{hi}}(D)$; hence, $\mathrm{Div}(D/P)$ is free if and only if all the maximal ideals of D/P are finitely generated. Since every maximal ideal M of D contains P, we have that M is finitely generated if and only M/P is finitely generated; thus $\mathrm{Div}(D/P)$ is free if and only if every maximal ideal of D is finitely generated. The claim is proved.

By induction, the equivalence holds for every semilocal Prüfer domain. □

Corollary 6.14 *Let D be a Prüfer domain that is finite-dimensional and locally finite. Suppose that $\Gamma(D_P)$ is free for every $P \in \mathrm{Spec}_{\mathrm{hi}}(D)\setminus \mathrm{Max}(D)$. Then, $\mathrm{Div}(D)$ is free if and only if every maximal ideal is finitely generated.*

Proof Note that every maximal ideal is branched since D has finite dimension.

Let Θ be the family defined in the proof of Corollary 6.11; then, $\mathrm{Div}(D)$ is free if and only if $\mathrm{Div}(T)$ is free for every $T \in \Theta$. Since each $T \in \Theta$ is a Jaffard overring, $N \in \mathrm{Max}(T)$ is finitely generated if and only if $N \cap D \in \mathrm{Max}(D)$ is finitely generated; using Proposition 6.13 we see that $\mathrm{Div}(D)$ is free if and only if each maximal ideal is finitely generated, as claimed. □

Proposition 6.6 can also be used to extend results proved by other means. For example, suppose that D is a strongly discrete 2-dimensional Prüfer domain with a single height-one prime ideal P. Then, $\Gamma(D_P) \simeq \mathbb{Z}$ is free, while D/P is an almost Dedekind domain and thus $\mathrm{Inv}(D/P)$ is free by [25, Proposition 5.3]. Using quotients (i.e., Proposition 6.6) we have that also $\mathrm{Inv}(D)$ is free. Extending this reasoning, we have the following result, where $X^k(D)$ indicates the set of prime ideals of D of height k.

Proposition 6.15 *Let D be a strongly discrete Prüfer domain of dimension $d < \infty$. If $\mathrm{Spec}(D)\setminus X^d(D)$ is finite, then $\mathrm{Inv}(D)$ is free.*

Proof If $d = 1$, then D is an almost Dedekind domain, and the claim follows from [25, Proposition 5.3]. Suppose that $d > 1$ and that the claim is true up to dimension $d - 1$, and let Θ be the standard decomposition of D. Since D is finite-dimensional, Θ is in bijective correspondence with the elements of $X^1(D) \subseteq \mathrm{Spec}(D)\setminus X^d(D)$, and thus it is finite; hence Θ is a Jaffard family of D and $\mathrm{Inv}(D) \simeq \bigoplus\{\mathrm{Inv}(T) \mid T \in \Theta\}$.

Any $T \in \Theta$ has a unique height-1 prime ideal, say P; then, D_P is a DVR since D is strongly discrete. If P is maximal, then $T = D_P$ and $\mathrm{Inv}(D_P) \simeq \Gamma(D_P) \simeq \mathbb{Z}$ is free. If P is not maximal, then P is divided; moreover, D/P is a strongly discrete Prüfer domain of dimension $d-1$, and $\mathrm{Spec}(D/P)\setminus X^{d-1}(D/P)$ is finite since it is in bijective correspondence with $V(P)\setminus X^d(D) \subseteq \mathrm{Spec}(D)\setminus X^d(D)$. By hypothesis, $\mathrm{Inv}(D/P)$ is free; by Corollary 6.7, it follows that $\mathrm{Inv}(D)$ is free. By induction, the claim is proved. □

Remark 6.16 The condition in the statement is stronger than requiring that $\mathrm{Spec}(D)\setminus \mathrm{Max}(D)$ is finite, since there may be maximal ideals of height lower than d. If we only required $|\mathrm{Spec}(D)\setminus \mathrm{Max}(D)| < \infty$, we would not be able to prove that the standard decomposition is finite, as there may be infinitely many maximal ideals of height 1.

The previous proposition and the earlier Corollary 6.5 suggest the following

Conjecture If D is a strongly discrete Prüfer domain, then $\mathrm{Inv}(D)$ is free.

We shall see another case of this conjecture in the next section.

