Università degli Studi di Udine
Ph. D. Course in Computer Science, Mathematics and Physics

Fuzzy algebraic theories and $\mathcal{M}, \mathcal{N}$-adhesive categories

Candidate
Davide Castelnovo

Supervisor
Marino Miculan

Cycle XXXV

Il ragno compie operazioni che assomigliano a quelle del tessitore, l'ape fa vergognare molti architetti con la costruzione delle sue cellette di cera. Ma ciò che fin da principio distingue il peggiore architetto dall'ape migliore è il fatto che egli ha costruito la celletta nella sua testa prima di costruirla in cera. Alla fine del processo lavorativo emerge un risultato che era già presente al suo inizio nella *idea del lavoratore*, che quindi era già presente *idealmente*. Non che egli *effettui* soltanto un cambiamento di forma dell'elemento naturale; egli *realizza* nell'elemento naturale, allo stesso tempo, il *proprio scopo*, da lui *ben conosciuto*, che determina come legge il modo del suo operare, e al quale deve subordinare la sua volontà.

K. Marx, *Il Capitale*, Libro I

In Friuli piove per coprire le lacrime.

Andrea P., comunicazione personale

Abstract

This thesis deals with two quite unrelated subjects in Computer Science: one is the relationship between algebraic theories and monads, the other one is the study of adhesivity properties of categories.

The first part of the thesis begins by revisiting some basic facts regarding monads. Specifically, we review the correspondence between monads, with rank, on the category of sets and functions, and algebraic theories in which the operations' arity is bounded by some regular cardinal.

Next, we move to the heart of this part of the thesis: the extension of this correspondence to the category $\mathbf{Fuz}(\mathbf{H})$ of *fuzzy sets*. This result is obtained by means of a formal system for *fuzzy algebraic reasoning*. We define a sequent calculus based on two types of propositions: those that establish the equality of terms, and those that assert the *membership degree* of a term. We establish a sound semantics for this calculus, and demonstrate the existence of a notion of *free model* for any theory in the system. This, in turn, allows us to prove a completeness result: a formula is derivable from a given theory if and only if it is satisfied by all models of the theory. Moreover, we also prove that, under certain restrictions, it is possible to recover models of a given theory as Eilenberg-Moore algebras for a monad on $\mathbf{Fuz}(\mathbf{H})$. Finally, leveraging the work of Milius and Urbat, we provide a HSP-like characterization of subcategories of algebras that are categories of models of specific types of theories.

The second part of the thesis is devoted to the study of adhesivity properties of various categories. *Adhesive* and *quasiadhesive* categories, and other generalizations such as $\mathcal{M},\mathcal{N}$*-adhesive ones*, marked a watershed moment for the algebraic approaches to the rewriting of graph-like structures, since they provide an abstract framework where many general results (on, e.g., parallelism) could be recast and uniformly proved. However, often checking that a model satisfies the adhesivity properties is far from immediate. After having recalled, the basic definitions, we present a new criterion giving a sufficient condition for $\mathcal{M},\mathcal{N}$-adhesivity.

It is known that in a quasiadhesive category the join of any two regular subobjects is also a regular subobject. Conversely, if regular monomorphisms are *adhesive*, the existence of a regular join for every pair of regular subobjects implies quasiadhesivity. Furthermore, (quasi)adhesive categories can be embedded in a Grothendieck topos via a functor that preserves pullbacks and pushouts along (regular) monomorphisms. In this thesis, we extend these results to $\mathcal{M},\mathcal{N}$-adhesive categories. To achieve this, we introduce the concept of an $\mathcal{N}$*-(pre)adhesive morphism*, which enables us to express $\mathcal{M},\mathcal{N}$-adhesivity as a condition on the poset of subobjects. Additionally, $\mathcal{N}$-adhesive morphisms allow us to demonstrate how a $\mathcal{M},\mathcal{N}$-adhesive category can be embedded into a Grothendieck topos, preserving pullbacks and $\mathcal{M},\mathcal{N}$-pushouts.

Finally, we exploit the previous results to establish adhesivity properties of several existing categories of graph-like structures, including hypergraphs, various kinds of *hierarchical graphs* (a formalism that is notoriously difficult to fit in the mould of algebraic approaches to rewriting), and combinations of them.

Acknowledgments

Undertaking a Ph.D. program, especially during a global pandemic, is a task that, as for any challenging and meaningful one, is difficult to navigate without a supportive network of relationships. I consider myself fortunate to have been surrounded by caring individuals who have helped me along the way. Without their support, I would not have been able to complete this journey.

My first thanks are for my father Pierangelo and my mother Gigliola, coming back to my home's mountains has always been a relief, even in my darkest moments.

Next, I'm grateful to my supervisor Marino Miculan: I don't know if, in the end, I learned to play the "Il giuoco delle perle di vetro", but what I know about its rules is due to him.

I also wish to thank Fabio Gadducci for the shared work and, especially, for a conversation on the stairs of Pisa's Math Department during a lunch break in the middle of an ItaCa workshop.

The months spent at Tallinn University of Technology have been a significant moment in my journey, and I am grateful to Pawel Sobociński and the Laboratory for Compositional Systems and Methods team for their warm hospitality during my stay. I also want to extend a special thank you to Fosco Loregian for our lengthy conversations on category theory and for sharing photos of cats with me. It was truly an honor and a pleasure to work alongside him and Greta Coraglia.

There is a twofold debt of gratitude that I owe to Anna Giordano Bruno: first, for taking me climbing during my initial months in Friuli, and secondly, for giving me the chance to continue my study of algebra.

My officemates Martina Iannella and Vittorio Cipriani have been a crucial part of my journey, and I cannot emphasize enough how much their support and companionship have meant to me throughout the years. I will always be grateful for the many glasses of wine we shared and the burdens we carried together. A thanks goes also to Martina's housemate Elisa and to Lucia, I'll say only this: ducks are important.

I would like to extend my best wishes to Fernando Barrera, Sebastiano Thei, and Andrea Volpi for their doctorates. I am particularly grateful to Andrea for giving me the most beautiful coffee cup I have ever owned. I also want to thank Fernando and Linda, my housemates, for making our shared apartment feel like a real home.

One of the best ways to alleviate the difficulties of a shared journey is by sharing them, so I want to thank Benedetta for the wonderful moments we shared, as well as for the mutual support and affection we provided each other over the past year (the photos of dogs were also greatly appreciated).

Meeting Valerio, Francesca, and Simone on my first days in Friuli was incredibly lucky, and I am grateful for the warm welcome they gave me and the many evenings we spent together. I would also like to thank Mattia and the entire Bloom group for the stimulating events they organized, and a special thank you to Carlo Federico for our thought-provoking conversations on politics, history, and modal logic. I also want to thank Andrea for the second epigraph and for the many beers and thought we shared when he was my neighbour.

I spent many evenings in the Cas'Aupa club: thanks to Sara, Jacopo and all the staff for their work, for the music and for the fun they bring to Udine.

Friuli borders the Balkans and has served as an entryway to Italy for many migrants in the past year. I would like to express my appreciation to the volunteers of Ospiti in Arrivo for their tireless energy and effort in making their arrival here a little easier. A big, and heartfelt thank goes to Laura, for the long nights together in the train station.

I would like to extend a special thank you to Simona, who has been a constant and significant presence in my life over the past few years. Despite the many kilometers that now separate us, her emotional and practical support has been invaluable to me.

Last, but definitely not least, I want to express my deepest gratitude and love to Arianna for her presence in my life. Thank you for bearing with me during these difficult months (not an easy task). Your unwavering support and understanding have meant more than I am able to express. I hope that you will continue to stand by me for a long time to come.

Contents

Introduction

CHAPTER 1

This thesis is divided into two distinct halves, the first of which focuses on algebraic theories and monads, while the second deals with graph rewriting and adhesive categories. Despite the disconnection between these fields, both are united by the use of category theory as a common framework. This can be seen as yet another testament to the power and flexibility of Category Theory, which is capable of bridging diverse areas of Mathematics and Computer Science using shared concepts.

In Part I, the focus is on the study of equations, algebraic theories, and algebraic structures, which are the fundamental concepts in Universal Algebra [115]. This field has a long-standing tradition in mathematics, dating back to the late 19$^{\text{th}}$ century [122], and it forms the basis of modern algebra. The observation that for (almost) every algebraic theory there is a *free structure* on a given set (a free monoid, a free group, a free R-module, etc.) establishes a connection between Universal Algebra and Category Theory. In particular, the construction of a free structure provides a left adjoint to the underlying set functor. Every adjunction gives rise to a monad, which, in turn, carries its own kind of algebras, called *Eilenberg-Moore algebras*. This led naturally to the idea of relating some kind of algebraic structures with the Eilenberg-Moore algebras of the corresponding monad. It turns out that models of a given algebraic theory correspond with the Eilenberg-Moore algebras of the induced monads, and vice versa: if a monad preserves certain kinds of colimits, called κ-filtered, then its Eilenberg-Moore algebras are, essentially, the models of an algebraic theory.

In the sixties, Lawvere and Linton [76, 78, 80] proposed a new approach to these problems, focusing on the concept of *Lawvere Theory* instead of equations. The key idea is to represent all desired operations and axioms as a category with natural numbers as objects. Endowing a set with a family of operations is then equivalent to defining a product-preserving functor from a given Lawvere theory to **Set**. Interestingly, this approach is equivalent to the traditional one based on equations: the correspondence between certain monads and algebraic theories also holds between the same class of monads and Lawvere theories.

This approach is particularly well-suited to introduce algebraic concepts in categories different from **Set** and indeed a wide range of different computational and algebraic notions have been accomodated into this framework [23, 63, 82, 83, 100, 107]. The idea is the following: algebraic structures in a (possibly enriched) category **X** correspond to some class of (enriched) monads on it, which, in turn, correspond to (enriched) Lawvere theories.

On the negative side, this approach does not provide a syntax describing this new kind of algebraic structures not based on **Set**, so we can wonder what is the analogous of equations in these new environment. We propose a solution for the case of the category of *fuzzy sets*: these are sets equipped with a function into some frame **H** and algebraic structures on them are well known and used since the seventies (see e.g., [8, 92, 98, 111]).

To substitute the traditional calculus of equations we introduce the *fuzzy sequent calculus*. While classical equations capture equalities, the membership function's information is captured using syntactic items called *membership propositions* of the form $\mathsf{m}(h, t)$, which can be interpreted as "the membership degree of term t is at least h". We can then define a notion of *fuzzy algebra*, which is a fuzzy set endowed with operations, providing a sound and complete semantics for our calculus.

As in the classical context, there is a notion of free model of a theory Λ and thus we get an associated monad. However, the correspondence between fuzzy algebraic theories and monads is not as straightforward as it is for classical ones. Only for a special class of theories, called basic, does the correspondence between Eilenberg-Moore algebras for the induced monad and models of a given theory hold. Moreover, the task of identifying a characterization of the monads that arise from fuzzy algebraic theories, either in terms of the preservation of certain colimits or by means of left Kan extensions, remains unsolved.

An important line of research in Computer Science since the nineties is given by so-called *graph rewriting* [42]: roughly speaking it is the study of how to get a new graph out of an old one according to some given set of rules. One of the main algebraic approaches to this issue is given by the so called *double-pushout approach* [22]: in this approach a rule is given by a pair of monomorphisms $l : K \to L$ and $r : K \to R$. We can then say that a graph H is obtained by G through an application of the rule (l, r), if we can build two pushout squares as in the following diagram

$$\begin{tikzcd} L \arrow[d, dotted, "f"'] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d, dotted, "h"] & R \arrow[d, dotted, "g"] \\ G & T \arrow[l, dotted, "p"] \arrow[r, dotted, "q"'] & H \end{tikzcd}$$

Informally, T is obtained deleting the image, the *match*, of L from G and the second pushout "fills" the resulting hole glueing R in it. This approach involves only categorical concepts such as monomorphisms and pushouts, we can then apply it to every category. Therefore, it is natural to inquire which properties a category $\mathbf{X}$ should possess in order to have a desirable rewriting system with useful properties, such as confluence. This leads, in order of increasing generality, to the notions of *adhesivity*, *quasiadhesivity*, $\mathcal{M}$*-adhesivity* and $\mathcal{M},\mathcal{N}$*-adhesivity* [13, 43, 73, 104]. Part II is devoted to the study of these concepts.

The works of Garner, Johnstone, Lack and Sobociński [52, 67, 73, 74] provide a link between adhesive categories and *toposes* showing that all elementary toposes are adhesive and that all quasiadhesive categories can be embedded into a Grothendieck topos via a functor which preserves all the relevant categorical structures. The first issue we tackle in the second half of this thesis is the generalization of this result to the context of $\mathcal{M},\mathcal{N}$-adhesive categories: we provide conditions guaranteeing that $\mathcal{M},\mathcal{N}$-adhesive category can be realized as a full subcategory of a topos, closed under the relevant limits and colimits.

Another problem is to actually prove that a given category is $\mathcal{M},\mathcal{N}$-adhesive. In order to do so, one can take either an *ad hoc* approach or a modular one. The latter involves constructing categories of graphs or hypergraphs from other categories using the comma and slice constructions, which under certain assumptions, preserve the adhesivity properties. This modular approach enables us to establish the $\mathcal{M},\mathcal{N}$-adhesivity of several interesting (hyper)graphical categories.

Structure of the thesis This thesis is structured into two parts, each containing two technical chapters and a conclusion. Part I focuses on algebraic theories. Chapter 2 covers the fundamentals of the theory of monads and demonstrates how monads are related to algebraic theories. In Chapter 3, a syntax for algebraic theories in the category of fuzzy sets is introduced and studied. Conclusions and directions for future work are in Chapter 4 . Part II discusses various concepts related to adhesivity. In the more theoretical Chapter 5, the concept of $\mathcal{M},\mathcal{N}$-adhesivity is introduced and several results about it are proven. Chapter 6 establishes adhesivity properties for various categories of graphs and hypergraphs. We summarize our findings in Chapter 7. Finally, in Appendix A we collect some useful categorical results.

Notation We end this introduction stipulating some notational conventions which will be used throughout this thesis.

Given a category $\mathbf{X}$ we will not distinguish notationally between $\mathbf{X}$ and its class of objects: so that "$X \in \mathbf{X}$" means that X belongs to the class of objects of $\mathbf{X}$.

If 1 is a terminal object in a category $\mathbf{X}$, the unique arrow $X \to 1$ from another object X will be denoted by $!_X$. Similarly, if 0 is initial in $\mathbf{X}$ then $?_X$ will denote the unique arrow $0 \to X$. When $\mathbf{X}$ is **Set** and 1 is a singleton, δ_x will denote the arrow $1 \to X$ with value $x \in X$.

Finally, we will use the following notation for some special classes of arrows of a category $\mathbf{X}$:

- $\mathcal{A}(\mathbf{X})$ will denote the class of all arrows of $\mathbf{X}$;
- $\mathcal{M}(\mathbf{X})$ will denote the class of all monos of $\mathbf{X}$;
- $\mathcal{R}(\mathbf{X})$ will denote the class of all regular monos of $\mathbf{X}$.

PART I
ALGEBRAIC THEORIES

Algebraic theories and monads

CHAPTER 2

Contents

The study of monads is one of the pillars of category theory since their invention in the fifties [84] and the discovery of their relation with adjunctions in the sixties [46, 70]. Also in the sixties, Lawvere and Linton's seminal works [76, 78] established the connection between monads and algebraic theories which, since then, has been the backbone of the "category theoretic understanding of universal algebra" [63].

On the other hand, one of the most fruitful and influential lines of research of Logic in Computer Science is the algebraic study of computation and, after Moggi's foundational work [97], monads, and their counterpart given by (enriched) Lawvere theories [69, 100, 110], lie at the heart of it (see also [106, 107]).

Our interest in monads stems from this relation between them, algebra and computer science. This chapter is devoted to recall some well known results of the theory of monads that will be needed in Chapter 3. There are various textbook accounts of monads which contain all these results (along many others), we refer the interested reader to [12, 20, 29, 85, 89].

Synopsis In Section 2.1.1 we will recall the definition of monad and of Eilenberg-Moore algebra; in it we will show how to compute limits and colimits of algebras and discuss regularity of monadic categories. Section 2.2 is devoted to the relationship between monads on **Set** and algebraic theories.

2.1 An introduction to monads

This first section is devoted to recall some well known facts of the theory of monads. The main aim of this section is to prove some basic categorical properties of the categories of Eilenberg-Moore algebras of a monad, we are in particular interested in the existence and computation of limits and colimits, and in regularity of such categories.

2.1.1 Monads and their algebras

In this section we will recall the basic notions about monads. We will also recall the concept of Elilenberg-Moore algebra and of monadic category.

Definition 2.1.1. A *monad* $\mathbf{T}$ on a category $\mathbf{X}$ is a triple (T, η, μ) where $T : \mathbf{X} \to \mathbf{X}$ is a functor and $\eta : \mathsf{id}_{\mathbf{X}} \to T$, $\mu : T \circ T \to T$ are natural transformations, called *unit* and *multiplication*, such that the following diagrams commute.

$$\begin{tikzcd} T\circ T\circ T \arrow[r, "T*\mu"] \arrow[d, "\mu*T"'] & T\circ T \arrow[d, "\mu"] \\ T\circ T \arrow[r, "\mu"'] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "T*\eta"] \arrow[dr, "\mathsf{id}_T"'] & T\circ T \arrow[d, "\mu"] & T \arrow[l, "\eta*T"'] \arrow[dl, "\mathsf{id}_T"] \\ & T & \end{tikzcd}$$

Example 2.1.2. On the category of **Set**, the powerset functor $\mathcal{P} \colon \mathbf{Set} \to \mathbf{Set}$ gives rise to a monad $\mathbf{P}$ where the component of unit and multiplication are given by

$$\eta_X \colon X \to \mathcal{P}(X) \qquad x \mapsto \{x\} \qquad\qquad \mu_X \colon \mathcal{P}(\mathcal{P}(X)) \to \mathcal{P}(X) \qquad A \mapsto \bigcup_{B\in A} B$$

For every cardinal κ, we can consider the functor $\mathcal{P}_\kappa \colon \mathbf{Set} \to \mathbf{Set}$ sending X to the set of its subset of cardinality strictly less then κ. If, moreover, we assume that κ is regular, then the monad structure we have just defined for $\mathcal{P}$ can be restricted to one on $\mathcal{P}_\kappa$.

Example 2.1.3. Let E be an object in a category $\mathbf{X}$ with binary coproducts. We can define the *exception monad* $\mathbf{T}$ taking as $T \colon \mathbf{X} \to \mathbf{X}$

$$\begin{tikzcd} X \arrow[r, mapsto] \arrow[d, "f"'] & X+E \arrow[d, "f+\mathsf{id}_E"] \\ Y \arrow[r, mapsto] & Y+E \end{tikzcd}$$

Then η is just the inclusion $X \to X+E$ and $\mu \colon X+E+E \to X+E$ is the arrow induced by id_X and the codiagonal $\nabla_E \colon E+E \to E$.

Example 2.1.4. Let $(\mathbf{X}, \otimes, I)$ be a monoidal category and (P, m, e) a monoid object in it, then the functor $T_P \colon \mathbf{X} \to \mathbf{X}$ given by $(-) \otimes P$ carries the structure of a monad, called the *writer monad*. If ρ and α are, respectively, the right unitor and the associator, then the components of η and μ are given by the compositions

$$X \xrightarrow{\rho_X^{-1}} X \otimes I \xrightarrow{\mathsf{id}_X \otimes e} X \otimes P \qquad\qquad (X\otimes P)\otimes P \xrightarrow{\alpha_{X,P,P}} X \otimes (P\otimes P) \xrightarrow{\mathsf{id}_X \otimes m} X\otimes P$$

We can get back the exception monad taking the monoidal structure given by the coproduct, e to be the unique arrow from the initial object and m to be ∇_E .

A rich (and exhaustive) source of examples is given by adjunctions.

Proposition 2.1.5. *Let* $U\colon \mathbf{X} \to \mathbf{Y}$ *be a functor with a left adjoint* F. *Let also* η *and* ϵ *be the unit and the counit of the adjunction, then* $(U \circ F, \eta, U * \epsilon * F)$ *is a monad on* Y.

Proof. The first square is obtained applying U to the naturality square

$$\begin{tikzcd} F(U(F(U(F(Y))))) \arrow[r, "F(U(\epsilon_{F(Y)}))"] \arrow[d, "\epsilon_{F(U(F(Y)))}"'] & F(U(F(Y))) \arrow[d, "\epsilon_{F(Y)}"] \\ F(U(F(Y))) \arrow[r, "\epsilon_{F(Y)}"'] & F(Y) \end{tikzcd}$$

For the two triangles, let us start with the triangular identities of the adjunction:

$$\begin{tikzcd} F(Y) \arrow[r, "F(\eta_Y)"'] \arrow[rr, bend left, "\mathsf{id}_{F(Y)}"] & F(U(F(Y))) \arrow[r, "\epsilon_{F(Y)}"'] & F(Y) \end{tikzcd} \qquad \begin{tikzcd} U(X) \arrow[r, "\eta_{U(X)}"'] \arrow[rr, bend left, "\mathsf{id}_{F(Y)}"] & U(F(U(X))) \arrow[r, "U(\epsilon_X)"'] & U(X) \end{tikzcd}$$

Then applying U to the first and instatiating the second with $X = F(Y)$ we get the thesis. □

Example 2.1.6. Let $(\mathbf{X}, \otimes, I)$ be a symmetric monoidal closed category, and let S be an object in it, then the adjunction $S \otimes - \dashv [S, -]$ induces the *state monad* sending an object X to $[S, S \otimes X]$.

Example 2.1.7. [71, 94] Let again S be an object of symmetric monoidal closed category $(X, \otimes, I)$, then, since $[-, S]\colon \mathbf{X}^{op} \to \mathbf{X}$ is adjoint to its opposite, Proposition 2.1.5 gives us a monad, called the *continuation monad*, sending an object X to $[[X, S], S]$.

Another example of monad is given by the *Kleene star* [10, 24, 114].

Example 2.1.8. Given a set X, define a *word* on X as a function $w\colon n \to X$ with domain $n \in \mathbb{N}$. The domain n will be also called the *length* of w and the value $w(i)$ at $i \in n$ its $(i+1)^{th}$ *letter*. Let $X^\star$ be the set of all words on X, if $f\colon X \to Y$ is a function, then we can define

$$f^\star\colon X^\star \to Y^\star \qquad w \mapsto f \circ w$$

obtaining a functor $(-)^\star\colon \mathbf{Set} \to \mathbf{Set}$. We want to endow it with a monad structure.

First of all notice that we can equip $X^\star$ with a structure of monoid. Given $v\colon n \to X$ and $w\colon m \to X$, since the number $n + m$ is also a coproduct of the sets n and m, we can define the *concatenation* $v \cdot w$ of v and w as the induced arrow $n + m \to X$. Explicitly,

$$v \cdot w\colon n + m \to X \qquad i \mapsto \begin{cases} v(i) & i \leq n \\ w(i-n) & n < i \end{cases}$$

Notice that, in particular, for every $w\colon n \to X$ with $n \neq 0$, we have

$$w = \prod_{i=1}^{n} \delta_{w(i)}$$

Since $(\mathbb{N}, +, 0)$ is a monoid, we get at once that $(X^\star, \cdot, ?_X)$ is a monoid too. We want to show that in this way we get a left adjoint $F_{\mathbf{Mon}}$ to $U_{\mathbf{Mon}} \colon \mathbf{Mon} \to \mathbf{Set}$, the forgetful functor from the category of monoids.

We have a function

$$\eta_X \colon X \to X^\star \qquad x \mapsto \delta_x$$

Now, if $(M, \cdot, e)$ is another monoid and $f \colon X \to M$ is a function we can put

$$\hat{f} \colon X^\star \to M \qquad w \mapsto \begin{cases} e & w = ?_X \\ \prod_{i=1}^{\mathsf{dom}(w)} f(w(i)) & \mathsf{dom}(w) \neq 0 \end{cases}$$

which, by construction, is the unique morphism of **Mon** fitting in the following diagram.

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, bend right, "f"'] & X^\star \arrow[d, dotted, "\hat{f}"] \\ & M \end{tikzcd}$$

Finally, we can notice that $f^\star$ is the unique morphism $(X^\star, \cdot, ?_X) \to (Y^\star, \cdot, ?_Y)$ such that

$$\eta_Y \circ f = f^\star \circ \eta_X$$

and thus we can conclude from Proposition 2.1.5 that $(-)^\star = U_{\mathbf{Mon}} \circ F_{\mathbf{Mon}}$ carries a monad structure.

Definition 2.1.9. Given a monad $\mathbf{T}$ on a category $\mathbf{X}$, an *Eilenberg-Moore algebra* for $\mathbf{T}$ is a pair (X, ξ) where X is an object of $\mathbf{X}$ and $\xi \colon T(X) \to X$ such that the following diagrams commute

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "\mathsf{id}_X"'] & T(X) \arrow[d, "\xi"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} T(T(X)) \arrow[r, "\mu_X"] \arrow[d, "T(\xi)"'] & T(X) \arrow[d, "\xi"] \\ T(X) \arrow[r, "\xi"'] & X \end{tikzcd}$$

A morphism between (X, ξ_1) and (Y, ξ_2) is an arrow $f \colon X \to Y$ such that the following square commutes

$$\begin{tikzcd} T(X) \arrow[r, "T(f)"] \arrow[d, "\xi_1"'] & T(Y) \arrow[d, "\xi_2"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

We will denote with $\mathbf{EM}(\mathbf{T})$ the resulting category of Eilenberg-Moore algebras. We will also denote by $U_{\mathbf{T}}$ the forgetful functor $U_{\mathbf{T}} \colon \mathbf{EM}(\mathbf{T}) \to \mathbf{X}$ which sends (X, ξ) to X and is the identity on arrows.

Example 2.1.10. Take a monoidal category $(\mathbf{X}, \otimes, I)$ and consider the monad of Example 2.1.4 associated to an internal monoid (P, m, e). A Eilenberg-Moore algebra (X, ξ) for such monad is given by an arrow

$\xi\colon X \otimes P \to X$ fitting in the diagrams below.

$$
\begin{tikzcd}
X \arrow[r, "\rho_X^{-1}"] \arrow[dr, bend right, "\mathsf{id}_X"'] & X \otimes I \arrow[r, "\mathsf{id}_X \otimes e"] \arrow[d, "\rho_X"] & X \otimes P \arrow[dl, bend left, "\xi"] \\
& X &
\end{tikzcd}
\qquad
\begin{tikzcd}
(X \otimes P) \otimes P \arrow[r, "\alpha_{X,P,P}"] \arrow[d, "\xi \otimes \mathsf{id}_P"'] & X \otimes (P \otimes P) \arrow[r, "\mathsf{id}_X \otimes m"] & X \otimes P \arrow[d, "\xi"] \\
X \otimes P \arrow[rr, "\xi"'] & & X
\end{tikzcd}
$$

Thus, the category of Eilenberg-Moore algebras for the writer monads, can be seen as the category of actions of the internal monoid (P, m, e) on objects of $\mathbf{X}$.

Proposition 2.1.11. *Let* $\mathbf{T}$ *be a monad on a category* $\mathbf{X}$*, then the following are true:*

1. $U_{\mathbf{T}}$ *reflects isomorphism;*
2. *for every* (X, ξ_1) *in* $\mathbf{EM}(\mathbf{T})$ *and isomorphism* $f\colon X \to Y$ *in* $\mathbf{X}$*, there exists a unique* $\xi_2\colon T(Y) \to Y$ *such that* (Y, ξ_2) *is in* $\mathbf{EM}(\mathbf{T})$ *isomorphic to* (X, ξ_1) *via* f.

Proof. 1. Let $f\colon (X, \xi_1) \to (Y, \xi_2)$ is a morphism in $\mathbf{EM}(\mathbf{T})$ which is an isomorphism in $\mathbf{X}$, then

$$
\begin{aligned}
f^{-1} \circ \xi_2 &= f^{-1} \circ f \circ \xi_1 \circ T(f)^{-1} \\
&= \mathsf{id}_X \circ \xi_1 \circ T(f)^{-1} \\
&= \xi_1 \circ T(f)^{-1}
\end{aligned}
$$

proving that f^{-1} is a morphism $(Y, \xi_2) \to (X, \xi_1)$ which is the inverse of f in $\mathbf{EM}(\mathbf{T})$.

2. Reasoning as before we see that the only possible choice is to define

$$\xi_2 := f \circ \xi_1 \circ T(f)^{-1}$$

Now, the previous equation entails at once that

$$\xi_2 \circ T(f) = f \circ \xi_1$$

This, in turn, allows us to build the following diagrams, entailing that (Y, ξ_2) is an object of $\mathbf{EM}(\mathbf{T})$.

$$
\begin{tikzcd}
T(T(Y)) \arrow[r, "\mu_Y"] \arrow[d, "T(T(f^{-1}))"] \arrow[ddd, bend right=60, "T(\xi_2)"'] & T(Y) \arrow[d, "T(f^{-1})"'] \arrow[ddd, bend left=60, "\xi_2"] \\
T(T(X)) \arrow[r, "\mu_X"] \arrow[d, "T(\xi_1)"] & T(X) \arrow[d, "\xi_1"'] \\
T(X) \arrow[r, "\xi_1"'] \arrow[d, "T(f)"] & X \arrow[d, "f"'] \\
T(Y) \arrow[r, "\xi_2"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r, "\eta_Y"] \arrow[d, "f^{-1}"] \arrow[dddr, bend right=60, "\mathsf{id}_Y"'] & T(Y) \arrow[d, "T(f^{-1})"'] \arrow[ddd, bend left=60, "\xi_2"] \\
X \arrow[r, "\eta_X"] \arrow[dr, "\mathsf{id}_X"] \arrow[ddr, bend right, "f"'] & T(X) \arrow[d, "\xi_1"'] \\
& X \arrow[d, "f^{-1}"'] \\
& Y
\end{tikzcd}
$$

From point 1 we can deduce that $f\colon (X, \xi_1) \to (Y, \xi_2)$ is an isomorphism of Eilenberg-Moore algebras and the thesis follows. □

The functor $U_{\mathbf{T}}$ has always a left adjoint, which sends an object to the *free algebra* on it.

Proposition 2.1.12. *Let* $\mathbf{T}$ *be a monad on the category* $\mathbf{Y}$*, then the forgetful functor* $U_{\mathbf{T}}\colon \mathbf{EM}(\mathbf{T}) \to \mathbf{X}$ *has a left adjoint* $F_{\mathbf{T}}\colon \mathbf{X} \to \mathbf{EM}(\mathbf{T})$ *which sends* X *to* $(T(X), \mu_X)$.

Proof. The axioms of monad entail at once that $(T(X), \mu_X)$ is an Eilenberg-Moore algebra. Let us show that η has the universal property of the unit of an adjunction. Given an Eilenberg-Moore algebra (Y, ξ) and a morphism $f\colon X \to Y$ of $\mathbf{X}$, we can consider the composition $\xi \circ T(f)\colon T(X) \to Y$. Pasting together the naturality diagrams of η, μ and those in the definition of Eilenberg-Moore algebras we get:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\eta_X"'] & Y \arrow[d, "\eta_Y"'] \arrow[dr, "\mathrm{id}_Y"] & \\ T(X) \arrow[r, "T(f)"'] & T(Y) \arrow[r, "\xi"'] & Y \end{tikzcd} \qquad \begin{tikzcd} T(T(X)) \arrow[r, "T(T(f))"] \arrow[d, "\mu_Y"'] & T(T(Y)) \arrow[r, "T(\xi)"] \arrow[d, "\mu_X"'] & T(Y) \arrow[d, "\xi"] \\ T(X) \arrow[r, "T(f)"'] & T(Y) \arrow[r, "\xi"'] & X \end{tikzcd}$$

showing that $\xi \circ T(f)$ is a morphism $(T(X), \mu_X) \to (Y, \xi)$ and that $U_T(\xi \circ T(f)) \circ \eta_X = \phi$. We are left with uniqueness. If $g\colon (T(X), \mu_X) \to (Y, \xi)$ is a morphism in $\mathbf{EM}(\mathbf{T})$ such that $U_T(g) \circ \eta_X = f$ then

$$\begin{tikzcd} T(X) \arrow[r, "T(\eta_X)"'] \arrow[rr, bend left, "T(f)"] \arrow[dr, "\mathrm{id}_{T(X)}"'] & T(T(X)) \arrow[r, "T(g)"'] \arrow[d, "\mu_X"] & T(Y) \arrow[d, "\xi"] \\ & T(X) \arrow[r, "g"'] & Y \end{tikzcd}$$

commutes and thus $g = \xi \circ T(f)$. □

Remark 2.1.13. It is worth to spell out explicitly the counit ϵ_T of $F_{\mathbf{T}} \dashv U_{\mathbf{T}}$. Given and algebra$(X, \xi)$, $\epsilon_{T,(X,\xi)}$ is the unique morphism $(T(X), \mu_X) \to (X, \xi)$ such that

$$\mathrm{id}_X = U_T\left(\epsilon_{T,(X,\xi)}\right) \circ \eta_X$$

But then the axioms of Definition 2.1.9 immediately entail that $\epsilon_{(X,\xi)} = \xi$. In particular, this implies that μ_X is the unique morphism $(T(T(X)), \mu_{T(X)}) \to (T(X), \mu_X)$ satisfying

$$\mathrm{id}_{T(X)} = \mu_Y \circ \eta_{T(X)}$$

Clearly $U_{\mathbf{T}} \circ F_{\mathbf{T}} = T$, moreover, whenever a functor $U\colon \mathbf{X} \to \mathbf{Y}$ has a left adjoint F such that $U \circ F = T$ we can canonically compare $\mathbf{X}$ with $\mathbf{EM}(\mathbf{T})$.

Proposition 2.1.14. *Let* $U\colon \mathbf{X} \to \mathbf{Y}$ *be a functor with a left adjoint* F *and* (T, η, μ) *the induced monad. Then there exists a* comparison functor $K\colon \mathbf{X} \to \mathbf{EM}(\mathbf{T})$ *which sends an object* X *to* $(U(X), U(\epsilon_X))$*, where* ϵ *is the counit of* $F \dashv U$.

Proof. First of all we have to verify that $(U(X), U(\epsilon_X))$ is an Eilenberg-Moore algebra. One of the axioms is just one of the triangular identities, the other is obtained applying U to the naturality square

$$\begin{tikzcd} F(U(F(U(X)))) \arrow[r, "\epsilon_{F(U(X))}"] \arrow[d, "F(U(\epsilon_X))"'] & F(U(X)) \arrow[d, "\epsilon_X"] \\ F(U(X)) \arrow[r, "\epsilon_X"'] & X \end{tikzcd}$$

Given $f\colon X \to Y$ in $\mathbf{X}$, if we apply U to the naturality square

$$\begin{tikzcd}
F(U(X)) \arrow[r, "F(U(f))"] \arrow[d, "\epsilon_X"'] & F(U(Y)) \arrow[d, "\epsilon_Y"] \\
X_1 \arrow[r, "f"'] & Y
\end{tikzcd}$$

to get that $U(f)$ is an arrow $K(X) \to K(Y)$, we can conclude the proof defining $K(f) := U(f)$. □

Remark 2.1.15. Notice that the comparison functor K is automatically faithful if U is so.

Definition 2.1.16. A functor $U\colon \mathbf{Y} \to \mathbf{X}$ is *(strictly) monadic* if it has a left adjoint F and the comparison functor of the previous lemma is an equivalence (isomorphism). A category $\mathbf{Y}$ will be called *(strictly) monadic over* $\mathbf{X}$ if there exists a (strictly) monadic functor $U : \mathbf{Y} \to \mathbf{X}$.

Example 2.1.17. The category **CSLat** of complete semilattices is the category which has as objects complete posets and functions preserving arbitrary suprema as arrows. We can see that the forgetful functor $U_{\mathbf{CSLat}}\colon \mathbf{CSLat} \to \mathbf{Set}$ is (strictly) monadic.

On the one hand, for every set X, $(\mathcal{P}(X), \subseteq)$ is an object of **CSLat** and we can consider

$$\eta_X\colon X \to \mathcal{P}(X) \qquad x \mapsto \{x\}$$

If $(Q, \leq)$ is another element of **CSLat** and given $f\colon X \to U_{\mathbf{CSLat}}(Q, \leq)$, we can define

$$g\colon \mathcal{P}(X) \to Q \qquad A \mapsto \bigvee_{x\in A} f(x)$$

which clearly preserves suprema, and so it defines $g\colon (\mathcal{P}(X), \subseteq) \to (Q, \leq)$. Moreover $g \circ \eta_X = f$ and if $h : (\mathcal{P}(X), \subseteq) \to (Q, \leq)$ has the same property, then, for every $A \in \mathcal{P}(X)$:

$$\begin{aligned}
h(A) &= h\left(\bigcup_{x\in A}\{x\}\right) \\
&= \bigvee_{x\in A} h(\{x\}) \\
&= \bigvee_{x\in A} f(x) \\
&= g(A)
\end{aligned}$$

which shows that $U_{\mathbf{CSLat}}$ has a left adjoint $F_{\mathbf{CSLat}}$.

On the other hand, $U_{\mathbf{CSLat}} \circ F_{\mathbf{CSLat}} = \mathcal{P}$, thus Proposition 2.1.14 and Remark 2.1.15 yield a faithful functor $K\colon \mathbf{CSLat} \to \mathbf{EM}(\mathbf{P})$. Notice that, for every $(X, \leq) \in \mathbf{CSLat}$, the component of the counit of the adjunction $F_{\mathbf{CSLat}} \dashv U_{\mathbf{CSLat}}$ is the morphism

$$\epsilon_{(X,\leq)}\colon (\mathcal{P}(X), \subseteq) \to (X, \leq) \qquad S \mapsto \mathsf{sup}(S)$$

Thus $K(X, \leq)$ is the Eilenberg-Moore algebra $(X, \xi_{\leq})$ in which

$$\xi_{\leq}\colon \mathcal{P}(X) \to X \qquad S \mapsto \mathsf{sup}(S)$$

Now, given $(X, \xi) \in \mathbf{EM}(\mathbf{P})$, we can define a relation $\leq_\xi$ on X putting $x \leq_\xi y$ if and only if

$$\xi(\{x, y\}) = y$$

This relation is actually a partial order:

- reflexivity follows from the first axiom of Eilenberg-Moore algebras: since $\xi \circ \eta_X = \mathsf{id}_X$ then, for every $x \in X$, $\xi(\{x\}) = x$, which shows $x \leq_\xi x$;
- for transitivity, let $x, y, z \in X$ be such that $x \leq_\xi y$ and $y \leq_\xi z$, using the second axiom of Eilenberg-Moore algebras we get

$$\begin{aligned}
\xi(\{x, z\}) &= \xi(\{\xi(\{x\}), \xi(\{y, z\})\}) \\
&= \xi(\mathcal{P}(\xi)(\{\{x\}, \{y, z\}\})) \\
&= \xi(\mu_X(\{\{x\}, \{y, z\}\}) \\
&= \xi(\{x, y, z\}) \\
&= \xi(\mu_X(\{\{x, y\}, \{z\}\})) \\
&= \xi(\mathcal{P}(\xi)(\{\{x, y\}, \{z\}\})) \\
&= \xi(\{\xi(\{x, y\}), \xi(\{z\})\}) \\
&= \xi(\{y, z\}) \\
&= z
\end{aligned}$$

  which shows that $x \leq_\xi z$;
- finally, if $x \leq_\xi y$ and $y \leq_\xi x$, then

$$\begin{aligned}
x &= \xi(\{y, x\}) \\
&= \xi(\{x, y\}) \\
&= y
\end{aligned}$$

  yielding antisimmetry.

Now let S be a subset of X, we can notice that $\xi(S)$ is a supremum for it:

- if $s \in S$ then we can compute

$$\begin{aligned}
\xi(\{s, \xi(S)\}) &= \xi(\{\xi(\{s\}), \xi(S)\}) \\
&= \xi(\mathcal{P}(\xi)(\{\{s\}, S\})) \\
&= \xi(\mu_X(\{\{s\}, S\})) \\
&= \xi(\{s\} \cup S) \\
&= \xi(S)
\end{aligned}$$

  and thus $\xi(S)$ is an upper bound for S;

- if y is another upper bound for S then, by definition $y = \xi(\{s, y\})$ for every $s \in S$, thus

$$\begin{aligned}\xi(\{\xi(S), y\}) &= \xi(\{\xi(S), \xi(\{y\})\}) \\ &= \xi(\mathcal{P}(\xi)(\{S, \{y\}\})) \\ &= \xi(\mu_X(\{S, \{y\}\})) \\ &= \xi(S \cup \{y\}) \\ &= \xi(\mu_X(\{\{s, y\}\}_{s\in S})) \\ &= \xi(\mathcal{P}(\xi)(\{\{s, y\}\}_{s\in S})) \\ &= \xi(\{\xi(\{s, y\})\}_{s\in S}) \\ &= \xi(\{y\}) \\ &= y\end{aligned}$$

  showing $\xi(S) \leq_\xi y$.

Now let $f\colon (X, \xi_1) \to (Y, \xi_2)$ be a morphism of $\mathbf{EM}(\mathbf{P})$, then, by construction, f defines also a morphism $(X, \leq_{\xi_1}) \to (Y, \leq_{\xi_2})$ of $\mathbf{CSLat}$, we can thus define a functor $H\colon \mathbf{EM}(\mathbf{P}) \to \mathbf{CSLat}$

$$\begin{tikzcd} (X, \xi_1) \arrow[r, mapsto] \arrow[d, "f"'] & (X, \leq_{\xi_1}) \arrow[d, "f"] \\ (Y, \xi_2) \arrow[r, mapsto] & (Y, \leq_{\xi_2}) \end{tikzcd}$$

It is now enough to show that H is the inverse of K.

- $K(H(X, \xi))$ is the Eilenberg-Moore algebra equipped with the arrow $\mathcal{P}(X) \to X$ which sends a subset S to its supremum, but we have already shown that this is just $\xi(S)$, thus $K \circ H = \mathsf{id}_{\mathbf{EM}(\mathbf{P})}$.
- $H(K(X, \leq))$ is the preorder $(X, \leq_{\xi_\leq})$, and, for every $x, y \in X$ we have a chain of equivalences

$$\begin{aligned} x \leq_{\xi_\leq} y &\iff \xi_\leq(\{x, y\}) = y \\ &\iff \mathsf{sup}(\{x, y\}) = y \\ &\iff x \leq y \end{aligned}$$

  This shows that $H \circ K = \mathsf{id}_{\mathbf{CSLat}}$.

Given a regular cardinal κ, the same argument applies also to κ-$\mathbf{CSLat}$: the category of *κ-complete semilattices*, i.e. posets in which every subset of cardinality strictly less then κ has a supremum. It is monadic over $\mathbf{Set}$ and the corresponding monad is $(\mathcal{P}_\kappa, \eta, \mu)$ defined at the end of Example 2.1.2.

Let us now examine a non example.

Example 2.1.18. Let $\mathbf{Ab}$ be the category of abelian groups and $\mathbf{Div}$ its full subcategory given by the divisible ones [75]. Then the forgetful functor $U_{\mathbf{Div}}\colon \mathbf{Div} \to \mathbf{Set}$ is not monadic. Take the quotient $\pi\colon \mathbf{Q} \to \mathbf{Q}/\mathbb{Z}$ and the zero morphism $z\colon \mathbf{Q} \to \mathbf{Q}/\mathbb{Z}$. If $f\colon G \to \mathbf{Q}$ is another morphism such that

$$z \circ f = \pi \circ f$$

then $f(G)$ must be a divisible subgroup of $\mathbb{Z}$, thus there is an equalizer diagram in $\mathbf{Div}$:

$$\begin{tikzcd} 0 \arrow[r, "i"] & \mathbf{Q} \arrow[r, shift left, "\pi"] \arrow[r, shift right, "g"'] & \mathbf{Q}/\mathbb{Z} \end{tikzcd}$$

Since an equalizer of $U_{\mathbf{Div}}(\pi)$ and $U_{\mathbf{Div}}(z)$ is given by the inclusion $\mathbb{Z} \to \mathbf{Q}$, this observation shows that $U_{\mathbf{Div}}$ cannot be a right adjoint.

Morphisms of monads

We introduce now the notion of morphism between monads on the same category. Our aim is to show that they corresponds exactly to functors between the categories of Eilenberg-Moore algebras which commutes with the forgetful functor. Let us start with an elementary observation.

Remark 2.1.19. Let $F, G\colon \mathbf{X} \rightrightarrows \mathbf{Y}$ be two functors and χ a natural transformation $F \to G$, then, for every $X \in \mathbf{X}$ we have a naturality square

$$\begin{tikzcd} F(F(X)) \arrow[r, "F(\chi_X)"] \arrow[d, "\chi_{F(X)}"'] & F(G(X)) \arrow[d, "\chi_{G(X)}"] \\ G(F(X)) \arrow[r, "G(\chi_X)"'] & G(G(X)) \end{tikzcd}$$

so that we can define $(\chi * \chi)_X$ as the diagonal of the above square. In this way we get a natural transformation $\chi * \chi\colon F \circ F \to G \circ G$ which coincides with both $(\chi * G) \circ (F * \chi)$ and $(G * \chi) \circ (\chi * F)$.

Definition 2.1.20. Let $\mathbf{T} = (T, \eta_T, \mu_T)$ and $\mathbf{S} = (S, \eta_S, \mu_S)$ be two monads on a category $\mathbf{X}$, a *morphism of monads* $\mathbf{T} \to \mathbf{S}$ is a natural transformation $\chi\colon T \to S$ such that the following diagrams commute:

$$\begin{tikzcd} \mathsf{id}_{\mathbf{X}} \arrow[r, "\eta_T"] \arrow[dr, "\eta_S"'] & T \arrow[d, "\chi"] \\ & S \end{tikzcd} \qquad \begin{tikzcd} T \circ T \arrow[r, "\mu_T"] \arrow[d, "\chi * \chi"'] & T \arrow[d, "\chi"] \\ S \circ S \arrow[r, "\mu_S"'] & S \end{tikzcd}$$

A morphism $\chi\colon \mathbf{T} \to \mathbf{S}$ will be called a *isomorphism* if it is a natural isomorphism $T \to S$.

Example 2.1.21. Take a monoidal category $(\mathbf{X}, \otimes, I)$ and consider two monoid objects (P, m, e) and (Q, n, f) in it. A *morphism of monoids* is an arrow $g\colon P \to Q$ such that the following diagrams commute.

$$\begin{tikzcd} & I \arrow[dl, "e"'] \arrow[dr, "f"] & \\ P \arrow[rr, "g"'] & & Q \end{tikzcd} \qquad \begin{tikzcd} P \otimes P \arrow[r, "g \otimes g"] \arrow[d, "m"'] & Q \otimes Q \arrow[d, "n"] \\ P \arrow[r, "g"'] & Q \end{tikzcd}$$

Such a g induces a morphism χ_g between the two associated writer monads. Indeed, if we define $\chi_{g,X}$ as $\mathsf{id}_X \otimes g$, then we have the following diagrams witness our claim.

$$\begin{tikzcd} & X \arrow[dl, "\rho_X^{-1}"'] \arrow[dr, "\rho_X^{-1}"] & \\ X \otimes I \arrow[rr, "\mathsf{id}_{X \otimes I}"] \arrow[d, "\mathsf{id}_X \otimes e"] & & X \otimes I \arrow[d, "\mathsf{id}_X \otimes f"'] \\ X \otimes P \arrow[rr, "\mathsf{id}_X \otimes g"'] & & X \otimes Q \end{tikzcd} \qquad \begin{tikzcd} (X \otimes P) \otimes P \arrow[r, "\alpha_{X,P,P}"] \arrow[d, "\mathsf{id}_{X \otimes P} \otimes g"'] & X \otimes (P \otimes P) \arrow[r, "\mathsf{id}_X \otimes m"] \arrow[dd, "\mathsf{id}_X \otimes (g \otimes g)"'] & X \otimes P \arrow[dd, "\mathsf{id}_X \otimes g"'] \\ (X \otimes P) \otimes Q \arrow[d, "(\mathsf{id}_X \otimes g) \otimes \mathsf{id}_Q"'] & & \\ (X \otimes Q) \otimes Q \arrow[r, "\alpha_{X,Q,Q}"'] & X \otimes (Q \otimes Q) \arrow[r, "\mathsf{id}_X \otimes n"'] & X \otimes Q \end{tikzcd}$$

Remark 2.1.22. Morphisms of monads compose. Let $\chi_1 \colon \mathbf{T} \to \mathbf{S}$ and $\chi_2 \colon \mathbf{S} \to \mathbf{R}$, then we have a diagram

$$\begin{tikzcd}
T(T(X)) \arrow[r, "T(\chi_{1,X})"] \arrow[rr, bend left, "T(\chi_{2,X}\circ\chi_{1,X})"] \arrow[d, "\chi_{1,T(X)}"'] \arrow[dr, "(\chi_1*\chi_1)_X"] & T(S(X)) \arrow[r, "T(\chi_{2,X})"] \arrow[d, "\chi_{1,S(X)}"] & T(R(X)) \arrow[d, "\chi_{1,R(X)}"] \\
S(T(X)) \arrow[r, "S(\chi_{1,X})"] \arrow[d, "\chi_{2,T(X)}"'] & S(S(X)) \arrow[r, "S(\chi_{2,X})"] \arrow[d, "\chi_{2,S(X)}"] \arrow[dr, "(\chi_2*\chi_2)_X"] & S(R(X)) \arrow[d, "\chi_{2,R(X)}"] \\
R(T(X)) \arrow[r, "R(\chi_{1,X})"'] \arrow[rr, bend right, "R(\chi_{2,X}\circ\chi_{1,X})"'] & R(S(X)) \arrow[r, "R(\chi_{2,X})"'] & R(R(X))
\end{tikzcd}$$

proving the, well known, interchange law

$$(\chi_2 * \chi_2) \circ (\chi_1 * \chi_1) = (\chi_2 \circ \chi_1) * (\chi_2 \circ \chi_1)$$

We can now construct the two diagrams below, showing that $\chi_2 \circ \chi_1$ is a morphism of monads.

$$\begin{tikzcd}
 & \mathsf{id}_{\mathbf{X}} \arrow[dl, bend right, "\eta_T"'] \arrow[d, "\eta_R"] \arrow[dr, bend left, "\eta_R"] & \\
T \arrow[r, "\chi_1"'] & S \arrow[r, "\chi_2"'] & R
\end{tikzcd}
\qquad
\begin{tikzcd}
T \circ T \arrow[r, "\chi_1*\chi_1"'] \arrow[rr, bend left, "(\chi_2*\chi_2)\circ(\chi_1*\chi_1)"] \arrow[d, "\mu_T"'] & S \circ S \arrow[r, "\chi_2*\chi_2"'] \arrow[d, "\mu_S"'] & R \circ R \arrow[d, "\mu_R"] \\
T \arrow[r, "\chi_1"'] & S \arrow[r, "\chi_2"'] & R
\end{tikzcd}$$

Remark 2.1.23. Notice that if $\chi \colon \mathbf{T} \to \mathbf{S}$ is an isomorphism of monads, then χ^{-1} is a morphism of monad too. First of all notice that, for every $X \in \mathbf{X}$:

$$\begin{aligned}
\left(\chi^{-1} * \chi^{-1}\right)_X &= \chi^{-1}_{T(X)} \circ S\left(\chi^{-1}_X\right) \\
&= \chi^{-1}_{T(X)} \circ (S(\chi_X))^{-1} \\
&= \left(S(\chi_X) \circ \chi_{T(X)}\right)^{-1} \\
&= (\chi * \chi)^{-1}
\end{aligned}$$

and thus we can further compute to get:

$$\begin{aligned}
\chi^{-1}_X \circ \eta_{S,X} &= \chi^{-1}_X \circ \chi_X \circ \eta_{T,X} \\
&= \mathsf{id}_{T(X)} \circ \eta_{T,X} \\
&= \eta_{T,X}
\end{aligned}
\qquad
\begin{aligned}
\chi_X \circ \chi^{-1}_X \circ \mu_{S,X} &= \mu_{S,X} \\
&= \mu_{S,X} \circ (\chi * \chi)_X \circ (\chi^{-1} * \chi^{-1})_X \\
&= \chi_X \circ \mu_{T,X} \circ (\chi^{-1} * \chi^{-1})_X
\end{aligned}$$

and the thesis now follows since χ_X is a mono.

Take now a morphism of monads $\chi \colon \mathbf{T} \to \mathbf{S}$, we can define a functor $F_\chi \colon \mathbf{EM}(\mathbf{S}) \to \mathbf{EM}(\mathbf{T})$ in the following way. Given an object (X, ξ) of $\mathbf{EM}(\mathbf{S})$, we can define ξ_χ as the composition

$$T(X) \xrightarrow{\chi_X} S(X) \xrightarrow{\xi} X$$

In this way we get a Eilenberg-Moore algebra for $\mathbf{T}$, as witnessed by the following two diagrams

$$
\begin{tikzcd}
X \arrow[r, "\eta_{T,X}"] \arrow[dr, "\eta_{S,X}"'] \arrow[ddr, bend right, "\mathsf{id}_X"'] & T(X) \arrow[d, "\chi_X"] \\
& S(X) \arrow[d, "\xi"] \\
& X
\end{tikzcd}
\qquad
\begin{tikzcd}
T(T(X)) \arrow[rr, "\mu_{T,X}"] \arrow[d, "T(\chi_X)"] \arrow[dr, "(\chi * \chi)_X"] \arrow[dd, bend right=60, "T(\xi_\chi)"'] & & T(X) \arrow[d, "\chi_X"'] \arrow[dd, bend left=60, "\xi_\chi"] \\
T(S(X)) \arrow[r, "\chi_{S(X)}"] \arrow[d, "T(\xi)"] & S(S(X)) \arrow[r, "\mu_{S,X}"] \arrow[d, "S(\xi)"'] & S(X) \arrow[d, "\xi"'] \\
T(X) \arrow[r, "\chi_X"'] & S(X) \arrow[r, "\xi"'] & X
\end{tikzcd}
$$

Moreover, if $f\colon (X, \xi_1) \to (Y, \xi_2)$ is an arrow in $\mathbf{EM}(\mathbf{S})$, then we the following diagram shows that the same f also induces an arrow $(X, \xi_{1,\chi}) \to (X, \xi_{2,\chi})$:

$$
\begin{tikzcd}
T(X) \arrow[r, "\chi_X"] \arrow[d, "T(f)"'] \arrow[rr, bend left, "\xi_{1,\chi}"] & S(X) \arrow[r, "\xi_1"] \arrow[d, "S(f)"'] & X \arrow[d, "f"] \\
T(Y) \arrow[r, "\chi_Y"'] \arrow[rr, bend right, "\xi_{1,\chi}"'] & S(Y) \arrow[r, "\xi_2"'] & Y
\end{tikzcd}
$$

Summing up, we have just built a functor $F_\chi\colon \mathbf{EM}(\mathbf{S}) \to \mathbf{EM}(\mathbf{T})$. We can also notice that this functor makes the following diagram commutative.

$$
\begin{tikzcd}
\mathbf{EM}(\mathbf{S}) \arrow[rr, "F_\chi"] \arrow[dr, "U_\mathbf{S}"'] & & \mathbf{EM}(\mathbf{T}) \arrow[dl, "U_\mathbf{T}"] \\
& \mathbf{X} &
\end{tikzcd}
$$

Every functor with this property arises in this way, as shown by the following proposition.

Proposition 2.1.24. *Let $\mathbf{T}$ and $\mathbf{S}$ be monads on the same category $\mathbf{X}$ and let also $F\colon \mathbf{EM}(\mathbf{S}) \to \mathbf{EM}(\mathbf{T})$ be a functor such that the following diagram commutes*

$$
\begin{tikzcd}
\mathbf{EM}(\mathbf{S}) \arrow[rr, "F"] \arrow[dr, "U_\mathbf{S}"'] & & \mathbf{EM}(\mathbf{T}) \arrow[dl, "U_\mathbf{T}"] \\
& \mathbf{X} &
\end{tikzcd}
$$

Then there exists a unique $\chi\colon \mathbf{T} \to \mathbf{S}$ such that $F_\chi = F$.

Proof. Take an object X of $\mathbf{X}$, by hypothesis we have

$$
\begin{aligned}
U_\mathbf{T}(F(F_\mathbf{S}(X))) &= U_\mathbf{S}(F_\mathbf{S}(X)) \\
&= S(X)
\end{aligned}
$$

Now, $F(F_{\mathbf{S}}(X))$ is an object of $\mathbf{EM}(\mathbf{T})$ and we have an arrow $\eta_{S,X}\colon X \to S(X)$. Thus there exists a unique $\chi_X\colon F_{\mathbf{T}}(X) \to F(F_{\mathbf{S}}(X))$ making the following diagram commutative

$$\begin{tikzcd}
X \arrow[r, "\eta_{S,X}"] \arrow[d, "\eta_{T,X}"'] & S(X) \\
T(X) \arrow[ur, dotted, "\chi_X"'] &
\end{tikzcd}$$

We claim that in this way we get a morphism of monads.

First of all we have to show naturality. Let $f\colon X \to Y$ be an arrow in $\mathbf{X}$. Since $S(f)$ is morphism $F_{\mathbf{S}}(X) \to F_{\mathbf{S}}(Y)$ in $\mathbf{EM}(\mathbf{S})$, we can use again the hypothesis on F to get

$$\begin{aligned}
U_{\mathbf{T}}(F(S(f))) &= U_{\mathbf{S}}(S(f)) \\
&= S(f)
\end{aligned}$$

showing that $S(f)$ also defines a morphism $F(F_{\mathbf{S}}(X)) \to F(F_{\mathbf{S}}(Y))$ in $\mathbf{EM}(\mathbf{T})$. Thus we have morphisms $S(f) \circ \chi_X, \chi_Y \circ T(f)\colon F_{\mathbf{T}}(X) \rightrightarrows F(F_{\mathbf{S}}(Y))$ in $\mathbf{EM}(\mathbf{T})$. On the other hand we have a diagram

$$\begin{tikzcd}
 & X \arrow[dl, bend right, "\eta_{T,X}"'] \arrow[dr, bend left, "\eta_{T,X}"] \arrow[d, "f"] \arrow[ddl, "\eta_{S,X}"] & \\
T(X) \arrow[dd, "\chi_X"'] & Y \arrow[ddr, phantom, ""] \arrow[dd, "\eta_{S,Y}"'] \arrow[dr, "\eta_{T,X}"] & T(X) \arrow[d, "T(f)"] \\
 & & T(Y) \arrow[d, bend left, "\chi_Y"] \\
S(X) \arrow[r, bend right, "S(f)"'] & S(Y) & \arrow[l]
\end{tikzcd}$$

which shows that

$$S(f) \circ \chi_X \circ \eta_{T,X} = \chi_Y \circ T(f) \circ \eta_{T,X}$$

This now implies that

$$S(f) \circ \chi_X = \chi_Y \circ T(f)$$

The first condition for being a morphism of monads is satisfied by construction, let us prove that also the other holds. Our line of argument is similar to the one used for naturality. We have the following list of morphisms in $\mathbf{EM}(\mathbf{T})$:

$$\begin{gathered}
\chi_{S(X)}\colon F_{\mathbf{T}}(S(X)) \to F(F_{\mathbf{S}}(S(X))) \quad \mu_{S,X}\colon F(F_{\mathbf{S}}(S(X))) \to F(F_{\mathbf{S}}(X)) \\
\mu_{T,X}\colon F_{\mathbf{T}}(T(X)) \to F_{\mathbf{T}}(X) \quad \chi_X\colon F_{\mathbf{T}}(X) \to F(F_{\mathbf{S}}(X)) \quad T(\chi_X)\colon F_{\mathbf{T}}(T(X)) \to F_{\mathbf{T}}(S(X))
\end{gathered}$$

and thus we have, $\chi_X \circ \mu_{T,X}, \mu_{S,X} \circ \chi_{S(X)} \circ T(\chi_X)\colon F_{\mathbf{T}}(T(X)) \rightrightarrows F(F_{\mathbf{S}}(X))$. We also have a diagram:

$$\begin{tikzcd}
T(T(X)) \arrow[d, "T(\chi_X)"'] & T(X) \arrow[l, "\eta_{T,T(X)}"'] \arrow[r, "\eta_{T,T(X)}"] \arrow[d, "\chi_X"'] \arrow[ddr, "\mathrm{id}_{T(X)}"] & T(T(X)) \arrow[dd, "\mu_{T(X)}"] \\
T(S(X)) \arrow[d, "\chi_{S(X)}"'] & S(X) \arrow[l, "\eta_{T,S(X)}"'] \arrow[dl, "\eta_{S,S(X)}"'] \arrow[d, "\mathrm{id}_{S(X)}"] & \\
S(S(X)) \arrow[r, "\mu_{S,X}"'] & S(X) & T(X) \arrow[l, "\chi_X"]
\end{tikzcd}$$

which entails

$$\begin{aligned}\chi_X \circ \mu_{T,X} \circ \eta_{T,T(X)} &= \chi_X \\ &= \mu_{S,X} \circ \chi_{S(X)} \circ T(\chi_X) \circ \eta_{T,T(X)}\end{aligned}$$

from which we can deduce that $\chi_X \circ \mu_{T,X} = \mu_{S,X} \circ (\chi * \chi)_X$.

We have now to show that $F_\chi = F$. The condition on F implies, in particular, that F must act as the identity on arrows, as F_χ. So it is enough to show that they are equal on objects. Let (X, ξ) be an object of $\mathbf{EM}(\mathbf{S})$ and (X, θ) be $F(X, \xi)$. By the definition of Eilenberg-Moore algebras, we know that θ defines a morphism $F_{\mathbf{T}}(X) \to (X, \theta)$ of $\mathbf{EM}(\mathbf{T})$. On the other hand, for the same reason, ξ also define a morphism $F_{\mathbf{S}}(X) \to (X, \xi)$ in $\mathbf{EM}(\mathbf{S})$ and thus also a morphism $F(F_{\mathbf{S}}(X)) \to (X, \theta)$ in $\mathbf{EM}(\mathbf{T})$. Precomposing with χ_X, which by definition is an arrow $F_{\mathbf{T}}(X) \to F(F_{\mathbf{S}}(X))$ we get a pair of parallel arrows $\theta, \xi \circ \chi_X : F_{\mathbf{T}}(X) \rightrightarrows (X, \theta)$. But now we can compute:

$$\begin{aligned}\xi \circ \chi_X \circ \eta_{T,X} &= \xi \circ \eta_{S,X} \\ &= \mathsf{id}_X \\ &= \theta \circ \eta_{T,X}\end{aligned}$$

and from this it follows that $\theta = \xi \circ \chi_X$, which is what we claimed.

Finally, we must prove uniqueness. Let $\chi' : \mathbf{T} \to \mathbf{S}$ be another morphism of monads such that $F = F_{\chi'}$. For every $X \in \mathbf{X}$ we have a diagram

$$\begin{tikzcd} T(T(X)) \arrow[rr, "\mu_{T,X}"] \arrow[d, "T(\chi'_X)"'] \arrow[dr, "(\chi' * \chi')_X"] & & T(X) \arrow[d, "\chi'_X"] \\ T(S(X)) \arrow[r, "\chi'_{S(X)}"'] & S(S(X)) \arrow[r, "\mu_{S,X}"'] & S(X) \end{tikzcd}$$

which shows that χ' is a morphism $F_{\mathbf{T}}(X) \to F_{\chi'}(F_{\mathbf{S}}(X))$, but, by hypothesis, the codomain of this arrow in $\mathbf{EM}(\mathbf{T})$ is just $F(X)$. On the other hand, we can precompose with $\eta_{T,X}$ to get

$$\begin{aligned}\chi'_X \circ \eta_{T,X} &= \eta_{S,X} \\ &= \chi_X \circ \eta_{T,X}\end{aligned}$$

and this now implies that $\chi_X = \chi'_X$. □

Remark 2.1.25. Notice that $F_{\mathsf{id}_{\mathbf{T}}} : \mathbf{EM}(\mathbf{T}) \to \mathbf{EM}(\mathbf{T})$ is the identity functor and that

$$F_{\chi' \circ \chi} = F_\chi \circ F_{\chi'}$$

for every $\chi : \mathbf{T} \to \mathbf{S}$ and $\chi' : \mathbf{S} \to \mathbf{R}$.

The following corollary now follows at once from the previous remark.

Corollary 2.1.26. *Two monads* $\mathbf{T}$ *and* $\mathbf{S}$ *on a category* $\mathbf{X}$ *are isomorphic if and only if there is an isomorphism* $F : \mathbf{EM}(\mathbf{S}) \to \mathbf{EM}(\mathbf{T})$ *such that the following triangle commutes.*

$$\begin{tikzcd} \mathbf{EM}(\mathbf{S}) \arrow[rr, "F"] \arrow[dr, "U_{\mathbf{S}}"'] & & \mathbf{EM}(\mathbf{T}) \arrow[dl, "U_{\mathbf{T}}"] \\ & \mathbf{X} & \end{tikzcd}$$

In general monads on a large category $\mathbf{X}$ do not form a category: there can be a proper class of morphisms between them. This can be somewhat solved by the following notion.

Definition 2.1.27. Let $J\colon \mathbf{Y} \to \mathbf{X}$ be functor, a monad $\mathbf{T} = (T, \eta, \mu)$ will be called a *J-monad*, if $(T, \mathsf{id}_{T\circ J})$ is the left Kan extension of $T \circ J$ along J.

Proposition 2.1.28. *Let $J\colon \mathbf{Y} \to \mathbf{X}$ be a functor with an essentially small domain (i.e. $\mathbf{Y}$ is equivalent to a small category), then there exists a category J-**Mnd** whose objects are J-monads and whose arrows are morphisms of monads.*

Proof. Since $(T, \mathsf{id}_{T\circ J})$ is a left Kan extension of $T \circ J$ along J, there is a bijection between $\mathbf{X}^{\mathbf{X}}(T, S)$ and $\mathbf{X}^{\mathbf{Y}}(T \circ J, S \circ J)$. Since morphisms of monads are natural transformations, the thesis now follows from essential smallness of $\mathbf{Y}$. $\square$

Remark 2.1.29. If the codomain of J is cocomplete, then we can use Corollary A.5.13 to get

$$T \simeq \int^{Y\in\mathbf{Y}} \mathbf{X}(J(Y), -) \bullet T(J(Y))$$

Moreover, by Theorem A.5.12, for every $X \in \mathbf{X}$ the component $\omega_{X,Y}\colon \mathbf{X}(J(Y), X) \bullet T(Y) \to T(X)$ of the universal cowedge ω_X can be described explicitly. Given $f \in J(Y) \to X$, if $\iota_f\colon T(J(Y)) \to \mathbf{X}(J(Y), X) \bullet T(X)$ is the corresponding coprojection, then $T(f) = \omega_{X,Y} \circ \iota_f$.

2.1.2 Limits and colimits in EM(T)

In this section we examine the existence of limits and colimits in categories of Eilenberg-Moore algebras. In particular we are interested in how to compute limits and colimits in categories monadic over **Set**.

The situation for limits is quite simple.

Proposition 2.1.30. *Let $\mathbf{T}$ be a monad on $\mathbf{X}$, the functor $U_{\mathbf{T}}\colon \mathbf{EM}(\mathbf{T}) \to \mathbf{X}$ creates limits.*

Proof. Given a functor $F\colon \mathbf{D} \to \mathbf{EM}(\mathbf{T})$, for every $D \in \mathbf{D}$ let $F(D)$ be the algebra (X_D, ξ_D). Suppose also that there exists a limit $(L, \{l_D\}_{D\in\mathbf{D}})$ of $U_T \circ F$. We are looking for an algebra (L, ξ) which makes all the l_D arrows of $\mathbf{EM}(\mathbf{T})$, so we must have a commutative square

$$\begin{tikzcd} T(L) \arrow[r, "\xi"] \arrow[d, "T(l_D)"'] & L \arrow[d, "l_D"] \\ T(X_D) \arrow[r, "\xi_D"'] & X_D \end{tikzcd}$$

Therefore ξ must be the unique arrow $T(L) \to L$ such that

$$l_D \circ \xi = \xi_D \circ T(l_D)$$

Let us check that (L,ξ) is really an object of $\mathbf{EM}(\mathbf{T})$. On the one hand

$$\begin{aligned} l_D \circ \xi \circ T(\xi) &= \xi_D \circ T(l_D) \circ T(\xi) \\ &= \xi_D \circ T(l_D \circ \xi) \\ &= \xi_D \circ T(\xi_D \circ T(l_D)) \\ &= \xi_D \circ T(\xi_D) \circ T(T(l_D)) \\ &= \xi_D \circ \mu_{X_D} \circ T(T(l_D)) \\ &= \xi_D \circ T(l_D) \circ \mu_L \\ &= l_D \circ \xi \circ \mu_L \end{aligned}$$

from which it follows that $\xi \circ T(\xi) = \xi \circ \mu_L$. On the other hand we have a commutative diagram

$$\begin{tikzcd} L \arrow[r, "\eta_L"] \arrow[d, "l_D"'] & T(L) \arrow[r, "\xi"] \arrow[d, "T(l_D)"'] & L \arrow[d, "l_D"] \\ X_D \arrow[r, "\eta_{X_D}"] \arrow[rr, bend right, "\mathsf{id}_{X_D}"'] & T(X_D) \arrow[r, "\xi_D"] & X_D \end{tikzcd}$$

therefore $l_D \circ (\xi \circ \eta_L) = l_D$ and thus $\xi \circ \eta_L = \mathsf{id}_L$.

We are left with the limiting property. Take a cone on F with vertex (Q,θ) and edges $f_D\colon (Q,\theta) \to (X_D,\xi_D)$, then $(Q,\{f_D\}_{D\in\mathbf{D}})$ is a cone for $U_\mathbf{T} \circ F$ and thus there is a unique $f\colon Q \to L$ in $\mathbf{X}$. If we show that f defines an arrow of $\mathbf{EM}(\mathbf{T})$, then we are done. We have

$$\begin{aligned} l_D \circ \xi \circ T(f) &= \xi_D \circ T(l_D) \circ T(f) \\ &= \xi_D \circ T(l_D \circ f) \\ &= \xi_D \circ T(f_D) \\ &= f_D \circ \theta \\ &= l_D \circ f \circ \theta \end{aligned}$$

from which it follows that $\xi \circ T(f) = f \circ \theta$. □

Corollary 2.1.31. *If* $\mathbf{T}$ *is a monad on a complete catgory* $\mathbf{X}$*, then* $\mathbf{EM}(\mathbf{T})$ *is complete.*

In particular we can specialize the previous result to **Set** to get the following.

Corollary 2.1.32. $\mathbf{EM}(\mathbf{T})$ *is complete for every monad* $\mathbf{T}$ *on* **Set**.

The situation for colimits is a bit more complicated.

Proposition 2.1.33. *Let* $\mathbf{T}$ *be a monad on* $\mathbf{X}$ *and* $F\colon \mathbf{D} \to \mathbf{EM}(\mathbf{T})$ *a functor such that* $U_\mathbf{T} \circ F$ *has a colimit* $(L,\{l_D\}_{D\in\mathbf{D}})$ *which is preserved by* T *and by* $T \circ T$*. Then there exists a unique* (L,ξ) *in* $\mathbf{EM}(\mathbf{T})$ *which makes every* l_D *an arrow of* $\mathbf{EM}(\mathbf{T})$ *and, moreover,* $((L,\xi),\{l_D\}_{D\in\mathbf{D}})$ *is colimiting for* F*.*

Remark 2.1.34. If T preserves all colimits of a certain shape $\mathbf{D}$, then the preservation of the same kind of colimits by $T \circ T$ follows for free.

Proof. By hypothesis $(T(L_D), \{T(l_D)\}_{D\in\mathbf{D}})$ is a colimit for $T \circ U_T \circ F$. Now if l_D is a morphism of $\mathbf{EM}(\mathbf{T})$ then we must have a commutative square

$$\begin{tikzcd} T(X_D) \arrow[r, "\xi_D"] \arrow[d, "T(l_D)"'] & X_D \arrow[d, "l_D"] \\ T(L) \arrow[r, "\xi"'] & L \end{tikzcd}$$

and thus ξ must be the unique arrow $T(L) \to L$ such that $\xi \circ T(l_D) = l_D \circ \xi_D$. As in Proposition 2.1.30 we have to show that (L, ξ) is in $\mathbf{EM}(\mathbf{T})$. On the one hand we have that:

$$\begin{aligned} \xi \circ T(\xi) \circ T(T(l_D)) &= \xi \circ T(\xi \circ T(l_D)) \\ &= \xi \circ T(l_D \circ \xi_D) \\ &= \xi \circ T(l_D) \circ T(\xi_D) \\ &= l_D \circ \xi_D \circ T(\xi_D) \\ &= l_D \circ \xi_D \circ \mu_{X_D} \\ &= \xi \circ T(l_D) \circ \mu_{X_D} \\ &= \xi \circ \mu_L \circ T(T(l_D)) \end{aligned}$$

and, since $(T(T(L)), \{T(T(l_D))\}_{D\in\mathbf{D}})$ is a colimit for $T \circ T \circ U_T \circ F$ we can deduce that $\xi \circ T(\xi) = \xi \circ \mu_L$. On the other hand the following diagram commutes

$$\begin{tikzcd} X_D \arrow[r, "\eta_{X_D}"'] \arrow[rr, bend left, "\mathsf{id}_{X_D}"] \arrow[d, "l_D"'] & T(X_D) \arrow[r, "\xi_D"'] \arrow[d, "T(l_D)"'] & X_D \arrow[d, "l_D"] \\ L \arrow[r, "\eta_L"'] & T(L) \arrow[r, "\xi"'] & L \end{tikzcd}$$

and $(L, \{l_D\}_{\mathbf{D}\in\mathbf{D}})$ is colimiting, so $\xi \circ \eta_L = \mathsf{id}_L$.

The colimiting property is proved as in Proposition 2.1.30: take a cocone on F with vertex (Q, θ) and edges $f_D\colon (X_D, \xi_D) \to (Q, \theta)$, then $(Q, \{f_D\}_{D\in\mathbf{D}})$ is a cocone for $U_T \circ F$ which induces a unique $f\colon L \to Q$, which is an arrow of $\mathbf{EM}(\mathbf{T})$ since we have

$$\begin{aligned} \theta \circ T(f) \circ T(l_D) &= \theta \circ T(f_D) \\ &= f_D \circ \xi_D \\ &= f \circ l_D \circ \xi_D \\ &= f \circ \xi \circ T(l_D) \end{aligned}$$

The thesis now follows. □

For an example of a non cocomplete category of Eilenberg-Moore algebras on a cocomplete category we refer the reader to [2]. In that paper a monad on the category **SGraph** of simple graphs (see Definition 6.1.2) is constructed and it is shown that its category of Eilenberg-Moore algebras does not have coequalizers.

Reflexive coequalizers and Linton's theorem

The remainder of this section is devoted to explore conditions on a monad $\mathbf{T}$, or on its base category, which can guarantee cocompleteness of $\mathbf{EM}(\mathbf{T})$. A pivotal role in this endeavour is played by a particular kind of coequalizers.

Definition 2.1.35. A pair of parallel arrows $f, g\colon X \rightrightarrows Y$ is *reflexive* if there exists an arrow $s\colon Y \to X$ such that

$$f \circ s = \mathsf{id}_Y \qquad g \circ s = \mathsf{id}_Y$$

A *reflexive coequalizer* is the coequalizer of a reflexive pair.

Remark 2.1.36. Every reflexive coequalizer in a category $\mathbf{X}$ is the colimit on a functor $\mathbf{D} \to \mathbf{X}$ where $\mathbf{D}$ is the category generated by the diagram

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & B \arrow[l, "s"'] \end{tikzcd}$$

and subjected to the equations

$$f \circ s = \mathsf{id}_B \qquad g \circ s = \mathsf{id}_B$$

Notice that $s \circ g$ is not equal to $s \circ f$ in $\mathbf{D}$.

It is well known [5, 85] that a category with (finite) coproducts and coequalizers admits all (finite) colimit. Actually coproducts and reflexive coequalizers are enough.

Lemma 2.1.37. *A category $\mathbf{X}$ with (finite) coproducts and reflexive coequalizers is (finitely) cocomplete.*

Proof. Let $f, g\colon X \rightrightarrows Y$ be parallel arrows in $\mathbf{X}$. We can consider the parallel pair $\langle f, \mathsf{id}_Y\rangle, \langle g, \mathsf{id}_Y\rangle\colon X + Y \rightrightarrows Y$, which is actually a reflexive pair: the common section to them is simply the inclusion $\iota_Y\colon Y \to X + Y$. Thus we have a coequalizer diagram

$$\begin{tikzcd} X + Y \arrow[r, shift left, "\langle f, \mathsf{id}_Y\rangle"] \arrow[r, shift right, "\langle g, \mathsf{id}_Y\rangle"'] & Y \arrow[r, "e"] & E \end{tikzcd}$$

Computing we have that

$$\begin{aligned} e \circ f &= e \circ \langle f, \mathsf{id}_Y\rangle \circ \iota_X \\ &= e \circ \langle g, \mathsf{id}_Y\rangle \circ \iota_X \\ &= e \circ g \end{aligned}$$

Moreover, if $q\colon Y \to Z$ is such that $q \circ f = q \circ g$ then

$$\begin{aligned} q \circ \langle f, \mathsf{id}_Y\rangle \circ \iota_X &= q \circ f \\ &= q \circ g \\ &= q \circ \langle g, \mathsf{id}_Y\rangle \circ \iota_X \end{aligned} \qquad \begin{aligned} q \circ \langle f, \mathsf{id}_Y\rangle \circ \iota_Y &= q \circ \mathsf{id}_Y \\ &= q \circ \langle g, \mathsf{id}_Y\rangle \circ \iota_Y \end{aligned}$$

Thus $q \circ \langle f, \mathsf{id}_Y\rangle = q \circ \langle g, \mathsf{id}_Y\rangle$ and we can conclude that e is the coequalizer of f and g. $\square$

We are now ready to prove the following classical result about cocompleteness of categories of Eilenberg-Moore algebras , due to Linton [79, Cor. 2].

Theorem 2.1.38. *Let* $\mathbf{T}$ *be a monad on a category* $\mathbf{X}$ *with (finite) coproducts , then the following are equivalent:*

1. $\mathbf{EM}(\mathbf{T})$ *is (finitely) cocomplete;*
2. $\mathbf{EM}(\mathbf{T})$ *admits reflexive coequalizers.*

Proof. $(1 \Rightarrow 2)$ This is obvious.

$(2 \Rightarrow 1)$ In light of Lemma 2.1.37 it is enough to show that $\mathbf{EM}(\mathbf{T})$ has (finite) coproducts. Let thus I be a (finite) set and, for every $i \in I$, suppose that an algebra (X_i, ξ_i) is given. Then we have

$$F_{\mathbf{T}}(X_i) = F_{\mathbf{T}}(U_{\mathbf{T}}(X_i, \xi_i)) \qquad F_{\mathbf{T}}(T(X_i)) = F_{\mathbf{T}}(U_{\mathbf{T}}(F_{\mathbf{T}}(U_{\mathbf{T}}(X_i, \xi_i))))$$

So, if $\epsilon\colon F_{\mathbf{T}} \circ U_{\mathbf{T}} \to \mathsf{id}_{\mathbf{EM}(\mathbf{T})}$ is the counit of $F_{\mathbf{T}} \dashv U_{\mathbf{T}}$, we can take $\epsilon_{F_{\mathbf{T}}(X_i)}$ and $F_{\mathbf{T}}(U_{\mathbf{T}}(\epsilon_{(X_i,\xi_i)}))$ to get a pair of parallel arrows $F_{\mathbf{T}}(T(X_i)) \rightrightarrows F_{\mathbf{T}}(X_i)$. These pairs are actually reflexive: indeed, by Remark 2.1.13 and the fact that $F_{\mathbf{T}}(f) = T(f)$ for every $f\colon X \to Y$ in $\mathbf{X}$, we have that

$$\begin{aligned} \epsilon_{F_{\mathbf{T}}(X_i)} &= \epsilon_{(T(X_i), \mu_{X_i})} \\ &= \mu_{X_i} \end{aligned} \qquad \begin{aligned} F_{\mathbf{T}}(U_{\mathbf{T}}(\epsilon_{(X_i,\xi_i)})) &= F_{\mathbf{T}}(U_{\mathbf{T}}(\xi_i)) \\ &= T(\xi_i) \end{aligned}$$

so $T(\eta_{X_i})$ is a section for both arrows.

Since $\mathbf{X}$ has (finite) coproducts we can define X and X' as the coproduct of $\{X_i\}_{i\in I}$ and $\{T(X_i)\}_{i\in I}$ respectively. $F_{\mathbf{T}}$ is a left adjoint, so $F_{\mathbf{T}}(X)$ and $F_{\mathbf{T}}(X')$ are the coproduct in $\mathbf{EM}(\mathbf{T})$ of $\{F_{\mathbf{T}}(X_i)\}_{i\in I}$ and $\{F_{\mathbf{T}}(T(X_i))\}_{i\in I}$, therefore we have a parallel pair

$$\begin{tikzcd} F_{\mathbf{T}}(X') \arrow[r, shift left, "\sum\limits_{i\in I}\mu_{X_i}"] \arrow[r, shift right, swap, "\sum\limits_{i\in I}T(\xi_i)"] & F_{\mathbf{T}}(X) \end{tikzcd}$$

which is still reflexive and so it has a coequalizer $e\colon F_{\mathbf{T}}(X) \to (E, \xi)$.

Now, the transposes $f, g\colon X' \to T(X)$ of $\sum_{i\in I} \mu_{X_i}$ and $\sum_{i\in I} T(\xi_i)$ are given by

$$f = U_{\mathbf{T}}\left(\sum_{i\in I} \mu_{X_i}\right) \circ \eta_{X'} \qquad g = U_{\mathbf{T}}\left(\sum_{i\in I} T(\xi_i)\right) \circ \eta_{X'}$$

and, since by construction

$$e \circ \sum_{i\in I} \mu_{X_i} = e \circ \sum_{i\in I} T(\xi_i)$$

we know that $U_{\mathbf{T}}(e) \circ f = U_{\mathbf{T}}(e) \circ g$ or, equivalently, $e \circ f = e \circ g$.

If we take $j_i\colon X_i \to X$ and $k_i\colon T(X_i) \to X'$ to be coprojections in $\mathbf{X}$ we can precompose f and g

with k_i to get diagrams

$$\begin{tikzcd}
T(X_i) \arrow[rrr, bend left, "T(j_i)"] & T(T(X_i)) \arrow[l, "\mu_{X_i}"'] \arrow[r, "T(k_i)"] & T(X') \arrow[r, "\sum_{i\in I}\mu_i"] & T(X) \\
& T(X_i) \arrow[ul, bend left, "\mathrm{id}_{T(X_i)}"] \arrow[u, "\eta_{T(X_i)}"] \arrow[r, "k_i"'] & X' \arrow[u, "\eta_{X'}"'] \arrow[ur, bend right, "f"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
X_i \arrow[d, "\eta_{X_i}"'] & T(X_i) \arrow[l, "\xi_i"'] \arrow[r, "k_i"] \arrow[d, "\eta_{T(X_i)}"'] & X' \arrow[d, "\eta_{X'}"] \arrow[dr, bend left, "g"] & \\
T(X_i) \arrow[rrr, bend right, "T(j_i)"'] & T(T(X_i)) \arrow[l, "T(\xi_i)"] \arrow[r, "T(k_i)"'] & T(X') \arrow[r, "\sum_{i\in I}T(\xi_i)"'] & T(X)
\end{tikzcd}$$

where the commutativity of the curved parts is justified because $T(j_i)\colon (T(X_i),\mu_{X_i}) \to (X,\mu_X)$ and $T(k_i)\colon (T(T(X_i)),\mu_{T(X_i)}) \to (T(X'),\mu_{X'})$ are coprojections in $\mathbf{EM}(\mathbf{T})$ by the left adjointness of F_T. Thus, from $e \circ f = e \circ g$ we can deduce that

$$\begin{aligned}
e \circ T(j_i) &= e \circ f \circ k_i \\
&= e \circ g \circ k_i \\
&= e \circ T(j_i) \circ T(\xi_i) \circ \eta_{T(X_i)} \\
&= e \circ T(j_i) \circ \eta_{X_i} \circ \xi_i \\
&= e \circ \eta_X \circ j_i \circ \xi_i
\end{aligned}$$

Therefore we have a commutative diagram

$$\begin{tikzcd}
T(X_i) \arrow[r, "T(j_i)"] \arrow[dd, "\xi_i"'] & T(X) \arrow[r, "T(\eta_X)"] \arrow[dr, "\mathrm{id}_{T(X)}"'] & T(T(X)) \arrow[r, "T(e)"] \arrow[d, "\mu_X"] & T(X) \arrow[dd, "\xi"] \\
& & T(X) \arrow[dr, "e"] & \\
X_i \arrow[r, "j_i"'] & X \arrow[r, "\eta_X"'] & T(X) \arrow[r, "e"'] & X
\end{tikzcd}$$

Which shows that, for every $i \in I$, $h_i\colon X_i \to E$ defined as the composition

$$X_i \xrightarrow{j_i} X \xrightarrow{\eta_X} T(X) \xrightarrow{e} E$$

is a morphism $(X_i,\xi_i) \to (E,\xi)$ of $\mathbf{EM}(\mathbf{T})$. We claim that the cocone $((E,\xi),\{h_i\}_{i\in I})$ is actually a coproduct for $\{(X_i,\xi_i)\}_{i\in I}$.

Let (Y,α) be an algebra and a morphism $a_i\colon (X_i,\xi_i)\to(Y,\alpha)$ for every $i\in I$ which induces an $a\colon X\to Y$. Then $T(a)$ is a morphism $(T(X),\mu_X)\to(T(Y),\mu_Y)$ in $\mathbf{EM}(\mathbf{T})$ and we can consider $\alpha\circ T(a)\colon (T(X),\mu_X)\to(Y,\alpha)$. Computing we get that, for every fixed $t\in I$

$$\begin{aligned}
\alpha\circ T(a)\circ\sum_{i\in I}\mu_{X_i}\circ T(k_t) &= \alpha\circ T(a)\circ T(j_t)\circ\mu_{X_t}\\
&= \alpha\circ T(a_t)\circ\mu_{X_t}\\
&= a_t\circ\xi_t\circ\mu_{X_t}\\
&= a_t\circ\xi_t\circ T(\xi_t)\\
&= \alpha\circ T(a_t)\circ T(\xi_t)\\
&= \alpha\circ T(a)\circ T(j_t)\circ T(\xi_t)\\
&= \alpha\circ T(a)\circ\sum_{i\in I}T(\xi_i)\circ T(k_t)
\end{aligned}$$

which implies that

$$\alpha\circ T(a)\circ\sum_{i\in I}\mu_{X_i} = \alpha\circ T(a)\circ\sum_{i\in I}T(\xi_i)$$

Thus there exists a unique $b\colon (E,\xi)\to(Y,\alpha)$ such that $b\circ e=\alpha\circ T(a)$. Now, for every $i\in I$:

$$\begin{aligned}
b\circ h_i &= b\circ e\circ\eta_X\circ j_i\\
&= \alpha\circ T(a)\circ\eta_X\circ j_i\\
&= \alpha\circ\eta_Y\circ a\circ j_i\\
&= \mathsf{id}_Y\circ a_i\\
&= a_i
\end{aligned}$$

We are left with uniqueness: let $c\colon (E,\xi)\to(Y,\alpha)$ another arrow such that $c\circ h_i=a_i$, we have that:

$$\begin{aligned}
c\circ e\circ T(j_i) &= c\circ e\circ\eta_X\circ j_i\circ\xi_i\\
&= h_i\circ\xi_i\\
&= c\circ\xi\circ T(h_i)\\
&= \alpha\circ T(c)\circ T(h_i)\\
&= \alpha\circ T(a_i)\\
&= \alpha\circ T(a)\circ T(j_i)
\end{aligned}$$

and thus $c\circ e=\alpha\circ T(a)$ which implies $c=b$. □

Using Proposition 2.1.33, the previous theorem gives us immediately the following result.

Corollary 2.1.39. *Let* $\mathbf{X}$ *be a category with (finite) coproducts and* $\mathbf{T}=(T,\eta,\mu)$ *a monad on it such that* T *preserves reflexive coequalizers. Then* $\mathbf{EM}(\mathbf{T})$ *is (finitely) cocomplete.*

2.1.3 Regularity of EM(T)

In the previous sections we showed how to compute limit and colimit in categories of Eilenberg-Moore algebras. In this one we will examine how regularity of $\mathbf{X}$ is inherited by categories monadic over it.

Factorization systems

Let us start by recalling the notion of a factorization system [32, 68, 113, 119].

Definition 2.1.40. Let $\mathbf{X}$ be a category and $\mathcal{E}$, $\mathcal{M}$ two classes of arrows, we will say that $(\mathcal{E}, \mathcal{M})$ is a (orthogonal) *factorization system* if:

1. every isomorphism is in both $\mathcal{E}$ and $\mathcal{M}$;
2. $\mathcal{E}$ and $\mathcal{M}$ are closed under composition;
3. every arrow $f\colon X \to Y$ of $\mathbf{X}$ admits a $(\mathcal{E}, \mathcal{M})$*-factorization*, i.e. there are arrows $e_f \in \mathcal{E}$ and $m_f \in \mathcal{M}$ with the property that $f = m_f \circ e_f$;
4. every $e \in \mathcal{E}$ has the *left lifting property* with respect to every $m \in \mathcal{M}$: for every commutative square

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "e"'] & Z \arrow[d, "m"] \\ Y \arrow[r, "f"'] \arrow[ur, dotted, "k"] & V \end{tikzcd}$$

   with $e \in \mathcal{E}$ and $m \in \mathcal{M}$ there exists a unique $k\colon Y \to Z$ such that

$$m \circ k = f \qquad k \circ e = g$$

A factorization system is *proper* if every $e \in \mathcal{E}$ is epi and every $m \in \mathcal{M}$ is mono; it's *stable* if for every pullback square ast the one below, $e \in \mathcal{E}$ implies $e' \in \mathcal{E}$.

$$\begin{tikzcd} P \arrow[r, "g"] \arrow[d, "e'"'] & X \arrow[d, "e"] \\ Z \arrow[r, "f"'] & Y \end{tikzcd}$$

The following proposition assures us that the factorization of an arrow is unique up to isomorphism.

Proposition 2.1.41. *Let $(\mathcal{E}, \mathcal{M})$ be a factorization system on a category $\mathbf{X}$. If $e\colon X \to Y$, $e'\colon X \to Y'$ and $m\colon Y \to Z$, $m'\colon Y' \to Z$ are arrows, respectively, in $\mathcal{E}$ and $\mathcal{M}$ such that $e' \circ m' = e \circ m$, then there exist a unique isomorphism $f\colon Y \to Y$ such that the following diagram commutes*

$$\begin{tikzcd} X \arrow[r, "e'"] \arrow[d, "e"'] & Y \arrow[d, "m'"] \\ Y \arrow[r, "m"'] \arrow[ur, dotted, "f"] & V \end{tikzcd}$$

Proof. Using the left lifting property we get two commutative diagrams:

$$\begin{tikzcd} X \arrow[r, "e'"] \arrow[d, "e"'] & Y' \arrow[d, "m'"] \\ Y \arrow[r, "m"'] \arrow[ur, "f"] & Z \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "e"] \arrow[d, "e'"'] & Y \arrow[d, "m"] \\ Y' \arrow[r, "m'"'] \arrow[ur, "g"] & Z \end{tikzcd}$$

thus

$$\begin{aligned} m' \circ f \circ g &= m \circ g \\ &= m' \end{aligned} \qquad \begin{aligned} f \circ g \circ e' &= f \circ e \\ &= e' \end{aligned} \qquad \begin{aligned} m \circ g \circ f &= m' \circ f \\ &= m \end{aligned} \qquad \begin{aligned} g \circ f \circ e &= g \circ e' \\ &= e \end{aligned}$$

So we have two square

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "e"'] & Y \arrow[d, "m"] \\ Y \arrow[r, "m"'] \arrow[ur, "g \circ f"] & Z \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "e'"] \arrow[d, "e'"'] & Y' \arrow[d, "m'"] \\ Y' \arrow[r, "m'"'] \arrow[ur, "f \circ g"] & Z \end{tikzcd}$$

and the thesis follows from the uniqueness half of the left lifting property. □

Corollary 2.1.42. *Given a factorization system $(\mathcal{E}, \mathcal{M})$ on a category $\mathbf{X}$, the following hold:*

1. *an arrow $f\colon X \to Y$ is in $\mathcal{E}$ (in $\mathcal{M}$) if and only if m_f (e_f) is an isomorphism;*
2. *$f \in \mathcal{E}$ and $f \in \mathcal{M}$ if and only if f is an isomorphism;*
3. *if $(\mathcal{E}, \mathcal{M})$ is proper, then $g \circ f$ is in $\mathcal{M}$ (in $\mathcal{E}$) implies $f \in \mathcal{M}$ ($g \in \mathcal{E}$).*

Proof. 1. ($\Rightarrow$) By hypothesis $f = \mathsf{id}_Y \circ f$ ($f = f \circ \mathsf{id}_X$) is a factorization with $\mathsf{id}_Y \in \mathcal{M}$ and $f \in \mathcal{E}$ ($\mathsf{id}_X \in \mathcal{E}, f \in \mathcal{M}$), so the thesis follows from Proposition 2.1.41.

($\Leftarrow$) $f = m_f \circ e_f$, thus if m_f (e_f) is an isomorphism then we have f is the composition of two arrows in $\mathcal{E}$ ($\mathcal{M}$) and we can conclude.