7 Algebras

Let $D \subseteq R$ be an extension of integral domains. Then, the inclusion map $D \longrightarrow R$ induces maps $\mathrm{Princ}(D) \longrightarrow \mathrm{Princ}(R)$ and $\mathrm{Inv}(D) \longrightarrow \mathrm{Inv}(R)$. It also induces a natural group homomorphism $\phi_p : \mathrm{Pic}(D) \longrightarrow \mathrm{Pic}(R)$ of their Picard groups. Following [26], we call the quotient $\mathrm{Pic}(R)/\phi_p(\mathrm{Pic}(D))$ the *local Picard group* of the extension $D \subseteq R$, and we denote it by $\mathrm{LPic}(R, D)$.

In this section, we use these facts to study the group of invertible ideals of R. To simplify the notation, we use $F^\bullet$ to denote the unit group of a field F.

Theorem 7.1 *Let D be an integral domain with quotient field K, and let R be a D-algebra that extends D and is an integral domain; let L be the quotient field of R. Suppose that:*

- $\mathcal{U}(D) = \mathcal{U}(R)$;
- *the quotient* $L^\bullet/K^\bullet$ *is free;*
- $\mathrm{LPic}(R, D)$ *and* $\mathrm{Inv}(D)$ *are free.*

Then, $\mathrm{Inv}(R)$ *is free.*

Proof Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{U}(D) \arrow[r] \arrow[d] & K^\bullet \arrow[r] \arrow[d] & \mathrm{Princ}(D) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{U}(R) \arrow[r] & L^\bullet \arrow[r] & \mathrm{Princ}(R) \arrow[r] & 0.
\end{tikzcd}$$

By hypothesis, the leftmost vertical map is an equality, and this equality also implies that the rightmost map $\mathrm{Princ}(D) \longrightarrow \mathrm{Princ}(R)$ is injective; moreover, the middle vertical map is injective since $K \subseteq L$. From the snake lemma, it follows that the cokernel of $\mathrm{Princ}(D) \longrightarrow \mathrm{Princ}(R)$ is isomorphic to the cokernel of $K^\bullet \longrightarrow L^\bullet$. Thus, we have an exact sequence

$$0 \longrightarrow \mathrm{Princ}(D) \longrightarrow \mathrm{Princ}(R) \longrightarrow L^\bullet/K^\bullet \longrightarrow 0,$$

that extends to a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Princ}(D) \arrow[r] \arrow[d] & \mathrm{Princ}(R) \arrow[r] \arrow[d] & L^\bullet/K^\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Inv}(D) \arrow[r] \arrow[d] & \mathrm{Inv}(R) \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Pic}(D) \arrow[r] \arrow[d] & \mathrm{Pic}(R) \arrow[r] \arrow[d] & \mathrm{LPic}(R, D) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where G is defined as the cokernel of the extension map $\mathrm{Inv}(D) \longrightarrow \mathrm{Inv}(R)$.

By hypothesis, $\mathrm{LPic}(R, D)$ is free; therefore, the rightmost column is a split exact sequence, and thus $G \simeq \mathrm{LPic}(R, D) \oplus L^\bullet/K^\bullet$ is free (using the hypothesis on $L^\bullet/K^\bullet$). Hence, also the middle row also splits. By hypothesis, $\mathrm{Inv}(D)$ is free; therefore, $\mathrm{Inv}(R) \simeq \mathrm{Inv}(D) \oplus G$ is free. □

We now show that, for rings of polynomials, the first two hypothesis of Theorem 7.1 often holds.

Lemma 7.2 *Let K be a field, and let L be a field containing K. If there is a Krull domain R containing K with quotient field L such that $\mathcal{U}(R) = K^\bullet$, then $L^\bullet/K^\bullet$ is free. In particular, if $\mathbf{X}$ is a set of indeterminates, then $K(\mathbf{X})^\bullet/K^\bullet$ is free.*

Proof Let $\phi : L^\bullet \longrightarrow \mathrm{Princ}(R)$ be the canonical map: then, $\ker\phi = \mathcal{U}(R) = K^\bullet$, and thus $L^\bullet/K^\bullet \simeq \mathrm{Princ}(R)$. Since R is a Krull domain, $\mathrm{Princ}(R)$ is free, and thus $L^\bullet/K^\bullet$ is free.

The last statement follows by considering $R = K[\mathbf{X}]$. □

Corollary 7.3 *Let D be an integral domain such that* $\mathrm{Inv}(D)$ *is free. Then:*

(a) if $\mathrm{LPic}(D[X], D)$ *is free, then* $\mathrm{Inv}(D[X])$ *is free;*
(b) if D is seminormal, then $\mathrm{Inv}(D[X])$ *is free.*

Proof The first point is a direct consequence of Theorem 7.1 and Lemma 7.2. The second one follows from the first one since $\mathrm{LPic}(D[X], D)$ is trivial if D is seminormal [13, Theorem 1.6]. □

The *ring of integer-valued polynomials* over a domain D, denoted by $\mathrm{Int}(D)$, is the ring of all polynomials $f \in K[X]$ (where K is the quotient field of D) such that $f(D) \subseteq D$. We say that $\mathrm{Int}(D)$ *behaves well under localization* if $\mathrm{Int}(D_P) = (D\setminus P)^{-1}\mathrm{Int}(D)$ for every prime ideal P. See [4] for information about integer-valued polynomials.

Corollary 7.4 *Let D be an almost Dedekind domain such that* $\mathrm{Int}(D)$ *behaves well under localization and* $\mathrm{Max}(D)$ *is scattered when endowed with the inverse topology. Then,* $\mathrm{Inv}(\mathrm{Int}(D))$ *is free.*

Proof The ring $\mathrm{Int}(D)$ is a D-algebra contained in $K[X]$; thus $\mathcal{U}(D) = \mathcal{U}(\mathrm{Int}(D))$ and $K(X)^\bullet/K^\bullet$ is free. The group $\mathrm{Inv}(D)$ is free since D is almost Dedekind [25, Proposition 5.3]. Under the hypothesis in the statement, $\mathrm{LPic}(\mathrm{Int}(D), D)$ is free [26, Corollary 7.6]. The claim follows from Theorem 7.1. □

Corollary 7.5 *Let D be a Dedekind domain. Then,* $\mathrm{Inv}(\mathrm{Int}(D))$ *is free.*

Proof If D is a Dedekind domain, each maximal ideal is finitely generated and thus an open point of $\mathrm{Max}(D)$, with respect to the inverse topology; thus $\mathrm{Max}(D)$ is discrete and hence scattered. Since D is Noetherian, $\mathrm{Int}(D)$ behaves well under localization [4, Theorem I.2.3]; the claim now follows from Corollary 7.4. □

We note that, if D is an almost Dedekind domain and $\mathrm{Int}(D)$ is Prüfer, then $\mathrm{Int}(D)$ is strongly discrete (see [4, Proposition VI.1.9 and the discussion above]); thus the two previous corollaries give some more weight to the conjecture advanced at the end of Sect. 6.

Corollaries 7.4 and 7.5 use in their proof the fact that the local Picard group is free; this is proved by localizing $\mathrm{LPic}(R, D)$ to the maximal ideals of R. We now show that, in some cases, we can use similar results for $\mathrm{Inv}(R)$ with a more direct approach.

Definition 7.6 Let D be an integral domain with quotient field K, and let R be a D-algebra containing D. We say that an integral ideal I of R is *unitary* if $I \cap D \neq (0)$. We denote by $\mathrm{Inv}_u(R, D)$ the subgroup of $\mathrm{Inv}(R)$ generated by the unitary ideals of R.

Theorem 7.7 *Let D be an integral domain and let Θ be a Jaffard family of D. Let R be a D-algebra that contains D and is an integral domain. Then,*

$$\mathrm{Inv}_u(R, D) \simeq \bigoplus_{T\in\Theta} \mathrm{Inv}_u(RT, T).$$

Proof We first note that, if $I \in \mathrm{Inv}_u(R, D)$, then $IT \in \mathrm{Inv}_u(RT, T)$, since $IT \cap T$ contains any element of $I \cap D$. Thus, we have extension maps $\phi_T : \mathrm{Inv}_u(R, D) \longrightarrow \mathrm{Inv}_u(RT, T)$, $\phi_T(I) = IT$, combining which we have a map

$$\Phi\colon \mathrm{Inv}_u(R, D) \longrightarrow \prod_{T \in \Theta} \mathrm{Inv}_u(RT, T),$$
$$I \longmapsto (IT)_{T \in \Theta}.$$

We claim that Φ is injective and that its range is exactly the direct sum.

To show that it is injective, consider the map $\star : I \mapsto \bigcap\{IRT \mid T \in \Theta\}$. Since $\bigcap\{RT \mid T \in \Theta\} = R$, the map $\star$ is a star operation on R, and thus $J^\star = J$ for every invertible ideal J of R. In particular, if $J \in \ker \Phi$ then

$$J = J^\star = \bigcap_{T \in \Theta} JRT = \bigcap_{T \in \Theta} RT = R.$$

Thus Φ is injective.