2. This follows immediately from the previous point.
3. Factor f and g as $m_f \circ e_f$ and $m_g \circ e_g$, let also h be $e_g \circ m_f$ and factor it as $m_h \circ e_h$ so that we get

$$\begin{tikzcd} & C \arrow[dr, "m_h"] & \\ A \arrow[ur, "e_h"] \arrow[rr, "h"] \arrow[dr, "m_f"] & & B \arrow[d, "m_g"] \\ X \arrow[u, "e_f"] \arrow[r, "f"'] & Y \arrow[ur, "e_g"] \arrow[r, "g"'] & Z \end{tikzcd}$$

Since $\mathcal{E}$ and $\mathcal{M}$ are closed under composition we know that $e_h \circ e_f \in \mathcal{E}$ and $m_g \circ m_h \in \mathcal{M}$, thus these arrows gives a $(\mathcal{E}, \mathcal{M})$-factorization of $g \circ f$. On the other hand $g \circ f \in \mathcal{E}$ ($g \circ f \in \mathcal{M}$), thus point 1 above implies that $e_h \circ e_f$ ($m_g \circ m_h$) is an isomorphism. In particular:

$$(e_h \circ e_f)^{-1} \circ e_h \circ e_f = \mathsf{id}_X \qquad (m_g \circ m_h \circ (m_g \circ m_h)^{-1} = \mathsf{id}_Z)$$

so e_f has a retraction (m_g has a section). The thesis now follows since e_f is epic (m_g is mono). □

Definition 2.1.43. Given a set I, a *source* (*sink*) is a family $\{f_i\}_{i \in I}$ of arrows $f_i\colon X \to Y_i$ ($f_i\colon Y_i \to X$) with the same (co)domain. A *wide pushout* (pullback) is the colimit (limit) of a source (sink). We will use c_i (p_i) to denote the coprojection from Y_i (the projection to Y_i) and c_X to denote the one from X.

Remark 2.1.44. Given a wide pushout $(C, \{c_i\}_{i \in I \cup \{X\}})$ on a source $\{f_i\}_{i \in I}$ with $f_i \colon X \to Y_i$, the coprojection c_X is such that, for every $i \in I$, the following diagram commute

$$\begin{tikzcd} & X \arrow[dl, "f_i"'] \arrow[dr, "c_X"] & \\ Y_i \arrow[rr, "c_i"'] & & C \end{tikzcd}$$

Proposition 2.1.45. *For every proper factorization system $(\mathcal{E}, \mathcal{M})$ on a category $\mathbf{X}$ the following hold:*

1. *for every pushout square as the one below, $e \in \mathcal{E}$ implies $n \in \mathcal{E}$*

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "e"'] & V \arrow[d, "n"] \\ Y \arrow[r, "f"'] & Z \end{tikzcd}$$

2. *if $(C, \{c_i\}_{i \in I \cup \{X\}})$ is a wide pushout on a source $\{e_i\}_{i \in I}$ such that $e_i \colon X \to Y_i$ is in $\mathcal{E}$ for every $i \in I$, then every coprojection is in $\mathcal{E}$ too.*

Proof. 1. Take a pushout square

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "e"'] & V \arrow[d, "n"] \\ Y \arrow[r, "f"'] & Z \end{tikzcd}$$

with e in $\mathcal{E}$. By hypothesis $n = m_n \circ e_n$ for $m_n \colon E \to Y$ in $\mathcal{M}$ and $e_n \colon V \to E$ in $\mathcal{E}$. If we show that m_n is an isomorphism we are done. We can apply again the left lifting property to get $l \colon E \to X$ which makes the following diagram commute.

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "e"'] & V \arrow[r, "e_n"] & E \arrow[d, "m_n"] \\ Z \arrow[urr, "l"] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

Therefore we get another diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "e"'] & V \arrow[d, "n"] \arrow[ddr, bend left, "e_n"] & \\ Z \arrow[r, "f"'] \arrow[drr, bend right, "l"'] & Y \arrow[dr, dotted, "k"] & \\ & & E \end{tikzcd}$$

and thus we can deduce the existence of the dotted $k\colon E \to V$. On the one hand, computing we get

$$\begin{aligned} m_n \circ k \circ n &= m_n \circ e_n \\ &= n \end{aligned} \qquad \begin{aligned} m_n \circ k \circ f &= m_n \circ l \\ &= f \end{aligned}$$

and so $m_n \circ k = \mathsf{id}_Y$. On the other hand

$$\begin{aligned} m_n \circ k \circ m_n &= \mathsf{id}_Y \circ m_n \\ &= m_n \end{aligned} \qquad \begin{aligned} k \circ m_n \circ e_n &= k \circ n \\ &= e_n \end{aligned}$$

so the following diagaram commutes

$$\begin{tikzcd} V \arrow[r, "e_n"] \arrow[d, "e_n"'] & E \arrow[d, "m_n"] \\ E \arrow[r, "m_n"'] \arrow[ur, "k \circ m_n"] & Y \end{tikzcd}$$

and thus $k \circ m_n = \mathsf{id}_E$ by the uniqueness clause of the left lifting property. Therefore e_n is an isomorphism and the thesis now follows from point 1 of Corollary 2.1.42.

2. By Remark 2.1.44 and point three of Corollary 2.1.42 it is enough to show that c_X is in $\mathcal{E}$. Since $(\mathcal{E}, \mathcal{M})$ is a factorization system then $c_X = m \circ e$ for some $m\colon E \to C$ in $\mathcal{M}$ and $e\colon X \to E$ in $\mathcal{E}$ to get, from Remark 2.1.44, a square

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "e_i"'] & V \arrow[d, "m"] \\ Y_i \arrow[r, "c_i"'] \arrow[ur, dotted, "k_i"] & Z \end{tikzcd}$$

and the left lifting property provides, for every $i \in I$, the dotted arrow $k_i\colon Y_i \to e$. Let k be the the induced arrow $Z \to V$. Then

$$\begin{aligned} m \circ k \circ c_i &= m \circ k_i \\ &= c_i \end{aligned}$$

hence $m \circ k = \mathsf{id}_Z$. By Corollary 2.1.42 $m \in \mathcal{M}$ thus it is an isomorphism and we can conclude. □

We are now going to show how, given a monad $\mathbf{T}$ on a category $\mathbf{X}$, is it possible to lift a factorization system on $\mathbf{X}$ to one on $\mathbf{EM}(\mathbf{T})$.

Theorem 2.1.46. *Let $(\mathcal{E}, \mathcal{M})$ be a proper factorization system on a category $\mathbf{X}$. Let also $\mathbf{T} = (T, \eta, \mu)$ be a monad on $\mathbf{X}$ and define*

$$\mathcal{E}_T := \{f \in \mathbf{EM}(\mathbf{T}) \mid U_{\mathbf{T}}(f) \in \mathcal{E}\} \qquad \mathcal{M}_T := \{f \in \mathbf{EM}(\mathbf{T}) \mid U_{\mathbf{T}}(f) \in \mathcal{M}\}$$

If $T(e) \in \mathcal{E}$ for every $e \in \mathcal{E}$ then $(\mathcal{E}_T, \mathcal{M}_T)$ is a proper factorization system on $\mathbf{EM}(\mathbf{T})$. Moreover, $(\mathcal{E}_T, \mathcal{M}_T)$ is stable if $(\mathcal{E}, \mathcal{M})$ is so.

Proof. First of all, let us notice that, since $U_{\mathbf{T}}$ is faithful, then every element of $\mathcal{E}_T$ is epi and every element of $\mathcal{M}_T$ is mono, thus properness comes for free. Moreover, take a pullbacks square

$$\begin{tikzcd}
(P,\xi_4) \arrow[r, "p_X"] \arrow[d, "p_Y"'] & (X,\xi_2) \arrow[d, "f"] \\
(Y,\xi_3) \arrow[r, "g"'] & (Z,\xi_1)
\end{tikzcd}$$

with $f \in \mathcal{E}_T$. By $U_{\mathbf{T}}$ is a right adjoint, thus we also have the following pullback square in $\mathbf{X}$:

$$\begin{tikzcd}
P \arrow[r, "p_X"] \arrow[d, "p_Y"'] & X \arrow[d, "f"] \\
Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

with $f \in \mathcal{E}$. So, if $(\mathcal{E}, \mathcal{M})$ is stable we get that p_Y is in $\mathcal{E}$ too, from which stability follows.

Let us now verify all the points of Definition 2.1.40.

1. If f is an isomorphism in $\mathbf{EM}(\mathbf{T})$, then $U_{\mathbf{T}}(f)$ is an isomorphism in $\mathbf{X}$ and thus it belongs to both $\mathcal{E}$ and $\mathcal{M}$.
2. $\mathcal{E}$ and $\mathcal{M}$ are closed under composition and thus also $\mathcal{E}_T$ and $\mathcal{M}_T$ are.
3. Let $f\colon (X,\xi_1) \to (Y,\xi_2)$ be a morphism in $\mathbf{EM}(\mathbf{T})$. We know that there exists $e\colon X \to I$ in $\mathcal{E}$ and $m\colon I \to Y$ in $\mathcal{M}$ such that $m \circ e = f$, we want to equip I with a structure of Eilenberg-Moore algebra which makes them arrows in $\mathbf{EM}(\mathbf{T})$. Consider now the following diagram

$$\begin{tikzcd}
T(X) \arrow[r, "\xi_1"] \arrow[d, "T(e)"'] & X \arrow[r, "e"] & I \arrow[d, "m"] \\
T(I) \arrow[r, "T(m)"'] \arrow[urr, dotted, "\xi"] & T(Y) \arrow[r, "\xi_2"'] & Y
\end{tikzcd}$$

   By hypothesis $T(e) \in \mathcal{E}$ and $m \in \mathcal{M}$, thus we get the wanted $\xi\colon T(I) \to I$. If we show that (I,ξ) is really an object of $\mathbf{EM}(\mathbf{T})$ we are done: the diagram above witnesses that both m and e are morphisms of Eilenberg-Moore algebras.

   On the one hand we can exploit the naturality of η to get

$$\begin{aligned}
\xi \circ \eta_I \circ e &= \xi \circ T(e) \circ \eta_X \\
&= e \circ \xi_1 \circ \eta_X \\
&= e \circ \mathsf{id}_X \\
&= e \\
&= \mathsf{id}_I \circ e
\end{aligned}$$

   from which it follows that $\xi \circ \eta_I = \mathsf{id}_I$ since e is epi.

On the other hand notice that $T(T(e)) \in \mathcal{E}$ and that we have diagrams

$$\begin{tikzcd}
T(T(X)) \arrow[r, "\mu_X"] \arrow[dd, "T(T(e))"'] & T(X) \arrow[r, "\xi_1"] \arrow[d, "T(e)"'] & X \arrow[r, "e"] & I \arrow[dd, "m"] \\
 & T(I) \arrow[urr, "\xi"'] \arrow[dr, "T(m)"] & & \\
T(T(I)) \arrow[r, "T(T(m))"'] \arrow[ur, "\mu_I"] & T(T(Y)) \arrow[r, "\mu_Y"'] & T(Y) \arrow[r, "\xi_2"'] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
T(T(X)) \arrow[r, "\mu_X"] \arrow[ddd, "T(T(e))"'] \arrow[dr, "T(\xi_1)"'] & T(X) \arrow[r, "\xi_1"] & X \arrow[r, "e"] & I \arrow[ddd, "m"] \\
 & T(X) \arrow[ur, "\xi_1"'] \arrow[d, "T(e)"'] & & \\
 & T(I) \arrow[uurr, "\xi"'] \arrow[r, "T(m)"] & T(Y) \arrow[dr, "\xi_2"] & \\
T(T(I)) \arrow[r, "T(T(m))"'] \arrow[ur, "T(\xi)"] & T(T(Y)) \arrow[r, "\mu_Y"'] \arrow[ur, "T(\xi_2)"'] & T(Y) \arrow[r, "\xi_2"'] & Y
\end{tikzcd}$$

The thesis follows from the uniqueness half of the left lifting property.

4. Let us start with the following squares, one in **EM**(**T**) and the other one in **X**:

$$\begin{tikzcd}
(A, \xi_A) \arrow[r, "f"] \arrow[d, "e"'] & (B, \xi_B) \arrow[d, "m"] \\
(C, \xi_C) \arrow[r, "g"'] \arrow[ur, dotted, "k"] & (D, \xi_D)
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "e"'] & B \arrow[d, "m"] \\
C \arrow[r, "g"'] \arrow[ur, dotted, "k"] & D
\end{tikzcd}$$

If $m\colon B \to D$ is in $\mathcal{M}$ and $e\colon A \to C$ is in $\mathcal{E}$, we get a unique k filling the diagram on the right, so, if we show that such κ is actually a morphism of **EM**(**T**) we are done. To see this, let us compute:

$$\begin{aligned}
m \circ k \circ \xi_C &= g \circ \xi_C \\
&= \xi_D \circ T(g) \\
&= \xi_D \circ T(m) \circ T(k) \\
&= m \circ \xi_B \circ T(k)
\end{aligned}$$

and we get the thesis since m is a monomorphism. $\square$

Regularity of EM(T)

We will start recalling the notion of regularity and some properties of regular categories.

Definition 2.1.47 ([19, 56]). We say that a category $\mathbf{X}$ is *regular* if

1. it has finite limits;
2. for every $f\colon X \to Y$, if the following square

$$\begin{tikzcd} P \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

   is a pullback (i.e. (P, p_1, p_2) is the *kernel pair* of f) then $p_1, p_2\colon P \rightrightarrows X$ have a coequalizer;
3. for every pullback square

$$\begin{tikzcd} P \arrow[r, "g"] \arrow[d, "e'"'] & X \arrow[d, "e"] \\ Z \arrow[r, "f"'] & Y \end{tikzcd}$$

   if e is a regular epi then e' is a regular epi too.

Remark 2.1.48. Let $f\colon X \to Y$ be an arrow of any category $\mathbf{X}$, then its kernel pair $p_1, p_2\colon P \rightrightarrows X$ (if it exists) is a reflexive pair. Indeed we have a diagram

$$\begin{tikzcd} X \arrow[dr, dotted, "s"] \arrow[drr, bend left, "\mathsf{id}_X"] \arrow[ddr, bend right, "\mathsf{id}_X"'] & & \\ & P \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, "f"] \\ & X \arrow[r, "f"'] & Y \end{tikzcd}$$

in which the existence of the dotted $s\colon X \to P$ is guaranteed by the definition of kernel pair.

Example 2.1.49. Every topos $\mathbf{X}$ is a regular category. This is a standard fact in topos theory [65, 86, 93] and its proof relies on two facts:

- every topos is finitely complete and cocomplete (proving items 1 and 2);
- given $f\colon X \to Y$, the *pullback functor* $f^*\colon \mathbf{X}/Y \to \mathbf{X}/X$ is a left adjoint (so item 3 follows) (see also Lemma A.3.13 for this).

Proposition 2.1.50. *Let $e\colon X \to Y$ be a regular epi in a category $\mathbf{X}$ with a kernel pair $p_1, p_2\colon P \rightrightarrows X$, then e is the coequalizer of p_1 and p_2.*

Proof. By hypothesis there exists a pair $f, g \colon Z \rightrightarrows X$ of which e is the coequalizer, since $e \circ f = e \circ g$ we have a diagram

$$\begin{tikzcd}
Z \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] \arrow[dr, dotted, "k"] & & \\
& P \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, "e"] \\
& X \arrow[r, "e"'] & Y
\end{tikzcd}$$

and thus there exists the dotted $k \colon Z \to P$. Let $h \colon Z \to V$ be an arrow such that $h \circ p_1 = h \circ p_2$, then

$$\begin{aligned} h \circ f &= h \circ p_1 \circ k \\ &= h \circ p_2 \circ k \\ &= h \circ g \end{aligned}$$

and thus there exists a unique $l \colon Y \to V$ such that $l \circ e = h$. □

Definition 2.1.51. Let $f \colon X \to Y$ be a morphisms in a category $\mathbf{X}$ with kernel pairs. The coequalizer of the kernel pair $p_1, p_2 \colon P \rightrightarrows X$ is called the *coimage* of f. We will denote such coequalizer by $(\mathrm{Coim}(f), e_f)$. In particular we have a coequalizer diagram

$$\begin{tikzcd}
X \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & Y \arrow[r, "e_f"] & \mathrm{Coim}(f)
\end{tikzcd}$$

Suppose now that $f \colon X \to Y$ has a kernel pair $p_1, p_2 \colon P \rightrightarrows X$ and a coimage, $e_f \colon X \to \mathrm{Coim}(f)$ then, since $f \circ p_1 = f \circ p_2$ we know that there exists a unique $m_f \colon \mathrm{Coim}(f) \to Y$ such that $f = m_f \circ e_f$.

$$\begin{tikzcd}
P \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & X \arrow[rr, "f"] \arrow[dr, "e_f"'] & & Y \\
& & \mathrm{Coim}(f) \arrow[ur, dotted, "m_f"'] &
\end{tikzcd}$$

Proposition 2.1.52. *Let $p_1, p_2 \colon P \rightrightarrows X$ be the kernel pair of an arrow $f \colon X \to Y$. Suppose also that f has a coimage $e_f \colon X \to \mathit{Coim}(f)$. Then $p_1, p_2 \colon P \rightrightarrows X$ is the kernel pair of e_f, too.*

Proof. Let $q_1, q_2 \colon Q \rightrightarrows X$ two arrows such that $e_f \circ q_1 = e_f \circ q_2$, then we have a diagram

$$\begin{tikzcd}
Q \arrow[r, dotted, "q"] \arrow[rr, bend left, "q_1"] \arrow[dr, "q_2"'] & P \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, "e_f"] \arrow[dr, "f"] & \\
& X \arrow[r, "e_f"] \arrow[rr, bend right, "f"'] & \mathrm{Coim}(f) \arrow[r, "m_f"] & Y
\end{tikzcd}$$

Since $p_1, p_2 \colon P \rightrightarrows X$ is a kernel pair for f there exists the unique dotted arrow q and we are done. □

If $\mathbf{X}$ is regular then every f has a coimage, so we can deduce that every f can be decomposed as $m_f \circ e_f$ with e_f a regular epi. We can say something more about m_f.

Proposition 2.1.53. *If $f\colon X \to Y$ is an arrow in a regular category $\mathbf{X}$, then m_f is a monomorphism.*

Proof. Take the diagram

$$\begin{tikzcd}
P \arrow[r, dotted, "h"] \arrow[rrr, bend left, "p_1"] \arrow[ddr, bend right, "p_1"'] & A \arrow[r, "a_1"] \arrow[d, "a_2"'] & B \arrow[r, "b_1"] \arrow[d, "b_2"'] & X \arrow[d, "e_f"] \arrow[dd, bend left=60, "f"] \\
 & C \arrow[r, "c_1"] \arrow[d, "c_2"'] & D \arrow[r, "d_1"] \arrow[d, "d_2"] & \mathrm{Coim}(f) \arrow[d, "m_f"] \\
 & X \arrow[r, "e_f"'] \arrow[rr, bend right, "f"'] & \mathrm{Coim}(f) \arrow[r, "m_f"'] & Y
\end{tikzcd}$$

in which every square is a pullback. This implies the existence of the dotted isomorphism h, therefore

$$\begin{aligned}
e_f \circ c_2 \circ a_2 \circ h &= e_f \circ p_1 \\
&= e_f \circ p_2 \\
&= e_f \circ b_1 \circ a_1 \circ h
\end{aligned}$$

and thus

$$e_f \circ c_2 \circ a_2 = e_f \circ b_1 \circ a_1$$

Now, c_1 and a_2 regular epis because they are pullbacks of regular epis, thus $c_1 \circ a_2$ is epi too and we have

$$\begin{aligned}
d_2 \circ c_1 \circ a_2 &= e_f \circ c_2 \circ a_2 \\
&= e_f \circ b_1 \circ a_1 \\
&= d_1 \circ b_2 \circ a_1 \\
&= d_1 \circ c_1 \circ a_2
\end{aligned}$$

hence $d_1 = d_2$ and we can conclude. □

We will now prove some important properties of regular epimorphisms in regular categories.

Lemma 2.1.54. *for an arrow $f\colon X \to Y$ in a regular category $\mathbf{X}$ the following are equivalent*

1. *f is a regular epi;*
2. *f has the left lifting property with respect to any mono (i.e. f is a* strong epi*).*

Proof. $(1 \Rightarrow 2)$ Suppose that f is the coequalizer of $g, h\colon Z \rightrightarrows X$. Take a diagram

$$\begin{tikzcd}
X \arrow[r, "t"] \arrow[d, "f"'] & A \arrow[d, "m"] \\
Y \arrow[r, "k"'] & B
\end{tikzcd}$$

with m a monomorphism, then:

$$\begin{aligned} m \circ t \circ g &= k \circ f \circ g \\ &= k \circ e \circ h \\ &= m \circ t \circ h \end{aligned}$$

from which it follows that $t \circ g = t \circ h$. Since f is the coequalizer of g and h there exists a unique $d\colon Y \to A$ such that $d \circ f = t$. Moreover

$$\begin{aligned} m \circ d \circ f &= m \circ t \\ &= k \circ f \end{aligned}$$

so $m \circ d = f$ since f is epi.

($2 \Rightarrow 1$) Let $f = m_f \circ e_f$ with m_f a mono and $e_f\colon X \to \mathrm{Coim}(e)$ its coimage, then we have a square

$$\begin{tikzcd} X \arrow[r, "e_f"] \arrow[d, "f"'] & \mathrm{Coim}(f) \arrow[d, "m_f"] \\ Y \arrow[r, "\mathrm{id}_Y"'] \arrow[ur, dotted, "k"] & T \end{tikzcd}$$

Since f is a strong epi and m_f a mono there exists the dotted $k\colon Y \to \mathrm{Coim}(f)$. Now $m_f \circ k = \mathrm{id}_Y$, so m_f is a mono with a section k, so m_f is an isomorphism with inverse k and thus $k \circ f = e_f$ implies that f is a regular epi. □

Corollary 2.1.55. *For every regular category* **X**, *if $\mathcal{E}$ is the class of regular epis and $\mathcal{M}$ the class af monos, then $(\mathcal{E}, \mathcal{M})$ is a proper and stable factorization system.*

Proof. Let us prove the four points of Definition 2.1.40.

1. Every isomorphism is mono and regular epi.
2. We already know that the class of monos is closed under composition. Let $e\colon X \to Y$ and $e'\colon Y \to Z$ be two regular epi, we are going to show that their composition is a strong epi, Lemma 2.1.54 will then deliver us the thesis. Take a diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "e"'] & A \arrow[dd, "m"] \\ Y \arrow[d, "e'"'] & \\ Z \arrow[r, "f"'] & B \end{tikzcd}$$

with m a monomorphism. We have to prove that there is a unique diagonal d that makes the diagram commute. Indeed, we can consider the diagram

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[d, "e"'] & & A \arrow[d, "m"] \\ Y \arrow[r, "e'"'] \arrow[urr, dotted, "k"] & Z \arrow[r, "f"'] & B \end{tikzcd}$$

and deduce the existence of the dotted $k\colon Y \to A$ from Lemma 2.1.54. Next we can use it to construct another diagram

$$\begin{tikzcd} Y \arrow[r, "k"] \arrow[d, "e'"'] & A \arrow[d, "m"] \\ Z \arrow[r, "f"'] \arrow[ur, dotted, "d"] & B \end{tikzcd}$$

to get, using again Lemma 2.1.54, another $d\colon Z \to A$ as in the diagram. Now, $m \circ d = f$ by construction, moreover:

$$\begin{aligned} d \circ e' \circ e &= k \circ e \\ &= g \end{aligned}$$

and thus we get the thesis.

3. We know that every $f\colon X \to Y$ is the composition of $e_f\colon X \to \mathrm{Coim}(f)$ and $m_f\colon \mathrm{Coim}(f) \to Y$, thus the thesis follows from Proposition 2.1.53.
4. Given Lemma 2.1.54 this is immediate.

Properness and stability follow by construction and from the regularity of $\mathbf{X}$. □

Example 2.1.56. Let **Cat** be the category of all small categories, and let $\mathbf{N}$, $\mathbf{Z}/2\mathbf{Z}$ be the 1-object categories associated to the monoids $\mathbb{N}$ and $\mathbb{Z}/2\mathbb{Z}$. Let also $\mathbf{2}$ be the category

$$\begin{tikzcd} A \arrow[loop left, "\mathrm{id}_A"] \arrow[r, "f"] & B \arrow[loop right, "\mathrm{id}_B"] \end{tikzcd}$$

Define $F\colon \mathbf{2} \to \mathbf{N}$ as the functor sending f to 1 and $G\colon \mathbf{N} \to \mathbf{Z}/2\mathbf{Z}$ the one sending n to its congruence class modulo 2. Notice thar F and G are regular epis:

- F is the coequalizer of $F_1, F_2\colon \mathbf{1} \rightrightarrows \mathbf{2}$ selecting, respectively, A and B;
- G is the coequalizer of $G_1, G_2\colon \mathbf{1} \rightrightarrows \mathbf{N}$ selecting, respectively, 0 and 2.

On the other hand, $H := G \circ F$ is the functor sending f to 1, which is not a regular epi. To see this, notice that if H is a regular epi then, by Proposition 2.1.50, H would be the coequalizer of its kernel pair. Now, the kernel pair of H is given by the two projections $P_1, P_2\colon \mathbf{P} \rightrightarrows \mathbf{2}$ where $\mathbf{P}$ is the subcategory of $\mathbf{2} \times \mathbf{2}$ containing all objects and in which the only non identity arrow is $(f, f)\colon (A, A) \to (B, B)$. Notice that

$$F \circ P_1 = F \circ P_2$$

but the only functor $K\colon \mathbf{Z}/2\mathbf{Z} \to \mathbf{N}$ is the one sending 1 to 0, so $K \circ H \neq F$, showing that H cannot be the coequalizer of its kernel pair.

Remark 2.1.57. The previous example shows that **Cat** is not regular.

We are now going to prove that, given a regular category $\mathbf{X}$, asking a form of the axiom of choice, i.e. that every regular epi has a section, is sufficient to guarantee the regularity of the category $\mathbf{EM}(\mathbf{T})$ for every monad (T, η, μ).

Definition 2.1.58. A *split coequalizer* of two parallel arrows $f, g\colon X \rightrightarrows Y$ is an $e\colon Y \to Z$ such that:

1. e has a section s;

2. there exists $t\colon Y \to X$ such that

$$f \circ t = \mathsf{id}_Y \qquad s \circ e = g \circ t$$

The following proposition justifies the name of split coequalizers.

Proposition 2.1.59. *If e is a split coequalizers for $f, g\colon X \rightrightarrows Y$, then it is a coequalizer for them.*

Proof. Let $h\colon Y \to W$ be an arrow such that $h \circ f = h \circ g$, then

$$\begin{aligned} h \circ s \circ e &= h \circ g \circ t \\ &= h \circ f \circ t \\ &= h \end{aligned}$$

On the other hand, if $k\colon Z \to W$ is another arrow such that $k \circ e = h$ then

$$\begin{aligned} k \circ e &= k \circ \mathsf{id}_Z \circ e \\ &= k \circ e \circ s \circ e \\ &= h \circ s \circ e \end{aligned}$$

so, since e is epi, $h \circ s = k$. ☐

Proposition 2.1.60. *Let $e\colon Y \to Z$ be a split coequalizer for $f, g\colon X \rightrightarrows Y$ in a category $\mathbf{X}$. Then for every every functor $F\colon \mathbf{X} \to \mathbf{Y}$, $F(e)$ is a split coequalizer for $F(f)$ and $F(g)$*

Proof. Let t and s be the sections of f and e, then $F(t)$ and $F(s)$ are sections for $F(f)$ and $F(e)$ and

$$\begin{aligned} F(s) \circ F(e) &= F(s \circ e) \\ &= F(g \circ t) \\ &= F(g) \circ F(t) \end{aligned}$$

and the thesis now follows at once. ☐

Kernel pairs provide a way to construct split coequalizers.

Proposition 2.1.61. *Let $p_1, p_2\colon P \rightrightarrows X$ be the kernel pair of an arrow $f\colon X \to Y$ with a coimage $e_f\colon X \to$ Coim(f). Suppose that e_f has a section s, then it is a split coequalizer.*

Proof. We have to construct a section t for p_1 such that $s \circ e_f = p_2 \circ t$. We have a diagram

$$\begin{tikzcd}
X \arrow[drr, bend left, "\mathsf{id}_X"] \arrow[dr, dotted, "t"] \arrow[ddr, bend right, "s\circ e_f"'] & & \\
& P \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, "e_f"] \\
& X \arrow[r, "e_f"'] & \mathrm{Coim}(f)
\end{tikzcd}$$

By Proposition 2.1.52 , $p_1, p_2\colon P \rightrightarrows X$ is a kernel pair for e_f, so the central square is a pullback and thus the dotted t exists. ☐

Corollary 2.1.62. *Let* $\mathbf{X}$ *be a category with kernel pairs in which every regular epi has a section, then every regular epi is a split coequalizer. In particular every functor* $F\colon \mathbf{X} \to \mathbf{Y}$ *preserves regular epis.*

Proof. By hypothesis a regular epi e has a kernel pair which, by Proposition 2.1.50, it coequalizes, so the thesis follows from Proposition 2.1.61. □

We can now start to apply what we have established about split coequalizers to categories of algebras.

Lemma 2.1.63. *Let* $\mathbf{T}$ *be a monad on a category* $\mathbf{X}$*, and* $f, g\colon (X, \xi) \rightrightarrows (Y, \xi_2)$ *two arrows such that* $U_{\mathbf{T}}(f)$ *and* $U_{\mathbf{T}}(g)$ *admit a split coequalizer* $e\colon Y \to Z$ *in* $\mathbf{X}$*. Then there exists a unique* $\xi\colon T(Z) \to Z$ *such that* $(Z, \xi) \in \mathbf{EM}(\mathbf{T})$ *and* $e\colon (Y, \xi_2) \to (Z, \theta)$ *is a coequalizer of* f *and* g*.*

Proof. Since e is split, by Proposition 2.1.60 we know that it is preserved by every functor. In particular it is preserved by T and $T \circ T$, so that we can conclude using Proposition 2.1.33. □

Now we have all the ingredients needed to show the main result of this section.

Theorem 2.1.64. *Let* $\mathbf{X}$ *be a regular category such that every regular epi has a section. Then* $\mathbf{EM}(\mathbf{T})$ *is regular for every monad* $\mathbf{T}$*.*

Proof. Let us prove the three points of Definition 2.1.47.

1. $\mathbf{EM}(\mathbf{T})$ is finitely complete by Proposition 2.1.30.
2. Let $p_1, p_2\colon (P, \theta) \rightrightarrows (X, \xi_1)$ be the kernel pair of $f\colon (X, \xi_1) \to (Y, \xi_2)$, since $U_{\mathbf{T}}$ preserves limits we know that $p_1, p_2\colon P \rightrightarrows X$ is a kernel pair for $f\colon X \to Y$ in $\mathbf{X}$. Let $e_f\colon X \to \mathrm{Coim}(f)$ be their coequalizer in $\mathbf{X}$, by hypothesis it has a sections s, thus by Proposition 2.1.61 it is a split coequalizer and Lemma 2.1.63 allows us to conclude.
3. Let $e\colon (X, \xi_1) \to (Y, \xi_2)$ be a regular epi in $\mathbf{EM}(\mathbf{T})$ and consider a pullback square in $\mathbf{EM}(\mathbf{T})$

$$\begin{tikzcd} (P, \xi) \arrow[r, "f'"] \arrow[d, "e'"'] & (X, \xi_1) \arrow[d, "e"] \\ (Z, \xi_3) \arrow[r, "f"'] & (Y\xi_2) \end{tikzcd}$$

Since $U_{\mathbf{T}}$ preserves limits then we also have a pullback diagram in $\mathbf{X}$

$$\begin{tikzcd} P \arrow[r, "f'"] \arrow[d, "e'"'] & X \arrow[d, "e"] \\ Z \arrow[r, "f"'] & Y \end{tikzcd}$$

and thus e' is a regular epi in $\mathbf{X}$. By Proposition 2.1.50 e' is the coequalizer of its kernel pair $q_1, q_2\colon Q \rightrightarrows P$. By Proposition 2.1.30 there exists a unique $\theta\colon T(Q) \to Q$ such that (Q, θ) is an object of $\mathbf{EM}(\mathbf{T})$ and q_1, q_2 are arrows $q_1, q_2\colon (Q, \theta) \rightrightarrows (O, \xi)$. By hypothesis e' has a section, so Proposition 2.1.61 and Lemma 2.1.63 entail that e' is the coequalizer of q_1 and q_2 in $\mathbf{EM}(\mathbf{T})$. □

We can also completely characterize regular epimorphisms between Eilenberg-Moore algebras.

Proposition 2.1.65. *Let* $\mathbf{T}$ *be a monad on a regular category* $\mathbf{X}$ *in which every regular epi has a section. Then* $U_{\mathbf{T}}$ *preserves and reflects regular epi.*

Proof.
- Preservation. Let $e\colon (X,\xi_1) \to (Y,\xi_2)$ be a regular epi, then by Proposition 2.1.50 e is the coequalizer of its kernel pair $p_1, p_2\colon (P,\theta) \rightrightarrows (X,\xi_1)$. Since $U_{\mathbf{T}}$ preserves limits then $p_1, p_2\colon P \rightrightarrows X$ is the kernel pair of e in $\mathbf{X}$ too. Let $e'\colon X \to Z$ be the coequalizer of p_1 and p_2 in $\mathbf{X}$. By Proposition 2.1.61 e' is a split coequalizer, so Lemma 2.1.63 implies that there exists a unique $\theta\colon T(Z) \to Z$ such that $e'\colon (X,\xi_1) \to (Z,\theta)$ is a coequalizer for p_1 and p_2 in $\mathbf{EM}(\mathbf{T})$. Then there exists an isomorphism $f\colon (Y,\xi_2) \to (Z,\theta)$ such that

$$\begin{tikzcd} & X \arrow[dl, "e"'] \arrow[dr, "e'"] & \\ Y \arrow[rr, "f"'] & & Z \end{tikzcd}$$

  Since f is an isomorphism also in $\mathbf{X}$ it follows that e is regular epi in $\mathbf{X}$ too.
- Reflection. Let $e\colon (X,\xi_1) \to (Y,\xi_2)$ be a morphism such that $e\colon X \to Y$ is a regular epi. Then e, by Proposition 2.1.50 is the coequalizer of its kernel pair $p_1, p_2\colon P \rightrightarrows X$ and, since by hypothesis it has a section, we also know by Proposition 2.1.61 that e is a split coequalizer of them. Now, from Proposition 2.1.30 there exists a unique $\theta\colon T(P) \to P$ such that $p_1, p_2\colon (P,\theta) \rightrightarrows (X,\xi_1)$ is the kernel pair of e in $\mathbf{EM}(\mathbf{T})$ and thus we conclude by Lemma 2.1.63 that e is the coequalizer of its kernel pair also in $\mathbf{EM}(\mathbf{T})$. □

Assuming the axiom of choice (i.e. that every epi has a section), **Set** satisfies the hypotheses of Theorem 2.1.64 and Proposition 2.1.65, therefore we get the following result at once.

Corollary 2.1.66. *Let* $\mathbf{T}$ *be a monad on* **Set**, *then:*

1. $\mathbf{EM}(\mathbf{T})$ *is regular;*
2. *an arrow* $f \in \mathbf{EM}(\mathbf{T})$ *is a regular epi if and only if* $U_{\mathbf{T}}(f)$ *is surjective.*

2.1.4 A cocompleteness theorem

We end this section showing how the interaction between monad and factorization system can guarantee cocompleteness for $\mathbf{EM}(\mathbf{T})$. We will prove a cocompleteness theorem due to Adámek [2] which encompasses and generalizes various other similar results [18, 29, 79].

Proposition 2.1.67. *Let* $\mathbf{X}$ *be a regular category in which every regular epi has a section. Then* $\mathbf{X}$ *is* $\mathcal{E}$*-cowellpowered, where* $\mathcal{E}$ *is the class of regular epis.*

Proof. By hypothesis every regular epi e has a (unique) section s_e, moreover, by Proposition 2.1.41 and Corollary 2.1.55

$$s_e \circ e = s_{e'} \circ e'$$

if and only if $e \equiv e'$. Thus there exists an injective function

$$\mathcal{E}\text{-}\mathsf{Quot}(X) \to \mathbf{X}(X,X) \qquad [e] \mapsto s_e \circ e$$

and the thesis follows since $\mathbf{X}(X,X)$ is a set. □

Theorem 2.1.68. *Let $(\mathcal{E}, \mathcal{M})$ be a proper factorization system on a cocomplete and $\mathcal{E}$-cowellpowered category* $\mathbf{X}$. *If* $\mathbf{T}$ *is a monad on* $\mathbf{X}$ *such that $T(e) \in \mathcal{E}$ for every $e \in \mathcal{E}$, then* $\mathbf{EM}(\mathbf{T})$ *is cocomplete.*

Proof. In light of Theorem 2.1.38 it is enough to show that $\mathbf{EM}(\mathbf{T})$ admits all coequalizers. Let $f, g\colon (X, \xi_1) \rightrightarrows (Y, \xi_2)$ be a pair of parallel arrows in $\mathbf{EM}(\mathbf{T})$. Since $\mathbf{X}$ is cowellpowered there exists a set $R(Y)$ of representatives for the relation $\equiv$ on $Y/\mathcal{E}$. Define I to be the set of all $e\colon Y \to Z_e$ in $R(Y)$ such that, for every $h\colon (Y, \xi_2) \to (V, \xi_3)$ satisfying

$$h \circ f = h \circ g$$

there exists $h_e\colon Z \to V$ such that $h_e \circ e = h$; we have a source given by all these $e \in I$, so that we can take its wide pushout $(C, \{c_i\}_{i \in I \cup \{Y\}})$. By Remark 2.1.44 we have

$$\begin{tikzcd} & Y \arrow[dl, "e"'] \arrow[dr, "c_Y"] & \\ Z_e \arrow[rr, "c_e"'] & & C \end{tikzcd}$$

Moreover, in I there exists $e\colon Y \to Z$ which is a coequalizer for f and g in $\mathbf{X}$, thus

$$\begin{aligned} c_Y \circ f &= c_e \circ e \circ f \\ &= c_e \circ e \circ g \\ &= c_Y \circ g \end{aligned}$$

By Proposition 2.1.45 we know that every coprojection c_D is in $\mathcal{E}$, in particular c_Y is in $\mathcal{E}$ and, by hypothesis, $T(c_Y) \in \mathcal{E}$ too. Take now a pushout square

$$\begin{tikzcd} T(Y) \arrow[r, "c_Y \circ \xi_2"] \arrow[d, "T(c_Y)"'] & C \arrow[d, "p_2"] \\ T(C) \arrow[r, "p_1"'] & P \end{tikzcd}$$

in which $p_2 \in \mathcal{E}$ as the pushout of $T(c_Y)$. In particular, since $(\mathcal{E}, \mathcal{M})$ is proper, this implies that p_2 is epi. Now, let $h\colon (Y, \xi_2) \to (V, \xi_3)$ be such that

$$h \circ f = h \circ g$$

then for every $e \in I$ there exist h_e such $h_e \circ e = h$, thus we have a cocone with vertex V and edges $\{h_e\}_{e \in I} \cup \{h\}$, so there exists the dotted k as in the diagram

$$\begin{tikzcd} Y \arrow[dr, "c_Y"] \arrow[drr, bend left, "h"] \arrow[dd, "e"'] & & \\ & C \arrow[r, dotted, "k"] & V \\ Z_e \arrow[ur, "c_e"'] \arrow[urr, bend right, "h_e"'] & & \end{tikzcd}$$

h is a morphism of $\mathbf{EM}(\mathbf{T})$, thus

$$\begin{aligned} k \circ c_Y \circ \xi_2 &= h \circ \xi_2 \\ &= \xi_3 \circ T(h) \\ &= \xi_3 \circ T(k) \circ T(c_Y) \end{aligned}$$

so that the dotted $t\colon P \to V$ in the following diagram exists

$$\begin{tikzcd}
T(Y) \arrow[r, "c_Y \circ \xi_2"] \arrow[d, "T(c_Y)"'] & C \arrow[d, "p_2"] \arrow[ddr, bend left, "k"] & \\
T(C) \arrow[r, "p_1"'] \arrow[drr, bend right, "\xi_3 \circ T(k)"'] & P \arrow[dr, dotted, "t"] & \\
 & & V
\end{tikzcd}$$

Therefore we have

$$\begin{aligned} h &= k \circ c_Y \\ &= t \circ p_2 \circ c_Y \end{aligned}$$

This, in turn, implies that there exists $e\colon Y \to Z$ in I such that $e \equiv p_2 \circ c_y$, i.e. there exists an isomorphism $p\colon P \to Z$ such that $e = p \circ p_2 \circ c_Y$, so that

$$\begin{aligned} c_Y &= c_e \circ e \\ &= c_e \circ p \circ p_2 \circ c_Y \end{aligned}$$

which, since c_Y is epi, implies

$$\mathsf{id}_Z = c_e \circ p \circ p_2$$

and we can conclude from point 2 and 3 of Corollary 2.1.42 that p_2 is an isomorphism.

Let $\xi\colon T(C) \to C$ be $p_2^{-1} \circ p_1$, by construction

$$p_2 \circ c_Y \circ \xi_2 = p_1 \circ T(c_Y)$$

and thus

$$\begin{aligned} c_Y \circ \xi_2 &= p_2^{-1} \circ p_1 \circ T(c_Y) \\ &= \xi \circ T(c_Y) \end{aligned}$$

This equation gives us the commutativity of

$$\begin{tikzcd}
C \arrow[dd, "\eta_C"'] & Y \arrow[l, "c_Y"'] \arrow[d, "\eta_Y"'] \arrow[dr, "\mathsf{id}_Y"] & \\
 & T(Y) \arrow[r, "\xi_2"'] \arrow[dl, "T(c_Y)"] & Y \arrow[d, "c_Y"] \\
T(C) \arrow[rr, "\xi"'] & & C
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & T(T(Y)) \arrow[dll, bend right, "T(T(c_Y))"'] \arrow[dl, "\mu_Y"] \arrow[dr, "T(\xi_2)"'] \arrow[drr, bend left, "T(T(c_Y))"] & & \\
T(T(C)) \arrow[d, "\mu_C"'] & T(Y) \arrow[dl, "T(c_Y)"] \arrow[dr, "\xi_2"'] & & T(Y) \arrow[dl, "\xi_2"] \arrow[dr, "T(c_Y)"'] & T(T(C)) \arrow[d, "T(\xi)"] \\
T(C) \arrow[drr, bend right, "\xi"'] & & Y \arrow[d, "c_Y"'] & & T(C) \arrow[dll, bend left, "\xi"] \\
 & & C & &
\end{tikzcd}$$

which in turn entails

$$\xi \circ \eta_C \circ c_Y = c_Y \qquad \xi \circ \mu_C \circ T(T(c_Y)) = \xi \circ T(\xi) \circ T(T(c_Y))$$

and thus (C, ξ) is an object of $\mathbf{EM}(\mathbf{T})$ since c_Y and $T(T(c_Y))$ are epi. Now it follows immediately that $c_Y : (Y, \xi_2) \to (C, \xi)$ is an arrow of $\mathbf{EM}(\mathbf{T})$.

We are left with the coequalizer property. We already proved that $c_Y \circ f = c_Y \circ g$ and that for every morphism $h\colon (Y, \xi_2) \to (V, \xi_3)$ such that $h \circ f = h \circ g$ there exists a unique $k\colon C \to V$ in $\mathbf{X}$ satisfying $k \circ_Y = h$, so it is enough to show that this k is an arrow of $\mathbf{EM}(\mathbf{T})$. If we consider the diagram

$$\begin{tikzcd}
T(C) \arrow[d, "p_1"] \arrow[dd, bend right=60, "\xi"'] & T(Y) \arrow[l, "T(c_Y)"'] \arrow[r, "T(c_Y)"] \arrow[d, "\xi_2"] \arrow[rr, bend left, "T(h)"] & T(C) \arrow[r, "T(k)"] & T(V) \arrow[ddd, "\xi_3"] \\
 & Y \arrow[d, "c_Y"] \arrow[ddrr, bend left=10, "h"] & & \\
P \arrow[d, "p_2^{-1}"] & C \arrow[l, "p_2"'] \arrow[dl, bend left, "\mathrm{id}_C"] & & \\
C \arrow[rrr, "k"'] & & & V
\end{tikzcd}$$

we get

$$k \circ \xi \circ T(c_Y) = \xi_3 \circ T(k) \circ T(c_Y)$$

and the thesis follows because $T(c_Y)$ is epi. □

If $\mathbf{X}$ is a cocomplete regular category satisfying the same form of the axiom of choice used in Theorem 2.1.64, i.e. that every regular epi has a section, we can use Corollary 2.1.62, Theorem 2.1.68, and Proposition 2.1.67 to get the following result (see also [29, Thm. 4.3.5])

Corollary 2.1.69. $\mathbf{EM}(\mathbf{T})$ *is a cocomplete category for every monad* $\mathbf{T}$ *on a cocomplete regular category* $\mathbf{X}$ *in which every regular epi has a section,.*

Assuming the axiom of choice the previous corollary can be immediately applied to **Set**.

Corollary 2.1.70. *For every monad* (T, η, μ) *on* **Set**, $\mathbf{EM}(\mathbf{T})$ *is cocomplete.*

2.2 Monads on Set

In this section we will explore the relationship between algebraic theories and monads on **Set**. This relationship was first developed with the approach of *Lawvere theories* in [76] and in [78, 80]. However, we are interested in a more syntactic approach, thus we will recall Lawvere's and Linton's results without using the technology of Lawvere theories.

2.2.1 Filtered categories, filtered colimits

In this section we take a brief detour to introduce the notion of rank of a functor which will be needed in the subsequent sections. Standard textbook references are [6, 7, 29]. Finally, let us warn the reader that, for us, a regular cardinal is always infinite.

Definition 2.2.1. Let κ be a regular cardinal, we say that a small category $\mathbf{D}$ is κ*-filtered* if:

1. $\mathbf{D}$ is non empty;

2. for each collection $\{D_i\}_{i\in I}$ with $|I| < \kappa$, there exist an object D and, for every $i \in I$, an arrow $f_i\colon D_i \to D$;
3. for every pair of objects D_1 and D_2 in $\mathbf{D}$, and every family $\{f_i\}_{i\in I} \subseteq \mathbf{D}(D_1, D_2)$ with $|I| < \kappa$, there exists a morphism $f\colon D_2 \to D$ such that, for every $i, j \in I$

$$f \circ f_i = f \circ f_j$$

A *κ-filtered colimit* is a colimit on a functor $F\colon \mathbf{D} \to \mathbf{X}$ with $\mathbf{D}$ κ-filtered.

Remark 2.2.2. Let $\mathbf{D}$ be a κ-filtered category, then $\mathbf{D}$ is also λ-filtered for every other regular cardinal λ such that $\lambda \leq \kappa$.

Remark 2.2.3. Let $(P, \leq)$ be a poset, we can specialize the previous definition to get the notion of κ-filtered (or *κ-directed*) poset. In this context point 3 becomes trivial and we get that $(P, \leq)$ is κ-filtered if and only if the following hold:

1. P is non empty;
2. every family $\{p_i\}_{i\in I}$ of cardinality less then κ has an upper bound.

Example 2.2.4. Let X be a set and κ be any regular cardinal. We can consider the poset $(\mathcal{P}_\kappa(X), \subseteq)$ which, since κ is regular, $(\mathcal{P}_\kappa(X), \subseteq)$ is κ-filtered by Remark 2.2.3. Now, $\mathcal{P}_\kappa(X)$ determines a diagram in **Set** whose κ-filtered colimit is X, with the inclusions as edges of the colimiting cone.

Example 2.2.5. Let $\mathbf{X}$ be a cartesian closed category, and (M, m, e) an internal monoid. The writer monad of Example 2.1.4 preserves all colimits since $(-) \times M$ is a left adjoint, in particular it is $\aleph_0$-filtered.

Lemma 2.2.6. *Let κ be a regular cardinal and $\mathbf{D}$ a small category, then the following are equivalent*

1. $\mathbf{D}$ *is κ-filtered;*
2. *every functor $F\colon \mathbf{X} \to \mathbf{D}$ with domain with strictly less than κ arrows, admits a cocone in* $\mathbf{D}$.

Remark 2.2.7. Notice that if the set of arrows of $\mathbf{X}$ has cardinality less then κ then its set of objects has the same property. A category with this property is said to be *κ-small*. A *κ-small* colimit is a colimit of a functor with a κ-small domain.

Proof. $(1 \Rightarrow 2)$ By the hypothesis on $\mathbf{X}$, the family $\{F(X)\}_{X\in\mathbf{X}}$ has cardinality strictly less then κ, so by point 2 of Definition 2.2.1 there exists an object $D \in \mathbf{D}$ with arrows $f_X\colon F(X) \to D$. Given $X \in \mathbf{X}$ can define I_X as the set of arrows with domain X and consider the family $\{f_{\mathsf{cod}(g)} \circ F(g)\}_{g\in I_X}$ which is a subset of $\mathbf{D}(F(X), D)$. By point 3 of Definition 2.2.1 there exists $e_X\colon D \to D_X$ such that for every $g\colon X \to Y$ and $h\colon X \to Z$

$$e_X \circ f_Y \circ F(g) = e_X \circ f_Z \circ F(h)$$

We can apply point 2 of the definition to the family $\{D_X\}_{X\in\mathbf{X}}$ to get an object E with an arrow $h_X\colon D_X \to E$ for every $X \in \mathbf{X}$ such that, for every $g\colon X \to Y$

$$\begin{aligned}
h_Y \circ e_Y \circ f_Y \circ F(g) &= h_X \circ e_X \circ f_X \circ F(\mathsf{id}_X)\\
&= h_X \circ e_X \circ f_X \circ \mathsf{id}_{F(X)}\\
&= h_X \circ e_X \circ f_X
\end{aligned}$$

showing that $(E, \{h_X \circ e_X \circ f_X\}_{X\in\mathbf{X}})$ is a cocone for F.

$(2 \Rightarrow 1)$ The three point of Definition 2.2.1 follow applying 1 to, respectively: the initial functor from the empty category, the functor from a discrete category associated to the family $\{D_i\}_{i\in I}$, the functor from the category with two objects and $|I|$ parallel arrows associated to the family $\{f_i\}_{i\in I}$. $\square$

Corollary 2.2.8. *Let $\mathbf{D}$ be a κ-filtered category and D an object in it. Then $D/\mathbf{D}$ is κ-filtered as well.*

Proof. Let $\mathbf{X}$ be a κ-small category and $F\colon \mathbf{X} \to D/\mathbf{D}$ a functor. If $\mathbf{X}$ is empty there is nothing to show. Otherwise, let us denote $F(X)$ by $g_X : D \to D_X$, we can consider the diagram $\mathbf{A}$ in $\mathbf{D}$ generated by the arrows $\{g_X\}_{X\in\mathbf{X}} \cup \{F(f)\}_{f\in\mathcal{A}(\mathbf{X})}$ which, since κ is regular, contains less then κ arrows. By the previous lemma there exists a cocone $(C, \{c_A\}_{A\in\mathbf{A}})$ on $\mathbf{A}$, in particular this implies that, for ever $X, Y \in \mathbf{X}$ we have

$$c_{D_X} \circ g_X = c_{D_Y} \circ g_Y$$

Let g be $c_{D_X} \circ g_X$ for some $X \in \mathbf{X}$. By construction c_{D_X} is a morphism $g_X \to g$. Moreover, if $f\colon X \to Y$ is an arrow in $\mathbf{X}$ then, using the cocone property of $(C, \{c_A\}_{A\in\mathbf{A}})$ we get

$$c_{D_X} = c_{D_Y} \circ F(f)$$

showing that $(g, \{c_{D_X}\}_{X\in\mathbf{X}})$ is a cocone on F as desired. □

κ-filtered colimits and limits in Set

We are now going to provide a more abstract characterization of κ-filtered categories in term of commutation of limits and colimits of sets.

Remark 2.2.9. Take a functor $F\colon \mathbf{D}\times\mathbf{X} \to \mathbf{Y}$, with $\mathbf{Y}$ a complete and cocomplete category, then we can perform two constructions on it.

- On the one hand for all $D \in \mathbf{D}$ we can first take the limit $(L(D), \{\alpha_{D,X}\}_{X\in\mathbf{X}})$ of $F(D,-)\colon \mathbf{X} \to \mathbf{Y}$. This defines a functor $L\colon \mathbf{D} \to \mathbf{Y}$

$$\begin{array}{ccc} D & \longmapsto & L(D) \\ f\big\downarrow & & \big\downarrow L(f) \\ E & \longmapsto & L(E) \end{array}$$

  where $L(f)$ is the unique arrow such that the following diagram commute

$$\begin{tikzcd} L(D) \arrow[r, "L(f)"] \arrow[d, "\alpha_{D,X}"'] & L(E) \arrow[d, "\alpha_{D,Y}"] \\ F(D,X) \arrow[r, "F(f{,}\mathrm{id}_X)"'] & F(E,X) \end{tikzcd}$$

  Then we can take the colimit $(C, \{i_D\}_{D\in\mathbf{D}})$ of this functor L.

- On the other hand we can first take the colimit $(C'(X), \{j_{D,X}\}_{D\in\mathbf{D}})$ of $F(-,X)\colon \mathbf{D} \to \mathbf{Y}$ getting a functor $C'\colon \mathbf{X} \to \mathbf{Y}$

$$\begin{array}{ccc} X & \longmapsto & C'(X) \\ g\big\downarrow & & \big\downarrow C'(g) \\ Y & \longmapsto & C'(Y) \end{array}$$

  with $C'(g)$ the unique arrows such that

$$\begin{tikzcd} F(D,X) \arrow[r, "F(\mathrm{id}_D{,}g)"] \arrow[d, "j_{D,X}"'] & F(D,Y) \arrow[d, "j_{D,Y}"] \\ C'(X) \arrow[r, "C'(g)"'] & C'(Y) \end{tikzcd}$$

commutes. Then we can define $(L', \{\beta_X\}_{X\in\mathbf{X}})$ as the limit of the functor C'.

These two construction are related: for every $D \in \mathbf{D}$ and $X \in \mathbf{X}$ we can consider the arrow $\Phi_{D,X} : L(D) \to C'(X)$ given by the composition

$$L(D) \xrightarrow{\alpha_{D,X}} F(D,X) \xrightarrow{j_{D,X}} C'(X)$$

Now, for every $X \in \mathbf{X}$ and $f : D \to E$ we have

$$\begin{tikzcd}
L(D) \arrow[r, "\alpha_{D,X}"] \arrow[d, "L(f)"'] \arrow[rr, bend left, "\Phi_{D,X}"] & F(D,X) \arrow[d, "F(f,\mathrm{id}_X)"'] \arrow[r, "j_{D,X}"] & C'(X) \\
L(E) \arrow[r, "\alpha_{E,X}"'] \arrow[rru, bend right, "\Phi_{E,X}"'] & F(E,X) \arrow[ru, "j_{E,X}"'] &
\end{tikzcd}$$

Therefore we have an induced $\Phi_X : C \to C'(X)$ and, given $g : X \to Y$ we get another diagram

$$\begin{tikzcd}
L(D) \arrow[r, "i_D"] \arrow[d, "\alpha_{D,X}"] \arrow[dr, "\Phi_{D,X}"] \arrow[dd, bend right=60, "\alpha_{E,X}"'] & C \arrow[d, "\Phi_X"] \\
F(D,X) \arrow[r, "j_{D,X}"'] \arrow[d, "F(\mathrm{id}_D,g)"] & C'(X) \arrow[d, "C'(g)"] \\
F(E,X) \arrow[r, "j_{E,Y}"'] & C'(Y)
\end{tikzcd}$$

showing that $(C, \{\Phi_X\}_{X\in\mathbf{D}})$ is a cone on C', so that there exists a unique *comparison morphism* $\Phi : C \to L'$ such that the following diagram commutes

$$\begin{tikzcd}
C \arrow[rr, "\Phi"] \arrow[rrd, "\Phi_X"] & & L' \arrow[d, "\beta_X"] \\
L(D) \arrow[u, "i_D"] \arrow[r, "\alpha_{D,X}"] \arrow[rr, bend right, "\Phi_{D,X}"'] & F(D,X) \arrow[r, "j_{D,X}"] & C'(X)
\end{tikzcd}$$

Remark 2.2.10. It is worth to point out explicitly that if Φ is an isomorphism, then L' is the vertex of a colimiting cocone on L, with coprojection $L(D) \to L'$ induced by the family $\{j_{D,X} \circ \alpha_{D,X}\}_{X\in\mathbf{X}}$.

We are now going to show that when $\mathbf{Y} = \mathbf{Set}$ and $\mathbf{X}$ is κ-small, then κ-filteredness of $\mathbf{D}$ is equivalent to this comparison morphism Φ being an isomorphism; in short that κ-filtered colimits commute with κ-small limits in **Set**. We start by describing κ-filtered colimits of sets.

Lemma 2.2.11. *Let $F\colon \mathbf{D} \to \mathbf{Set}$ be a functor with a κ-filtered domain, and, for every $D \in \mathbf{D}$ consider the coprojection $i_D\colon F(D) \to \sum_{D\in\mathbf{D}} F(D)$. In addition, let $\sim$ be the relation on $\sum_{D\in\mathbf{D}} F(D)$ defined by $i_{D_1}(x) \sim i_{D_2}(y)$ if and only if $x \in F(D_1)$, $y \in F(D_2)$ and there exists $f\colon D_1 \to D$, $g\colon D_2 \to D$ such that*

$$F(f)(x) = F(g)(y)$$

Then the following hold true:

1. *$\sim$ is an equivalence relation;*
2. *if C is the quotient $\sum_{D\in\mathbf{D}} F(D)/\!\sim$ and $\pi\colon \sum_{D\in\mathbf{D}} F(D) \to C$ is the quotient function, then a colimiting cocone for F is given by $(C, \{j_D\}_{D\in\mathbf{D}})$ where $j_D := \pi \circ i_D$.*

Proof. 1. Symmetry and reflexivity of $\sim$ follows at once from the definition, We have to show transitivity. Let $x \in F(D_1)$, $y \in F(D_2)$, $z \in F(D_3)$ be such that $i_{D_1}(x) \sim i_{D_2}(y)$ and $i_{D_2}(y) \sim i_{D_3}(z)$. Then in $\mathbf{D}$ we have a diagram

$$\begin{tikzcd}
D_1 \arrow[dr, "f_1"'] & & D_2 \arrow[dl, "g_1"] \arrow[dr, "f_2"'] & & D_3 \arrow[dl, "g_2"] \\
& D & & D' &
\end{tikzcd}$$

such that

$$F(f_1)(x) = F(g_1)(y) \qquad F(f_2)(y) = F(g_2)(z)$$

By Lemma 2.2.6 such a diagram admits a cocone, thus there exist morphisms $h_1\colon D \to E$ and $h_2\colon D' \to E$ such that

$$h_1 \circ g_1 = h_2 \circ f_2$$

But then

$$\begin{aligned}
F(h_1 \circ f_1)(x) &= F(h_1 \circ g_1)(y) \\
&= F(h_2 \circ f_2)(y) \\
&= F(h_2 \circ g_2)(z)
\end{aligned}$$

Therefore $i_{D_1}(x) \sim i_{D_3}(z)$.

2. Let $(X, \{t_D\}_{D\in\mathbf{D}})$ be a cocone on F. Then we have an arrow $t\colon S \to X$ such that

$$\begin{tikzcd}
F(D) \arrow[r, "i_D"] \arrow[rr, bend left, "t_D"] \arrow[dr, bend right, "j_D"'] & \sum\limits_{D\in\mathbf{D}} F(D) \arrow[r, "t"] \arrow[d, "\pi"] & X \\
& C \arrow[ur, dotted, "k"'] &
\end{tikzcd}$$

commutes. Now, if $i_{D_1}(x) \sim i_{D_2}(y)$, then there exist $f_1\colon D_1 \to D$ and $f_2\colon D_2 \to D$ such that

$$F(f_1)(x) = F(f_2)(y)$$

Thus we have

$$\begin{aligned} t(i_{D_1}(x)) &= t_{D_1}(x) \\ &= t_D(F_{f_1}(x)) \\ &= t_D(F(f_2)(y)) \\ &= t_{D_2}(y) \\ &= t(i_{D_2}(y)) \end{aligned}$$

showing the existence of the dotted k. For uniqueness: if k' is another arrow such that $k' \circ j_D = t_D$ for every $D \in \mathbf{D}$, then

$$k' \circ \pi \circ i_D = t \circ i_D$$

Hence $k' \circ \pi = t$, therefore $k' = k$. □

Corollary 2.2.12. *Let $F \colon \mathbf{D} \to \mathbf{Set}$ be a functor with a κ-filtered domain, then a cocone $(C, \{c_D\}_{D \in \mathbf{D}})$ is colimiting for F if and only if the following hold*

1. *for every $c \in C$ there exists $D \in \mathbf{D}$ and x_D in $F(D)$ such that $c_D(x_D) = c$;*
2. *if $c_{D_1}(x_{D_1}) = c_{D_2}(x_{D_2})$, then there exist arrows $f \colon D_1 \to D$ and $g \colon D_2 \to D$ such that*

$$F(f)(x_{D_1}) = F(g)(x_{D_2})$$

Remark 2.2.13. Now let F be a functor $\mathbf{D} \times \mathbf{X} \to \mathbf{Y}$ with $\mathbf{D}$ κ-filtered. Then, using the notation of Remark 2.2.9, the previous lemma yields a surjection

$$\pi_X \colon \sum_{D \in \mathbf{D}} F(D, X) \to C'(X)$$

for every $X \in \mathbf{X}$. These surjections form a natural transformation $\pi \colon \sum_{D \in \mathbf{D}} F(D, -) \to C'$. Indeed, given an arrow $g \colon X \to Y$, for every $D \in \mathbf{D}$ we have a diagram

$$\begin{tikzcd}
\sum\limits_{D \in \mathbf{D}} F(D,X) \arrow[rr, "\sum\limits_{D \in \mathbf{D}} F(\mathrm{id}_D, g)"] \arrow[dd, bend right, "\pi_X"'] & & \sum\limits_{D \in \mathbf{D}} F(D,Y) \arrow[dd, bend left, "\pi_Y"] \\
F(D,X) \arrow[u, "i_{D,X}"'] \arrow[rr, "F(\mathrm{id}_D, g)"] \arrow[d, "j_{D,X}"] & & F(D,Y) \arrow[u, "i_{D,Y}"] \arrow[d, "j_{D,Y}"'] \\
C'(X) \arrow[rr, "C'(g)"'] & & C'(Y)
\end{tikzcd}$$

in which the two inner squares commute, and thus the outer one is commutative too.

The next theorem gives us the promised characterization of κ-filtered categories.

Theorem 2.2.14. *Let κ be a regular cardinal and $\mathbf{D}$ be a small category, then the following are equivalent:*

1. $\mathbf{D}$ *is κ-filtered;*

2. *for every category $\mathbf{X}$ with strictly less then κ arrows and functor $F\colon \mathbf{D}\times\mathbf{X}\to\mathbf{Set}$, the comparison morphism Φ is an isomorphism.*

Proof. Throughout this proof will use the notation of Remark 2.2.9.

(1 ⇒ 2) As for every limit, the families $\{\beta_X\}_{X\in\mathbf{X}}$ and $\{\alpha_{D,X}\}_{X\in\mathbf{X}}$ induces injections

$$\beta\colon L'\to\prod_{X\in\mathbf{X}} C'(X)\qquad \alpha_D\colon L(D)\to\prod_{X\in\mathbf{X}} F(D,X)$$

which have as images, respectively

$$\{(c_X)_{X\in\mathbf{X}}\in\prod_{X\in\mathbf{X}} C'(X)\mid C'(g)(c_{X_1})=c_{X_2}\text{ for every } g\colon X_1\to X_2\}$$
$$\{(a_X)_{x\in\mathbf{X}}\in\prod_{X\in\mathbf{X}} F(D,X)\mid F(\mathsf{id}_D,g)(a_{X_1})=a_{X_2}\text{ for every } g\colon X_1\to X_2\}$$

In addition, Lemma 2.2.11 provides surjections

$$\pi\colon\sum_{D\in\mathbf{D}} L(D)\to C\qquad \pi_X\colon\sum_{D\in\mathbf{D}} F(D,X)\to C'(X)$$

These functions fit in the diagram

$$\begin{tikzcd}
 & C \arrow[r, "\Phi"] & L' \arrow[rr, "\beta_X"] \arrow[rrr, bend left, "\beta"] & & C'(X) & \prod_{X\in\mathbf{X}} C'(X) \arrow[l, "t_X"'] \\
\sum_{D\in\mathbf{D}} L(D) \arrow[ur, bend left, "\pi"] & L(D) \arrow[l, "k_D"] \arrow[u, "i_D"] \arrow[r, "\alpha_D"'] \arrow[rr, bend left, "\alpha_{D,X}"] & \prod_{X\in\mathbf{X}} F(D,X) \arrow[r, "p_X"'] & F(D,X) \arrow[ur, "j_{D,X}"] \arrow[r, "h_{D,X}"'] & \sum_{D\in\mathbf{D}} F(D,X) \arrow[u, "\pi_X"'] & \prod_{X\in\mathbf{X}}\sum_{D\in\mathbf{D}} F(D,X) \arrow[l, "q_X"] \arrow[u, "\prod_{X\in\mathbf{X}}\pi_X"']
\end{tikzcd}$$

where p_X, q_X and t_X are projections, while k_D and $h_{D,X}$ are coprojections.

We are going to show that the comparison morphism Φ is injective and surjective.

- Φ is injective. Let $c_1, c_2\in C$ such that $\Phi(c_1)=\Phi(c_2)$, since π is surjective there exist $d_1\in L(D_1)$ and $d_2\in L(D_2)$ such that

$$\pi(k_{D_1}(d_1))=c_1\qquad \pi(k_{D_2}(d_2))=c_2$$

  Now, by the commutativity of the diagram above, we can deduce that, for every $X\in\mathbf{X}$, we have

$$\pi_X(h_{D_1,X}(p_X(\alpha_{D_1}(d_1))))=\pi_X(h_{D_2,X}(p_X(\alpha_{D_2}(d_2))))$$

  Thus by Lemma 2.2.11 we know that there exist $f\colon D_1\to D$ and $g\colon D_2\to D$ such that

$$F(f,\mathsf{id}_X)(\alpha_{D_1,X}(d_1))=F(g,\mathsf{id}_X)(\alpha_{D_2,X}(d_2))$$

but then

$$\begin{aligned}\alpha_{D,X}(L(f)(d_1)) &= F(f, \mathsf{id}_X)(\alpha_{D_1,X}(d_1))\\ &= F(g, \mathsf{id}_X)(\alpha_{D_2,X}(d_2))\\ &= \alpha_{D,X}(L(g)(d_2))\end{aligned}$$

which in turn implies that

$$\alpha_D(L(f)(d_1)) = \alpha_D(L(g)(d_2))$$

and the thesis now follows from Lemma 2.2.11 and the injectivity of α_D.

- Φ is surjective. Let l be an element of L', applying β we get an element $(\beta_X(l))_{X\in X}$ of $C'(X)$. Now, for every component $X \in \mathbf{X}$ there exists an object D_X of $\mathbf{D}$ and an element $d_X \in F(D_X, X)$such that

$$\beta_X(l) = \pi_X(h_{D_X,X}(d_X))$$

Since $\mathbf{D}$ is κ-filtered and $\mathbf{X}$ has less then κ objects, there exists an object D with arrows $f_X\colon D_X \to D$ for each $X \in \mathbf{X}$. Let $e_X \in F(D, X)$ be the element $F(f_X, \mathsf{id}_X)(d_X)$, by Lemma 2.2.11 we have

$$\pi_X(h_{D_X,X}(d_X)) = \pi_X(h_{D,X}(e_X))$$

Now let $g\colon X_1 \to X_2$ be an arrow in $\mathbf{X}$, by Remark 2.2.13

$$\begin{aligned}\pi_{X_2}(h_{D_{X_2},X_2}(d_{X_2})) &= \beta_{X_2}(l)\\ &= C'(g)(\beta_{X_1}(l))\\ &= C'(g)(\pi_{X_1}(h_{D,X_1}(e_{X_1})))\\ &= \pi_{X_2}(h_{D,X_2}(F(\mathsf{id}_D, g)(e_{X_1})))\\ &= \pi_{X_2}(h_{D,X_2}(F(f_{X_1}, g)(d_{X_1})))\end{aligned}$$

Applying Lemma 2.2.11 we can deduce the existence of $v_g, u_g\colon D \rightrightarrows D_g$ such that

$$F(v_g \circ f_{X_1}, g)(d_{X_1}) = F(u_g \circ f_{X_2}, \mathsf{id}_{X_2})(d_{X_2})$$

Take now the diagram defined by the family $\{v_g, u_g\}_{g\in X(X_1,X_2)}$ which has less then κ arrows and thus there a cone $(E, \{z_g\}_{g\in\mathbf{X}(X_1,X_2)})$. In particular this implies that there exists an arrow $z\colon D \to E$ satisfying, for every $g\colon X_1 \to X_2$:

$$\begin{aligned}F(z \circ f_{X_1}, g)(d_{X_1}) &= F(z_g \circ v_g \circ f_{X_1}, g)(d_{X_1})\\ &= F(z_g \circ u_g \circ f_{X_2}, \mathsf{id}_{X_2})(d_{X_2})\\ &= F(z \circ f_{X_2}, \mathsf{id}_{X_2})(d_{X_2})\end{aligned}$$

This shows that there eists $a \in L(E)$ such that

$$\alpha_E(a) = (F(z \circ f_X, \mathsf{id}_X)(d_X))_{X\in\mathbf{X}}$$

but then, using again Lemma 2.2.11

$$\begin{aligned}\beta_X(\Phi(i_E(a))) &= \pi_X(h_{E,X}(F(z \circ f_X, \mathsf{id}_X)(d_X)))\\ &= \pi_X(h_{D_X,X}(d_X))\\ &= \beta_X(l)\end{aligned}$$

which implies

$$\beta(\Phi(i_E(a))) = \beta(l)$$

and we can conclude since β is injective.

($2 \Rightarrow 1$) Let us show the three points of Definition 2.2.1.

1. $\mathbf{D}$ is non empty. Suppose $\mathbf{D}$ is empty, we can take $\mathbf{X}$ to be the empty category as well. Then $L\colon \mathbf{D} \to Y$ is given by the initial and $C'\colon \mathbf{X} \to \mathbf{Set}$ are given by the initial functor and thus the comparison morphism $\Phi\colon C \to L'$ is the unique arrow $\emptyset \to 1$ which is not an isomorphism.
2. Let $\{D_i\}_{i\in I}$ a family of objects of $\mathbf{D}$ with $|I| < \kappa$ and consider $\mathbf{X}$ the discrete category with them as objects; we can take the functor $F\colon \mathbf{D} \times \mathbf{X} \to \mathbf{Set}$ sending a pair (D, D_i) to the set $\mathbf{D}(D_i, D)$. We have that, for every $D_i \in \mathbf{D}$, $F(-, D_i)$ is simply $\mathbf{D}(D_i, -)$, so $C'(D_i)$ is a singleton. Now, $C'\colon \mathbf{X} \to \mathbf{Set}$ is a functor on a discrete category, thus L' is a product of singletons and therefore it is non empty. By hypothesis $\Phi\colon C \to L'$ is an isomorphism, hence C is non empty too. But C is the colimit of the functor $L\colon \mathbf{D} \to \mathbf{Set}$ given by

   $$L(D) = \prod_{i\in I} \mathbf{D}(D_i, D)$$

   and since C is non empty L cannot be the constant functor in $\emptyset$, i.e. there exists a D such that $\mathbf{D}(D_i, D) \neq \emptyset$ for every $i \in I$, but this is exactly the thesis.
3. Let $\{f_i\}_{i\in I}$ a family of arrows $D_1 \to D_2$ with $|I| < \kappa$ and take as $\mathbf{X}$ the subcategory of $\mathbf{D}$ generated by it. We can again define $F\colon \mathbf{D} \times \mathbf{X}^{op} \to \mathbf{Set}$ sending (D, D_j) to $\mathbf{D}(D_j, D)$, where $j \in \{1, 2\}$. The argument now is similar to the one in the previous point: $C'(D_1)$ and $C'(D_2)$ are the colimits of $\mathbf{D}(D_1, -)$ and $\mathbf{D}(D_2, -)$ so they are singletons, i.e. C' is equivalent to the constant functor in 1. This implies that L' is the singleton too, which, in turn, implies that also $|C| = 1$. But C is the colimit of the functor L which we can compute explicitly, indeed:

   $$L(D) \simeq \{g \in \mathbf{D}(D_2, D) \mid g \circ f_i = g \circ f_j \text{ for every } i, j \in I\}$$

   Therefore, since $C \neq \emptyset$, L cannot be the functor constant in $\emptyset$, and the thesis follows. □

Locally κ-presentable categories

To proceed further, we need to introduce the concept of local κ-presentability [6, 29, 51, 87].

Definition 2.2.15. Let $\mathbf{X}$ and Y be categories, a functor $F\colon \mathbf{X} \to \mathbf{Y}$ has *rank* κ if preserves κ-filtered colimits. An object $X \in \mathbf{X}$ is said *κ-presentable* if $\mathbf{X}(X, -)\colon \mathbf{X} \to \mathbf{Set}$ has rank κ, we will denote by $\mathbf{X}_\kappa$ the full subcategory given by κ-presentable objects and by $J_\kappa\colon \mathbf{X}_\kappa \to \mathbf{X}$ the associated inclusion functor.

Remark 2.2.16. Let λ and κ be regular cardinals such that $\lambda \leq \kappa$. Then Remark 2.2.2 implies that a functor F with rank λ also has rank κ; this in turn entails that, in every category $\mathbf{X}$, $\mathbf{X}_\lambda$ is a subcategory of $\mathbf{X}_\kappa$.

Example 2.2.17. Let $(P, \leq)$ be a poset and κ a regular cardinal, an element $p \in P$ is *κ-compact* [1, 55] if for every κ-directed subset S of P (i. e. a subset which is κ-directed with the induced order) with supremum s such that $p \leq s$, there exists $s' \in S$ such that $p \leq s'$. κ-compact elements are exactly the κ-presentable objects of the category associated to $(P, \leq)$.

Example 2.2.18. For every regular κ, let $\boldsymbol{\kappa}$ be the category associated with the (total) order $(\kappa, \subseteq)$ we can consider the diagram $I\colon \boldsymbol{\kappa} \to \mathbf{Set}$ sending $\mu \in \kappa$ to itself and $\mu \subseteq \lambda$ to the inclusion $\iota_{\mu,\lambda}\colon \mu \to \lambda$. By Remark 2.2.3 this diagram is κ-filtered and we have a colimiting cocone

$$\begin{tikzcd} \mu \arrow[rr, "\iota_{\mu,\lambda}"] \arrow[dr, "i_\mu"'] & & \lambda \arrow[dl, "i_\lambda"] \\ & \kappa & \end{tikzcd}$$

in which $i_\lambda\colon \mu \to \kappa$ is again given by the inclusions. On the other hand, a colimiting cocone for $\mathcal{P} \circ I$ is given by $(Q(\kappa), \{j_\mu\}_{\mu\in\kappa})$ where

$$Q(\kappa) := \bigcup_{\mu\in\kappa} \mathcal{P}(\mu)$$

and $j_\mu\colon \mathcal{P}(\mu) \to Q(\kappa)$ is the inclusion, so that we have a diagram

$$\begin{tikzcd} \mathcal{P}(\mu) \arrow[dd, "\mathcal{P}(\iota_{\mu,\lambda})"'] \arrow[dr, "l_\mu"] \arrow[drr, bend left, "\mathcal{P}(i_\mu)"] & & \\ & Q(\kappa) \arrow[r, dotted, "i"] & \mathcal{P}(\kappa) \\ \mathcal{P}(\lambda) \arrow[ur, "l_\lambda"'] \arrow[urr, bend right, "\mathcal{P}(\kappa)"'] & & \end{tikzcd}$$

But the dotted arrow $i\colon Q(\kappa) \to \mathcal{P}(\kappa)$ is, again, simply the inclusion, so, since $\kappa \notin Q(\kappa)$, it follows that i is not an isomorphism and thus that $\mathcal{P}$ doesn't have rank κ.

Proposition 2.2.19. *Let $G\colon \mathbf{B} \to \mathbf{X}_\kappa$ be a diagram such that $\mathbf{B}$ has strictly less then κ arrows and suppose that $(X, \{c_B\}_{B\in\mathbf{B}})$ is a colimiting cone for $J_\kappa \circ G$. Then X is κ-presentable.*

Proof. Let $(C, \{d_D\}_{D\in\mathbf{D}})$ be a colimiting cocone for a functor $H\colon \mathbf{D} \to \mathbf{X}$ with κ-filtered domain. For simplicity , given $D \in \mathbf{D}$ and $B \in \mathbf{B}$, set

$$X_B := J_\kappa(G(B)) \qquad C_D := H(D)$$

We can define a functor $F\colon \mathbf{D} \times \mathbf{B}^{op} \to \mathbf{Set}$

$$\begin{tikzcd} (D_1, B_1) \arrow[r, mapsto] \arrow[d, "{(f,g)}"'] & \mathbf{X}(X_{B_1}, C_{D_1}) \arrow[d, "H(f)\circ(-)\circ J_\kappa(G(g))"] \\ (D_2, B_2) \arrow[r, mapsto] & \mathbf{X}(X_{B_2}, C_{D_2}) \end{tikzcd}$$

Now, for every $B \in \mathbf{B}$, since X_B is κ-presentable, the κ-filtered colimit of $H(-, B) = \mathbf{X}(X_B, -)$ is given by $\mathbf{X}(X_B, C)$ with coprojections

$$j_{D,B}\colon \mathbf{X}(X_B, C_D) \to \mathbf{X}(X_B, C) \qquad f \mapsto d_D \circ f$$

and we also know that the limit of the functor sending B to $\mathbf{X}(X_B, C)$ is $\mathbf{X}(X, C)$ with projections

$$\beta_X\colon \mathbf{X}(X, C) \to \mathbf{X}(X_B, C) \qquad f \mapsto f \circ c_B$$

On the other hand, the limit of $H(-, D)$ is given by $\mathbf{X}(X, C_D)$

$$\alpha_{D,B}; \mathbf{X}(X, C_D) \to \mathbf{X}(X_B, C_D) \qquad f \mapsto f \circ c_B$$

The thesis now follows from Remark 2.2.10 and Theorem 2.2.14. □

Corollary 2.2.20. *The representable functor* $\mathbf{Set}(X, -)$ *has rank* κ *if and only if* $|X| < \kappa$.

Proof. ($\Rightarrow$) By Example 2.2.4 $(X, \{i_A\}_{A \in \mathcal{P}_\kappa(X)})$, where $i_A \colon A \to X$ is the inclusion of $A \in \mathcal{P}_\kappa(X)$, is a κ-filtered colimit, thus $(\mathbf{Set}(X, X), \{i_A \circ (-)\}_{A \in \mathcal{P}_\kappa(X)})$ is again colimiting. Lemma 2.2.11 now implies that $\mathsf{id}_X = i_A \circ f$ for some $A \in \mathcal{P}_\kappa(X)$ and $f \colon X \to A$, showing $|X| < \kappa$.
($\Leftarrow$) $X \simeq \sum_{|X|} 1$, and 1 represents $\mathsf{id}_{\mathbf{Set}}$, thus Proposition 2.2.19 yields the thesis. □

Example 2.2.21. If S is a set with cardinality less then κ then the state monad $\mathbf{Set}(S, S \times -)$ has rank κ: indeed $S \times -$ preserves all colimits since it is a left adjoint, while the previous corollary entails that $\mathbf{Set}(S, -)$ preserves κ-filtered colimits.

Before turning to the central concept of this section we need to introduce the notion of generator.

Definition 2.2.22. [28, 31] Let $\mathcal{G}$ be a set of objects of a category $\mathbf{X}$. We say that $\mathcal{G}$ is a *generator*, if for each pair $f, g \colon X \rightrightarrows Y$ with $f \neq g$, there exist $G \in \mathcal{G}$ and $h \colon G \to X$, such that $f \circ h = g \circ h$. A generator is called *strong* (or *extremal*) provided that, for every mono $m \colon M \to X$ which is not an isomorphism, there exists $g \colon G \to X$, with $G \in \mathcal{G}$ which does not factor through m.

Remark 2.2.23. Let $\mathcal{G}$ be a (strong) generator and $\mathcal{H}$ be another set of objects fo $\mathbf{X}$. Then if $\mathcal{G} \subseteq \mathcal{H}$, we get that $\mathcal{H}$ is a (strong) generator too.

Example 2.2.24. The family containing only the terminal object provides a generator for **Set** and **Top**, which is strong only in the first case: any bijection which is not an homeomorphism provides a counterexamples to strongness in the latter case.

In the following we will need to extend a given generator adding to it some colimits. This is done in the following way: let $\mathcal{G}$ be a generator for a cocomplete category $\mathbf{X}$, then, for every cardinal κ, we can construct another set $\mathcal{G}^\kappa$, adding to $\mathcal{G}$ representatives for all κ-small colimits, this is done taking

$$\mathcal{G}^\kappa := \bigcup_{i \in \mathbb{N}} \mathcal{G}_i$$

where the family $\{\mathcal{G}_i\}_{i \in \mathbb{N}}$ is defined as follows:

- $\mathcal{G}_0 := \mathcal{G} \cup \{0\}$, where 0 is an initial object of $\mathbf{X}$;
- $\mathcal{G}_{i+1}$ is the obtained from $\mathcal{G}_i$ adding a representative for each κ-small coproduct and one for each coequalizer diagram.

Proposition 2.2.25. *Let* $\mathbf{X}$ *be a cocomplete category with a (strong) generator* $\mathcal{G}$. *Then, for every cardinal* κ, $\mathcal{G}^\kappa$ *is a (strong) generator.*

Proof. First of all we can notice that, by construction, $\mathcal{G}^\kappa$ is a set: this follows at once since $\mathcal{G}_0$ is a set and $\mathcal{G}_{i+1}$ is obtained from $\mathcal{G}$ adding a set of new objects. The thesis now follow at once from Remark 2.2.23 □

Definition 2.2.26. Let κ be a regular cardinal, a category $\mathbf{X}$ is *locally* κ*-presentable* if:

1. $\mathbf{X}$ is cocomplete;
2. there exists a strong generator $\mathcal{G}$ for $\mathbf{X}$, such that every object G in $\mathcal{G}$ is κ-presentable.

Remark 2.2.27. From Remark 2.2.16 it follows immediately that if $\mathbf{X}$ is locally κ-presentable category, then it is also locally λ-presentable for every regular cardinal λ greater then κ.

Example 2.2.28. The first half of Example 2.2.24 entails that **Set** is locally $\aleph_0$-presentable.

Example 2.2.29. Let $(P, \leq)$ be a poset, then cocompletenes is tantamount to asking for the existence of a supremum for every subset of P, in particular $(P, \leq)$ must be a complete lattice. On the other hand, since there are no parallel arrows the notion of generator becomes trivial: every subset of P is a generator. This is not the case for strongness as shown by the following facts.

- Let $G \subseteq P$ be a strong generator, then for every $p \in P$, p is the supremum of the set
$$G{\downarrow}p := \{g \in G \mid g \leq p\}$$
Indeed, let s be the supremum of this family and suppose $s \neq p$, then strongness implies the existence of $g \in G$ with $g \leq p$ and such that $g \nleq s$, which is absurd.
- Let $G \subseteq P$ be a set such that, for every $p \in P$ there exists $S_p \subseteq G$ with the property that p is the supremum of S_p, then G is a strong generator: every $q \in P$ with $q < p$ cannot be a upper bound for S_p, thus there must exists $g \in S_p$ such that $g \nleq q$.

Summing up, a strong generator for a cocomplete $(P, \leq)$ is a subset G such that every element of P is the supremum of a family S_p contained in G. On the other hand, Example 2.2.17 implies that the κ-presentable objects of $(P, \leq)$ are exactly its κ-compact elements, thus a cocomplete $(P, \leq)$ is locally κ-presentable if and only if every elements is the supremum of a family of κ-compact objects. This is exactly the notion of *κ-algebraic lattice* [1, 55, 109].

We can categorify Example 2.2.29 to provide an alternative criterion for local κ-presentability.

Lemma 2.2.30. *Let κ be a regular cardinal, then for every cocomplete category $\mathbf{X}$ the following are equivalent:*

1. *$\mathbf{X}$ locally κ-presentable;*
2. *there exists a small subcategory $\mathbf{Y}$ of $\mathbf{X}$, which objects are all κ-presentable in $\mathbf{X}$ and such that for every object $X \in \mathbf{X}$ there exists a functor $F_X : \mathbf{D} \to \mathbf{Y}$ with κ-filtered domain, with the property that X is the vertex of a colimiting cocone for $I \circ F_X$, where I is the inclusion functor $\mathbf{Y} \to \mathbf{X}$.*

Proof. ($1 \Rightarrow 2$) Let $\mathcal{G}$ be a strong generator, by Proposition 2.2.25 $\mathcal{G}^\kappa$ is a strong generator too. Moreover, Proposition 2.2.19 entails that every object in $\mathcal{G}^\kappa$ is κ-presentable. Now, given an object $X \in \mathbf{X}$, we can define $\mathcal{G}^\kappa{\downarrow}X$ as the category in which:

- objects are pair (G, g) made by an object $G \in \mathcal{G}^\kappa$ and an arrow $g\colon G \to X$;
- an arrow $(G, g) \to (H, h)$ is an arrow $f\colon G \to H$ such that the following diagram commutes.

$$\begin{tikzcd} G \arrow[rr, "f"] \arrow[dr, "g"'] & & H \arrow[dl, "h"] \\ & X & \end{tikzcd}$$

There is an obvious functor $U_X : \mathcal{G}^\kappa{\downarrow}X \to \mathbf{X}$ defined as

$$\begin{array}{ccc} (G,g) & \longmapsto & G \\ f\big\downarrow & & \big\downarrow f \\ (H,h) & \longmapsto & H \end{array}$$

We can also notice that Lemma 2.2.6 implies that $\mathcal{G}^\kappa{\downarrow}X$ is κ-filtered: indeed given a diagram $F\colon \mathbf{D} \to \mathcal{G}^\kappa$ with a κ-small domain, then there exists a colimiting cone $(G, \{c_D\}_{D\in\mathbf{D}})$ for $U_X \circ F$. Now, let $F(D)$ be (G_D, g_D) with $g_D\colon G_D \to X$, then for every $d\colon D_1 \to D_2$ we have

$$g_{D_2} \circ F(d) = g_{D_1}$$

which shows that $(X, \{g_D\}_{D\in\mathbf{D}})$ is a cocone on $U_X \circ F$ and thus there exists $g\colon G \to X$ such that

$$g \circ c_D = g_D$$

showing that $((G,g), \{c_D\}_{D\in\mathbf{D}})$ is a cocone on F. It is now enough to show that X is the vertex of a colimiting cocone for U_X.

For every $(G,g) \in \mathcal{G}^\kappa{\downarrow}X$ we can defince $d_{(G,g)}\colon G \to X$ simply as g, by construction this defines a cocone $(X, \{d_{(G,g)}\}_{(G,g)\in\mathcal{G}^\kappa\downarrow X})$ on U_X, let also $(C, \{c_{(G,g)}\}_{(G,g)\in\mathcal{G}^\kappa\downarrow X})$ be a colimiting cocone for such functor, there exists $m\colon C \to X$ such that

$$m \circ c_{(G,g)} = g$$

If we show that m is an isomorphism we are done. Notice that, every $g\colon G \to X$ with $G \in \mathcal{G}^\kappa$ factors trhough m, thus, since $\mathcal{G}^\kappa$ is a strong generator, it is enough to show that m is a monomorphism.

Let $p, q\colon Y \rightrightarrows C$ be two arrows such that $m \circ p = m \circ q$. Since $\mathcal{G}$ is a generator, if we show that

$$p \circ g = q \circ g$$

for any arrow $g\colon G \to X$ with domain in $\mathcal{G}$, we can conclude. G is κ-presentable, thus there exists $(H,h) \in \mathcal{G}^\kappa$ and $p', q'\colon G \rightrightarrows G$ such that the following diagrams commute

$$\begin{tikzcd} & & H \arrow[d, "c_{(H,h)}"] \\ G \arrow[urr, bend left, "p'"] \arrow[r, "g"'] & Y \arrow[r, "p"'] & C \end{tikzcd} \qquad \begin{tikzcd} & & H \arrow[d, "c_{(H,h)}"] \\ G \arrow[urr, bend left, "q'"] \arrow[r, "g"'] & Y \arrow[r, "q"'] & C \end{tikzcd}$$

There is a coequalizer diagram

$$\begin{tikzcd} G \arrow[r, shift left, "p'"] \arrow[r, shift right, "q'"'] & H \arrow[r, "e"] & Q \end{tikzcd}$$

with $Q \in \mathcal{G}^\kappa$. My hypothesis we have

$$\begin{aligned} h \circ p' &= m \circ c_{(H,h)} \circ p' \\ &= m \circ p \circ g \\ &= m \circ q \circ g \\ &= m \circ c_{(H,h)} \circ q' \\ &= h \circ q' \end{aligned}$$

and thus there exists a unique $k\colon Q \to X$ such that the following square commutes

$$\begin{tikzcd} H \arrow[r, "e"] \arrow[d, "c_{(H,h)}"'] \arrow[dr, "h"] & Q \arrow[d, "k"] \\ C \arrow[r, "m"'] & X \end{tikzcd}$$

(Q, k) is an object of $\mathcal{G}^\kappa{\downarrow}X$ and e is an arrow $(H, h) \to (Q, k)$, thus

$$\begin{aligned} p \circ g &= c_{(H,h)} \circ p' \\ &= c_{(Q,k)} \circ e \circ p' \\ &= c_{(Q,k)} \circ e \circ q' \\ &= c_{(H,h)} \circ q' \\ &= q \circ g \end{aligned}$$

$(2 \Rightarrow 1)$ Let $\mathcal{G}$ be the set of objects of $\mathbf{Y}$. Let also $f, g\colon X \rightrightarrows Y$ be two parallel arrows, by hypothesis X is the vertex of a colimiting cocone $(X, \{c_D\}_{D\in\mathbf{D}})$ with $c_D\colon X_D \to X$ with X_D in $\mathbf{Y}$. If $f \neq g$, there must be a $D \in \mathbf{D}$ such that $f \circ c_D = g \circ c_D$, proving that $\mathcal{G}$ is a generator. For strongness: let $m\colon M \to X$ be a mono and suppose that every $g\colon G \to X$ with domain in $\mathcal{G}$ factors through it. In particular, for every $D \in \mathbf{D}$ there exists $d_D\colon X_D \to M$ such that $m \circ d_D = c_D$ and thus we have an induced $n\colon X \to M$ with the property that $n \circ c_D = d_D$, therefore

$$\begin{aligned} m \circ n \circ c_D &= m \circ d_D \\ &= c_D \end{aligned}$$

proving $m \circ n = \mathsf{id}_X$. It follows that m is mono and split epi, hence an isomorphism. □

We can now obtain a characterization for endofunctors with rank κ on a locally λ-presentable category.

Theorem 2.2.31. *Let $\mathbf{X}$ be a locally λ-presentable category, let also κ be a regular cardinal greater or equal than λ. Then for every functor $F\colon \mathbf{X} \to \mathbf{X}$, the following are equivalent:*

1. *F has rank κ;*
2. *$(F, \mathsf{id}_{F\circ J_\kappa})$ is a left Kan extension of $F \circ J_\kappa$ along J_κ;*
3. *the following isomorphism hold*

$$F \simeq \int^{X\in\mathbf{X}_\kappa} \mathbf{X}(X, -) \bullet F(X)$$

Proof. $(1 \Rightarrow 2)$ Let us show that $(F, \mathsf{id}_{F\circ J_k})$ enjoy the universal property of a left Kan extension. Let $G\colon \mathbf{X} \to \mathbf{X}$ be a functor and η a natural transformation $F \circ J_\kappa \to G \circ J_\kappa$. We are going to construct a $\overline{\eta}\colon F \to G$ such that $\overline{\eta}_X = \eta_X$ for every $X \in \mathbf{X}_\kappa$.