Let $I \in \mathrm{Inv}_u(R, D)$. Then, $I = JL^{-1}$ for some unitary invertible ideals J, L. There are at most finitely many $T \in \Theta$ such that $JT \neq T$, and finitely many $S \in \Theta$ such that $LS \neq S$; hence, $JT = T = LT$ for all but finitely many $T \in \Theta$, and so $IT = T$ for almost all T. Hence the range of Φ is contained in the direct sum.

To show that the range of Φ is the whole direct sum, it is enough to show that for every $J \in \mathrm{Inv}_u(RT, T)$ there is an $I \in \mathrm{Inv}_u(R, D)$ such that $IRT = J$ and $IRS = RS$ for all $S \in \Theta$, $S \neq T$. Writing $J = J_1(J_2)^{-1}$, we can suppose without loss of generality that $J \subseteq T$. We claim that $I = J \cap R$ is the right choice. Indeed, by [26, Lemma 7.2], we have $IT = (J \cap R)T = JT \cap RT = J$ and $IS = (J \cap R)S = JS \cap RS = RS$ if $S \neq T$, since $JS = JRTS = JRK = RK$ as J is unitary. We show that I is invertible: by [26, Lemma 7.3], I is finitely generated. Let M be a maximal ideal of R. If $M \cap D = (0)$, then R_M contains RK and thus RS for all $S \in \Theta$; in particular, if $S \neq T$, then $IR_M \supseteq IRS = RS$, and so $IR_M = R_M$ is principal. If $M \cap D = P \neq (0)$, then R_M contains D_P: therefore, R_M contains RS, where $S \in \Theta$ is such that $PS \neq S$. If $S \neq T$ the claim holds as above. If $S = T$, then R_M is a localization of RT, and $IR_M = JR_M$; since J is invertible, it is locally principal, and thus IR_M is principal. Therefore, I is finitely generated and locally principal, and thus it is invertible. Hence the range of Φ is the direct sum, and the claim follows. □

The previous theorem requires to look at the distance between $\mathrm{Inv}(R)$ and $\mathrm{Inv}_u(R, D)$.

Proposition 7.8 *Let D be an integral domain with quotient field K and let R be a D-algebra that contains D and is an integral domain. Then, $\dfrac{\mathrm{Inv}(R)}{\mathrm{Inv}_u(R, D)}$ is isomorphic to a subgroup of* $\mathrm{Inv}(RK)$.

Proof Consider the extension map $\phi : \mathrm{Inv}(R) \longrightarrow \mathrm{Inv}(RK)$, $\phi(I) = IRK$; we claim that $\ker \phi = \mathrm{Inv}_u(R, D)$.

Indeed, if I is unitary then $IRK = RK$ since I contains some $d \in D$, $d \neq 0$; thus $\mathrm{Inv}_u(R, D) \subseteq \ker \phi$. Conversely, if $I \in \ker \phi$, then $IRK = RK$, and thus $I \cap S \neq (0)$, where $S := D\setminus\{0\}$. If J is the inverse of I, then also $J \in \ker \phi$, and thus $JRK = RK$ and $J \cap S \neq (0)$; however, $J = (R : I)$, and thus there is a $d \in D$, $d \neq 0$ such that $dI \subseteq R$. In particular, dI is unitary (if $d' \in I \cap D$, $d' \neq 0$, then $dd' \in dI \cap D$), and thus $I = (dR)^{-1}(dI) \in \mathrm{Inv}_u(R, D)$. Hence $\ker \phi = \mathrm{Inv}_u(R, D)$.

It follows that ϕ induces an injective map $\overline{\phi} : \dfrac{\mathrm{Inv}(R)}{\mathrm{Inv}_u(R, D)} \longrightarrow \mathrm{Inv}(RK)$. The claim is proved. □

Proposition 7.9 *Let D be an integral domain and let Θ be a Jaffard family of D. Let R be a D-algebra that contains D and is an integral domain. If* $\mathrm{Inv}(RT)$ *is free for every* $T \in \Theta$ *and* $\mathrm{Inv}(RK)$ *is free, then also* $\mathrm{Inv}(R)$ *is free.*

Proof By Proposition 7.8, $\frac{\mathrm{Inv}(R)}{\mathrm{Inv}_u(R,D)} \simeq G \subseteq \mathrm{Inv}(RK)$; in particular, if $\mathrm{Inv}(RK)$ is free then G is free and $\mathrm{Inv}(R) \simeq \mathrm{Inv}_u(R, D) \oplus G$. Thus we only need to show that $\mathrm{Inv}_u(R, D)$ is free. By Theorem 7.7, $\mathrm{Inv}_u(R, D) \simeq \bigoplus\{\mathrm{Inv}_u(RT, T) \mid T \in \Theta\}$, and each $\mathrm{Inv}_u(RT, T)$ is free since it is a subgroup of the free group $\mathrm{Inv}(RT)$. Hence $\mathrm{Inv}_u(R, D)$ is free and so is $\mathrm{Inv}(R)$. □