Let X be an object of $\mathbf{X}$, by hypothesis $\mathbf{X}$ is locally λ-presentable so, by Remark 2.2.27, it is locally κ-presentable too, therefore Lemma 2.2.30 implies that X is the vertex of a colimiting cocone $(X, \{c_D\}_{D\in\mathbf{D}})$ with $\mathbf{D}$ a κ-filtered category and every $c_D\colon X_D \to X$ has a domain lying in $\mathbf{X}_\kappa$, so $(F(X), \{F(c_D)\}_{D\in\mathbf{D}}$ is colimiting too. This implies that there exists a unique $\overline{\eta}_X \colon F(X) \to G(X)$ making the following diagram commute

$$\begin{tikzcd}
F(J_\kappa(X_D)) \arrow[r, "\eta_{X_D}"] \arrow[d, "F(c_D)"'] & G(J_\kappa(X_D)) \arrow[d, "G(c_D)"] \\
F(X) \arrow[r, "\overline{\eta}_X"'] & G(X)
\end{tikzcd}$$

Notice that, by construction, if X is an object of $\mathbf{X}_\kappa$ then $\bar{\eta}_X = \eta_X$, so we only have to show the naturality of the family $\{\overline{\eta}_X\}_{X\in\mathbf{X}}$. Take an arrow $f\colon X \to Y$, then Y is again a vertex of a colimiting cocone $(Y, \{d_B\}_{B\in\mathbf{B}})$ with $\mathbf{B}$ κ-filtered and such that $d_B \colon Y_B \to Y$ has a κ-presentable domain. Since $\mathbf{X}(X_D, -)$ has rank κ, it follows from Lemma 2.2.11 that there exists $B_D \in \mathbf{B}$ and $f_D \colon X_D \to Y_{B_D}$ such that the following square commutes.

$$\begin{tikzcd}
X_D \arrow[r, "f_D"] \arrow[d, "c_D"'] & Y_{B_D} \arrow[d, "d_{B_D}"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Since J_κ is simply an inclusion, for every $D \in \mathbf{D}$ we get a commutative diagram in $\mathbf{X}$

$$\begin{tikzcd}
 & & & F(X_D) \arrow[dlll, bend right, "F(c_D)"'] \arrow[dl, "F(f_D)"] \arrow[dr, "\eta_{X_D}"] \arrow[drrr, bend left, "F(c_D)"] & & & \\
F(X) \arrow[d, "F(f)"'] & & F(Y_{B_D}) \arrow[dll, "F(d_{B_D})"] \arrow[dr, "\eta_{Y_{B_D}}"] & & G(X_D) \arrow[dl, "G(f_D)"] \arrow[drr, "G(c_D)"] & & F(X) \arrow[d, "\overline{\eta}_X"] \\
F(Y) \arrow[drrr, bend right, "\overline{\eta}_Y"'] & & & G(Y_{B_D}) \arrow[d, "G(d_{B_D})"] & & & G(X) \arrow[dlll, bend left, "G(f)"] \\
 & & & G(Y) & & &
\end{tikzcd}$$

which, by the colimiting property of $(F(X), \{F(c_D)\}_{D\in\mathbf{D}})$, shows that

$$G(f) \circ \overline{\eta}_X = \overline{\eta}_Y \circ F(f)$$

We are left with uniqueness. If $\epsilon\colon F \to G$ is a natural transformation such that $\epsilon_Y = \eta_Y$ for every $Y \in \mathbf{X}_\kappa$, then, for every $D \in \mathbf{D}$ we have

$$\begin{aligned}
\epsilon_X \circ F(c_D) &= G(c_D) \circ \epsilon_{X_D} \\
&= G(c_D) \circ \eta_{X_D} \\
&= \overline{\eta}_X \circ F(c_D)
\end{aligned}$$

from which the thesis follows using again the fact that $(F(X), \{F(c_D)\}_{D\in\mathbf{D}})$ is colimiting.

$(2 \Rightarrow 3)$ This follows from the explicit formula for left Kan extensions.

$(3 \Rightarrow 1)$ $(-) \bullet F(X)$ is a left adjoint, so it preserves all colimits, $\mathbf{X}(X, -)$ preserves κ-filtered colimits by hypothesis. Thus the thesis follows since coends commute with all colimits. $\square$

Remark 2.2.32. Take as $\mathbf{X}$ the category of **Set**, then for every $S \in \mathbf{Set}$ $(-) \bullet S$, being the left adjoint to $\mathbf{Set}(S,-)$, coincides, up to isomorphism, with $(-) \times S$. Thus if a functor $F\colon \mathbf{Set} \to \mathbf{Set}$ has rank κ, we must have the following isomorphism

$$F \simeq \int^{Y \in \mathbf{Set}_\kappa} \mathbf{Set}(Y,-) \times F(Y)$$

Moreover, the coproduct structure $T \times S$ is given by

$$\iota_t \colon S \to T \times S \qquad s \mapsto (t,s)$$

so that we can write the components $\omega_{X,Y}\colon \mathbf{Set}(Y,X) \times F(Y) \to F(X)$ of the initial cowedge ω_X as

$$\omega_{X,Y}\colon \mathbf{Set}(Y,X) \times F(Y) \to F(X) \qquad (f,t) \mapsto T(f)(t)$$

We end this section with a brief discussion of the results obtained applying the notion of rank to monads.

Definition 2.2.33. Let κ be a regular cardinal we will say that a monad $\mathbf{T} = (T, \eta, \mu)$ on a category $\mathbf{X}$ has *rank* κ if κ is the rank of T.

Let J_κ be the inclusion $\mathbf{Set}_\kappa \to \mathbf{Set}$, by Remark Remark 2.2.27 and 2.2.28, Corollary 2.2.20 and Theorem 2.2.31, monads with rank κ are exactly J_κ-monad as defined in Definition 2.1.27. Take now two monads $\mathbf{T}$ and $\mathbf{S}$ with rank, respectively, κ and λ and let also μ be the maximum between them, by Remark 2.2.16 they have both rank μ, thus we can apply Proposition 2.1.28 to get the next result.

Proposition 2.2.34. *There exists a category* **RMnd** *in which objects are monads* $\mathbf{T}$ *on* **Set** *with rank, and arrows are morphism of monads.*

Finally, we point out the following two results .

Proposition 2.2.35. *Let* $L\colon \mathbf{Y} \to \mathbf{X}$ *and* $R\colon \mathbf{X} \to \mathbf{Y}$ *be functor such that* $L \dashv R$*, and suppose that* R *has rank* κ*. Then* $R \circ L$ *has rank* κ *too.*

Proof. This follows at once since L, being a left adjoint, preserves all colimits. □

Corollary 2.2.36. *The following are equivalent for a monad* $\mathbf{T}$ *on a cocomplete category* $\mathbf{X}$*:*

1. $\mathbf{T}$ *has rank* κ*;*
2. $U_{\mathbf{T}}$ *has rank* κ*.*

Proof. $(1 \Rightarrow 2)$ Let $F\colon \mathbf{D} \to \mathbf{EM}(\mathbf{T})$ be a functor with κ-filtered domain, since T preserves κ-filtered colimits, then the thesis follows applying Proposition 2.1.33 and Remark 2.1.34.

$(2 \Rightarrow 1)$ This is a consequence of Proposition 2.2.35. □

2.2.2 Algebraic theories

Let us start recalling the traditional notion of algebraic theory from universal algebra [88, 89, 115].

Definition 2.2.37. Let **Card** be the class of all cardinals, an *algebraic signature* Σ is a pair $(O_\Sigma, \mathsf{ar}_\Sigma)$, where O_Σ is a class of *operations* and ar_Σ is a function $O_\Sigma \to \mathbf{Card}$ such that, for every cardinal κ,

$$O_{\Sigma,\kappa} := \{o \in O_\Sigma \mid \mathsf{ar}_\Sigma(o) = \kappa\}$$

is a set, called the set of *operations of arity* κ. Given a regular cardinal κ, we will say that Σ is κ*-bounded* if $O_{\Sigma,\lambda} = \emptyset$ for every cardinal λ such that $\lambda \geq \kappa$.

The category $\mathbf{Sign}_\kappa$ is defined as the category with κ-bounded signatures as objects and in which a morphism $f\colon \Sigma_1 \to \Sigma_2$ is a function $O_{\Sigma_1} \to O_{\Sigma_2}$ such that the following triangle commutes.

$$\begin{tikzcd} O_{\Sigma_1} \arrow[rr, "f"] \arrow[dr, "\mathsf{ar}_{\Sigma_1}"'] & & O_{\Sigma_2} \arrow[dl, "\mathsf{ar}_{\Sigma_2}"] \\ & \mathbf{Card} & \end{tikzcd}$$

Remark 2.2.38. If Σ is κ-bounded, then O_Σ is a set, not a proper class, so that $\mathbf{Sign}_\kappa(\Sigma_1, \Sigma_2)$ is a set too, proving that $\mathbf{Sign}_\kappa$ is really a category.

Example 2.2.39. The signature Σ_S of *semigroups* is given by $(O_{\Sigma_S}, \mathsf{ar}_{\Sigma_S})$ where

$$O_{\Sigma_S} = \{\cdot\} \qquad \mathsf{ar}_{\Sigma_S}(\cdot) = 2$$

Example 2.2.40. The signature Σ_M of *monoids* is given by $(O_{\Sigma_M}, \mathsf{ar}_{\Sigma_M})$ where $O_{\Sigma_G} = \{\cdot, e\}$ and

$$\mathsf{ar}_{\Sigma_G}(\cdot) = 2 \qquad \mathsf{ar}_{\Sigma_G}(e) = 0$$

Example 2.2.41. The signature Σ_G of *groups* is $(O_{\Sigma_G}, \mathsf{ar}_{\Sigma_G})$ where $O_{\Sigma_G} = \{\cdot, e, (-)^{-1}\}$ and

$$\mathsf{ar}_{\Sigma_G}(\cdot) = 2 \quad \mathsf{ar}_{\Sigma_G}(e) = 0 \quad \mathsf{ar}_{\Sigma_G}\left((-)^{-1}\right) = 1$$

Definition 2.2.42. Let Σ be an algebraic signature, a Σ*-algebra* $\mathcal{A}$ is a pair $(A, \{o^{\mathcal{A}}\}_{o \in O_\Sigma})$ where A is a set and, for every $o \in O_\Sigma$, $o^{\mathcal{A}}$ is a function $A^{\mathsf{ar}_\Sigma(o)} \to A$. A Σ*-homomorphism* $f\colon \mathcal{A} \to \mathcal{B}$ is a function $f\colon A \to B$ such that, for every $o \in O_\Sigma$, the following rectangle commutes

$$\begin{tikzcd} A^{\mathsf{ar}_\Sigma(o)} \arrow[r, "f^{\mathsf{ar}_\Sigma(o)}"] \arrow[d, "o^{\mathcal{A}}"'] & B^{\mathsf{ar}_\Sigma(o)} \arrow[d, "o^{\mathcal{B}}"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

We will denote by Σ-**Alg** the category of Σ-algebras and Σ-homomorphisms, and by U_Σ the functor Σ-**Alg** $\to$ **Set** defined by

$$\begin{tikzcd} (A, \{o^{\mathcal{A}}\}_{o \in O_\Sigma}) \arrow[r, mapsto] \arrow[d, "f"'] & A \arrow[d, "f"] \\ (B, \{o^{\mathcal{B}}\}_{o \in O_\Sigma}) \arrow[r, mapsto] & B \end{tikzcd}$$

Now let $f\colon \Sigma_1 \to \Sigma_2$ be a morphism of $\mathbf{Sign}_\kappa$ and take a Σ_2-algebra $\mathcal{A} = (A, \{p^{\mathcal{A}}\}_{p\in O_{\Sigma_2}})$, then we can define a Σ_1-algebra on $f^*(\mathcal{A}) = (A, \{o^{f^*(\mathcal{A})}\}_{o\in O_{\Sigma_1}})$ putting, for every $o \in O_{\Sigma_1}$

$$o^{f^*(\mathcal{A})} := (f(o))^{\mathcal{A}}$$

which is well-defined since $\mathsf{ar}_{\Sigma_1}(o) = \mathsf{ar}_{\Sigma_2}(f(o))$. This construction can be easily extended to a functor.

Proposition 2.2.43. *For every morphism $f\colon \Sigma_1 \to \Sigma_2$ of $\mathbf{Sign}_\kappa$ there is a functor $f^*\colon \Sigma_2\text{-}\mathbf{Alg} \to \Sigma_1\text{-}\mathbf{Alg}$ sending a Σ_1-algebra $\mathcal{A}$ to $f^*(\mathcal{A})$.*

Proof. We have to extend the previous contruction to morphism. Let $g\colon \mathcal{A} \to \mathcal{B}$ be a Σ_2-homomorphism, then for every $p \in O_{\Sigma_2}$ we have a commutative rectangle

$$\begin{tikzcd}
A^{\mathsf{ar}_\Sigma(p)} \arrow[r, "g^{\mathsf{ar}_\Sigma(p)}"] \arrow[d, "p^{\mathcal{A}}"'] & B^{\mathsf{ar}_\Sigma(p)} \arrow[d, "p^{\mathcal{B}}"] \\
A \arrow[r, "g"'] & B
\end{tikzcd}$$

In particular this holds when $p = f(o)$ for some $o \in O_{\Sigma_1}$, which gives us the thesis. $\square$

Remark 2.2.44. Notice that, for every κ-bounded signature Σ, id_Σ^* is the identity functor on Σ-**Alg**. Moreover, given $f\colon \Sigma_1 \to \Sigma_2$ and $g\colon \Sigma_2 \to \Sigma_3$, then

$$(g \circ f)^* = f^* \circ g^*$$

Remark 2.2.45. Given $f\colon \Sigma_1 \to \Sigma_2$, the induced $f^*\colon \Sigma_2\text{-}\mathbf{Alg} \to \Sigma_1\text{-}\mathbf{Alg}$ commutes with the forgetful functor, i.e. the following diagram is commutative.

$$\begin{tikzcd}
\Sigma_2\text{-}\mathbf{Alg} \arrow[rr, "f^*"] \arrow[dr, "U_{\Sigma_2}"'] & & \Sigma_1\text{-}\mathbf{Alg} \arrow[dl, "U_{\Sigma_1}"] \\
& \mathbf{Set} &
\end{tikzcd}$$

The free Σ-algebra

Let us look more closely at the forgetful functor $U_\Sigma\colon \Sigma\text{-}\mathbf{Alg} \to \mathbf{Set}$. The following results show that the boundedness of Σ is encoded into its rank.

Lemma 2.2.46. *Let Σ be a κ-bounded signature and $F\colon \mathbf{D} \to \Sigma\text{-}\mathbf{Alg}$ be a functor with a κ-filtered domain, let also $(A, \{c_D\}_{D\in\mathbf{D}})$ be a colimiting cocone for $U_\Sigma \circ F$. Then there exists a unique $\mathcal{A}$ in Σ-**Alg** such that $U_\Sigma(\mathcal{A}) = A$, and which makes every c_D a Σ-homomorphism $F(D) \to \mathcal{A}$. Moreover, the cocone $(\mathcal{A}, \{c_D\}_{D\in\mathbf{D}})$ is colimiting for F.*

Proof. Since $\mathsf{ar}_\Sigma(o) < \kappa$ for every $o \in O_\Sigma$, Corollary 2.2.20 entails that $(A^{\mathsf{ar}_\Sigma(o)}, \{c_D^{\mathsf{ar}_\Sigma(o)}\}_{D\in\mathbf{D}})$ is colimiting for the functor $(U_\Sigma(F(-)))^{\mathsf{ar}_\Sigma(o)}$. Let $f\colon D_1 \to D_2$ be an arrow of $\mathbf{D}$, then $F(f)$ is a Σ-

homomorphism, so that we have a diagram

$$\begin{tikzcd}
(U_\Sigma(F(D_1)))^{\mathsf{ar}_\Sigma(o)} \arrow[r, "o^{F(D_1)}"] \arrow[d, "F(f)^{\mathsf{ar}_\Sigma(o)}"'] & U_\Sigma(F(D_1)) \arrow[d, "F(f)"'] \arrow[dr, bend left, "c_{D_1}"] & \\
& & A \\
(U_\Sigma(F(D_2)))^{\mathsf{ar}_\Sigma(o)} \arrow[r, "o^{F(D_2)}"'] & U_\Sigma(F(D_2)) \arrow[ur, bend right, "c_{D_2}"'] &
\end{tikzcd}$$

and thus there exists a unique $o^{\mathcal{A}}\colon A^{\mathsf{ar}_\Sigma(o)} \to A$ such that

$$\begin{tikzcd}
(U_\Sigma(F(D)))^{\mathsf{ar}_\Sigma(o)} \arrow[r, "o^{F(D)}"] \arrow[d, "c_D^{\mathsf{ar}_\Sigma(o)}"'] & U_\Sigma(F(D)) \arrow[d, "c_D"] \\
A^{\mathsf{ar}_\Sigma} \arrow[r, "o^{\mathcal{A}}"'] & A
\end{tikzcd}$$

commutes. Let $\mathcal{A}$ be $(A, \{o^{\mathcal{A}}\}_{o\in O_\Sigma})$ the resulting Σ-algebra, we are going to show that $(\mathcal{A}, \{c_D\}_{D\in\mathbf{D}})$ is colimiting for F. Let $(\mathcal{B}, \{d_D\}_{\mathbf{D}})$ be another cocone on F, we already know that there is a unique $d\colon A \to B$, where $B = U_\Sigma(\mathcal{B})$, such that $d \circ c_D = d_D$, if we show that it is a Σ-homomorphism we are done. Since each d_D is an arrow of Σ-**Alg** we have

$$\begin{aligned}
d \circ o^{\mathcal{A}} \circ c_D^{\mathsf{ar}_\Sigma(o)} &= d \circ c_D \circ o^{F(D)} \\
&= d_D \circ o^{F(D)} \\
&= o^{\mathcal{B}} \circ d_D^{\mathsf{ar}_\Sigma(o)} \\
&= o^{\mathcal{B}} \circ d^{\mathsf{ar}_\Sigma(o)} \circ c_D^{\mathsf{ar}_\Sigma(o)}
\end{aligned}$$

and the thesis follows from the colimiting property of $(A^{\mathsf{ar}_\Sigma(o)}, \{c_D^{\mathsf{ar}_\Sigma(o)}\}_{D\in\mathbf{D}})$. □

Corollary 2.2.47. *Let Σ be a κ-bounded signature for some regular cardinal κ, then the following hold*

1. Σ-**Alg** *has all κ-filtered colimits;*
2. U_Σ *has rank κ.*

Our next step is to show that U_Σ is a right adjoint whenever Σ is κ-bounded (see, for instance [7, 88, 89]). Thus let Σ be κ-bounded. By Remark 2.2.38, O_Σ is a set, hence given $X \in$ **Set** we can define

$$S(X) := \sum_{o\in O_\Sigma} X^{\mathsf{ar}_\Sigma(o)}$$

which provides us with a functor $S\colon$ **Set** $\to$ **Set**. Let $\boldsymbol{\kappa}$ be the category associated with the (total) order $(\kappa, \subseteq)$, we can use S to inductively define a functor $D_X\colon \kappa \to$ **Set**. We will denote by $t_{\mu,\lambda}\colon D_X(\mu) \to D_X(\lambda)$ the image of an inequality $\mu \leq \lambda$.

- If λ is a limit ordinal, suppose that the functor D_X is defined for all $\mu < \lambda$, that is to say that we have a diagram $D_X^\lambda\colon \boldsymbol{\lambda} \to$ **Set** and we can define $D_X(\lambda)$ and $t_{\mu,\lambda}\colon D_X(\mu) \to D_X(\lambda)$ as, respectively, the vertex and the coprojections of a colimiting cocone for D_X^λ.

- If $\lambda = \mu + 1$ is a successor, we can put $D_X(\lambda) := X + S(D_X(\mu))$. By induction, to construct $t_{\alpha,\lambda}$ for an $\alpha \leq \lambda$ it is enough to define $t_{\mu,\lambda}$. We have two cases.
  - μ is a successor too. Then $\mu = \beta + 1$ for some β and $D_X(\mu) = X + S(D_X(\beta))$ and we can define $t_{\mu,\lambda}$ as $\mathsf{id}_X + S(t_{\beta,\mu})$.
  - μ is a limit ordinal. Then for every $\beta < \mu$ we can define $t_{\beta,\lambda}$ as the composition

$$D_X(\beta) \xrightarrow{t_{\beta,\beta+1}} X + S(D_X(\beta)) \xrightarrow{\mathsf{id}_X + S(t_{\beta,\mu})} X + S(D_X(\mu))$$

Now, for every $\gamma \in \mu$ such that $\beta \leq \gamma$ we have a diagram

$$\begin{tikzcd}
D_X(\beta) \arrow[r, "t_{\beta,\beta+1}"] \arrow[d, "t_{\beta,\gamma}"'] & X + S(D_X(\beta)) \arrow[d, "t_{\gamma,\gamma+1}"] \arrow[dr, bend left, "\mathsf{id}_X + S(t_{\beta,\mu})"] & \\
& & X + S(D_X(\mu)) \\
D_X(\gamma) \arrow[r, "t_{\gamma,\gamma+1}"'] & X + S(D_X(\gamma)) \arrow[ur, bend right, "\mathsf{id}_X + S(t_{\gamma,\mu})"'] &
\end{tikzcd}$$

which commutes since, by the previous point, $t_{\gamma,\gamma+1} = \mathsf{id}_X + S(t_{\beta,\gamma})$. But this commutativity entails that $(D_X(\lambda), \{t_{\beta,\lambda}\}_{\beta<\mu})$ is a cocone on D^{μ}_X and we get $t_{\mu,\lambda}$ as the induced arrow.

Remark 2.2.48. We shall remark two things about the construction of D_X.

- The first item of the previous induction yields $D_X(0) = \emptyset$.
- For every λ, if $\mu \leq \lambda$, then $t_{\mu+1,\lambda+1}$ is given by $\mathsf{id}_X + S(t_{\mu,\lambda})$.

Definition 2.2.49. Given a κ-bounded algebraic signature κ, the set $T_\Sigma(X)$ of Σ-*terms* on the set X is the vertex of a colimiting cocone $(T_\Sigma(X), \{j_{X,\lambda}\}_{\lambda\in\kappa})$ for the functor $D_X \colon \boldsymbol{\kappa} \to \mathbf{Set}$ defined above. Given $o \in O_\Sigma$ and $\sigma \colon \mathsf{ar}_\Sigma(o) \to T_{\Sigma,\lambda}(X)$, $o(\sigma)$ will denote the image of (o, σ) under the composition

$$S(D_X(\lambda)) \xrightarrow{s_\lambda} D_X(\lambda + 1) \xrightarrow{j_{X,\lambda+1}} T_\Sigma(X)$$

where s_λ is the inclusion $S(D_X(\lambda)) \to D_X(\lambda + 1)$.

Notation. When $\mathsf{ar}_\Sigma(o) = 0$, there is only one arrow $?_{T_{\Sigma,\lambda}(X)} \colon \emptyset \to T_{\Sigma,\lambda}(X)$. In such a case we will write simply o for $o(?_{T_{\Sigma,\lambda}(X)})$.

Take an operation $o \in O_\Sigma$, then for every $\lambda \in \kappa$ an element of $(D_X(\lambda))^{\mathsf{ar}_\Sigma(o)}$ is just a function $\sigma \colon \mathsf{ar}_\Sigma(o) \to D_X(\lambda)$ and

$$D_X(\lambda + 1) = X + \sum_{o \in O_\Sigma} (D_X(\lambda))^{\mathsf{ar}_\Sigma(o)}$$

So we can define $o^{F_\Sigma(X)}_\lambda \colon (D_X(\lambda))^{\mathsf{ar}_\Sigma(o)} \to D_X(\lambda + 1)$ simply as the inclusion on the component given by o. Now, if $\alpha \leq \beta$ then

$$t_{\alpha+1,\beta+1} = \mathsf{id}_X + \sum_{o \in O_\Sigma} t^{\mathsf{ar}_\Sigma(o)}_{\alpha,\beta}$$

thus the following diagram is commutative

$$\begin{tikzcd}
(D_X(\alpha))^{\mathsf{ar}_\Sigma(o)} \arrow[r, "o_\alpha^{F_\Sigma(X)}"] \arrow[d, "t_{\alpha,\beta}^{\mathsf{ar}_\Sigma(o)}"'] & X + S(D_X(\alpha)) \arrow[d, "t_{\alpha+1,\beta+1}"] \\
(D_X(\beta))^{\mathsf{ar}_\Sigma(o)} \arrow[r, "o_\beta^{F_\Sigma(X)}"'] & X + S(D_X(\beta))
\end{tikzcd}$$

and this implies that $(T_\Sigma(X), \{j_{X,\lambda+1} \circ o_\lambda^{F_\Sigma(X)}\}_{\lambda\in\kappa})$ is a cocone on the composition of D^X and $(-)^{\mathsf{ar}_\Sigma(o)}$.

Now, from Remark 2.2.3 and Corollary 2.2.20 it follows that $\left((T_\Sigma(X))^{\mathsf{ar}_\Sigma(o)}, \left\{j_\lambda^{\mathsf{ar}_\Sigma(o)}\right\}_{\lambda\in\kappa}\right)$ is a colimiting cocone for the composite functor $(-)^{\mathsf{ar}_\Sigma(o)} \circ D^X$, therefore there exists a unique function $o^{F_\Sigma(X)} \colon (T_\Sigma(X))^{\mathsf{ar}_\Sigma(X)} \to T_\Sigma(X)$ making the following diagram commutes.

$$\begin{tikzcd}
(D_X(\lambda))^{\mathsf{ar}_\Sigma(o)} \arrow[r, "o_\lambda^{F_\Sigma(X)}"] \arrow[d, "j_i^{\mathsf{ar}_\Sigma(o)}"'] & D_X(\lambda) \arrow[d, "j_{X,\lambda+1}"] \\
(T_\Sigma(X))^{\mathsf{ar}_\Sigma(o)} \arrow[r, "o^{F_\Sigma(X)}"'] & T_\Sigma(X)
\end{tikzcd}$$

Remark 2.2.50. Since $T_\Sigma(X)$ arises as the vertex of a κ-filtered colimit and $(-)^{\mathsf{ar}_\Sigma(o)}$ has rank κ for every $o \in O_\Sigma$, it follows from Lemma 2.2.11 that every $\sigma \colon \mathsf{ar}_\Sigma(o) \to T_\Sigma(X)$ factors through $D_X(\lambda)$ for some $\lambda \in \kappa$. Moreover, given $\sigma \colon \mathsf{ar}_\Sigma(o) \to D_X(\lambda)$, then, by definition, $o(\sigma)$ coincides with $o^{F_\Sigma(X)}(j_{X,\lambda} \circ \sigma)$. Therefore, we can conclude that, for every $\sigma \colon \mathsf{ar}_\Sigma(o) \to T_\Sigma(X)$

$$o^{F_\Sigma(X)}(\sigma) = o(\sigma)$$

Theorem 2.2.51. *Let Σ be a κ-bounded algebraic signature, then $U_\Sigma \colon \Sigma$-**Alg** $\to$ **Set** has a left adjoint.*

Proof. Let X be a set and define $F_\Sigma(X)$ as $(T_\Sigma(X), \{o^{F_\Sigma(X)}\}_{o\in O_\Sigma})$. By definition $D_X(1) = X + S(\emptyset)$, so we can take $\eta_{\Sigma,X} \colon X \to T_\Sigma(X)$ as the composition of an inclusion with the coprojection $j_{X,1} \colon D_1(X) \to T_\Sigma(X)$. Take also a function $f \colon X \to A$, where $A = U_\Sigma(\mathcal{A})$; for every $\lambda \in \kappa$ we are going to use induction in order to define an arrow $f_\lambda \colon D_X(\lambda) \to A$ such that, for every $\mu \leq \lambda$

$$f_\lambda \circ t_{\mu,\lambda} = f_\mu$$

and the following rectangle commutes

$$\begin{tikzcd}
(D_X(\lambda))^{\mathsf{ar}_\Sigma(o)} \arrow[r, "o_\lambda^{F_\Sigma(X)}"] \arrow[d, "f_\lambda^{\mathsf{ar}_\Sigma(o)}"'] & D_X(\lambda+1) \arrow[d, "f_{\lambda+1}"] \\
A^{\mathsf{ar}_\Sigma(o)} \arrow[r, "o^{\mathcal{A}}"'] & A
\end{tikzcd}$$

- If λ is a limit ordinal and f_μ is defined for all $\mu < \lambda$, then $(A, \{f_\mu\}_{\mu<\lambda})$ is a cocone (empty if $\lambda = 0$) and we can take $f_\lambda \colon D_X(\lambda) \to A$ to be the induced arrow.

- Let λ be $\mu+1$ for some μ, given $o \in O_\Sigma$, let also $k_{\mu,o}\colon (D_X(\mu))^{\mathsf{ar}_\Sigma(o)} \to D_X(\mu)$ be the corresponding coprojection. We can define $h_\lambda\colon S(D_X(\lambda)) \to A$ as the unique arrow such that the following diagram commutes

$$\begin{tikzcd}
(D_X(\mu))^{\mathsf{ar}_\Sigma(o)} \arrow[r, "k_{\mu,o}"] \arrow[d, "f_\mu^{\mathsf{ar}_\Sigma(o)}"'] & S(D_X(\mu)) \arrow[d, "h_\lambda"] \\
A^{\mathsf{ar}_\Sigma(o)} \arrow[r, "o^{\mathcal{A}}"'] & A
\end{tikzcd}$$

commutes, and use it to define $f_\lambda\colon X + D_X(\lambda) \to A$ as $\langle f, h_\lambda\rangle$. Notice that we get a diagram

$$\begin{tikzcd}
(D_X(\mu))^{\mathsf{ar}_\Sigma(o)} \arrow[r, "k_{\mu,o}"] \arrow[d, "f_\mu^{\mathsf{ar}_\Sigma(o)}"'] \arrow[rr, bend left, "o_\mu^{F_\Sigma(X)}"] & S(D_X(\mu)) \arrow[r, "s_\mu"] \arrow[d, "h_\lambda"] & D_X(\lambda) \arrow[dl, bend left, "f_\lambda"] \\
A^{\mathsf{ar}_\Sigma(o)} \arrow[r, "o^{\mathcal{A}}"'] & A &
\end{tikzcd}$$

so we only need to check that $f_\lambda \circ t_{\mu,\lambda} = f_\mu$ to conclude our induction.

  - Suppose $\mu = \beta + 1$ is a successor too. Then $t_{\mu,\lambda} = \mathsf{id}_X + S(t_{\beta,\mu})$ and thus

$$f_\lambda \circ t_{\mu,\lambda} = \langle f, h_\lambda \circ S(t_{\beta,\mu})\rangle$$

  thus if $h_\lambda \circ S(t_{\beta,\mu} = h_\mu$ we are done, but this follows from the commutativity of the following diagram for each $o \in O_\Sigma$.

$$\begin{tikzcd}
(D_X(\beta))^{\mathsf{ar}_\Sigma(o)} \arrow[r, "k_{\beta,o}"] \arrow[d, "t_{\mu,\beta}^{\mathsf{ar}_\Sigma(o)}"] \arrow[dd, bend right=60, "f_\beta^{\mathsf{ar}_\Sigma(o)}"'] & S(D_X(\beta)) \arrow[d, "S(t_{\beta,\mu})"] \\
(D_X(\mu))^{\mathsf{ar}_\Sigma(o)} \arrow[r, "k_{\mu,o}"] \arrow[d, "f_\mu^{\mathsf{ar}_\Sigma(o)}"] & S(D_X(\mu)) \arrow[d, "h_\lambda"] \\
A^{\mathsf{ar}_\Sigma(o)} \arrow[r, "o^{\mathcal{A}}"'] & A
\end{tikzcd}$$

  - If μ is a limit, take $\beta < \mu$, then we have a diagram

$$\begin{tikzcd}
(D_X(\beta))^{\mathsf{ar}_\Sigma(o)} \arrow[r, "t_{\beta,\mu}^{\mathsf{ar}_\Sigma(o)}"] \arrow[d, "k_{\beta,o}"'] \arrow[rr, bend left, "f_\beta^{\mathsf{ar}_\Sigma(o)}"] & (D_X(\mu))^{\mathsf{ar}_\Sigma(o)} \arrow[r, "f_\mu^{\mathsf{ar}_\Sigma(o)}"] \arrow[d, "k_{\mu,o}"'] & A^{\mathsf{ar}_\Sigma(o)} \arrow[d, "o^{\mathcal{A}}"] \\
S(D_X(\beta)) \arrow[r, "S(t_{\beta,\mu})"'] & S(D_X(\mu)) \arrow[r, "h_\lambda"'] & A
\end{tikzcd}$$

which shows that $h_\lambda \circ S(t_{\beta,\mu}) = h_{\beta+1}$. This in turn entails that

$$\begin{aligned} f_\lambda \circ t_{\beta,\lambda} &= \langle f, h_\lambda \rangle \circ \mathsf{id}_X + S(t_{\beta,\mu}) \circ t_{\beta,\beta+1} \\ &= \langle f, h_\lambda \circ S(t_{\beta,\mu}) \rangle \circ t_{\beta,\beta+1} \\ &= \langle f, h_{\beta+1} \rangle \circ t_{\beta,\beta+1} \\ &= f_{\beta+1} \circ t_{\beta,\beta+1} \\ &= f_\beta \end{aligned}$$

But then we also have

$$\begin{aligned} f_\lambda \circ t_{\mu,\lambda} \circ t_{\beta,\mu} &= f_\lambda \circ t_{\beta,\lambda} \\ &= f_\beta \\ &= f_\mu \circ t_{\beta,\mu} \end{aligned}$$

from which $f_\lambda \circ t_{\mu,\lambda} = f_\mu$ follows at once.

Now, by construction we have a cone $(A, \{f_\lambda\}_{\lambda\in\kappa})$ which induces $f_{\Sigma,*} \colon T_\Sigma(X) \to A$ such that

$$f = f_{\Sigma,*} \circ \eta_{\Sigma,X}$$

Moreover all the internal rectangles and triangles of the diagram below are commutative, so that we can conclude that $f_{\Sigma,*}$ is a Σ-homomorphism $F_\Sigma(X) \to \mathcal{A}$.

$$\begin{tikzcd}
(T_\Sigma(X))^{\mathsf{ar}_\Sigma(o)} \arrow[rrr, "o^{F_\Sigma(X)}"] \arrow[d, "f_*^{\mathsf{ar}_\Sigma(o)}"'] & & & T_\Sigma(X) \arrow[d, "f_*"] \\
& (D_X(\lambda))^{\mathsf{ar}_\Sigma(o)} \arrow[ul, "j_\lambda^{\mathsf{ar}_\Sigma(o)}"'] \arrow[dl, "f_\lambda^{\mathsf{ar}_\Sigma(o)}"] \arrow[r, "o_\lambda^{F_\Sigma(X)}"] & D_X(\lambda+1) \arrow[ur, "j_{\lambda+1}"] \arrow[dr, "f_{\lambda+1}"'] & \\
A^{\mathsf{ar}_\Sigma(o)} \arrow[rrr, "o^{\mathcal{A}}"'] & & & A
\end{tikzcd}$$

We are left with uniqueness: let $k \colon F_\Sigma(X) \to \mathcal{A}$ such that $k \circ \eta_{\Sigma,X} = f$, we can proceed by induction to show that $k \circ j_{X,\lambda} = f_\lambda$ for every $\lambda \in \kappa$.

- Let λ be a limit ordinal, and suppose that $k \circ j_\mu = f_\mu$ for every $\mu < \lambda$, then

$$\begin{aligned} k \circ j_{X,\lambda} \circ t_{\mu,\lambda} &= k \circ j_{X,\mu} \\ &= f_\mu \\ &= f_{X,\lambda} \circ t_{\mu,\lambda} \end{aligned}$$

  and we can conclude since $(D_X(\lambda), \{t_{\mu,\lambda}\}_{\mu<\lambda})$ is a colimiting cocone.

- If $\lambda = \mu + 1$ for some ordinal μ, since k is a Σ-homomorphism we get diagrams

$$
\begin{tikzcd}
(D_X(\mu))^{\mathsf{ar}_\Sigma(o)} \arrow[r, "o_\mu^{F_\Sigma(X)}"] \arrow[d, "j_{X,\mu}^{\mathsf{ar}_\Sigma(o)}"] \arrow[dd, bend right=60, "f_\mu^{\mathsf{ar}_\Sigma(o)}"'] & D_X(\lambda) \arrow[d, "j_{X,\lambda}"] \\
(T_\Sigma(X))^{\mathsf{ar}_\Sigma(o)} \arrow[r, "o^{F_\Sigma(X)}"] \arrow[d, "k^{\mathsf{ar}_\Sigma(o)}"] & T_\Sigma(X) \arrow[d, "k"] \\
A^{\mathsf{ar}_\Sigma(o)} \arrow[r, "o^{\mathcal{A}}"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "\eta_{\Sigma,X}"] \arrow[d, "l_1"'] \arrow[dd, bend right=60, "l_\lambda"'] \arrow[rr, bend left, "f"] & T_\Sigma(X) \arrow[r, "k"] & A \\
D_X(1) \arrow[ur, "j_{X,1}"] \arrow[d, "t_{1,\lambda}"'] & & \\
D_X(\lambda) \arrow[uur, "j_{X,\lambda}"'] & &
\end{tikzcd}
$$

where l_1 and l_λ are coprojections. Notice that the commutativity of the diagram on the right is guaranteed by Remark 2.2.48. We can conclude that

$$f_\lambda \circ o_\mu^{F_\Sigma(X)} = k \circ j_{X,\lambda} \circ o_\mu^{F_\Sigma(X)} \qquad k \circ j_{X,\lambda} \circ l_\lambda = f_\lambda \circ l_\lambda$$

which entail the thesis. □

Let T_Σ be $U_\Sigma \circ F_\Sigma$, using Corollary 2.2.47 and Proposition 2.2.35 we can deduce at once the following.

Corollary 2.2.52. *Let Σ be a κ-bounded signature, then the T_Σ has rank κ.*

The calculus of Σ-equations

We have now all the ingredients needed to introduce equations and their calculus.

Definition 2.2.53. Given Σ be a κ-bounded algebraic signature, the set $\mathsf{Eq}(\Sigma)$ of Σ-*equations* (or simply an equation) is defined as

$$\mathsf{Eq}(\Sigma) := \sum_{\lambda \in \kappa} T_\Sigma(\lambda) \times T_\Sigma(\lambda)$$

For every $\lambda \in \kappa$, the image of $(t_1, t_2) \in T_\Sigma(\lambda) \times T_\Sigma(\lambda)$ in $\mathsf{Eq}(\sigma)$ will be denoted by $\lambda \mid t_1 \equiv t_2$ and we will call λ the *context* of the equation.

For every $S \subseteq \mathsf{Eq}(\Sigma)$, its *deductive closure* $S^{\vdash}$ is the smallest subset of $\mathsf{Eq}(\Sigma)$ which contains S and it is closed under the rules of Fig. 2.1, i.e. if all the premises of a rule are in it, then so is the conclusion. An equation is *derivable* from S (or simply derivable if $S = \emptyset$) if it belongs to $S^{\vdash}$.

Notation. We will always use 0 to denote $\emptyset$ when it appears as a context.

Remark 2.2.54. Let μ and λ be two cardinals in κ such that $\mu \leq \lambda$, so that we can consider the inclusion $\iota_{\mu,\lambda} \colon \mu \to \lambda$. Applying Subst to $\eta_{\Sigma,\lambda} \circ \iota_{\mu,\lambda}$ we get the following rule

$$\frac{\mu \leq \lambda \qquad \mu \mid t_1 \equiv t_2}{\lambda \mid T_\Sigma(\iota_{\mu,\lambda})(t_1) \equiv T_\Sigma(\iota_{\mu,\lambda})(t_2)}\ \textsc{Incl}$$

which can be interpreted as a form of weakening.

$$\frac{}{\lambda \mid t \equiv t}\ \textsc{Refl} \qquad \frac{\lambda \mid t_1 \equiv t_2}{\lambda \mid t_2 \equiv t_1}\ \textsc{Sym}$$

$$\frac{\lambda \mid t_1 \equiv t_2 \qquad \lambda \mid t_2 \equiv t_3}{\lambda \mid t_1 \equiv t_3}\ \textsc{Trans} \qquad \frac{\sigma\colon \lambda_1 \to T_\Sigma(\lambda_2) \qquad \lambda_1 \mid t_1 \equiv t_2}{\lambda_2 \mid \sigma_{\Sigma,*}(t_1) \equiv \sigma_{\Sigma,*}(t_2)}\ \textsc{Subst}$$

$$\frac{o \in O_\Sigma \qquad \sigma_1, \sigma_2 \colon \mathsf{ar}_\Sigma(o) \rightrightarrows T_\Sigma(\lambda) \qquad \{\lambda \mid \sigma_1(\alpha) \equiv \sigma_2(\alpha)\}_{\alpha \in \mathsf{ar}_\Sigma(o)}}{\lambda \mid o(\sigma_1) \equiv o(\sigma_2)}\ \textsc{Cong}$$

Figure 2.1: Derivation rules for the calculus of Σ-equations.

Proposition 2.2.55. *Let Σ be a κ-bounded signature, then the following hold:*

1. *if S_1 and S_2 are subsets of $\mathsf{Eq}(\Sigma)$ and $S_1 \subseteq S_2$, then $S_1^{\vdash} \subseteq S_2^{\vdash}$;*
2. *for every $S \subseteq \mathsf{Eq}(\Sigma)$, $\left(S^{\vdash}\right)^{\vdash} = S^{\vdash}$.*

Proof. 1. This follows at once since $S_2^{\vdash}$ contains S_2.

2. Clearly $S \subseteq S^{\vdash}$, so $S^{\vdash} \subseteq \left(S^{\vdash}\right)^{\vdash}$. On the other hand $S^{\vdash}$ is closed under the rules of our calculus by definition, so $\left(S^{\vdash}\right)^{\vdash} \subseteq S^{\vdash}$. □

Now let $f\colon \Sigma_1 \to \Sigma_2$ be a morphism in $\mathbf{Sign}_\kappa$. We wish to have a way to translate a Σ_1-equation to a Σ_2 equation. Now, if we denote by $\eta_{\Sigma_1,\lambda}\colon \lambda \to T_{\Sigma_1}(\lambda)$ and $\eta_{\Sigma_2,\lambda}\colon \lambda \to T_{\Sigma_2}(\lambda)$ the components in $\lambda \in \kappa$ of the units of, respectively, the adjunctions $F_{\Sigma_1} \dashv U_{\Sigma_1}$ and $F_{\Sigma_2} \dashv U_{\Sigma_2}$, we know that there exists a unique $\left(\eta_{\Sigma_2,\lambda}\right)_{\Sigma_1,*} : F_{\Sigma_1}(\lambda) \to f^*(F_{\Sigma_2}(\lambda))$ such that the following diagram commutes in **Set**.

$$\begin{tikzcd} & \lambda \arrow[dl, "\eta_{\Sigma_1,\lambda}"'] \arrow[dr, "\eta_{\Sigma_2,\lambda}"] & \\ T_{\Sigma_1}(\lambda) \arrow[rr, "\left(\eta_{\Sigma_2,\lambda}\right)_{\Sigma_1,*}"'] & & T_{\Sigma_2}(\lambda) \end{tikzcd}$$

We can use this arrow to extend the construction of equations to a functor.

Proposition 2.2.56. *There exists a functor $\mathsf{Eq}\colon \mathbf{Sign}_\kappa \to \mathbf{Set}$ sending a signature Σ to the set of Σ-equations.*

Proof. Let f be a morphism $\Sigma_1 \to \Sigma_2$ in $\mathbf{Sign}_\kappa$, then we can define

$$\mathsf{tr}_{f,\lambda}\colon T_{\Sigma_1}(\lambda) \times T_{\Sigma_1}(\lambda) \to T_{\Sigma_2}(\lambda) \times T_{\Sigma_2}(\lambda)$$

putting $\mathsf{tr}_{f,\lambda} := \left(\eta_{\Sigma_2,\lambda}\right)_{\Sigma_1,*} \times \left(\eta_{\Sigma_2,\lambda}\right)_{\Sigma_1,*}$. To get the thesis it is now enough to define the image of f as the *translating function* $\mathsf{tr}_f\colon \mathsf{Eq}(\Sigma_1) \to \mathsf{Eq}(\Sigma_2)$ given by the sum of the family $\{\mathsf{tr}_{f,\lambda}\}_{\lambda\in\kappa}$. □

Definition 2.2.57. A subset $\Lambda \subseteq \mathsf{Eq}(\Sigma)$ is a Σ-*theory* (or simply a theory) if $\Lambda = S^{\vdash}$ for some $S \subseteq \mathsf{Eq}(\Sigma)$. An *axiom* for a Σ-theory Λ is simply an element of such an S.

We say that Σ-algebra $\mathcal{A} = (A, \{o^{\mathcal{A}}\}_{o\in O_\Sigma})$, *satisfies* a Σ-equation $\lambda \mid t_1 \equiv t_2$ if, for every $f\colon \lambda \to A$, the induced morphism $f_{\Sigma,*}\colon F_\Sigma(\lambda) \to \mathcal{A}$ satisfies

$$f_{\Sigma,*}(t_1) = f_{\Sigma,*}(t_2)$$

Finally, the category $\mathbf{Mod}(\Lambda)$ of *models* of a Σ-theory Λ is the full subcategory of Σ-**Alg** given by algebras satisfying all the equations in Λ. We will denote by $U_\Lambda \colon \mathbf{Mod}(\Lambda) \to \mathbf{Set}$ the restriction of U_Σ.

Lemma 2.2.58. *For every Σ-algebra $\mathcal{A} = (A, \{o^{\mathcal{A}}\}_{o\in O_\Sigma})$, if $\mathcal{A}$ satisfies all the premises of a rule of the calculus of equations, then it satisfies also its conclusion.*

Proof. The thesis follows at once for rules Refl, Sym and Trans, let us examine the other two.
Subst. Take $f \colon \lambda_2 \to A$, then

$$f_{\Sigma,*} \circ \sigma_{\Sigma,*} \circ \eta_{\Sigma,\lambda_1} = f_{\Sigma,*} \circ \sigma$$

and thus $f_{\Sigma,*} \circ \sigma_{\Sigma,*} = (f_{\Sigma,*} \circ \sigma)_{\Sigma,*}$. From this we can compute and get

$$\begin{aligned} f_{\Sigma,*}\left(\sigma_{\Sigma,*}(t_1)\right) &= \left(f_{\Sigma,*} \circ \sigma\right)_{\Sigma,*}(t_1) \\ &= \left(f_{\Sigma,*} \circ \sigma\right)_{\Sigma,*}(t_2) \\ &= f_{\Sigma,*}\left(\sigma_{\Sigma,*}(t_2)\right) \end{aligned}$$

Cong. Since $\mathcal{A}$ satisfies the family of equations $\{\lambda \mid \sigma_1(\alpha) \equiv \sigma_2(\alpha)\}_{\alpha\in\mathsf{ar}_\Sigma(o)}$ it follows that

$$f_{\Sigma,*} \circ \sigma_1 = f_{\Sigma,*} \circ \sigma_2$$

for every $f \colon \lambda \to A$. Now, since $f_{\Sigma,*}$ is a Σ-homomorphism, we have a diagram

$$\begin{tikzcd} (T_\Sigma(\lambda))^{\mathsf{ar}_\Sigma(o)} \arrow[r, "f_{\Sigma,*}^{\mathsf{ar}_\Sigma(o)}"] \arrow[d, "o^{F_\Sigma(\lambda)}"'] & A^{\mathsf{ar}_\Sigma(o)} \arrow[d, "o^{\mathcal{A}}"] \\ T_\Sigma(\lambda) \arrow[r, "f_{\Sigma,*}"'] & A \end{tikzcd}$$

which, by Remark 2.2.50, entails that

$$\begin{aligned} f_{\Sigma,*}(o(\sigma_1) &= f_{\Sigma,*}\left(o^{F_\Sigma(\lambda)}(\sigma_1)\right) \\ &= o^{\mathcal{A}}\left(f_{\Sigma,*}^{\mathsf{ar}_\Sigma(o)}(\sigma_1)\right) \\ &= o^{\mathcal{A}}(f_{\Sigma,*} \circ \sigma_1) \\ &= o^{\mathcal{A}}(f_{\Sigma,*} \circ \sigma_2) \\ &= o^{\mathcal{A}}\left(f_{\Sigma,*}^{\mathsf{ar}_\Sigma(o)}(\sigma_2)\right) \\ &= f_{\Sigma,*}\left(o^{F_\Sigma(\lambda)}(\sigma_2)\right) \\ &= f_{\Sigma,*}(o(\sigma_2)) \end{aligned}$$

and we are done. □

Corollary 2.2.59. *Let Λ be a Σ-theory and S a set of axioms for it, then a Σ-algebra is a model of Λ if and only if it satisfies every equation in S.*

Notation. In order to improve readability, we will use x, y, z (possibly with a subscript), to denote variables coming from some λ. We will also use infix notation for operations of arity 2. For instance, given a signature $O_\Sigma = \{+\}$ with $\mathsf{ar}_\Sigma(+) = 2$, we will write $x + y$ instead of $+(\eta_{\Sigma,2})$.

Example 2.2.60. The first example of a Σ-theory is the one with empty set of axioms: its models are all the Σ-algebras.

Example 2.2.61. Take the signature Σ_S of Example 2.2.39, the theory Λ_S of *semigroups* is the Σ_S-theory with axiom

$$3 \mid x \cdot (y \cdot z) \equiv (x \cdot y) \cdot z$$

The models for this theory are precisely the semigroups.

Example 2.2.62. The theory Λ_M of *monoids* is the Σ_M-theory given by the axioms

$$3 \mid x \cdot (y \cdot z) \equiv (x \cdot y) \cdot z \qquad 1 \mid e \cdot x \equiv x \qquad 1 \mid x \cdot e \equiv x$$

Taking $\mathbf{Mod}(\Lambda_M)$ we recover the classical category of monoids and their homomorphisms.

Example 2.2.63. In the signature Σ_G of Example 2.2.41, we can define the theory of *groups* Λ_G as the one generated by the following axioms

$$1 \mid x \cdot x^{-1} \equiv e \quad 1 \mid x^{-1} \cdot x \equiv e \quad 1 \mid e \cdot x \equiv x \quad 1 \mid x \cdot e \equiv x \quad 3 \mid (x \cdot y) \cdot z \equiv x \cdot (y \cdot z)$$

In this case, $\mathbf{Mod}(\Lambda_G)$ coincides with $\mathbf{Grp}$, the category of groups.

Let us take a closer look to $U_\Lambda \colon \mathbf{Mod}(\Lambda) \to \mathbf{Set}$, proving that it preserves some colimits.

Lemma 2.2.64. *Let Σ be a κ-bounded algebraic signature and Λ a Σ-theory. In addition, let I_Λ be the inclusion $\mathbf{Mod}(\Lambda) \to \Sigma$-$\mathbf{Alg}$ and $F \colon \mathbf{D} \to \mathbf{Mod}(\Lambda)$ a functor with κ-filtered domain. If $(\mathcal{A}, \{c_D\}_{D \in \mathbf{D}})$ is a colimiting cocone for $I_\Lambda \circ F$ then $\mathcal{A}$ is a model for Λ.*

Proof. Let $\lambda \mid t_1 \equiv t_2$ be an equation in Λ and $f \colon \lambda \to U_\Sigma(\mathcal{A})$. Since $\lambda < \kappa$, Corollary 2.2.47 implies that there exists $D \in \mathbf{D}$ and $g \colon \lambda \to U_\Sigma(I_\Lambda(F(D)))$ such that $f = c_D \circ g$. Now

$$c_D \circ g_{\Sigma,*} \circ \eta_{\Sigma,\lambda} = c_D \circ g$$

thus $f_{\Sigma,*} = c_D \circ g_{\Sigma,*}$. By hypothesis $F(D)$ is a model of Λ, so that

$$\begin{aligned} f_{\Sigma,*}(t_1) &= c_D(g_{\Sigma,*}(t_1)) \\ &= c_D(g_{\Sigma,*}(t_2)) \\ &= f_{\Sigma,*}(t_2) \end{aligned}$$

from which we can deduce that $\mathcal{A}$ belongs to $\mathbf{Mod}(\Lambda)$. $\square$

Corollary 2.2.65. *For every κ-bounded signature Σ and Σ-theory Λ, $U_\Lambda \colon \mathbf{Mod}(\Lambda) \to \mathbf{Set}$ has rank κ.*

The free model of a theory

We have ended the last section by establishing that the forgetful functor $U_\Lambda \colon \mathbf{Mod}(\Lambda) \to \mathbf{Set}$ has rank κ whenever Λ is a theory in a κ-bounded signature. We are now going to show that U_Λ has a left adjoint.

Definition 2.2.66. Let $\mathcal{A} = (A, \{o^{\mathcal{A}}\}_{o\in O_\Sigma})$ be a Σ-algebra for an algebraic signature Σ. A Σ-*congruence* (or simply a congruence) is an equivalence relation $\sim$ on A, such that, for every $o \in O_\Sigma$ and functions $\sigma_1, \sigma_2 \colon \mathsf{ar}_\Sigma(o) \rightrightarrows A$, if $\sigma_1(\alpha) \sim \sigma_2(\alpha)$ for every $\alpha \in \mathsf{ar}_\Sigma(o)$, then $o^{\mathcal{A}}(\sigma_1) \sim o^{\mathcal{A}}(\sigma_2)$.

Proposition 2.2.67. *Let $e \colon \mathcal{A} \to \mathcal{B}$ be a Σ-homomorphism such that $U_\Sigma(e)$ is surjective, let also $f \colon \mathcal{A} \to \mathcal{C}$ be another Σ-homomorphism such that $f(a_1) = f(a_2)$ whenever $e(a_1) = e(a_2)$, then the unique arrow $g \colon U_\Sigma(\mathcal{B}) \to U_\Sigma(\mathcal{C})$ fitting in the following diagram is a Σ-homomorphism.*

$$\begin{tikzcd} U_\Sigma(\mathcal{A}) \arrow[r, "f"] \arrow[d, "(U_\Sigma)(e)"'] & U_\Sigma(\mathcal{C}) \\ U_\Sigma(\mathcal{B}) \arrow[ur, dotted, "g"'] & \end{tikzcd}$$

Proof. For every $o \in O_\Sigma$ have the following chain of equalities:

$$\begin{aligned} o^{\mathcal{C}} \circ g^{\mathsf{ar}_\Sigma(o)} \circ e^{\mathsf{ar}_\Sigma(o)} &= o^{\mathcal{C}} \circ f^{\mathsf{ar}_\Sigma(o)} \\ &= f \circ o^{\mathcal{A}} \\ &= g \circ e \circ o^{\mathcal{A}} \\ &= g \circ o^{\mathcal{B}} \circ e^{\mathsf{ar}_\Sigma(o)} \end{aligned}$$

and the thesis follows since $e^{\mathsf{ar}_\Sigma(o)}$ is epi in **Set**. $\square$

Lemma 2.2.68. *Let $\mathcal{A} = (A, \{o^{\mathcal{A}}\}_{o\in O_\Sigma})$ be a Σ-algebra and $\sim$ a congruence on it. Let $\pi \colon A \to B$ be the projection on the quotient. Then the following hold:*

1. *there exists a unique Σ-algebra $\mathcal{B} = (B, \{o^{\mathcal{B}}\}_{o\in O_\Sigma})$, called the* quotient Σ-algebra, *which makes the function π a Σ-homomorphism;*
2. *if $f \colon \mathcal{A} \to \mathcal{C}$ is a Σ-homomorphism such that $f(a_1) = f(a_2)$ for every a_1, a_2 satisfying $\pi(a_1) = \pi(a_2)$, then the unique arrow $g \colon B \to U_\Sigma(\mathcal{C})$ is a Σ-homomorphism.*

Proof. 1. Take $o \in O_\Sigma$ and $\sigma_1, \sigma_2 \colon \mathsf{ar}_\Sigma(o) \rightrightarrows A$ such that

$$\pi \circ \sigma_1 = \pi \circ \sigma_2$$

then for every $\alpha \in \mathsf{ar}_\Sigma(o)$ we have $\sigma_1(\alpha) \sim \sigma_2(\alpha)$, and thus, since $\sim$ is a Σ-congruence

$$\pi(o^{\mathcal{A}}(\sigma_1)) = \pi(o^{\mathcal{A}}(\sigma_2))$$

By the axiom of choice, π has a section, thus $\pi^{\mathsf{ar}_\Sigma(o)}$ is surjective, and the equation above implies the existence of a unique $o^{\mathcal{B}} \colon B^{\mathsf{ar}_\Sigma(o)} \to B$ making the following rectangle commutative

$$\begin{tikzcd} A^{\mathsf{ar}_\Sigma(o)} \arrow[r, "o^{\mathcal{A}}"] \arrow[d, "\pi^{\mathsf{ar}_\Sigma(o)}"'] & A \arrow[d, "\pi"] \\ B^{\mathsf{ar}_\Sigma(o)} \arrow[r, "o^{\mathcal{B}}"'] & B \end{tikzcd}$$

which is precisely what we had to show.

2. This follows from Proposition 2.2.67. □

Definition 2.2.69. Let Λ be a Σ-theory for a κ-bounded signature Σ. For every cardinal $\lambda < \kappa$, we define a relation $\sim_{\Lambda,\lambda}$ on $T_\Sigma(\lambda)$ putting $t_1 \sim_{\Lambda,\lambda} t_2$ if and only if $\lambda \mid t_1 \equiv t_2$ belongs to Λ.

Proposition 2.2.70. *Given a κ-bounded signature Σ, $\lambda < \kappa$ and a Σ-theory Λ, the relation $\sim_{\Lambda,\lambda}$ is a Σ-congruence on $F_\Sigma(\lambda)$.*

Proof. Rules Refl, Sym and Trans imply that $\sim_{\Lambda,\lambda}$ is an equivalence relation. To see that it is a congruence, take $o \in O_\Sigma$, $\sigma_1, \sigma_2 \colon \mathsf{ar}_\Sigma(o) \rightrightarrows T_\Sigma(\lambda)$ and suppose that, for every $\alpha \in \mathsf{ar}_\Sigma(o)$, the equation $\lambda \mid \sigma_1(\alpha) \equiv \sigma_2(\alpha)$ belongs to Λ. Then we can apply rule Cong and get

$$\frac{\sigma_1, \sigma_2 \colon \mathsf{ar}_\Sigma(o) \rightrightarrows T_\Sigma(\lambda) \qquad \{\lambda \mid \sigma_1(\alpha) \equiv \sigma_2(\alpha)\}_{\alpha \in \mathsf{ar}_\Sigma(o)}}{\lambda \mid o(\sigma_1) \equiv o(\sigma_2)}\ \textsc{Cong}$$

which, by Remark 2.2.50, means exactly that

$$o^{F_\Sigma(\lambda)}(\sigma_1) \sim_{\Lambda,\lambda} o^{F_\Sigma(\lambda)}(\sigma_2)$$

and we can conclude at once. □

Since $\sim_{\Lambda,\lambda}$ is a Σ-congruence we can use Lemma 2.2.68 to obtain, for every $\lambda < \kappa$, the quotient Σ-algebra $F_\Lambda(\lambda)$. Equations satisfied by this Σ-algebra are exactly the ones belonging to Λ, as shown by the following proposition.

Proposition 2.2.71. *Let Σ be a κ-bounded signature Σ, Λ a Σ-theory and $\lambda\kappa$. Then an equation $\lambda \mid t_1 \equiv t_2$ belongs to Λ if and only if it is satisfied by $F_\Lambda(\lambda)$.*

Notation. We will denote $U_\Sigma(F_\Lambda(\lambda))$ with $T_\Lambda(\lambda)$ and use $\pi_{\Lambda,\lambda}$ to denote the quotient arrow.

Remark 2.2.72. In particular, the second half of the thesis entails that $F_\Lambda(\lambda)$ is a model for Λ.

Proof. ($\Rightarrow$) Take an equation $\lambda \mid t_1 \equiv t_2$ belonging to Λ and a function $f \colon \lambda \to T_\Lambda(\lambda)$. Fix also a section $s \colon T_\Lambda(\lambda) \to T_\Sigma(\lambda)$ of π_λ, this yields a function $s \circ f \colon \lambda \to T_\Sigma(\lambda)$. Notice that

$$\begin{aligned} \pi_{\Lambda,\lambda} \circ (s \circ f)_{\Sigma,*} \circ \eta_{\Sigma,\lambda} &= \pi_\lambda \circ s \circ f \\ &= f \end{aligned}$$

Thus $\pi_{\Lambda,\lambda} \circ (s \circ f)_{\Sigma,*} = f_{\Sigma,*}$. Now, we can apply rule Subst to get

$$\frac{s \circ f \colon \lambda \to T_\Sigma(\lambda) \qquad \lambda \mid t_1 \equiv t_2}{\lambda \mid (s \circ f)_{\Sigma,*}(t_1) \equiv (s \circ f)_{\Sigma,*}(t_2)}\ \textsc{Subst}$$

Therefore

$$\begin{aligned} f_{\Sigma,*}(t_1) &= \pi_{\Lambda,\lambda}((s \circ f)_{\Sigma,*}(t_1)) \\ &= \pi_{\Lambda,\lambda}((s \circ f)_{\Sigma,*}(t_2)) \\ &= f_{\Sigma,*}(t_2) \end{aligned}$$

($\Leftarrow$) Suppose that $\lambda \mid t_1 \equiv t_2$ is satisfied by $F_\Lambda(\lambda)$ and consider the arrow $\pi_{\Lambda,\lambda} \circ \eta_{\Sigma,\lambda} \colon \lambda \to T_\Lambda(\lambda)$. Since $\pi_{\Lambda,\lambda}$ is a Σ-homomorphism we have

$$\begin{aligned}(\pi_{\Lambda,\lambda} \circ \eta_{\Sigma,\lambda})_{\Sigma,*} &= \pi_{\Lambda,\lambda} \circ (\eta_{\Sigma,\lambda})_{\Sigma,*} \\ &= \pi_{\Lambda,\lambda} \circ \mathsf{id}_{T_{\Sigma,\lambda}} \\ &= \pi_{\Lambda,\lambda}\end{aligned}$$

Thus $\pi_{\Lambda,\lambda}(t_1) = \pi_{\Lambda,\lambda}(t_2)$, which means exactly that $\lambda \mid t_1 \equiv t_2$ belongs to Λ. $\square$

The second half of the previous proposition allows us to deduce the following completeness result.

Corollary 2.2.73. *For every κ-bounded signature Σ, a Σ-equation $\lambda \mid t_1 \equiv t_2$ is satisfied by all models of Λ if and only if it belongs to Λ.*

Now let X be a set, by Example 2.2.4 we know that $\big(X, \{i_A\}_{A \in \mathcal{P}_\kappa(X)}\big)$ is a colimiting cocone. For every $A \in \mathcal{P}_\kappa(X)$ we can fix a bijection $\phi_A \colon |A| \to A$, and composing with the inclusion $i_A \colon A \to X$ we get another colimiting cocone $\big(X, \{j_A\}_{A \in \mathcal{P}_\kappa(X)}\big)$. Let $j_{A,B} \colon |A| \to |B|$ be the arrow associated to an inclusion $A \subseteq B$, given t_1,$t_2 \in T_\Sigma(|A|)$ such that $|A| \mid t_1 \equiv t_2$ is in Λ we can derive

$$\frac{\eta_{\Sigma,|B|} \circ j_{A,B} \colon |A| \to T_\Sigma(|B|) \qquad |A| \mid t_1 \equiv t_2}{|B| \mid T_\Sigma(j_{A,B})(t_1) \equiv T_\Sigma(j_{A,B})(t_2)}\ \textsc{Subst}$$

Thus there exists a unique $T_\Lambda(j_{A,B}) \colon T_\Lambda(|A|) \to T_\Lambda(|B|)$ such that the following square commutes

$$\begin{tikzcd} T_\Sigma(|A|) \arrow[r, "T_\Sigma(j_{A,B})"] \arrow[d, "\pi_{\Lambda,|A|}"'] & T_\Sigma(|B|) \arrow[d, "\pi_{\Lambda,|B|}"] \\ T_\Lambda(|A|) \arrow[r, "T_\Lambda(j_{A,B})"'] & T_\Lambda(|B|) \end{tikzcd}$$

Since $\pi_{\Lambda,|B|} \circ T_\Sigma(j_{A,B})$ is a Σ-homomorphism, Lemma 2.2.68 assures us that $T_\Lambda(j_{A,B})$ is a Σ-homomorphism. T_Σ is a functor and we have equations

$$j_{B,C} \circ j_{A,B} = j_{A,C} \qquad j_{A,A} = \mathsf{id}_{|A|}$$

Hence,there is a diagram in Σ-**Alg** made by the family $\{T_\Lambda(|A|)\}_{A \in \mathcal{P}_\kappa(X)}$ with edges given by all the functions of the form $T_\Lambda(j_{A,B})$ for $A \subseteq B$ in $\mathcal{P}_\kappa(X)$. In light of Corollary 2.2.47 we can consider a colimiting cocone $\big(F_\Lambda(X), \{T_\Lambda(j_A)\}_{A \in \mathcal{P}_\kappa(X)}\big)$ for this diagram and put

$$T_\Lambda(X) := U_\Sigma(F_\Lambda(X))$$

Now, for every A, $B \in \mathcal{P}_\kappa(X)$ such that $A \subseteq B$ we have

$$\begin{aligned}T_\Lambda(j_B) \circ \pi_{\Lambda,|B|} \circ T_\Sigma(j_{A,B}) &= T_\Lambda(j_B) \circ T_\Lambda(j_{A,B}) \circ \pi_{\Lambda,|A|} \\ &= T_\Lambda(j_A) \circ \pi_{\Lambda,|A|}\end{aligned}$$

yielding a cocone $\big(F_\Lambda(X), \{T_\Lambda(j_A) \circ \pi_{\Lambda,|A|}\}_{A \in \mathcal{P}_\kappa(X)}\big)$ which, by Corollary 2.2.52, implies the existence of a unique Σ-homomorphism $\pi_{\Lambda,X} \colon F_\Sigma(X) \to F_\Lambda(X)$ making the following square commutative.

$$\begin{tikzcd} T_\Sigma(|A|) \arrow[r, "T_\Sigma(j_A)"] \arrow[d, "\pi_{\Lambda,|A|}"'] & T_\Sigma(X) \arrow[d, "\pi_{\Lambda,X}"] \\ T_\Lambda(|A|) \arrow[r, "T_\Lambda(j_A)"'] & T_\Lambda(X) \end{tikzcd}$$

Remark 2.2.74. Notice that $\pi_{\Lambda,X}$ is epi in **Set**, and thus a surjective function. Indeed if $f, g\colon T_\Lambda(X) \rightrightarrows A$ are arrows such that $f \circ \pi_{\Lambda,X} = g \circ \pi_{\Lambda,X}$, then, for every $A \in \mathcal{P}_\kappa(X)$ we have

$$\begin{aligned} f \circ T_\Lambda(j_A) \circ \pi_{\Lambda,|A|} &= f \circ \pi_{\Lambda,X} \circ T_\Sigma(j_A) \\ &= g \circ \pi_{\Lambda,X} \circ T_\Sigma(j_A) \\ &= g \circ T_\Lambda(j_A) \circ \pi_{\Lambda,|A|} \end{aligned}$$

and we know that every $\pi_{\Lambda,|A|}$ is epi, thus

$$f \circ T_\Lambda(j_A) = g \circ T_\Lambda(j_A)$$

from which the thesis follows. U_Σ is faithful, so $\pi_{\Lambda,X}$ is epi in Σ-**Alg** too.

Theorem 2.2.75. *Let Σ be a κ-bounded signature, then the forgetful functor $U_\Lambda\colon$ **Mod**$(\Lambda) \to$ **Set** has a left adjoint $F_\Lambda\colon$ **Set** $\to$ **Mod**(Λ) for every Σ-theory Λ.*

Proof. For every set X, we notice that, by Proposition 2.2.71 and Remark 2.2.72, $F_\Lambda(X)$ arises as a κ-filtered colimit of objects of **Mod**(Λ), thus Lemma 2.2.64 implies that $F_\Lambda(X) \in$ **Mod**(Λ). Define $\eta_{\Lambda,X}\colon X \to T_\Lambda(X)$ as the composition

$$X \xrightarrow{\eta_{\Sigma,X}} T_\Sigma(X) \xrightarrow{\pi_{\Lambda,X}} T_\Lambda(X)$$

Take a Σ-algebra $\mathcal{C} = \left(C, \{o^{\mathcal{C}}\}_{o\in O_\Sigma}\right)$ which is a model for Λ and a function $f\colon X \to C$. Then for every $A \in \mathcal{P}_\kappa(X)$, we have a Σ-homomorphism $F_\Sigma(|A|) \to \mathcal{C}$ given by $f_{\Sigma,*} \circ T_\Sigma(j_A)$. Moreover

$$\begin{aligned} f_{\Sigma,*} \circ T_\Sigma(j_A) \circ \eta_{\Sigma,|A|} &= f_{\Sigma,*} \circ \eta_{\Sigma,X} \circ j_A \\ &= f \circ j_A \end{aligned}$$

so that

$$f_{\Sigma,*} \circ T_\Sigma(j_A) = (f \circ j_A)_{\Sigma,*}$$

In particular, this identity entails that for every $t_1, t_2 \in T_\Sigma(|A|)$ such that $|A| \mid t_1 \equiv t_2$ is in Λ

$$f_{\Sigma,*}(T_\Sigma(j_A)(t_1)) = f_{\Sigma,*}(T_\Sigma(j_A)(t_2))$$

We can then deduce the existence of a unique $g_A\colon F_\Lambda(|A|) \to \mathcal{C}$ such that

$$g_A \circ \pi_{\Lambda,|A|} = f_{\Sigma,*} \circ T_\Sigma(j_A)$$

Notice that, if B is another element of $\mathcal{P}_\kappa(X)$ such that $A \subseteq B$, then

$$\begin{aligned} g_B \circ T_\Lambda(j_{A,B}) \circ \pi_{\Lambda,|A|} &= g_B \circ \pi_{\Lambda,|B|} \circ T_\Sigma(j_{A,B}) \\ &= f_{\Sigma,*} \circ T_\Sigma(j_B) \circ T_\Sigma(j_{A,B}) \\ &= f_{\Sigma,*} \circ T_\Sigma(j_A) \\ &= g_A \circ \pi_{\Lambda,|A|} \end{aligned}$$

showing that $\left(\mathcal{C}, \{g_A\}_{A\in\mathcal{P}_\kappa(X)}\right)$ is a cocone in Σ-**Alg** and entailing the existence of a unique Σ-homomorphism $f_{\Lambda,*}\colon F_\Lambda(X) \to \mathcal{C}$ satisfying $g_A = f_{\Lambda,*} \circ T_\Lambda(j_A)$. Therefore

$$\begin{aligned} f_{\Lambda,*} \circ \pi_{\Lambda,X} \circ T_\Sigma(j_A) &= f_{\Lambda,*} \circ T_\Lambda(j_A) \circ \pi_{\Lambda,|A|} \\ &= g_A \circ \pi_{\Lambda,|A|} \\ &= f_{\Sigma,*} \circ T_\Sigma(j_A) \end{aligned}$$

which shows that $f_{\Lambda,*} = f_{\Sigma,*} \circ \pi_{\Lambda,X}$ and thus $f = f_{\Lambda,*} \circ \eta_{\Lambda,X}$.

For uniqueness, let g be a morphism $F_\Lambda(X) \to \mathcal{C}$ such that $g \circ \eta_{\Lambda,X} = f$, then we must have

$$\begin{aligned} f \circ j_A &= g \circ \eta_{\Lambda,X} \circ j_A \\ &= g \circ \circ \pi_{\Lambda,X} \circ \eta_{\Sigma,X} \circ j_A \\ &= g \circ \pi_{\Lambda,X} \circ T_\Sigma(j_A) \circ \eta_{\Sigma,A} \end{aligned}$$

showing

$$f_{\Sigma,*} \circ T_\Sigma(j_A) = g \circ \pi_{\Lambda,X} \circ T_\Sigma(j_A)$$

from which it follows that

$$f_{\Lambda,*} \circ \pi_{\Lambda,X} = g \circ \pi_{\Lambda,X}$$

We can now conclude since, by Remark 2.2.74, $\pi_{\Lambda,X}$ is an epimorphism. □

Finally, as in the case of Corollary 2.2.52, we can define $T_\Lambda \colon \mathbf{Set} \to \mathbf{Set}$ as the composition $U_\Lambda \circ F_\Lambda$, and deduce from Corollary 2.2.65 the following result.

Corollary 2.2.76. *Let Σ be a κ-bounded signature then, for every Σ-theory Λ, the functor T_Λ has rank κ and*

$$T_\Lambda \simeq \int^{Y \in \mathbf{Set}_\kappa} \mathbf{Set}(Y, -) \times T_\Lambda(Y) \qquad T_\Lambda \simeq \int^{\lambda < \kappa} X^\lambda \times T_\Lambda(\lambda)$$

Proof. This follows from Theorem 2.2.31, Remark 2.2.32, and Corollary 2.2.36. □

2.2.3 Algebraic theories and monads

We have seen in Theorem 2.2.75 that, given a κ-bounded signature Σ and a Σ-theory Λ, the forgetful functor $U_\Lambda \colon \mathbf{Mod}(\Lambda) \to \mathbf{Set}$ has a left adjoint F_Λ. By Proposition 2.1.5 we also known that we can equip $T_\Lambda = U_\Lambda \circ F_\Lambda$ with a monad structure, obtaining $\mathbf{T}_\Lambda := (T_\Lambda, \eta_\Lambda, \mu_\Lambda)$. We are now going to prove that U_Λ is actually a monadic functor, showing $\mathbf{EM}(\mathbf{T}_\Lambda)$ and $\mathbf{Mod}(\Lambda)$ are equivalent.

Remark 2.2.77. By Corollary 2.2.76, we already know that $\mathbf{T}_\Lambda$ has rank κ.

Remark 2.2.78. When Λ is the theory with no axioms, $\mathbf{T}_\Lambda$ is isomorphic, as a monad, to $\mathbf{T}_\Sigma$, where $\mathbf{T}_\Sigma := (T_\Sigma, \eta_\Sigma, \mu_\Sigma)$ is obtained from the adjunction $F_\Sigma \dashv U_\Sigma$.

Let us look more closely at the counit ϵ_Λ of the adjunction $F_\Lambda \dashv U_\Lambda$. Given $\mathcal{A} = \left(A, \{o^{\mathcal{A}}\}_{o \in O_\Sigma}\right)$ in $\mathbf{Mod}(\Lambda)$, the component $\epsilon_{\Lambda,\mathcal{A}}$ is given by $(\mathsf{id}_A)_{\Lambda,*} \colon F_\Lambda(A) \to \mathcal{A}$. This observation, together with Propositions 2.1.5 and 2.1.14, allows us to establish the following two things:

- for every set X, $\mu_{\Lambda,X} \colon T_\Lambda(T_\Lambda(X)) \to T_\Lambda(X)$ is $\left(\mathsf{id}_{T_\Lambda(X)}\right)_{\Lambda,*}$, in particular this also entails that $\mu_{\Lambda,X}$ defines a Σ-homomorphism $F_\Lambda(T_\Lambda(X)) \to F_\Lambda(X)$;
- the comparison functor $K_\Lambda \colon \mathbf{Mod}(\Lambda) \to \mathbf{EM}(\mathbf{T}_\Lambda)$ is defined by

$$\begin{array}{ccc} \mathcal{A} & \longmapsto & (A, (\mathsf{id}_A)_{\Lambda,*}) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \mathcal{B} & \longmapsto & (B, (\mathsf{id}_B)_{\Lambda,*}) \end{array}$$

Our next step is to construct an inverse to K_Λ.

Definition 2.2.79. Let Λ be a Σ-theory, given an Eilenberg-Moore algebra (X, ξ) for $\mathbf{T}_\Lambda$, its *associated Σ-algebra* $H_\Lambda(X,\xi) = \left(X, \left\{o^{H_\Lambda(X,\xi)}\right\}_{o \in O_\Sigma}\right)$ is defined taking as $o^{H_\Lambda(X,\xi)}$ the composition

$$X^{\mathsf{ar}_\Sigma(o)} \xrightarrow{\eta_{\Lambda,X}^{\mathsf{ar}_\Sigma(o)}} (T_\Lambda(X))^{\mathsf{ar}_\Sigma(o)} \xrightarrow{o^{F_\Lambda(X)}} T_\Lambda(X) \xrightarrow{\xi} X$$

In order to extend the construction just defined to a functor $\mathbf{EM}(\mathbf{T}_\Lambda) \to \mathbf{Mod}(\Lambda)$, the first thing that we have to prove is that $H_\Lambda(X, \xi)$ is really a model of Λ. Let us start with a preliminary result.

Proposition 2.2.80. *For every Σ-theory Λ, with $\Sigma \in \mathbf{Sign}_\kappa$, if (X, ξ) is an Eilenberg-Moore algebra for $\mathbf{T}_\Lambda$, then the arrow ξ itself is a Σ-homomorphism $F_\Lambda(X) \to H(X, \xi)$. Moreover, $\xi = (\mathsf{id}_X)_{\Lambda,*}$.*

Proof. The thesis is equivalent to the commutativity of the outside of the diagram:

$$\begin{tikzcd}
X^{\mathsf{ar}_\Sigma(o)} \arrow[r, "\eta_{\Lambda,X}^{\mathsf{ar}_\Sigma(o)}"] & (T_\Lambda(X))^{\mathsf{ar}_\Sigma(o)} \arrow[drr, bend left, "o^{F_\Lambda(X)}"] & & \\
(T_\Lambda(X))^{\mathsf{ar}_\Sigma(o)} \arrow[u, "\xi^{\mathsf{ar}_\Sigma(o)}"] \arrow[r, "\eta_{\Lambda,T_\Lambda(X)}^{\mathsf{ar}_\Sigma(o)}"] \arrow[dr, bend right, "\mathsf{id}_{T_\Lambda(X)}^{\mathsf{ar}_\Sigma(o)}"'] & (T_\Lambda(T_\Lambda(X)))^{\mathsf{ar}_\Sigma(o)} \arrow[u, "(T_\Lambda(\xi))^{\mathsf{ar}_\Sigma(o)}"'] \arrow[r, "o^{F_\Lambda(T_\Lambda(X))}"] \arrow[d, "\mu_{\Lambda,X}^{\mathsf{ar}_\Sigma(o)}"'] & T_\Lambda(T_\Lambda(X)) \arrow[r, "T_\Lambda(\xi)"] \arrow[d, "\mu_{\Lambda,X}"'] & T_\Lambda(X) \arrow[d, "\xi"] \\
& (T_\Lambda(X))^{\mathsf{ar}_\Sigma(o)} \arrow[r, "o^{F_\Lambda(X)}"'] & T_\Lambda(X) \arrow[r, "\xi"'] & X
\end{tikzcd}$$

But this follows at once since we already know that all the internal subdiagrams commute. We get the second half from the identity $\xi \circ \eta_{\Lambda,X} = \mathsf{id}_X$. □

Now we are ready to show that $H_\Lambda(X, \xi)$ is indeed an object of $\mathbf{Mod}(\Lambda)$.

Lemma 2.2.81. *Let Σ be a κ-bounded signature and Λ a theory in it. Then, for every object (X, ξ) of $\mathbf{EM}(\mathbf{T}_\Lambda)$, the Σ-algebra $H_\Lambda(X, \xi)$ is a model of Λ.*

Proof. Let $\lambda \mid t_1 \equiv t_2$ be an equation in Λ and let $f \colon \lambda \to X$ be a function. We can notice that

$$\begin{aligned}
\xi \circ T_\Lambda(f) \circ \pi_{\Lambda,\lambda} \circ \eta_{\Sigma,\lambda} &= \xi \circ T_\Lambda(f) \circ \eta_{\Lambda,\lambda} \\
&= \xi \circ \eta_{\Lambda,X} \circ f \\
&= \mathsf{id}_X \circ f \\
&= f
\end{aligned}$$

By Proposition 2.2.80, ξ is a Σ-homorphism, thus the previous chain of equalities entails that

$$f_{\Sigma,*} = \xi \circ T_\Lambda(f) \circ \pi_{\Lambda,\lambda}$$

We can now conclude since $\pi_{\Lambda,\lambda}(t_1)$ and $\pi_{\Lambda,\lambda}(t_2)$ are equal. □

Consider now a morphism $f\colon (X,\xi_1) \to (Y,\xi_2)$ in $\mathbf{EM}(\mathbf{T}_\Lambda)$, then we have a diagram

$$\begin{tikzcd}
X^{\mathsf{ar}_\Sigma(o)} \arrow[r, "\eta^{\mathsf{ar}_\Sigma(o)}_{\Lambda,X}"] \arrow[d, "f^{\mathsf{ar}_\Sigma(o)}"'] & (T_\Lambda(X))^{\mathsf{ar}_\Sigma(o)} \arrow[r, "o^{F_\Lambda(X)}"] \arrow[d, "(T_\Lambda(f))^{\mathsf{ar}_\Sigma(o)}"'] & T_\Lambda(X) \arrow[r, "\xi_1"] \arrow[d, "T_\Lambda(f)"'] & X \arrow[d, "f"] \\
Y^{\mathsf{ar}_\Sigma(o)} \arrow[r, "\eta^{\mathsf{ar}_\Sigma(o)}_{\Lambda,Y}"'] & (T_\Lambda(Y))^{\mathsf{ar}_\Sigma(o)} \arrow[r, "o^{F_\Lambda(X)}"'] & T_\Lambda(Y) \arrow[r, "\xi_2"'] & Y
\end{tikzcd}$$

which is made by commutative rectangles, thus, f is a Σ-homomorphism $H_\Lambda(X,\xi_1) \to H_\Lambda(Y,\xi_2)$. In particular, this allows usto define a functor $H_\Lambda\colon \mathbf{EM}(\mathbf{T}_\Lambda) \to \mathbf{Mod}(\Lambda)$

$$\begin{tikzcd}
(X,\xi_1) \arrow[r, mapsto] \arrow[d, "f"'] & H(X,\xi_1) \arrow[d, "f"] \\
(Y,\xi_2) \arrow[r, mapsto] & H(Y,\xi_2)
\end{tikzcd}$$

Theorem 2.2.82. *For every object Σ-theory Λ, the previously defined functor $H_\Lambda\colon \mathbf{EM}(\mathbf{T}_\Lambda) \to \mathbf{Mod}(\Lambda)$ is the inverse of the comparison functor $K_\Lambda\colon \mathbf{Mod}(\Lambda) \to \mathbf{EM}(\mathbf{T}_\Lambda)$.*

Proof. H_Λ and K_Λ both act on arrows as the identity, so, if we show that they are one the inverse of the other on objects we get the thesis.

On the one hand, let (X,ξ) be an Eilenberg-Moore algebra for $\mathbf{T}_\Lambda$, by construction

$$K_\Lambda(H_\Lambda(X,\xi)) = (X, (\mathsf{id}_X)_{\Lambda,*})$$

and, by Proposition 2.2.80, $\xi = (\mathsf{id}_X)_{\Lambda,*}$ so that $K_\Lambda \circ H_\Lambda = \mathsf{id}_{\mathbf{EM}(\mathbf{T}_\Lambda)}$.

On the other hand, if $\mathcal{A} = \left(A, \left\{o^{\mathcal{A}}\right\}_{o\in O_\Sigma}\right)$ is a model of Λ, then we have a diagram

$$\begin{tikzcd}
A^{\mathsf{ar}_\Sigma(o)} \arrow[r, "\eta^{\mathsf{ar}_\Sigma(o)}_{\Lambda,A}"] \arrow[dr, bend right, "\mathsf{id}_A^{\mathsf{ar}_\Sigma(o)}"'] & (T_\Lambda(A))^{\mathsf{ar}_\Sigma(o)} \arrow[r, "o^{F_\Lambda(A)}"] \arrow[d, "(\mathsf{id}_A)^{\mathsf{ar}_\Sigma(o)}_{\Lambda,*}"] & T_\Lambda(A) \arrow[d, "(\mathsf{id}_A)_{\Lambda,*}"] \\
& A^{\mathsf{ar}_\Sigma(o)} \arrow[r, "o^{\mathcal{A}}"'] & A
\end{tikzcd}$$

which is commutative since $K_\Lambda(\mathcal{A})$ is an object of $\mathbf{EM}(\mathbf{T}_\Lambda)$ and $(\mathsf{id}_A)_{\Lambda,*}$ is a Σ-homomorphism. In particular this shows that $o^{\mathcal{A}} = o^{H_\Lambda(K_\Lambda(\mathcal{A}))}$, and thus $H_\Lambda \circ K_\Lambda = \mathsf{id}_{\mathbf{Mod}(\Lambda)}$. ☐

Corollary 2.2.83. *Let Σ be a κ-bounded signature and Λ a Σ-theory, then U_Λ is strictly monadic.*

Let $I_\Lambda\colon \mathbf{Mod}(\Lambda) \to \Sigma\text{-}\mathbf{Alg}$ be the inclusion of models of Λ into the category of Σ-algebras. By Corollary 2.2.83 we know that there is a functor $F\colon \mathbf{EM}(\mathbf{T}_\Lambda) \to \mathbf{EM}(\mathbf{T}_\Sigma)$ fitting in the diagram below

$$\begin{tikzcd}
\mathbf{Mod}(\Lambda) \arrow[rr, "I_\Lambda"] \arrow[dr, "U_\Lambda"] \arrow[dd, "K_\Lambda"'] & & \Sigma\text{-}\mathbf{Alg} \arrow[dl, "U_\Sigma"'] \arrow[dd, "K_\Sigma"] \\
& \mathbf{Set} & \\
\mathbf{EM}(\mathbf{T}_\Lambda) \arrow[ur, "U_{\mathbf{T}_\Lambda}"] \arrow[rr, dotted, "F"'] & & \mathbf{EM}(\mathbf{T}_\Sigma) \arrow[ul, "U_{\mathbf{T}_\Sigma}"']
\end{tikzcd}$$

We can also notice that, for every $\mathcal{A} \in \mathbf{Mod}(\Lambda)$, $(\mathsf{id}_A)_{\Lambda,*} \circ \pi_{\Lambda,A}$ is the unique Σ-homomorphism which makes the following diagram commute

$$\begin{tikzcd} & A \arrow[dl, bend right, "\eta_{\Sigma,A}"'] \arrow[d, "\eta_{\Lambda,A}"] \arrow[dr, bend left, "\mathsf{id}_A"] & \\ T_\Sigma(A) \arrow[r, "\pi_{\Lambda,A}"'] & T_\Lambda(A) \arrow[r, "(\mathsf{id}_A)_*"'] & A \end{tikzcd}$$

Applying this argument to $I_\Lambda(H_\Lambda(X,\xi))$, and using Proposition 2.2.80 we get that F is given by

$$\begin{array}{ccc} (X,\xi_1) & \longmapsto & (X,\xi_1 \circ \pi_{\Lambda,X}) \\ f \big\downarrow & & \big\downarrow f \\ (Y,\xi_2) & \longmapsto & (Y,\xi_2 \circ \pi_{\Lambda,Y}) \end{array}$$

If we apply Proposition 2.1.24, the previous observations now yield the following result.

Proposition 2.2.84. *Given $\Sigma \in \mathbf{Sign}_\kappa$ and a Σ-theory Λ, there exists a morphism of monads $\pi_\Lambda \colon \mathbf{T}_\Sigma \to \mathbf{T}_\Lambda$ whose component at X is given by $\pi_{\Lambda,X}$.*

We can now exploit the newly established naturality of π_Λ to prove the following result.

Proposition 2.2.85. *For every set X, $\Sigma \in \mathbf{Sign}_\kappa$ and Σ-theory Λ, the next are equivalent for elements t_1, t_2 of $T_\Lambda(X)$:*

1. *t_1 and t_2 are equal;*
2. *there exist $\mu < \kappa$, s_1, $s_1 \in T_\Sigma(\mu)$ and a function $f \colon \mu \to X$ such that*

$$t_1 = \pi_{\Lambda,X}(T_\Sigma(f)(s_1)) \qquad t_2 = \pi_{\Lambda,X}(T_\Sigma(f)(s_2))$$

and $\mu \mid s_1 \equiv s_2$ belongs to Λ.

Proof. $(1 \Rightarrow 2)$ By Remark 2.2.74, we know that there exists $s'_1, s'_2 \in T_\Sigma(X)$ such that

$$t_1 = \pi_{\Lambda,X}(s'_1) \qquad t_2 = \pi_{\Lambda,X}(s'_2)$$

Using Example 2.2.4 and Corollary 2.2.65 we also know that $\Big(T_\Sigma(X), \{T_\Sigma(j_A)\}_{A \in \mathcal{P}_\kappa(X)}\Big)$ is a colimiting cocone. Thus , by Lemma 2.2.11, there exist $A_1, A_2 \in \mathcal{P}_\kappa(X)$, $p_1 \in T_\Sigma(|A_1|)$, $p_2 \in T_\Sigma(|A_2|)$ such that

$$s'_1 = T_\Sigma(j_{A_1})(p_1) \qquad s'_2 = T_\Sigma(j_{A_2})(p_2)$$

Computing we have

$$\begin{aligned} T_\Lambda(j_{A_1})(\pi_{\Lambda,|A_1|}(p_1)) &= \pi_{\Lambda,X}(T_\Sigma(j_{A_1})(p_1)) & T_\Lambda(j_{A_2})(\pi_{\Lambda,|A_2|}(p_2)) &= \pi_{\Lambda,X}(T_\Sigma(j_{A_2})(p_2)) \\ &= \pi_{\Lambda,X}(s'_1) & &= \pi_{\Lambda,X}(s'_2) \\ &= t_1 & &= t_2 \end{aligned}$$

Using Corollary 2.2.12 we can deduce that there exists $A \in \mathcal{P}_\kappa(X)$ containing A_1 and A_2 such that

$$T_\Lambda(j_{A_1,A})(\pi_{\Lambda,|A_1|}(p_1)) = T_\Lambda(j_{A_2,A})(\pi_{\Lambda,|A_2|}(p_2))$$

But then we also have the chain of identities

$$\begin{aligned}\pi_{\Lambda,|A|}(T_\Sigma(j_{A_1,A}(p_1))) &= T_\Lambda(j_{A_1,A})(\pi_{\Lambda,|A_1|}(p_1)) \\ &= T_\Lambda(j_{A_2,A})(\pi_{\Lambda,|A_2|}(p_2)) \\ &= \pi_{\Lambda,|A|}(T_\Sigma(j_{A_2,A})(p_2))\end{aligned}$$

which, by definition, entails that $|A| \mid T_\Sigma(j_{A_1,A})(p_1)) \equiv T_\Sigma(j_{A_2,A})(p_2)$ is in Λ. Let s_1 and s_2 be, respectively $T_\Sigma(j_{A_1,A})(p_1)$ and $T_\Sigma(j_{A_2,A})(p_2)$ and compute:

$$\begin{aligned}T_\Sigma(j_A)(s_1) &= T_\Sigma(j_A)(T_\Sigma(j_{A_1,A}(p_1))) \\ &= T_\Sigma(j_A \circ j_{A_1,A})(p_1) \\ &= T_\Sigma(j_{A_1})(p_1) \\ &= s_1'\end{aligned} \qquad \begin{aligned}T_\Sigma(j_A)(s_2) &= T_\Sigma(j_A)(T_\Sigma(j_{A_2,A}(p_2))) \\ &= T_\Sigma(j_A \circ j_{A_2,A})(p_2) \\ &= T_\Sigma(j_{A_2})(p_2) \\ &= s_2'\end{aligned}$$

so the thesis follows taking $j_A \colon |A| \to X$ as f.

($2 \Rightarrow 1$) Using naturality and the definition of $\pi_{\Lambda,\mu}$ we get

$$\begin{aligned}t_1 &= \pi_{\Lambda,X}(T_\Sigma(f)(s_1)) \\ &= T_\Lambda(f)(\pi_{\Lambda,\mu}(s_1)) \\ &= T_\Lambda(f)(\pi_{\Lambda,\mu}(s_2)) \\ &= \pi_{\Lambda,X}(T_\Sigma(f)(s_2)) \\ &= t_2\end{aligned}$$

which is precisely our thesis. □

Remark 2.2.86. Examples 2.1.17 and 2.2.18 show that there exist interesting algebraic structures, like complete semilattices, which arise as Eilenberg-Moore algebras that cannot be studied using κ-bounded signatures. On the other hand, it can be shown that other useful algebraic structures like complete lattices and complete boolean algebras do *not* arise as Eilenberg-Moore algebras for any monads on **Set** (see, for instance, [40, 61, 64, 89]). We will not dwell further in the unbounded case.

An adjunction between algebraic theories and monads

Let $\mathbf{T}_\Lambda$ be the monad associated to a Σ-theory Λ. By Corollary 2.2.76 we know that, if Σ is in $\mathbf{Sign}_\kappa$, then $\mathbf{T}_\Lambda$ has rank κ, so that it is an object of **RMnd**. We can wonder if assigning $\mathbf{T}_\Lambda$ to the pair (Σ, Λ) is somehow functorial. To do so, first of all we have to organize algebraic theories into a category.

Definition 2.2.87. The category **ATh** is the category in which

- objects are pairs (Σ, Λ) made by a signature Σ which is κ-bounded for some κ and a Σ-theory Λ;
- arrows between (Σ_1, Λ_1) and (Σ_2, Λ_2) are morphisms of monads $\mathbf{T}_{\Lambda_1} \to \mathbf{T}_{\Lambda_2}$.

We can now easily define the *semantic functor* $\mathsf{Sem} \colon \mathbf{ATh} \to \mathbf{RMnd}$ putting

$$\begin{tikzcd}(\Sigma_1, \Lambda_1) \arrow[r, mapsto] \arrow[d, "\chi"'] & \mathbf{T}_{\Lambda_1} \arrow[d, "\chi"] \\ (\Sigma_2, \Lambda_2) \arrow[r, mapsto] & \mathbf{T}_{\Lambda_2}\end{tikzcd}$$

Our final aim for this chapter is to show that the functor $\mathsf{Sem}\colon \mathbf{ATh} \to \mathbf{RMnd}$ admits a right adjoint $\mathsf{Syn}\colon \mathbf{RMnd} \to \mathbf{ATh}$. This last functor can be thought of as a *syntactic functor*: it assigns to a monad an algebraic theory "axiomatising" its category of Eilenberg-moore algebras.