Note that these results allow to use the results obtained on locally finite families even on families that are not locally finite: indeed, a Jaffard family Θ is locally finite, but in general $R \cdot \Theta := \{RT \mid T \in \Theta\}$ is not. For example, if $D = \mathbb{Z}$, $\Theta := \{D_M \mid M \in \mathrm{Max}(D)\}$ and $R = \mathrm{Int}(\mathbb{Z})$, then $R \cdot \Theta$ is not locally finite (indeed, all nonconstant polynomials survive in each $\mathrm{Int}(\mathbb{Z})\mathbb{Z}_{(p)} = \mathrm{Int}(\mathbb{Z}_{(p)})$).

We conclude the paper with an example that is very far from the other cases we considered.

Example 7.10 Let E be the ring of all entire functions, i.e., the ring of all functions $\mathbb{C} \longrightarrow \mathbb{C}$ that are analytical everywhere. The ring E is an infinite-dimensional Bézout domain such that every nonzero prime is contained in only one maximal ideal; we refer to [14] and [8, Section 8.1] for generalities on this ring. We claim that $\mathrm{Inv}(E)$ is not free.

Indeed, let $f \in E$ be a function with infinitely many zeros. Then, f induces a Jaffard family $\{E_1, E_2\}$, where $E_1 := \bigcap\{E_M \in \mathrm{Max}(E) \mid f \in M\}$ and $E_2 := \bigcap\{E_M \in \mathrm{Max}(E) \mid f \notin M\} = E[1/f]$, and so $\mathrm{Inv}(E) \simeq \mathrm{Inv}(E_1) \oplus \mathrm{Inv}(E_2)$. We claim that $\mathrm{Inv}(E_1)$ is not free.

Note first that, since E is a Bézout domain, so is E_1, and thus $\mathrm{Inv}(E_1) = \mathrm{Princ}(E_1)$. For every $g \in Q(E)$ and every $x \in \mathbb{C}$, let $v_x(g)$ be the order of the zero of g in x (with $v_x(g)$ negative if x is a pole), or equivalently let $v_x(g)$ be the order of g in the ring $E_{(X-x)}$ (which is a discrete valuation ring). Let $Z(g)$ be the set of zeros of a $g \in E$, and set $Z := Z(f)$. Consider the map

$$\begin{aligned} \psi\colon \mathrm{Inv}(E_1) &\longrightarrow \prod_{x \in Z} \mathbb{Z}, \\ (g) &\longmapsto (v_x(g))_{x \in Z}. \end{aligned}$$

It's clear that ψ is a group homomorphism; we claim that it is actually an isomorphism.

To show that it is surjective, it is enough to note that, since Z is discrete (being the zero set of f), by Weierstrass' theorem (see e.g. [1, Chapter 5, Theorem 7]) for every sequence $(e_x)_{x\in Z}$ of integers there is an entire function h such that $v_x(h) = e_x$; then, $\psi(h) = (e_x)_{x\in Z}$.

Suppose now that $h \in \ker\psi$. Then, h has no zeros nor poles in Z; write $h = h_1/h_2$, where $h_1, h_2 \in E$ have no common zero. Let M be a maximal ideal containing f; then, the set $Z(M) := \{Z(g) \mid g \in M\}$ is a filter (actually, an ultrafilter). Since $Z(f) \cap Z(h_2) = \emptyset$, we have $Z(h_2) \notin Z(M)$ and thus $h_2 \notin M$; therefore, $h \in E_M$. Since M was arbitrary, $h \in E_1$. Likewise, $Z(f) \cap Z(h_1) = \emptyset$, and thus $h^{-1} = h_2/h_1 \in E_1$. Hence h is a unit of E_1, and thus $hE_1 = E_1$. Therefore, ψ is injective.

It follows that $\mathrm{Inv}(E_1)$ is isomorphic to the direct product of an infinite family of copies of $\mathbb{Z}$. The latter is not free [11, Theorem 8.2], and thus neither $\mathrm{Inv}(E)$ is free.

We note that E is not strongly discrete (see Examples 4.16–4.19 of [7]).

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