Definition 2.2.88. Let $\mathbf{T} = (T, \eta, \mu)$ be a monad in **RMnd**, and let also κ be smallest regular cardinal such that $\mathbf{T}$ has rank κ. The *algebraic signature* $\Sigma_{\mathbf{T}}$ *associated to* $\mathbf{T}$ has as set of operations

$$O_{\Sigma_{\mathbf{T}}} := \sum_{\lambda \in \kappa} T(\lambda)$$

and, $\mathsf{ar}_{\Sigma_{\mathbf{T}}}$ is the arrow induced by the constant functions

$$f_\lambda \colon T(\lambda) \to \mathbf{Card} \qquad x \mapsto \lambda$$

Take now a set X, we can endow $T(X)$ with a $\Sigma_{\mathbf{T}}$-algebra structure $L(X)$. Given $t \in T(\lambda)$, there is a corresponding operation $\iota_\lambda(t)$ in $O_{\Sigma_{\mathbf{T}}}$ for which we can define $\iota_\lambda(t)^{L(X)}$ as

$$(\iota_\lambda(t))^{L(X)} \colon T(X)^\lambda \to T(X) \qquad \sigma \mapsto \mu_X(T(\sigma)(t))$$

Since $L(X)$ is a $\Sigma_{\mathbf{T}}$-algebra, we know that there exists the unique dotted $\Sigma_{\mathbf{T}}$-homomorphism $\pi_{\mathbf{T},X} \colon F_{\Sigma_{\mathbf{T}}}(X) \to L(X)$ in the diagram below

$$\begin{tikzcd}
 & X \arrow[dl, "\eta_{\Sigma_{\mathbf{T}},X}"'] \arrow[dr, "\eta_X"] & \\
T_{\Sigma_{\mathbf{T}}}(X) \arrow[rr, dotted, "\pi_{\mathbf{T},X}"'] & & T(X)
\end{tikzcd}$$

Lemma 2.2.89. *Given a monad* $\mathbf{T}$ *of rank* κ*, the following hold true:*

1. *for every set* X*,* μ_X *defines a* $\Sigma_{\mathbf{T}}$*-homomorphism* $L(T(X)) \to L(X)$*;*
2. *for every* $f\colon X \to Y$*,* $T(f)$ *is a* $\Sigma_{\mathbf{T}}$*-homomorphism* $L(X) \to L(Y)$*;*
3. *for every* $f\colon X \to T(Y)$ *be the following diagram commutes*

$$\begin{tikzcd}
T_{\Sigma_{\mathbf{T}}}(X) \arrow[r, "f_{\Sigma_{\mathbf{T}},*}"] \arrow[d, "\pi_{\mathbf{T},X}"'] & T(Y) \\
T(X) \arrow[r, "T(f)"'] & T(T(Y)) \arrow[u, "\mu_Y"']
\end{tikzcd}$$

4. *there exists a natural transformation* $\pi_{\mathbf{T}} \colon T_{\Sigma_{\mathbf{T}}} \to T$ *having* $\pi_{\mathbf{T},X}$ *as component in* X*;*
5. *for every set* X*,* $\pi_{\mathbf{T},X}$ *is surjective.*

Proof. 1. Given $\lambda < \kappa$ and $t \in T(\lambda)$, for every $\sigma\colon \lambda \to T(T(X))$ we compute to get

$$\begin{aligned}
\mu_X\Big((\iota_\lambda(t))^{L(T(X))}(\sigma)\Big) &= \mu_X\big(\mu_{T(X)}(T(\sigma)(t))\big) \\
&= \mu_X(T(\mu_X)(T(\sigma)(t))) \\
&= \mu_X(T(\mu_X \circ \sigma)(t)) \\
&= (\iota_\lambda(t))^{L(X)}(\mu_X \circ \sigma) \\
&= (\iota_\lambda(t))^{L(X)}\big(\mu_X^\lambda(\sigma)\big)
\end{aligned}$$

which is precisely our claim.

2. As before, fix $\lambda < \kappa$ and $t \in T(\Lambda)$, given $\sigma \colon \lambda \to T(X)$ we have

$$
\begin{aligned}
T(f)\Big((\iota_\lambda(t))^{L(X)}(\sigma)\Big) &= T(f)(\mu_X(T(\sigma)(t))) \\
&= \mu_Y(T(T(f))(T(\sigma)(t))) \\
&= \mu_Y(T(T(f)\circ\sigma)(t)) \\
&= (\iota_\lambda(t))^{L(Y)}(T(f)\circ\sigma) \\
&= (\iota_\lambda(t))^{L(Y)}\big(T(f)^\lambda(\sigma)\big)
\end{aligned}
$$

and we can conclude.

3. Let us compute

$$
\begin{aligned}
\mu_Y \circ T(f) \circ \pi_{\mathbf{T},X} \circ \eta_{\Sigma_{\mathbf{T}},X} &= \mu_Y \circ T(f) \circ \eta_X \\
&= \mu_Y \circ \eta_{T(Y)} \circ f \\
&= \mathsf{id}_{T(Y)} \circ f \\
&= f \\
&= f_{\Sigma_{\mathbf{T}}} \circ \eta_{\Sigma_{\mathbf{T}},X}
\end{aligned}
$$

The thesis now follows from the previous two points.

4. Given $f \colon X \to Y$ we have

$$
\begin{aligned}
T(f) \circ \pi_{\mathbf{T},X} \circ \eta_{\mathbf{T},X} &= T(f) \circ \eta_X \\
&= \eta_Y \circ f \\
&= \pi_{\mathbf{T},Y} \circ \eta_{\mathbf{T},Y} \circ f \\
&= \pi_{\mathbf{T},X} \circ T(f) \circ \eta_{\mathbf{T},X}
\end{aligned}
$$

and the thesis now follows because $T(f)$ is a $\Sigma_{\mathbf{T}}$-homomorphism.

5. We know, by Theorem 2.2.31 and Remark 2.2.32, that

$$
T(X) \simeq \int^{Y \in \mathbf{Set}_\kappa} \mathbf{Set}(Y, X) \times T(Y)
$$

In particular, for every $s \in T(X)$, there exists $\lambda < \kappa$, $f \colon \lambda \to X$ and $t \in T(\lambda)$ such that

$$
\begin{aligned}
s &= \omega_{X,\lambda}(f, t) \\
&= T(f)(t)
\end{aligned}
$$

where ω_X is the initial cowedge. Now, take the element $(j_\lambda(t))^{F_{\Sigma_{\mathbf{T}}}(X)}(T_{\Sigma_{\mathbf{T}}}(f) \circ \eta_{\Sigma_{\mathbf{T}},\lambda})$ of $T_{\Sigma_{\mathbf{T}}}(X)$, since $\pi_{\mathbf{T},X}$ is a $\Sigma_{\mathbf{T}}$-homomorphism and using the previous point we have:

$$\begin{aligned}
\pi_{\mathbf{T},X}\Big((\iota_\lambda(t))^{F_{\Sigma_{\mathbf{T}}}(X)}(T_{\Sigma_{\mathbf{T}}}(f)\circ\eta_{\Sigma_{\mathbf{T}},\lambda})\Big) &= (\iota_\lambda(t))^{L(X)}\big(\pi^{\lambda}_{\mathbf{T},X}\,(T_{\Sigma_{\mathbf{T}}}(f)\circ\eta_{\Sigma_{\mathbf{T}},\lambda})\big)\\
&= (\iota_\lambda(t))^{L(X)}(\pi_{\mathbf{T},X}\circ T_{\Sigma_{\mathbf{T}}}(f)\circ\eta_{\Sigma_{\mathbf{T}},\lambda})\\
&= (\iota_\lambda(t))^{L(X)}(T(f)\circ\pi_{\mathbf{T},\lambda}\circ\eta_{\Sigma_{\mathbf{T}},\lambda})\\
&= (\iota_\lambda(t))^{L(X)}(T(f)\circ\eta_\lambda)\\
&= (\iota_\lambda(t))^{L(X)}(\eta_X\circ f)\\
&= \mu_X(T(\eta_X\circ f)(t))\\
&= \mu_X(T(\eta_X)(T(f)(t)))\\
&= (\mu_X\circ T(\eta_X))(T(f)(t))\\
&= \mathsf{id}_{T(X)}(T(f)(t))\\
&= T(f)(t)
\end{aligned}$$

which is what we wished to show. □

Using the natural transformation $\pi_{\mathbf{T}}\colon T_{\Sigma_{\mathbf{T}}}\to T$ we can now define a set $\Lambda_{\mathbf{T}}$ of $\Sigma_{\mathbf{T}}$-equations saying that, for every λ strictily less then the rank of $\mathbf{T}$, $\lambda \mid t_1\equiv t_2$ is in $\Lambda_{\mathbf{T}}$ if and only if

$$\pi_{\mathbf{T},\lambda}(t_1)=\pi_{\mathbf{T},\lambda}(t_2)$$

Proposition 2.2.90. *For every* $\mathbf{T}\in\mathbf{RMnd}$, $\Lambda_{\mathbf{T}}$ *is a* $\Sigma_{\mathbf{T}}$*-theory. Moreover, for every* $X\in\mathbf{Set}$, $L(X)$ *is an object of* $\mathbf{Mod}(\Lambda_{\mathbf{T}})$.

Proof. Closure under rules Refl, Sym and Trans it's obvious. Let us show the other two.

Subst. Suppose that $\lambda_1\mid t_1\equiv t_2$ is in $\Lambda_{\mathbf{T}}$ and take $\sigma\colon\lambda_1\to T_{\Sigma_{\mathbf{T}}}(\lambda_2)$. Since $\pi_{\mathbf{T},\lambda_2}$ is a $\Sigma_{\mathbf{T}}$-homomorphism we must have that

$$(\pi_{\mathbf{T},\lambda_2}\circ\sigma)_{\Sigma_{\mathbf{T}},*}=\pi_{\mathbf{T},\lambda_2}\circ\sigma_{\Sigma_{\mathbf{T}},*}$$

Thus the third point Lemma 2.2.89 yields the diagram

$$\begin{tikzcd}
\lambda_1 \arrow[rr, "\sigma"] \arrow[d, "\eta_{\Sigma_{\mathbf{T}}}"'] & & T_{\Sigma_{\mathbf{T}}}(\lambda_2) \arrow[d, "\pi_{\mathbf{T},\lambda_2}"] \\
T_{\Sigma_{\mathbf{T}}}(\lambda_1) \arrow[urr, "\sigma_{\Sigma_{\mathbf{T}},*}"] \arrow[rr, "(\pi_{\mathbf{T},\lambda_2}\circ\sigma)_{\Sigma_{\mathbf{T}},*}"'] \arrow[d, "\pi_{\mathbf{T},\lambda_1}"'] & & T(\lambda_2) \\
T(\lambda_1) \arrow[rr, "T(\pi_{\mathbf{T},\lambda_2}\circ\sigma)"'] & & T(T(\lambda_2)) \arrow[u, "\mu_{\lambda_2}"']
\end{tikzcd}$$

Therefore we have equalities

$$\begin{aligned}
\pi_{\mathbf{T},\lambda_2}(\sigma_{\Sigma_{\mathbf{T}},*}(t_1)) &= \mu_{\lambda_2}(T(\pi_{\mathbf{T},\lambda_2}\circ\sigma)(\pi_{\mathbf{T},\lambda_1}(t_1)))\\
&= \mu_{\lambda_2}(T(\pi_{\mathbf{T},\lambda_2}\circ\sigma)(\pi_{\mathbf{T},\lambda_1}(t_2)))\\
&= \pi_{\mathbf{T},\lambda_2}(\sigma_{\Sigma_{\mathbf{T}},*}(t_2))
\end{aligned}$$

CONG. Take $t \in T(\lambda_1)$ and $\sigma_1, \sigma_2 \colon \lambda_1 \rightrightarrows T_{\Sigma_{\mathbf{T}}}(\lambda_2)$ and suppose that $\{\lambda \mid \sigma_1(\alpha) \equiv \sigma_2(\alpha)\}_{\alpha\in\lambda}$ are contained in $\Lambda_{\mathbf{T}}$, then

$$\pi_{\mathbf{T},\lambda_2} \circ \sigma_1 = \pi_{\mathbf{T},\lambda_2} \circ \sigma_2$$

and, since $\pi_{\mathbf{T},\lambda_2}$ is a $\Sigma_{\mathbf{T}}$-homomorphism, we get

$$\begin{aligned}
\pi_{\mathbf{T},\lambda_2}(\iota_{\lambda_1}(t)(\sigma_1)) &= \pi_{\mathbf{T},\lambda_2}\Big((\iota_{\lambda_1}(t))^{F_{\Sigma_{\mathbf{T}}}(\lambda_2)}(\sigma_1)\Big)\\
&= (\iota_{\lambda_1}(t))^{L(\lambda_2)}(\pi_{\mathbf{T},\lambda_2}\circ\sigma_1)\\
&= (\iota_{\lambda_1}(t))^{L(\lambda_2)}(\pi_{\mathbf{T},\lambda_2}\circ\sigma_2)\\
&= \pi_{\mathbf{T},\lambda_2}\Big((\iota_{\lambda_1}(t))^{F_{\Sigma_{\mathbf{T}}}(\lambda_2)}(\sigma_2)\Big)\\
&= \pi_{\mathbf{T},\lambda_2}(\iota_{\lambda_1}(t)(\sigma_2))
\end{aligned}$$

Finally, let $\lambda \mid t_1 \equiv t_2$ be an equation in $\Lambda_{\mathbf{T}}$ and $f \colon \lambda \to T(X)$. By point 3 of Lemma 2.2.89 we have

$$f_{\Sigma_{\mathbf{T}},*} = \mu_X \circ T(f) \circ \pi_{\mathbf{T},\lambda}$$

so that

$$\begin{aligned}
f_{\Sigma_{\mathbf{T}},*}(t_1) &= \mu_X(T(f)(\pi_{\mathbf{T},\lambda}(t_1)))\\
&= \mu_X(T(f)(\pi_{\mathbf{T},\lambda}(t_2)))\\
&= f_{\Sigma_{\mathbf{T}},*}(t_2)
\end{aligned}$$

proving the thesis. $\square$

Proposition 2.2.91. *Let* $\mathbf{T}$ *be a monad of rank* κ*, then there exists an isomorphism* $\theta_{\mathbf{T}} \colon \mathbf{T}_{\Lambda_{\mathbf{T}}} \to \mathbf{T}$.

Proof. For every set X, by Proposition 2.2.90 we know that there exists $\theta_{\mathbf{T},X} \colon F_{\Lambda_{\mathbf{T}}}(X) \to L(X)$ such that the triangle below commutes.

$$\begin{tikzcd}
& T_{\Sigma_{\mathbf{T}}}(X) \arrow[dl, "\pi_{\Lambda_{\mathbf{T}},X}"'] \arrow[dr, "\pi_{\mathbf{T},X}"] & \\
T_{\Lambda_{\mathbf{T}}}(X) \arrow[rr, dotted, "\theta_{\mathbf{T},X}"'] & & T(X)
\end{tikzcd}$$

We can immediately notice that this definition gives us a diagram

$$\begin{tikzcd}
& X \arrow[d, "\eta_{\Sigma_{\mathbf{T}},X}"'] \arrow[ddl, bend right, "\eta_{\Lambda_{\mathbf{T}},X}"'] \arrow[ddr, bend left, "\eta_X"] & \\
& T_{\Sigma_{\mathbf{T}}}(X) \arrow[dl, "\pi_{\Lambda_{\mathbf{T}},X}"'] \arrow[dr, "\pi_{\mathbf{T},X}"] & \\
T_{\Lambda_{\mathbf{T}}}(X) \arrow[rr, "\theta_{\mathbf{T},X}"'] & & T(X)
\end{tikzcd}$$

On the other hand, given $f\colon X \to Y$ we have

$$\begin{aligned} T(f) \circ \theta_{\mathbf{T},X} \circ \eta_{\Lambda_\mathbf{T},X} &= T(f) \circ \eta_X \\ &= \eta_Y \circ f \\ &= \theta_{\mathbf{T},Y} \circ \eta_{\Lambda_\mathbf{T},Y} \circ f \\ &= \theta_{\mathbf{T},Y} \circ T_{\Lambda_\mathbf{T},Y}(f) \circ \eta_{\Lambda_\mathbf{T},Y} \end{aligned}$$

which, by the second point of Lemma 2.2.89 gives us the equality

$$T(f) \circ \theta_{\mathbf{T},X} = \theta_{\mathbf{T},Y} \circ T_{\Lambda_\mathbf{T},Y}(f)$$

Summing up we have constructed a natural transformation $\theta_\mathbf{T}\colon T_{\Lambda_\mathbf{T}} \to T$ such that $\eta = \theta_\mathbf{T} \circ \eta_{\Lambda_\mathbf{T}}$. By point 5 of Lemma 2.2.89 we already know that, for every set X, $\theta_{\mathbf{T},X}$ is surjective. To see that it is injective, let $s_1, s_2 \in T_{\Lambda_\mathbf{T}}(X)$ be such that

$$\theta_{\mathbf{T},X}(s_1) = \theta_{\mathbf{T},X}(s_2)$$

Using Lemma 2.2.11, Example 2.2.4, and Corollary 2.2.76 we can deduce that there are $A_1, A_2 \in \mathcal{P}_\kappa(X)$, $p_1 \in T_{\Lambda_\mathbf{T}}(|A_1|)$ and $p_2 \in T_{\Lambda_\mathbf{T}}(|A_2|)$ satisfying

$$s_1 = T_{\Lambda_\mathbf{T}}(j_{A_1})(p_1) \qquad s_2 = T_{\Lambda_\mathbf{T}}(j_{A_2})(p_2)$$

Let A be a set in $\mathcal{P}_\kappa(X)$ containing both A_1 and A_2 and define $q_1, q_2 \in T_{\Lambda_\mathbf{T}}(|A|)$ as, respectively, $T_{\Lambda_\mathbf{T}}(j_{A_1,A})(p_1)$ amd $T_{\Lambda_\mathbf{T}}(j_{A_2,A})(p_2)$. By construction, q_1 and $q_2$are such that

$$\begin{aligned} s_1 &= T_{\Lambda_\mathbf{T}}(j_{A_1})(p_1) & s_2 &= T_{\Lambda_\mathbf{T}}(j_{A_2})(p_2) \\ &= T_{\Lambda_\mathbf{T}}(j_A \circ j_{A_1,A})(p_1) & &= T_{\Lambda_\mathbf{T}}(j_A \circ j_{A_2,A})(p_2) \\ &= T_{\Lambda_\mathbf{T}}(j_A)(T_{\Lambda_\mathbf{T}}(j_{A_1,A})(p_1)) & &= T_{\Lambda_\mathbf{T}}(j_A)(T_{\Lambda_\mathbf{T}}(j_{A_2,A})(p_2)) \\ &= T_{\Lambda_\mathbf{T}}(j_A)(q_1) & &= T_{\Lambda_\mathbf{T}}(j_A)(q_2) \end{aligned}$$

Since, by Remark 2.2.74, each component of the natural transformation $\pi_{\Lambda_\mathbf{T}}$ is surjective, there exist $t_1, t_2 \in T_{\Sigma_\mathbf{T}}(|A|)$ such that $q_1 = \pi_{\Lambda_\mathbf{T},|A|}(t_1)$ and $q_2 = \pi_{\Lambda_\mathbf{T},|A|}(t_2)$. A computation now yields

$$\begin{aligned} T(j_A)(\pi_{\mathbf{T},|A|}(t_1)) &= T(j_A)(\theta_{\mathbf{T},|A|}(\pi_{\Lambda_\mathbf{T},|A|}(t_1))) \\ &= T(j_A)(\theta_{\mathbf{T},|A|}(q_1)) \\ &= \theta_{\mathbf{T},X}(T_{\Lambda_\mathbf{T}}(j_A)(q_1)) \\ &= \theta_{\mathbf{T},X}(s_1) \\ &= \theta_{\mathbf{T},X}(s_2) \\ &= \theta_{\mathbf{T},X}(T_{\Lambda_\mathbf{T}}(j_A)(q_2)) \\ &= T(j_A)(\theta_{\mathbf{T},|A|}(q_2)) \\ &= T(j_A)(\theta_{\mathbf{T},|A|}(\pi_{\Lambda_\mathbf{T},|A|}(t_2))) \\ &= T(j_A)(\pi_{\mathbf{T},|A|}(t_2)) \end{aligned}$$

By hypothesis T has rank κ, thus by Lemma 2.2.11 there is $B \in \mathcal{P}_\kappa(X)$ containing A and such that

$$T(j_{A,B})(\pi_{\mathbf{T},|A|}(t_1)) = T(j_{A,B})(\pi_{\mathbf{T},|A|}(t_2))$$

but $\pi_{\mathbf{T}}$ is a natural transformation, therefore we also have

$$\pi_{\mathbf{T},B}(T_{\Sigma_{\mathbf{T}}}(j_{A,B})(t_1)) = \pi_{\mathbf{T},B}(T_{\Sigma_{\mathbf{T}}}(j_{A,B})(t_2))$$

By definition the previous identity implies that $|B| \mid T_{\Sigma_{\mathbf{T}}}(j_{A,B})(t_1) \equiv T_{\Sigma_{\mathbf{T}}}(j_{A,B})(t_2)$ is in $\Lambda_{\mathbf{T}}$ and

$$\begin{aligned} s_1 &= T_{\Sigma_{\mathbf{T}}}(j_A)(t_1) & s_2 &= T_{\Sigma_{\mathbf{T}}}(j_A)(t_2) \\ &= T_{\Sigma_{\mathbf{T}}}(j_B \circ j_{A,B})(t_1) & &= T_{\Sigma_{\mathbf{T}}}(j_B \circ j_{A,B})(t_2) \\ &= T_{\Sigma_{\mathbf{T}}}(j_B)(T_{\Sigma_{\mathbf{T}}}(j_{A,B})(t_1)) & &= T_{\Sigma_{\mathbf{T}}}(j_B)(T_{\Sigma_{\mathbf{T}}}(j_{A,B})(t_2)) \end{aligned}$$

so we can conclude that $s_1 = s_2$ applying Proposition 2.2.85. By point 1 of Proposition 2.1.11 and by Corollary 2.2.83, U_Σ reflects isomorphisms and so we deduce that $\theta_{\mathbf{T}}$ is a natural isomorphism.

Finally, for every $X \in \mathbf{Set}$, consider the following diagram, which is commutative because, by construction and our previous remarks all the internal subdiagrams commute:

$$\begin{tikzcd}
 & & & T_{\Lambda_{\mathbf{T}}}(X) \arrow[dlll, bend right, "\eta_{\Lambda_{\mathbf{T}},T_{\Lambda_{\mathbf{T}}}(X)}"'] \arrow[d, "\theta_{\mathbf{T},X}"'] \arrow[drrr, bend left, "\eta_{\Lambda_{\mathbf{T}},T_{\Lambda_{\mathbf{T}}}(X)}"] \arrow[ddrrr, "\mathsf{id}_{T_{\Lambda_{\mathbf{T}}}(X)}"] & & & \\
T_{\Lambda_{\mathbf{T}}}(T_{\Lambda_{\mathbf{T}}}(X)) \arrow[d, "T_{\Lambda_{\mathbf{T}}}(\theta_{\mathbf{T},X})"'] & & & T(X) \arrow[dlll, "\eta_{\Lambda_{\mathbf{T}},T(X)}"'] \arrow[ddl, "\eta_{T(X)}"'] \arrow[ddr, "\mathsf{id}_{T(X)}"] & & & T_{\Lambda_{\mathbf{T}}}(T_{\Lambda_{\mathbf{T}}}(X)) \arrow[d, "\mu_{\Lambda_{\mathbf{T}},X}"] \\
T_{\Lambda_{\mathbf{T}}}(T(X)) \arrow[drr, bend right, "\theta_{\mathbf{T},T(X)}"'] & & & & & & T_{\Lambda_{\mathbf{T}}}(X) \arrow[dll, bend left, "\theta_{\mathbf{T},X}"] \\
 & & T(T(X)) \arrow[rr, "\mu_X"'] & & T(X) & &
\end{tikzcd}$$

The commutativity of this whole diagrams yields

$$\mu_X \circ \theta_{\mathbf{T},T(X)} \circ T_{\Lambda_{\mathbf{T}}}(\theta_{\mathbf{T},X}) \circ \eta_{\Lambda_{\mathbf{T}},T_{\Lambda_{\mathbf{T}}}(X)} = \theta_{\mathbf{T},X} \circ \mu_{\Lambda_{\mathbf{T}},X} \circ \eta_{\Lambda_{\mathbf{T}},T_{\Lambda_{\mathbf{T}}}(X)}$$

Now, notice that $\theta_{\mathbf{T},T(X)}$ is a $\Sigma_{\mathbf{T}}$-homomorphism $L(T(X)) \to F_{\Lambda_{\mathbf{T}}}(T(X))$ and $\theta_{\mathbf{T},X}$ is an arrow in $\Sigma_{\mathbf{T}}$-**Alg** between $F_{\Lambda_{\mathbf{T}}}(X)$ and $T(X)$. Points 1 and 2 of Lemma 2.2.89 entail that we also have $\Sigma_{\mathbf{T}}$-homomorphisms $\mu_X \colon L(T(1X)) \to L(X)$ and $T_{\Lambda_{\mathbf{T}}}(\theta_{\mathbf{T},X}) \colon F_{\Lambda_{\mathbf{T}}}(T_{\Lambda_{\mathbf{T}}}(X)) \to F_{\Lambda_{\mathbf{T}}}(T(X))$ and we already observed that $\mu_{\Lambda_{\mathbf{T}},X}$ is an arrow $F_{\Lambda_{\mathbf{T}}}(F_{\Lambda_{\mathbf{T}}}(X)) \to F_{\Lambda_{\mathbf{T}}}(X)$. We can therefore conclude

$$\mu_X \circ \theta_{\mathbf{T},T(X)} \circ T_{\Lambda_{\mathbf{T}}}(\theta_{\mathbf{T},X}) = \theta_{\mathbf{T},X} \circ \mu_{\Lambda_{\mathbf{T}},X}$$

which entails that $\theta_{\mathbf{T}}$ is an isomorphism of monads $\mathbf{T}_{\Lambda_{\mathbf{T}}} \to \mathbf{T}$. $\square$

Corollary 2.2.92. *The functor* $\mathsf{Sem} \colon \mathbf{ATh} \to \mathbf{RMnd}$ *has a right adjojnt* $\mathsf{Syn} \colon \mathbf{RMnd} \to \mathbf{ATh}$.

$$\begin{array}{ccc} \mathbf{T}_1 & \longmapsto & (\Sigma_{\mathbf{T}_1}, \Lambda_{\mathbf{T}_1}) \\ \chi \big\downarrow & & \big\downarrow \theta_{\mathbf{T}_2}^{-1} \circ \chi \circ \theta_{\mathbf{T}_1} \\ \mathbf{T}_2 & \longmapsto & (\Sigma_{\mathbf{T}_2}, \Lambda_{\mathbf{T}_2}) \end{array}$$

Proof. By construction, for every $\mathbf{T}$ in **RMnd** we have an isomorphism $\theta_{\mathbf{T}} \colon \mathbf{T}_{\Lambda_{\mathbf{T}}} \to \mathbf{T}$, so, for every $\chi \colon \mathsf{Sem}(\Sigma, \Lambda) \to \mathbf{T}$, $\theta_{\mathbf{T}}^{-1} \circ \chi$ is the unique morphism $(\Sigma, \Lambda) \to (\Sigma_{\mathbf{T}}, \Lambda_{\mathbf{T}})$ such that

$$\chi = \theta_{\mathbf{T}} \circ \theta_{\mathbf{T}}^{-1} \circ \chi$$

But this proves that $\theta_{\mathbf{T}}$ is the component in $\mathbf{T}$ of the counit of an adjunction $\mathsf{Sem} \dashv \mathsf{Syn}$. $\square$

Fuzzy algebraic theories

CHAPTER 3

Contents

The work of Lawvere [76], has inspired the development of various extensions of Lawvere theory, aiming to connect monads with an increasing number of computational notions [23, 63, 82, 83, 100, 107]. In the previous chapter, a relationship was established between (ranked) monads on **Set** and algebraic theories based on syntactic constructs such as equations. However, Lawvere theories, even enriched ones, are syntax-free. Therefore, a question naturally arises: what kind of syntactic constructs are suitable for describing "algebraic structures" on categories that are different from **Set**?

Recently a framework for *quantitative algebraic reasoning* has been introduced [15, 16, 90, 91]. In its syntax equations are decorated with a rational number, to be interpreted as the distance between the two sides of a given equation. This kind of structures have a natural semantics given by *quantitative algebras*: (extended) metric spaces equipped with operations. Quantitative algebras and quantitative algebraic theories, in turn, are linked, to *metric monads* [112] and a correspondence between such monads and quantitative algebraic theories, similar to the one examined in Chapter 2 can be shown [3, 4].

Along this line of research, in this work we study algebraic reasoning on *fuzzy sets*. Algebraic structures on fuzzy sets are well known since the seventies (see e.g., [8, 92, 98, 111]). Fuzzy sets are very important in computer science, with applications ranging from pattern recognition to decision making, from system modeling to artificial intelligence. So, it is natural to ask if it is possible to use an approach similar to the one above for *fuzzy algebraic reasoning*.

In this chapter we answer this question positively. We propose a sequent calculus based on two kinds of propositions, one expressing equality of terms and the other the existence of a term as a member of a fuzzy set. These sequents have a natural interpretation in categories of fuzzy sets endowed with operations. This calculus is sound and complete for such a semantics: a formula is satisfied by all the models of a given theory if and only if it is derivable from it.

It is possible to go further. Both in the classical and in the quantitative settings there is a notion of free model for a theory; we show that is also true for theories in our formal system for fuzzy sets. In general the category of models of a given theory will not be equivalent to the category of Eilenberg-Moore algebras for the induced monad, but we will show that this equivalence holds for theories with sufficiently simple axioms. Finally we will use the techniques developed in [95] to prove two results analogous to the classical Birkhoff's *HSP theorem* [25].

This chapter is an expanded and revised version of [37].

Synopsis In Section 3.1 we define the category $\mathbf{Fuz}(\mathbf{H})$ of fuzzy sets over a frame $(H, \leq)$ and investigate some of its categorical properties. Section 3.2 introduces syntax and semantics of fuzzy algebraic theories. We will show that the proposed calculus is sound and complete. Moreover, we will show in Section 3.2.2 that if a theory is *basic* then its category of models arose as the category of Eilenberg-Moore algebras for a monad on $\mathbf{Fuz}(\mathbf{H})$. Finally, in Section 3.3 we recall the results of [95] and use them to prove two HSP theorems for our calculus.

3.1 An introduction to fuzzy sets

In this first section we are going to recall the definition and some well-known properties of the category of fuzzy sets over a frame $\mathbf{H}$ [123, 124].

3.1.1 Heyting algebras and frames

To begin, we will review the definitions of Heyting and Boolean algebra and introduce the concept of a frame (i.e. a complete Heyting algebra [28, 47, 64]).

Definition 3.1.1. A bounded lattice $\mathbf{H} := (H, \leq)$ is a *Heyting algebra* if for every element h of H the function $(-) \wedge h\colon (H, \leq) \to (H, \leq)$ has a right adjoint $h \to (-)$, called *implication operator*.

Remark 3.1.2. In particular, for every two elements h, k of a Heyting algebra $(H, \leq)$, the unit of the adjunction $(-) \wedge h \vdash h \to (-)$ yields the inequality

$$(h \to k) \wedge h \leq k$$

Let us prove some properties of implication.

Proposition 3.1.3. *Let* $\mathbf{H} = (H, \leq)$ *be a Heyting algebra, then the following hold true:*

1. *for every* h_1, h_2 *and* k *in* H*, if* $h_1 \leq h_2$ *then* $(h_2 \to k) \leq (h_1 \to k)$*;*
2. *for every* $h, k \in H$*,* $h \to k$ *is the supremum of the set*

$$S_{h,k} := \{x \in H \mid x \wedge h \leq k\}$$

Proof. 1. Using Remark 3.1.2 we have

$$\begin{aligned}(h_2 \to k) \wedge h_1 &\leq (h_2 \to k) \wedge h_2 \\ &\leq k\end{aligned}$$

The thesis follows by adjointness.

2. Let us start noticing that, by adjointness, every $x \in S_{h,k}$ is less or equal than $h \to k$. To conclude it is enough to notice that Remark 3.1.2 entails that $h \to k$ belongs to $S_{h,k}$. □

Definition 3.1.4. Let $\mathbf{H} = (H, \leq)$ be a Heyting algebra. For every element $h \in H$, we define its*negation* $\neg h$ as $h \to \bot$. h is said to be *regular* if $\neg(\neg h) = h$. $(H, \leq)$ is a *boolean algebra* if every $h \in H$ is regular.

Remark 3.1.5. By Remark 3.1.2 we have the following identities

$$\begin{aligned}\neg h \wedge h &= (h \to \bot) \wedge h \\ &\leq \bot\end{aligned}$$

Thus, for every $h \in H$, $\neg h \wedge h = \bot$. In particular we have that

$$\begin{aligned}\neg \top &= \neg \top \wedge \top \\ &= \bot\end{aligned}$$

Remark 3.1.6. Let h and k be elements of a Heyting algebra $(H, \leq)$ such that $h \leq k$. Then point 1 of Proposition 3.1.3 entails $\neg k \leq \neg h$. This means that $\neg$ defines a morphism $(H, \leq) \to (H, \leq)^{op}$, where $(H, \leq)^{op}$ is the set H equipped with the reverse order. Take now $(H, \leq)$ to be boolean, then $\neg \circ \neg = \mathsf{id}_{(H,\leq)}$, and thus $\neg$ is an isomorphism. In particular, in every boolean algebra the following equations hold true for every $h, k \in H$:

$$\neg(h \vee k) = \neg h \wedge \neg k \quad \neg(h \wedge k) = \neg h \vee \neg k \quad \neg h \vee h = \top$$

The previous remark yields at once the following result.

Lemma 3.1.7. *Let $(H, \leq)$ be a boolean algebra, then for every $h, k \in H$ we have*

$$h \to k = k \vee \neg h$$

Proof. We can start noticing that, using Remark 3.1.5 we have

$$\begin{aligned}(k \vee \neg h) \wedge h &= k \vee (\neg h \wedge h) \\ &= k \vee \bot \\ &= k\end{aligned}$$

This shows that $\neg h \vee k$ is less or equal than $h \to k$. For the other inequality, let x be an element of $S_{h,k}$, then, using $Remark$ 3.1.6

$$\begin{aligned}x &= x \wedge \top \\ &= x \wedge (h \vee \neg h) \\ &= (x \wedge h) \vee \neg h \\ &\leq k \vee \neg h\end{aligned}$$

Point 2 of Proposition 3.1.3 gives us the thesis. □

We are now ready to introduce frames.

Definition 3.1.8. A *frame* or **H**, is a complete lattice $(H, \leq)$ such that, for every element $h \in H$ and family $\{h_i\}_{i\in I} \subseteq H$ the following equation hold

$$h \wedge \bigvee_{i\in I} h_i = \bigvee_{i\in I}(h \wedge h_i)$$

The next proposition shows that frames are exactly complete Heyting algebras. This result can be seen as an application of Freyd's Adjoint Functor Theorem [28, 41, 49, 50, 85]. However, we will still present a proof for the sake of completeness.

Proposition 3.1.9 ([28])**.** *Let $(H, \leq)$ be a complete lattice, then the following are equivalent*

1. *$(H, \leq)$ is a frame;*
2. *$(H, \leq)$ is a Heyting algebra.*

Proof. $(1 \Rightarrow 2)$ Given $h, k \in H$, we can consider again the set $S_{h,k}$ of elements x such that $x \wedge h \leq k$. As $h \to k$ we take the supremum of $S_{h,k}$. If $k_1 \leq k_2$ then $S_{h,k_1} \subseteq S_{h,k_2}$ so that we get a monotone function $h \to -\colon (H, \leq) \to (H, \leq)$. Let us show that this function is right adjoint to $- \wedge h$.

- Suppose that $k_1 \wedge h \leq k_2$. Then k_1 belongs to S_{h,k_2}, hence $k_1 \leq h \to k_2$
- Suppose that $k_1 \leq h \to k_2$, then we have

$$\begin{aligned} k_1 \wedge h &\leq (h \to k_2) \wedge h \\ &= h \wedge (h \to k_2) \\ &= h \wedge \bigvee_{x \in S_{h,k_2}} x \\ &= \bigvee_{x \in S_{h,k_2}} (h \wedge x) \\ &\leq k_2 \end{aligned}$$

$(2 \Rightarrow 1)$ This follows from the general fact that left adjoints preserve colimits. □

Example 3.1.10. Let $(L, \leq)$ be a complete linear order, then $(L, \leq)$ it is a frame. Indeed, in a linear order the inequality

$$h \leq \bigvee_{i\in I} h_i$$

holds if and only if $h \leq h_j$ for some $j \in I$. Thus

$$\begin{aligned} h \wedge \bigvee_{i\in I} h_i &= \begin{cases} h & h \leq h_j \text{ for some } j \in I \\ \bigvee_{i\in I} h_i & h_i < h \text{ for every } i \in I \end{cases} \\ &= \bigvee_{i\in I}(h \wedge h_i) \end{aligned}$$

In this case we can describe explicitly $h \to -$. Let $k \in L$, we have two cases.

- $h \leq k$. Then $\top$ belongs to $S_{h,k}$, so that $h \to k = \top$.
- $k < h$. Let $l \in L$, then

$$l \wedge h = \begin{cases} h & h \leq l \\ l & l < h \end{cases}$$

  In particular this means that every $l \in S_{h,k}$ is less or equal than k. Since $h \wedge k \leq k$ we deduce that $h \to k$ must be k.

Summing up we have proved that, in a complete linear order the implication operator is given by

$$h \to - : (L, \leq) \to (L, \leq) \qquad k \mapsto \begin{cases} \top & h \leq k \\ k & k < h \end{cases}$$

Example 3.1.11. Let X be a set. Then $(\mathcal{P}(X), \subseteq)$ is a frame, in which, for every $A \subseteq X$, $\neg A = X \smallsetminus A$. To see this just notice that $S_{A,\emptyset}$ is the set of al subsets which are disjoint from S. In particular, $(\mathcal{P}(X), \subseteq)$ is boolean and $A \to B$ coincides with $(X \smallsetminus A) \cup B$.

Example 3.1.12. Consider again a set X. Then every topology $\Theta \subseteq \mathcal{P}(X)$ is a frame when ordered by the inclusion. Indeed, suprema are given by arbitrary unions, while finite infima coincide with intersection. Moreover, given $U \in \Theta$ and $\{U_i\}_{i \in I} \subseteq \Theta$ we have

$$U \cap \bigcup_{i \in I} U_i = \bigcup_{i \in I} (U \cap U_i)$$

In this setting, for every $U \in \Theta$, $S_{U,\emptyset}$ is the family of opens cointained in $X \smallsetminus U$, so that $\neg U$ is the interior of the complement of U.

3.1.2 Topological functors

Before going into the concept of fuzzy sets we will introduce some classical result about topological functors [5, Ch. 21] which will be useful in the rest of this section.

Definition 3.1.13. Let $U \colon \mathbf{X} \to \mathbf{Y}$ be a functor and I be a class, a *U-structured source* is a (possibly large) family $\{f_i\}_{i \in I}$ of arrows $f_i \colon Y \to U(X_i)$. We say that a U-structured source has an *initial lift* if there exist an object X in $\mathbf{X}$ and arrows $m_i \colon X \to Y_i$ for every $i \in I$, such that:

1. $U(X) = Y$;
2. for every $i \in I$, $U(m_i) = f_i$;
3. given arrows $g \colon U(Z) \to Y$ and $n_i \colon Z \to X_i$ such that, for every $i \in I$, $U(n_i) = f_i \circ g$, there exists a unique $h \colon Z \to X$ such that $U(h) = g$ and $n_i = m_i \circ h$.

$$\begin{tikzcd} Z \arrow[dr, dotted, "h"'] \arrow[drr, bend left, "n_i"] & & \\ & X \arrow[r, "m_i"'] & X_i \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} U(Z) \arrow[dr, "g"'] \arrow[drr, bend left, "U(n_i)"] & & \\ & Y \arrow[r, "f_i"'] & U(X_i) \end{tikzcd}$$

U is a *topological functor* if every U-structured source has an initial lift.

Dually, an *U-structured sink* is a (large) family $\{f_i\}_{i \in I}$ of arrows $f_i \colon U(X_i) \to Y$ and a *final lift* for it is given by an object X in $\mathbf{X}$ and arrows $m_i \colon X_i \to X$, such that:

1. $U(X) = Y$;
2. for every $i \in I$, $U(m_i) = f_i$;
3. given $g \colon Y \to U(Z)$ and $n_i \colon X_i \to Z$ such that, for every $i \in I$, $U(n_i) = g \circ f_i$, there exists a unique $h \colon X \to Z$ such that $U(h) = g$ and $n_i = h \circ m_i$.

$$\begin{tikzcd} X_i \arrow[r, "m_i"] \arrow[drr, bend right, "n_i"'] & X \arrow[dr, dotted, "h"] & \\ & & Z \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} U(X_i) \arrow[r, "f_i"] \arrow[drr, bend right, "U(n_i)"'] & Y \arrow[dr, "g"] & \\ & & U(Z) \end{tikzcd}$$

A functor U is *cotopological* if every U-structures sink admits a final lift.

Example 3.1.14. The paradigmatic example of a topological functor is the forgetful functor from the category of topological spaces to the category of sets.

Remark 3.1.15. If we take $I = \emptyset$ in the previous definition, then a U-structured source (sink) is just an object of $\mathbf{Y}$, and a lift of it is just an object X of $\mathbf{X}$ such that $U(X) = Y$.

Remark 3.1.16. Inital lifts, and thus also final ones, are unique up to isomorphism. Indeed if $\{m_i\}_{i\in I}$ and $\{n_i\}_{\in I}$ ar two lifting for a U-structured source $f_{i_{i\in I}}$ then we have diagrams

$$\begin{tikzcd} X_i \arrow[r, "m_i"] \arrow[drr, bend right, "n_i"'] & X \arrow[dr, dotted, "h_1"] & \\ & & Z \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} U(X_i) \arrow[r, "f_i"] \arrow[drr, bend right, "U(n_i)"'] & Y \arrow[dr, "\mathsf{id}_Y"] & \\ & & Y \end{tikzcd} \quad \leftarrow\!\shortmid \quad \begin{tikzcd} X_i \arrow[r, "n_i"] \arrow[drr, bend right, "m_i"'] & Z \arrow[dr, dotted, "h_2"] & \\ & & X \end{tikzcd}$$

Then $h_2 \circ h_1$ and $h_1 \circ h_2$ are the unique arrows sent by U to id_Y such that all the triangles in the following diagram commute

$$\begin{tikzcd} X_i \arrow[r, "m_i"] \arrow[drr, bend right, "m_i"'] & X \arrow[dr, dotted, "h_2\circ h_1"] & \\ & & X \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} U(X_i) \arrow[r, "f_i"] \arrow[drr, bend right, "U(n_i)"'] & Y \arrow[dr, "\mathsf{id}_Y"] & \\ & & Y \end{tikzcd} \quad \leftarrow\!\shortmid \quad \begin{tikzcd} X_i \arrow[r, "n_i"] \arrow[drr, bend right, "n_i"'] & Z \arrow[dr, dotted, "h_2\circ h_1"] & \\ & & Z \end{tikzcd}$$

and this in turn implies that $h_2 = h_1^{-1}$.

Proposition 3.1.17. *If $U \colon \mathbf{X} \to \mathbf{Y}$ is topological, then it is faithful.*

Proof. Let $f, g \colon X \rightrightarrows V$ be two arrows such that $U(f) = U(g)$, we can define a (constant) U-structured source indexed on the class of arrows of $\mathbf{X}$ simply defining f_h as $U(f) \colon U(X) \to U(V)$ for any arrow h in $\mathbf{X}$. By hypothesis we have an initial lift for this U-structured source, thus we get a class of arrows $m_h \colon W \to V$ which can be used to define another source putting

$$s_h := \begin{cases} f & \mathsf{cod}(h) = W \text{ and } m_h \circ h = g \\ g & \text{otherwise} \end{cases}$$

By construction we have two diagrams:

$$\begin{tikzcd} X \arrow[dr, dotted, "k"'] \arrow[drr, bend left, "s_h"] & & \\ & W \arrow[r, "m_h"'] & V \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} U(X) \arrow[dr, "\mathsf{id}_{U(X)}"'] \arrow[drr, bend left, "U(s_h)"] & & \\ & Y \arrow[r, "U(f)"'] & U(V) \end{tikzcd}$$

so, by initiality, we get the dotted $k\colon X \to W$. In particular this implies that $s_k = m_k \circ k$ and we have two cases:

- if $s_k = f$, then by definition $m_k \circ k = g$ and thus $f = g$;
- if $s_k = g$ then $m_k \circ k = g$, so $s_k = f$ and again we can conclude that $f = g$. □

The following lemma shows that the property of being topological is autodual.

Lemma 3.1.18. *A functor $U\colon \mathbf{X} \to \mathbf{Y}$ is topological if and only if is cotopological.*

Proof. ($\Rightarrow$) Let $\{f_i\}_{i\in I}$ with $f_i\colon U(X_i) \to Y$ be a U-structured sink , we must construct a lift of it. Take H to be the class of all pairs (h, V) such that

- $V \in \mathbf{X}$ and $h\colon Y \to U(V)$;
- for every $i \in I$, there exists $h_i\colon X_i \to V$ such that $h \circ f_i = U(h_i)$.

Putting $g_{(h,V)} := h$ we get a U-source $\left\{g_{(h,V)}\right\}_{(h,V)\in H}$ which, by hypothesis, has an initial lift $\{m_{(h,V)}\}_{(h,V)\in H}$ with $m_{(h,V)}\colon X \to V$, in particular we have $U(X) = Y$. By definition, for every $i \in I$ we have the solid part of the following diagram

$$\begin{tikzcd} X_i \arrow[dr, dotted, "a_i"'] \arrow[drr, bend left, "h_i"] & & \\ & X \arrow[r, "m_{(h,V)}"'] & V \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} U(X_i) \arrow[dr, "f_i"'] \arrow[drr, bend left, "U(h_i)"] & & \\ & Y \arrow[r, "h"'] & U(V) \end{tikzcd}$$

from which we can deduce the existence of the dotted $a_i\colon X_i \to X$, which provides a lift $\{a_i\}_{i\in I}$ for the family $\{f_i\}_{i\in I}$. We are left with finality of such a lift. Suppose that there exists $g\colon Y \to U(Z)$ and for every $i \in I$ an arrow $n_i\colon X_i \to Z$ such that the following triangle commutes

$$\begin{tikzcd} U(X_i) \arrow[r, "f_i"] \arrow[drr, bend right, "U(n_i)"'] & Y \arrow[dr, "g"] & \\ & & U(Z) \end{tikzcd}$$

Then (g, Z) belongs to the family H, so there exists $m_{(g,Z)}\colon X \to Z$ such that $U(m_{(g,Z)}) = g$. By Proposition 3.1.17 we know that such lift of g is unique and so we get the thesis.
($\Leftarrow$) U is cotopological if and only if U^{op} is topological, by the previous point this implies U^{op} is cotopological too, so $U = (U^{op})^{op}$ is topological. □

The existence of a topological functor $U\colon \mathbf{X} \to \mathbf{Y}$ allows us to lift many properties from $\mathbf{Y}$ to $\mathbf{X}$.

Proposition 3.1.19. *Let $U\colon \mathbf{X} \to \mathbf{Y}$ be a topological functor, then the following hold:*

1. *U is a right adjoint;*
2. *U is a left adjoint;*
3. *given a diagram $F\colon \mathbf{D} \to \mathbf{X}$ and a limiting cone $(L, \{l_D\}_{D\in\mathbf{D}})$ for $U \circ F$, then the initial lift $\{m_D\}_{D\in\mathbf{D}}$ of $\{l_D\}_{D\in\mathbf{D}}$ induces a limiting cone $(X, \{m_D\}_{D\in\mathbf{D}})$ for F;*
4. *given a diagram $F\colon \mathbf{D} \to \mathbf{X}$ and a colimiting cocone $(C, \{l_D\}_{D\in\mathbf{D}})$ for $U \circ F$, then the final lift $\{m_D\}_{D\in\mathbf{D}}$ of $\{l_D\}_{D\in\mathbf{D}}$ induces a colimiting cocone $(X, \{m_D\}_{D\in\mathbf{D}})$ for F.*

Proof. 1. For every $Y \in \mathbf{Y}$, let $L(Y)$ be the common domain of a final lift of the empty U-sink with domain X. By definition $U(L(Y)) = Y$ and for every arrow $g\colon Y \to U(Z)$ there is a unique arrow $h\colon L(Y) \to Z$ such that $U(h) = g$, showing that id_X is the unit of an adjunction $L \dashv U$.

2. By Lemma 3.1.18 U^{op} is topological, thus the previous point implies the existence of a functor $L\colon \mathbf{Y}^{op} \to \mathbf{X}^{op}$ which is its left adjoint, therefore L^{op} is a right adjoint for U.

3. Let $f\colon D_1 \to D_2$ be an arrow of $\mathbf{D}$, then

$$\begin{aligned} U(m_{D_2} \circ F(f)) &= U(m_{D_2}) \circ U(F(f)) \\ &= l_{D_2} \circ U(F(f)) \\ &= l_{D_1} \\ &= U(m_{D_1}) \end{aligned}$$

which shows that $(X, \{m_D\}_{D\in\mathbf{D}})$ is a cone for F. Now let $(Z, \{n_D\}_{D\in\mathbf{D}})$ be another cone, then $(U(Z), \{U(n_D)\}_{D\in\mathbf{D}})$ is a cone on $U \circ F$, so there exists a g as in the right-hand triangle of the following diagram

$$\begin{tikzcd} Z \arrow[dr, dotted, "h"'] \arrow[drr, bend left, "n_D"] & & & & U(Z) \arrow[dr, "g"'] \arrow[drr, bend left, "U(n_D)"] & & \\ & X \arrow[r, "m_D"'] & X_D & \mapsto & & Y \arrow[r, "f_D"'] & U(X_D) \end{tikzcd}$$

and, by initiality, we can deduce the existence and uniqueness of the dotted h.

4. This follows from Lemma 3.1.18 and the previous point. □

Corollary 3.1.20. *Given a topological functor $U\colon \mathbf{X} \to \mathbf{Y}$ and an arrow $f\colon X \to Y$ in $\mathbf{X}$, the following facts hold true:*

1. *f is a monomorphism (epimorphism) if and only if $U(f)$ is mono (epi);*
2. *f is a regular monomorphism (regular epimorphism) if and only if $U(f)$ is a regular mono (regular epi) and m is its initial (final) lift.*

Finally, we can show that also factorization systems can be lifted along topological functors.

Definition 3.1.21. Let $U\colon \mathbf{X} \to \mathbf{Y}$ be a topological functor, and suppose that a proper and stable factorization system $(\mathcal{E}, \mathcal{M})$ on $\mathbf{Y}$ is given. We define the following four classes of arrows of $\mathbf{X}$:

$$\begin{aligned} \mathcal{E}_U &:= \{e \in \mathbf{X} \mid U(e) \in \mathcal{E}\} & \mathcal{E}_{fin} &:= \{e \in \mathbf{X} \mid U(e) \in \mathcal{E} \text{ and } e \text{ is its final lift}\} \\ \mathcal{M}_U &:= \{m \in \mathbf{X} \mid U(m) \in \mathcal{M}\} & \mathcal{M}_{in} &:= \{m \in \mathbf{X} \mid U(m) \in \mathcal{M} \text{ and } m \text{ is its initial lift}\} \end{aligned}$$

Lemma 3.1.22. *If $U\colon \mathbf{X} \to \mathbf{Y}$ is a topological functor and $(\mathcal{E}, \mathcal{M})$ is a proper and stable factorization system on $\mathbf{Y}$ then:*

1. *$(\mathcal{E}_U, \mathcal{M}_{in})$ is a proper and stable factorization system on $\mathbf{X}$;*
2. *$(\mathcal{E}_{fin}, \mathcal{M}_U)$ is a proper and stable factorization system on $\mathbf{X}$.*

Proof. 1. Let us show the four points of Definition 2.1.40.

(a) If $f\colon X \to Y$ is an isomorphism in $\mathbf{X}$, then $U(f)$ lies both in $\mathcal{E}$ and $\mathcal{M}$, thus $f \in \mathcal{E}_U$. On the other hand f is also the initial lift of the U-source given by $U(f)$: given a diagram

$$\begin{tikzcd} Z \arrow[dr, dotted, "h"'] \arrow[drr, bend left, "n"] & & \\ & X \arrow[r, "f"'] & Y \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} U(Z) \arrow[dr, "g"'] \arrow[drr, bend left, "U(n)"] & & \\ & Y \arrow[r, "U(f)"'] & U(Y) \end{tikzcd}$$

then we can take $f^{-1} \circ n$ as h.

(b) Closure under composition of $\mathcal{E}_U$ follows at once. Let $f\colon X \to Y$ and $g\colon Y \to Z$ arrows in $\mathcal{M}_{in}$, then $U(g \circ f) \in \mathcal{M}$. For initiality, take the diagram

$$\begin{tikzcd} V \arrow[dr, dotted, "h"'] \arrow[drr, dotted, bend left, "k"] \arrow[drrr, bend left, "n"] & & & \\ & X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} U(V) \arrow[dr, "u"'] \arrow[drr, bend left, "U(f)\circ u"] \arrow[drrr, bend left, "U(n)"] & & & \\ & Y \arrow[r, "U(f)"'] & U(Y) \arrow[r, "U(g)"'] & U(Z) \end{tikzcd}$$

The arrow k comes from the initiality of f, while the arrow h comes from the one of g.

(c) For every arrow $f\colon X \to Y$, there exist $m\colon C \to U(Y)$ in $\mathcal{M}$ and $e\colon U(X) \to C$ in $\mathcal{E}$ such that $U(f) = m \circ e$. Take $n\colon V \to Y$ to be an initial lift of $\{m\}$, then we have a diagram

$$\begin{tikzcd} X \arrow[dr, dotted, "h"'] \arrow[drr, bend left, "f"] & & \\ & V \arrow[r, "n"'] & Y \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} U(X) \arrow[dr, "e"'] \arrow[drr, bend left, "U(f)"] & & \\ & C \arrow[r, "m"'] & U(Y) \end{tikzcd}$$

which, by initiality, entails the existence of the dotted $h\colon X \to V$, belonging to $\mathcal{E}_U$.

(d) For the left lifting property, let us start with the square on the left in the diagram:

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "e"'] & Z \arrow[d, "m"] \\ Y \arrow[r, "f"'] \arrow[ur, dotted, "h"] & V \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} U(X) \arrow[r, "U(g)"] \arrow[d, "U(e)"'] & U(Z) \arrow[d, "U(m)"] \\ U(Y) \arrow[r, "U(f)"'] \arrow[ur, dotted, "k"] & U(V) \end{tikzcd}$$

By hypothesis in the right-hand square $U(m) \in \mathcal{M}$ and $U(e) \in \mathcal{E}$, so the dotted k exists. By the initiality of m we can deduce the existence of a unique $h\colon Y \to Z$ such that $U(h) = k$, moreover

$$U(m \circ k) = U(f) \qquad U(k \circ e) = U(g)$$

thus Proposition 3.1.17 entails

$$m \circ k = f \qquad k \circ e = g$$

Stability follows immediately from Proposition 3.1.19 and the stability of $(\mathcal{E}, \mathcal{M})$.

2. Follows from point 1 and Lemma 3.1.18. □

3.1.3 The category $\mathbf{Fuz}(\mathbf{H})$

We are now ready to introduce the definition of fuzzy sets [123, 124].

Definition 3.1.23. Given a frame $\mathbf{H} = (H, \leq)$, a $\mathbf{H}$*-fuzzy set* (or simply a fuzzy set) is a pair (X, μ_X) consisting in a set X and a *membership degree function* $\mu_X : X \to H$. The *support of* μ_X is the set

$$\mathsf{supp}(X, \mu_X) := \{x \in X \mid \mu_X(x) \neq \bot\}$$

A *morphism of* $\mathbf{H}$*-fuzzy sets* $f : (X, \mu_X) \to (Y, \mu_Y)$ is a function $f : X \to Y$ such that

$$\mu_X(x) \leq \mu_Y(f(x))$$

for every $x \in X$. The resulting category of $\mathbf{H}$-fuzzy sets will be denoted by $\mathbf{Fuz}(\mathbf{H})$.

We have a forgetful functor $V_{\mathbf{H}} : \mathbf{Fuz}(\mathbf{H}) \to \mathbf{Set}$ which simply forgets the membership function. We are going to show that this functor is topological allowing us to recover many informations on $\mathbf{Fuz}(\mathbf{H})$.

Lemma 3.1.24. *The functor* $V_{\mathbf{H}} : \mathbf{Fuz}(\mathbf{H}) \to \mathbf{Set}$ *is topological.*

Proof. Take a $V_{\mathbf{H}}$-source $\{f_i\}_{i \in I}$ with $X \to V_H(X_i, \mu_{X_i})$ and define

$$\mu_X : X \to H \qquad x \mapsto \bigwedge_{i \in I} \mu_{X_i}(f_i(x))$$

Clearly $V_{\mathbf{H}}(X, \mu_X) = X$ and, for every $i \in I$, f_i itself becomes a morphism $(X, \mu_X) \to (X_i, \mu_{X_i})$, let us prove initiality. Given the solid part of the following diagram

$$\begin{tikzcd}
(Z,\mu_Z) \arrow[dr, dotted, "g"'] \arrow[drr, bend left, "n_i"] & & & & Z \arrow[dr, "g"'] \arrow[drr, bend left, "n_i"] & & \\
& (X,\mu) \arrow[r, "f_i"'] & (X_i,\mu_{X_i}) & \mapsto & & Y \arrow[r, "f_i"'] & X_i
\end{tikzcd}$$

it is enough to prove that g itself is a morphism of $\mathbf{Fuz}(\mathbf{H})$. To see this we can compute to get:

$$\begin{aligned} \mu_Z(z) &\leq \mu_{X_i}(n_i(z)) \\ &= \mu_{X_i}(f_i(g(z))) \end{aligned}$$

This now implies that $\mu_Z(z) \leq \mu_X(g(z))$ which is precisely the thesis. $\square$

By Lemma 3.1.18 we already know that V_H is cotopological, for the sake of completeness we will spell out the explicit construction of final lifts.

Proposition 3.1.25. *Let* $\{f_i\}_{i \in I}$ *be a* $V_{\mathbf{H}}$*-structured sink with arrows* $f_i : V_H(X_i, \mu_{X_i}) \to Y$*. For every element* i *of* I*, define a function*

$$\mu_i : Y \to H \qquad y \mapsto \bigvee_{x \in f_i^{-1}(y)} \mu_{X_i}(x)$$

Then a final lift for $\{f_i\}_{i \in I}$ *is given by the collection of arrows* $f_i : (X_i, \mu_{X_i}) \to (Y, \mu_Y)$ *where*

$$\mu_Y : Y \to H \qquad y \mapsto \bigvee_{i \in I} \mu_i(y)$$

Proof. First of all notice that every $f_i\colon X_i \to Y$ becomes a morphism $(X_i, \mu_{X_i}) \to (Y, \mu_Y)$ of $\mathbf{Fuz}(\mathbf{H})$: every $x \in X_i$ is in the preimage of $f_i(x)$, thus we have

$$\begin{aligned} \mu_{X_i}(x) &\leq \mu_i(f_i(x)) \\ &\leq \mu_Y(f_i(x)) \end{aligned}$$

Now let $\{n_i\}_{i\in I}$ be a family of arrows $n_i\colon (X_i, \mu_{X_i}) \to Z$ such that $n_i = g \circ f_i$ for some $g\colon Y \to Z$, we have to show that g defines a morphism of fuzzy sets $(Y, \mu_Y) \to (Z, \mu_Z)$. For every $y \in Y$ and $i \in I$, computing we get

$$\begin{aligned} \mu_i(y) &= \bigvee_{x\in f_i^{-1}(y)} \mu_{X_i}(x) \\ &\leq \bigvee_{x\in f_i^{-1}(y)} \mu_Z(n_i(x)) \\ &= \bigvee_{x\in f_i^{-1}(y)} \mu_Z(g(f_i(x))) \\ &= \bigvee_{x\in f_i^{-1}(y)} \mu_Z(g(y)) \\ &= \mu_Z(g(y)) \end{aligned}$$

□

Now we are ready to exploit the results of the previous section, namely Proposition 3.1.19 and Corollary 3.1.20, paired with Proposition 3.1.25, to get the following results at once.

Corollary 3.1.26. *Given a frame* $\mathbf{H}$*, the following hold true:*

1. *there exist functors* $\Delta_{\mathbf{H}}, \nabla_{\mathbf{H}}\colon \mathbf{Set} \to \mathbf{Fuz}(\mathbf{H})$ *such that* $\nabla_{\mathbf{H}} \dashv V_{\mathbf{H}} \dashv \Delta_{\mathbf{H}}$*, moreover, for every set* $X \neq \emptyset$ *the following equalities hold*

   $$\nabla_{\mathbf{H}}(X) = (X, c_\perp) \qquad \Delta_{\mathbf{H}}(X) = (X, c_\top)$$

   where $c_\perp, c_\top\colon X \rightrightarrows H$ *are the functions constant in* $\perp$ *and* $\top$ *respectively;*
2. *an arrow* $f\colon (X, \mu_X) \to (Y, \mu_Y)$ *is mono (epi) if and only it* $V_H(f)$ *is injective (surjective);*
3. *every diagram* $F\colon \mathbf{D} \to \mathbf{Fuz}(\mathbf{H})$ *has a limiting cone* $((L, \mu_L), \{l_D\}_{D\in\mathbf{D}})$ *where* $(L, \{l_D\}_{D\in\mathbf{D}})$ *is a limiting cone for* $V_H \circ F$ *and*

   $$\mu_L\colon L \to H \qquad x \mapsto \bigwedge_{D\in\mathbf{D}} \mu_{F(D)}(l_D(x))$$

4. *given a diagram* $F\colon \mathbf{D} \to \mathbf{Fuz}(\mathbf{H})$*, if* $(C, \{c_D\}_{D\in\mathbf{D}})$ *is colimiting for* $V_H \circ F$*,* $F(D) = (X_D, \mu_{X_D})$ *and for every* $D \in \mathbf{D}$

   $$\mu_D\colon C \to H \qquad y \mapsto \bigvee_{x\in c_D^{-1}(y)} \mu_{X_D}(x)$$

   then F *has a colimiting cocone* $((C, \mu_C), \{c_D\}_{D\in\mathbf{D}})$ *where*

   $$\mu_C\colon C \to H \qquad y \mapsto \bigvee_{D\in\mathbf{D}} \mu_D(y)$$

5. *an arrow* $f\colon (X,\mu_X)\to(Y,\mu_Y)$ *is a regular mono if and only if* $V_{\mathbf{H}}(f)$ *is injective and*

$$\mu_X(x)=\mu_Y(f(x))$$

for every $x\in X$.

Remark 3.1.27. Let F be a functor $\mathbf{Fuz}(\mathbf{H})\to\mathbf{Fuz}(\mathbf{H})$ and $e\colon (X,\mu_X)\to(Y,\mu_Y)$ be an epimorphism, then $F(e)$ is surjective too. To see this, define $G\colon \mathbf{Set}\to\mathbf{Set}$ as the composition

$$\mathbf{Set}\xrightarrow{\Delta_{\mathbf{H}}}\mathbf{Fuz}(\mathbf{H})\xrightarrow{F}\mathbf{Fuz}(\mathbf{H})\xrightarrow{V_{\mathbf{H}}}\mathbf{Set}$$

and notice that

$$G(V_{\mathbf{H}}(e))=V_{\mathbf{H}}(F(e))$$

By point 2 of the previous lemma $V_{\mathbf{H}}(e)$ is surjective, thus, assuming the axiom of choice, $F(e)$ must be surjective too.

We can use Example 2.2.4 and point 4 of Corollary 3.1.26 to get at once the following results.

Corollary 3.1.28. *Let* (X,μ_X) *be a* $\mathbf{H}$*-fuzzy sets. Then the following hold true:*

1. *for every regular cardinal* κ, $((X,\mu_X),\{i_A\}_{A\in\mathcal{P}_\kappa(X)})$, *is a colimiting cocone for the functor sending* $A\in\mathcal{P}_\kappa(X)$ *to* $(A,\mu_{X|A})$, *and* $A\subseteq B$ *to the inclusion arrow* $i_{A,B}\colon (A,\mu_{X|A})\to(B,\mu_{X|B})$*;*
2. (X,μ_X) *is the coproduct of the family* $\left\{\left(1,\delta_{\mu_X(x)}\right)\right\}_{x\in X}$.

We can also further exploit point 4 of Corollary 3.1.26 specializing it to the case of κ-filtered colimits.

Proposition 3.1.29. *Let* $F\colon \mathbf{D}\to\mathbf{Fuz}(\mathbf{H})$ *be a functor with a* κ*-filtered domain and with colimiting cocone* $((C,\mu_C),\{c_D\}_{D\in\mathbf{D}})$*, then, for every* $x\in V_{\mathbf{H}}(F(D))$ *the following equality holds*

$$\mu_C(c_D(x))=\bigvee_{f\in D/\mathbf{D}}\mu_{X_{\mathrm{cod}(f)}}(F(f)(x))$$

Proof. Let D' be an object of $\mathbf{D}$, and $d\in F(D')$ be an element such that $c_{D'}(d)=c_D(x)$, by Lemma 2.2.11 there exist arrows $g\colon D'\to D''$, $f\colon D\to D''$ in $\mathbf{D}$ such that $F(g)(d)=F(f)(x)$, therefore

$$\begin{aligned}\mu_{X_{D'}}(d)&\le\mu_{X_{D''}}(F(g)(d))\\&=\mu_{X_{D''}}(F(f)(x))\end{aligned}$$

and we can conclude that

$$\begin{aligned}\mu_C(c_D(x))&=\bigvee_{D'\in\mathbf{D}}\mu_{D'}(c_D(x))\\&\le\bigvee_{f\in D/\mathbf{D}}\mu_{X_{\mathrm{cod}(f)}}(F(f)(x))\end{aligned}$$

On the other hand, for every $f\colon D\to D'$ in $\mathbf{D}$ we have $c_{D'}(F(f)(x))=c_D(x)$ so that

$$\mu_{X_{D'}}(F(f)(x))\le\mu_{D'}(c_D(x))$$

from which the other inequality follows. □

In Section 3.3 we will need a description of split epimorphisms which we can easily provide here.

Proposition 3.1.30. *An arrow $f\colon (X,\mu_X)\to(Y,\mu_Y)$ is a split epimorphism if and only if for any $y\in Y$ there exists x_y such that $f(x_y)=y$ and $\mu_Y(y)=\mu_X(x_y)$.*

Proof. ($\Rightarrow$) Let $m\colon (Y,\mu_Y)\to(X,\mu_X)$ be the right inverse of f, then $\mu_Y(y)\le\mu_X(m(y))$ because m is an arrow of **Fuz**(**H**), while

$$\begin{aligned}\mu_X(m(y)) &\le \mu_Y(f(m(y)))\\ &= \mu_Y(y)\end{aligned}$$

($\Leftarrow$) It is enough to define

$$m\colon (Y,\mu_Y)\to(X,\mu_X)\qquad y\mapsto x_y$$

by hypothesis $\mu_Y(y)=\mu_X(m(y))$ and $f\circ m=\mathsf{id}_Y$. □

We can also instantiate Lemma 3.1.22 to get the following

Corollary 3.1.31. *There exists a factorization system $(\mathcal{E},\mathcal{M})$ on* **Fuz**(**H**) *where $\mathcal{E}$ and $\mathcal{M}$ are, respectively, the class of all epimorphisms and the one of all regular monomorphisms.*

Proof. It is enough to notice that the proof of Lemma 3.1.24 entails that a monomorphism $f\colon (X,\mu_X)\to(Y,\mu_Y)$ is the initial lift of $V_H(f)$ if and only if

$$\mu_X(x)=\mu_Y(f(x))$$

for every $x\in X$ and then apply points 2 and 4 of Corollary 3.1.26. □

The next step is showing that **Fuz**(**H**) has a notion of exponentials.

Theorem 3.1.32. *For every frame* **H**, **Fuz**(**H**) *is cartesian closed.*

Proof. We have already proved that **Fuz**(**H**) is complete, so it is enough to show that, for every fuzzy set (X,μ_X), the functor $(-)\times(X,\mu_X)$ has a right adjoint $(-)^{(X,\mu_X)}$. For every $(Y,\mu_Y)\in$ **Fuz**(**H**), we can exploit the implication operator of **H** to define

$$\mu_{Y^X}: Y^X\to H\qquad f\mapsto\bigwedge_{x\in X}(\mu_X(x)\to\mu_Y(f(x)))$$

Take now the evaluation arrow $\mathsf{ev}_{X,Y}: Y^X\times X\to Y$, then for every $f\in Y^X$ and $x'\in X$ we have

$$\begin{aligned}\mu_{Y^X}(f)\wedge\mu_X(x') &= \mu_X(x')\wedge\bigwedge_{x\in X}(\mu_X(x)\to\mu_Y(f(x)))\\ &\le \mu_X(x')\wedge(\mu_X(x')\to\mu_Y(f(x')))\\ &\le \mu_Y(f(x'))\end{aligned}$$

which shows that $\mathsf{ev}_{X,Y}$ is an arrow $(X,\mu_X)\times(Y^X,\mu_{Y^X})\to(Y,\mu_Y)$. Now, take an arrow $g\colon (Z,\mu_Z)\times(X,\mu_X)\to(Y,\mu_Y)$, then we know that, in **Set**, there is a unique $h\colon Z\to Y^X$ such that the diagram

$$\begin{tikzcd} Y^X\times X \arrow[r, "\mathsf{ev}_{X,Y}"] & Y \\ Z\times X \arrow[u, "h\times\mathsf{id}_X"] \arrow[ur, "g"'] & \end{tikzcd}$$

commutes. We also know that, for every $z \in Z$

$$h(z)\colon X \to Y \qquad x \mapsto g(z,x)$$

If we show that h is actually a morphism $(Z, \mu_Z) \to (Y^X, \mu_{Y^X})$ of $\mathbf{Fuz}(\mathbf{H})$ we are done. For every $z \in Z$ we can compute and get

$$\begin{aligned}
\mu_{Y^X}(h(z)) &= \bigwedge_{x \in X} (\mu_X(x) \to \mu_Y(g(z,x))) \\
&\geq \bigwedge_{x \in X} (\mu_X(x) \to (\mu_X(x) \wedge \mu_Z(z))) \\
&= \bigwedge_{x \in X} ((\mu_X(x) \to \mu_X(x)) \wedge (\mu_X(x) \to \mu_Z(z))) \\
&= \bigwedge_{x \in X} (\mu_X(x) \to \mu_Z(z)) \\
&\geq \bigwedge_{x \in X} \mu_Z(z) \\
&= \mu_Z(z)
\end{aligned}$$

so that we conclude. □

Remark 3.1.33. Let us point out two things:

- an element $f \in (Y^X, \mu_{Y^X})$ is a morphism of fuzzy sets if and only if $\mu_{Y^X}(f) = \top$;
- if $(X, \mu_X) = \Delta_{\mathbf{H}}(X)$, then $(Y, \mu_Y)^{(X,\mu_X)}$ is isomorphic to $(Y, \mu_Y)^{|X|}$: to see this it is enough to notice that, for every $f\colon X \to Y$, the following equalities hold:

$$\begin{aligned}
\mu_{Y^X}(f) &= \bigwedge_{x \in X} (\mu_X(x) \to \mu_Y(f(x))) \\
&= \bigwedge_{x \in X} (\top \to \mu_Y(f(x))) \\
&= \bigwedge_{x \in X} \mu_Y(f(x))
\end{aligned}$$

Our next problem is to characterize κ-presentable objects in $\mathbf{Fuz}(\mathbf{H})$. Let us start with the following preliminary result.

Proposition 3.1.34. *Let κ be a regular cardinal, if (X, μ_X) is κ-presentable in $\mathbf{Fuz}(\mathbf{H})$, then $X \in \mathbf{Set}_\kappa$.*

Proof. This is done as in Corollary 2.2.20: by Corollary 3.1.28 we know that $((X, \mu_X), \{i_A\}_{A \in \mathcal{P}_\kappa(X)})$ is a κ-filtered colimit, thus $(\mathbf{Fuz}(\mathbf{H})(X, X), \{i_A \circ (-)\}_{A \in \mathcal{P}_\kappa(X)})$ is again colimiting. Lemma 2.2.11 this implies that $\mathsf{id}_{(X,\mu_X)} = i_A \circ f$ for some $A \in \mathcal{P}_\kappa(X)$ and $f\colon (X, \mu_X) \to (A, \mu_{X|A})$, showing $|X| < \kappa$. □

The following example shows that the converse does not hold.

Example 3.1.35. Let $\mathbf{H}$ be $([0,1], \leq)$, for every $i \in \mathbb{N}$ we can consider $(1, \delta_{h_i})$, where

$$h_i := 1 - \frac{1}{i+1}$$

If $i \leq j$, then id_1 defines a morphism $(1, \delta_{h_i}) \to \left(1, \delta_{h_j}\right)$, thus we get a $\aleph_0$-filtered diagram in $\mathbf{Fuz}(\mathbf{H})$, which has a colimiting cocone $((1, \delta_\top), \{a_i\}_{i\in\mathbb{N}})$, where $a_i \colon (1, \delta_{h_i}) \to (1, \delta_\top)$ is simply the identity. Take now $\mathsf{id}_{(1,\delta_\top)} \colon (1, c_\top) \to (1, c_\top)$, it does not factor through any of the a_i, thus $(1, c_\top)$ is not $\aleph_0$-presentable.

Lemma 3.1.36. *Let κ be a regular cardinal then the following are equivalent for an object (X, μ_X) of $\mathbf{Fuz}(\mathbf{H})$:*

1. *(X, μ_X) is κ-presentable;*
2. *$|X| < \kappa$ and $\mu_X(x)$ is κ-compact for every $x \in X$.*

Proof. $(1 \Rightarrow 2)$ Half of the thesis follows from Proposition 3.1.34. For the other half, fix $x_0 \in X$ and suppose that $\mu_X(x_0) \leq s_0$, where s_0 is the supremum of a κ-directed family $S \subseteq H$. For every $s \in S$ we can define a fuzzy set (X, μ_s) putting

$$\mu_s \colon X \to H \qquad x \mapsto \begin{cases} \mu_X(x) & x \neq x_0 \\ s & x = x_0 \end{cases}$$

If $s \leq t$, then id_X defines an arrow $(X, \mu_s) \to (X, \mu_t)$, thus we have a diagram in $\mathbf{Fuz}(\mathbf{H})$ whose colimit is, by Corollary 3.1.26, given by $((X, \mu_S), \{b_s\}_{s\in S})$ where $b_s = \mathsf{id}_X$ and

$$\mu_S \colon X \to H \qquad x \mapsto \begin{cases} \mu_X(x) & x \neq x_0 \\ s_0 & x = x_0 \end{cases}$$

Now, since $\mu_X(x_0) \leq s_0$, id_X defines an arrow $(X, \mu_X) \to (X, \mu_S)$, since (X, μ_X) is κ-presentable there must exists $s' \in S$ such that id_X factors through $(X, \mu_{s'})$, showing that $\mu_X(x_0) \leq s'$.

$(2 \Rightarrow 1)$ Let h be an element of H, with a corresponding $\delta_h \colon 1 \to H$. By Proposition 2.2.19 and the second point of Corollary 3.1.28 it is enough to show that $(1, \delta_h)$ is κ-presentable whenever h is κ-compact. Let $((A, \mu_A), \{a_D\}_{D\in\mathbf{D}})$ be a colimiting cocone for a functor $F \colon \mathbf{D} \to \mathbf{Fuz}(\mathbf{H})$ with κ-filtered domain, we are going to show that $(\mathbf{Fuz}(\mathbf{H})\left((1, \delta_h), (A, \mu_A)\right), \{a_D \circ (-)\}_{D\in\mathbf{D}})$ satisfies both points of Corollary 2.2.12.

1. Take a morphism $g \colon (1, \delta_h) \to (A, \mu_A)$, and let $x \in A$ be the image of $\emptyset$ through it. By definition of morphism $h \leq \mu_A(x)$, on the other hand Proposition 3.1.29 entails that

$$\mu_A(x) = \bigvee_{f\in D/\mathbf{D}} \mu_{X_{\mathsf{cod}(f)}}(F(f)(y))$$

for some $D \in \mathbf{D}$ and $y \in F(D)$ such that $a_D(y) = x$. The family $\left\{\mu_{X_{\mathsf{cod}(f)}}(F(f)(y))\right\}_{f\in D/\mathbf{D}}$ is κ-filtered: take a subfamily $\{\mu_{X_{\mathsf{cod}(f_i)}}(F(f_i)(y))\}_{i\in I}$ for some I with cardinality strictly less than κ. Then by Lemma 2.2.6 there exists a cocone on the source $\{f_i\}_{i\in I}$, that is arrows $b \colon D \to D'$ and $b_i \colon \mathsf{cod}(f_i) \to D'$ such that the following diagram commutes for every $i \in I$

$$\begin{tikzcd} & \mathsf{cod} f_i \arrow[dr, "b_i"] & \\ D \arrow[ur, "f_i"] \arrow[rr, "b"'] & & D' \end{tikzcd}$$

and this, in particular, entails that $\mu_{X_{D'}}(F(b)(y))$ is an upper bound for $\{\mu_{X_{\mathsf{cod}(f_i)}}(F(f_i)(y))\}_{i\in I}$. By hypothesis h is κ-compact, thus there exists $f \in D/\mathbf{D}$ such that

$$h \leq \mu_{X_{\mathsf{cod}(f)}}(F(f)(y))$$

and thus we have

$$f' \colon (1, \delta_h) \to F(\mathsf{cod}(f)) \qquad \emptyset \mapsto F(f)(y)$$

Since $a_{\mathsf{cod}(f)} \circ F(f) = a_D$ the thesis follows.

2. Let $f \colon (1, \delta_h) \to F(D_1)$ and $g \colon (1, \delta_h) \to F(D_2)$ be arrows such that

$$a_{D_1} \circ f = a_{D_2} \circ g$$

Since $(A, \{a_D\}_{D \in \mathbf{D}})$ is colimiting for $V_{\mathbf{H}} \circ F$, Corollary 2.2.12 entails that there exist $g_1 \colon D_1 \to D_3$ and $g_2 \colon D_2 \to D_3$ such that

$$F(g_1)(f(\emptyset)) = F(g_2)(g(\emptyset))$$

but this is exactly the thesis. □

We are now ready to prove the following theorem.

Theorem 3.1.37. *Let κ be a regular cardinal, and $\mathbf{H}$ be a frame, then $\mathbf{Fuz}(\mathbf{H})$ is locally κ-presentable if and only if $\mathbf{H}$ is a κ-algebraic lattice.*

Proof. ($\Rightarrow$) Let h be an element of H, by Lemma 2.2.30 $((1, \delta_h), \{c_D\}_{D \in \mathbf{D}})$ is the colimiting cocone on some functor $F \colon \mathbf{D} \to \mathbf{Fuz}(\mathbf{H})$ such that $F(D) = (X_D, \mu_{X_D})$ is κ-presentable for every $D \in \mathbf{D}$. By Lemma 3.1.36 this means that $|X_D| < \kappa$ and $\mu_{X_D}(x)$ is κ-compact for every $x \in X_D$. Define

$$s_D := \bigvee_{x \in X_D} \mu_{X_D}(x)$$

By Proposition 2.2.19 each s_D is κ-compact and by Corollary 3.1.26

$$h = \bigvee_{D \in \mathbf{D}} s_D$$

($\Leftarrow$) Let $\mathbf{H}_\kappa$ be the set of κ-compact elements of $\mathbf{H}$, and define

$$\mathcal{G} := \{(1, \delta_h) \in \mathbf{Fuz}(\mathbf{H}) \mid h \in \mathbf{H}_\kappa\}$$

By Lemma 3.1.36 every element of $\mathcal{G}$ is κ-presentable, let us show that $\mathcal{G}$ is a strong generator.

- $\mathcal{G}$ is a generator. Given two arrows $f, g \colon (X, \mu_X) \rightrightarrows (Y, \mu_Y)$ such that $f \neq g$, there exists $x \in X$ such that $f(x) \neq g(x)$ and thus $\delta_x \colon (1, c_\perp) \to (X, \mu_X)$ is such that such that

$$f \circ \delta_x \neq g \circ \delta_x$$

  The thesis follows since $\perp$ is κ-compact for every regular cardinal κ.
- $\mathcal{G}$ is strong. Let $f \colon (M, \mu_M) \to (X, \mu_X)$ be a monomorphism which is not an isomorphism, by Corollary 3.1.26 there exists $x \in X \smallsetminus f(M)$, and, by hypothesis, there exists $h \in \mathbf{H}_\kappa$ such that $h \leq \mu_X(x)$, then the morphism $\delta_x \colon (1, \delta_h) \to (X, \mu_X)$ does not factor through f. □

Remark 3.1.38. As shown by the previous theorem, $\mathbf{Fuz}(\mathbf{H})$ is locally κ-presentable category only in the case in which $\mathbf{H}$ is κ-algebraic. Nonetheless, we can still express a fuzzy set over any frame $\mathbf{H}$ as a κ-filtered colimit of the family of its subobjects of cardinality less then κ. Indeed, the first point of Corollary 3.1.28 shows that every (X, μ_X) is the colimit of the functor $D_{\kappa,(X,\mu_X)}$ assigning to each $A \in \mathcal{P}_\kappa(X)$ the fuzzy set $\left(A, \mu_{X|A}\right)$, where $\mu_{X|A}$ is the restriction of μ_X to A, and to each inclusion $A \subseteq B$ the corresponding arrow $i_{A,B} \colon \left(A, \mu_{X|A}\right) \to \left(B, \mu_{X|b}\right)$.

For every regular cardinal κ, let $\mathbf{Fuz}_k(\mathbf{H})$ be the category of fuzzy sets whose underlying set has cardinality strictly less than κ. Let also $J_\kappa \colon \mathbf{Fuz}_\kappa(\mathbf{H}) \to \mathbf{Fuz}(\mathbf{H})$ be the inclusion functor, Remark 3.1.38 allows us to prove an analog of Theorem 2.2.31.

Theorem 3.1.39. *For every regular cardinal κ and for every functor $F \colon \mathbf{Fuz}(\mathbf{H}) \to \mathbf{Fuz}(\mathbf{H})$, the following are equivalent:*

1. *for every object (X, μ_X), the cocone $\big(F(X, \mu_X), \{F(i_A)\}_{A \in \mathcal{P}_\kappa(X)}\big)$ is colimiting for $F \circ D_{\kappa,(X,\mu_X)}$;*
2. *$(F, \mathsf{id}_{F \circ J_\kappa})$ is a left Kan extension of $F \circ J_\kappa$ along J_κ;*
3. *the following isomorphism hold*

$$F \simeq \int^{(Y,\mu_Y) \in \mathbf{Fuz}_\kappa(\mathbf{H})} \mathbf{Fuz}(\mathbf{H})((Y, \mu_Y), -) \bullet F(Y, \mu_Y)$$

Proof. $(1 \Rightarrow 2)$ Let us show that $(F, \mathsf{id}_{F \circ J_k})$ enjoys the universal property of a left Kan extension. Take a functor $G \colon \mathbf{Fuz}(\mathbf{H}) \to \mathbf{Fuz}(\mathbf{H})$ a natural transformation $\eta \colon F \circ J_\kappa \to G \circ J_\kappa$, we need to construct a $\overline{\eta} \colon F \to G$ such that $\overline{\eta}_{(Y,\mu_Y)} = \eta_{(Y,\mu_Y)}$ for every $(Y, \mu_Y) \in \mathbf{Fuz}_\kappa(\mathbf{H})$.

Take another fuzzy set (X, μ_X), given $A, B \in \mathcal{P}_\kappa(X)$ such that $A \subseteq B$. Then

$$\begin{aligned} G(i_B) \circ \eta_{(B,\mu_B)} \circ F(i_{A,B}) &= G(i_B) \circ G(i_{A,B}) \circ \eta_{(A,\mu_A)} \\ &= G(i_B \circ i_{A,B}) \circ \eta_{(A,\mu_A)} \\ &= G(i_A) \circ \eta_{(A,\mu_A)} \end{aligned}$$

Therefore we have a cocone $\big(G(X, \mu_X), \{G(i_A) \circ \eta_{(A,\mu_A)}\}_{A \in \mathcal{P}_\kappa(X)}\big)$ which, by hypothesis, entails the existence of a unique $\overline{\eta}_{(X,\mu_X)}$ fitting in the diagram

$$\begin{tikzcd} F\Big(A, \mu_{X|A}\Big) \arrow[r, "\eta_{(A,\mu_{X|A})}"] \arrow[d, "F(i_A)"'] & G\Big(A, \mu_{X|A}\Big) \arrow[d, "G(i_A)"] \\ F(X, \mu_X) \arrow[r, dotted, "\overline{\eta}_{(X,\mu_X)}"'] & G(X, \mu_X) \end{tikzcd}$$

By construction, if $|X| < \kappa$ then $\bar{\eta}_{(X,\mu_X)} = \eta_{(X,\mu_X)}$, so we only have to show the naturality of the family $\{\overline{\eta}_{(X,\mu_X)}\}_{(X,\mu_X) \in \mathbf{Fuz}(\mathbf{H})}$. Now, notice that for every morphism $f \colon (X, \mu_X) \to (Y, \mu_Y)$ and $A \in \mathcal{P}_\kappa(X)$, we have $f(A)$ in $\mathcal{P}_\kappa(Y)$ and, for every $x \in A$:

$$\begin{aligned} \mu_{X|A}(x) &= \mu_X(x) \\ &\leq \mu_Y(f(x)) \\ &\leq \mu_{Y|f(A)}(f(x)) \end{aligned}$$

Hence, restricting and corestricting f we get a morphism $f' \colon \Big(A, \mu_{X|A}\Big) \to \Big(f(A), \mu_{Y|f(A)}\Big)$ which makes the following square commutative

$$\begin{tikzcd} \Big(A, \mu_{X|A}\Big) \arrow[r, dotted, "f'"] \arrow[d, "i_A"'] & \Big(f(A), \mu_{Y|f(A)}\Big) \arrow[d, "i_{f(A)}"] \\ (X, \mu_X) \arrow[r, "f"'] & (Y, \mu_Y) \end{tikzcd}$$

Applying F, this square in turn yields a bigger diagram

$$
\begin{tikzcd}
 & & F\Big(A,\mu_{X|A}\Big) \arrow[dll, bend right, "F(i_A)"'] \arrow[dl, "F(f')"] \arrow[dr, "\eta_{(A,\mu_{X|A})}"] \arrow[drr, bend left, "F(i_A)"] & & \\
F(X,\mu_X) \arrow[d, "F(f)"'] & F\Big(f(A),\mu_{Y|f(A)}\Big) \arrow[dl, "F(i_{f(A)})"] \arrow[dr, "\eta_{(f(A),\mu_{Y|f(A)})}"] & & G\Big(A,\mu_{X|A}\Big) \arrow[dl, "G(f')"'] \arrow[dr, "G(i_A)"'] & F(X,\mu_X) \arrow[d, "\overline{\eta}_{(X,\mu_X)}"] \\
F(Y,\mu_Y) \arrow[drr, bend right, "\overline{\eta}_{(Y,\mu_Y)}"'] & & G\Big(f(A),\mu_{Y|f(A)}\Big) \arrow[d, "G(i_{f(A)})"] & & G(X,\mu_X) \arrow[dll, bend left, "G(f)"] \\
 & & G(Y,\mu_Y) & &
\end{tikzcd}
$$

which, since $\big(F(X,\mu_X), \{F(i_A)\}_{A\in\mathcal{P}_\kappa(X)}\big)$ is colimiting, shows that

$$G(f)\circ\overline{\eta}_{(X,\mu_X)} = \overline{\eta}_{(Y,\mu_Y)}\circ F(f)$$

We are left with uniqueness. If $\epsilon\colon F\to G$ is a natural transformation such that $\epsilon_{(Y,\mu_Y)} = \eta_{(Y,\mu_Y)}$ for every $(Y,\mu_Y)\in \mathbf{Fuz}_\kappa(\mathbf{H})$, then, for every $A\in\mathcal{P}_\kappa(X)$ we have

$$
\begin{aligned}
\epsilon_{(X,\mu_X)}\circ F(i_A) &= G(i_A)\circ \epsilon_{(A,\mu_{X|A})}\\
&= G(i_A)\circ \eta_{(A,\mu_{X|A})}\\
&= \overline{\eta}_{(X,\mu_X)}\circ F(i_A)
\end{aligned}
$$

from which the thesis follows using again the colomiting property of $\big(F(X,\mu_X), \{F(i_A)\}_{A\in\mathcal{P}_\kappa(X)}\big)$.

$(2\Rightarrow 3)$ This follows from the formula for left Kan extensions.

$(3\Rightarrow 1)$ As in the proof of Theorem 2.2.31, since $(-)\bullet F(Y,\mu_Y)$ is a left adjoint, it is enough to show that $\big(\mathbf{Fuz}(\mathbf{H})((Y,\mu_Y),(X,\mu_X)), \{i_A\circ(-)\}_{A\in\mathcal{P}_\kappa(X)}\big)$ is colimiting for $\mathbf{Fuz}(\mathbf{H})((Y,\mu_Y),-)\circ D_{\kappa,(X,\mu_X)}$ whenever $|Y|<\kappa$. To see this, let $\Big(C, \{f_A\}_{A\in\mathcal{P}_\kappa(X)}\Big)$ be a cocone. Notice that for every $g\colon (Y,\mu_Y)\to (X,\mu_X)$, $g(Y)$ belongs to $\mathcal{P}_\kappa(X)$ and there exists a unique $g'\colon (Y,\mu_Y)\to \Big(g(Y),\mu_{X|g(Y)}\Big)$ such that $g = i_{g(Y)}\circ g'$, so that we can define

$$f\colon \mathbf{Fuz}(\mathbf{H})((Y,\mu_Y),(X,\mu_X))\to C \qquad g\mapsto f_{g(Y)}(g')$$

By construction, for every $h\colon (Y,\mu_Y)\to \Big(A,\mu_{X|A}\Big)$ we have a unique arrow $(Y,\mu_Y)\to\Big(h(Y),\mu_{X|h(Y)}\Big)$

as in the diagram below

$$\begin{tikzcd}
(Y,\mu_Y) \arrow[r, "h"] \arrow[d, dotted, "h'"'] & \left(A,\mu_{X|A}\right) \arrow[d, "i_A"] \\
\left(h(Y),\mu_{X|h(Y)}\right) \arrow[ur, "i_{h(Y),A}"] \arrow[r, "i_{h(Y)}"'] & (X,\mu_X)
\end{tikzcd}$$

and therefore

$$\begin{aligned} f(i_A \circ h) &= f_{h(Y)}(h') \\ &= f_A(i_{h(Y),A} \circ h') \\ &= f_A(h) \end{aligned}$$

If k is another function $\mathbf{Fuz}(\mathbf{H})((Y,\mu_Y),(X,\mu_X)) \to C$ such that $f_A = k(i_A \circ (-))$ for every $A \in \mathcal{P}_\kappa(X)$, then, since every $g\colon (Y,\mu_Y) \to (X,\mu_X)$ is equal to $i_{g(Y)} \circ g'$, we have

$$k(g) = f_{g(Y)}(g')$$

showing uniqueness of f and the thesis. □

On the rank of exponentials

The previous results settle the questions of computing the rank of the functor $\mathbf{Fuz}(\mathbf{H})((X,\mu_X),-)$, and of locally κ-presentability of $\mathbf{Fuz}(\mathbf{H})$. We can also wonder if there is a way to compute the rank of $(-)^{(X,\mu_X)}$. The situation is less clear but we can still provide some positive result.

Proposition 3.1.40. *Let $\mathbf{H}$ be a frame, and (X,μ_X) an object of $\mathbf{Fuz}(\mathbf{H})$. Then the following hold true:*

1. *if $(-)^{(X,\mu_X)}$ has rank κ then $|X| < \kappa$;*
2. *suppose that $|X| < \kappa$, given a functor $F\colon \mathbf{D} \to \mathbf{Fuz}(\mathbf{H})$ with a κ-filtered domain, a colimiting cocone $((C,\mu_C),\{c_D\}_{D\in\mathbf{D}})$ for it and putting $F(D) = (X_D,\mu_{X_D})$, then $\left((C,\mu_C)^{(X,\mu_X)},\left\{c_D^{(X,\mu_X)}\right\}_{D\in\mathbf{D}}\right)$ is colimiting for $(-)^{(X,\mu_X)} \circ F$ if and only if, for every $f\colon C \to X_D$ the following equality holds*

$$\bigvee_{g\in D/\mathbf{D}} \left(\bigwedge_{x\in X} (\mu_X(x) \to \mu_{X_D}(F(g)(f(x))))\right) = \bigwedge_{x\in X}\left(\mu_X(x) \to \bigvee_{g\in D/\mathbf{D}} \mu_{X_D}(F(g)(f(x)))\right)$$

Proof. 1. We have a commutative diagram

$$\begin{tikzcd}
\mathbf{Fuz}(\mathbf{H}) \arrow[r, "(-)^{(X,\mu_X)}"] & \mathbf{Fuz}(\mathbf{H}) \arrow[d, "V_{\mathbf{H}}"] \\
\mathbf{Set} \arrow[u, "\nabla_{\mathbf{H}}"] \arrow[r, "\mathbf{Set}(X,-)"'] & \mathbf{Set}
\end{tikzcd}$$

which, by hypothesis, implies that $\mathbf{Set}(X,-)$ has rank κ, so Corollary 2.2.20 yields the thesis.

2. We already know that $V_{\mathbf{H}}\left((C,\mu_C)^{(X,\mu_X)}\right) = C^X$, thus, by Corollary 2.2.20, $\left(C^X, \{c_D^X\}_{D\in\mathbf{D}}\right)$ is colimiting and the thesis follows from point 4 of Corollary 3.1.26 and from Proposition 3.1.29. □

Corollary 3.1.41. *Let X be a finite set, then:*

1. *$(-)^{\Delta_{\mathbf{H}}(X)}$ has rank $\aleph_0$;*
2. *if $\mathbf{H}$ is boolean, then $(-)^{(X,\mu_X)}$ has rank $\aleph_0$.*

Proof. 1. The equality of Proposition 3.1.40 becomes

$$\bigvee_{g\in D/\mathbf{D}}\left(\bigwedge_{x\in X}\mu_{X_D}\left(F(g)(f(x))\right)\right) = \bigwedge_{x\in X}\left(\bigvee_{g\in D/\mathbf{D}}\mu_{X_D}\left(F(g)(f(x))\right)\right)$$

which holds by the cartesian closedness of $\mathbf{H}$.

2. Let $\{h_i\}_{i\in I}$ be a family of elements of H and h another element of it. Since $\mathbf{H}$ is boolean, we can use Lemma 3.1.7 to get

$$\begin{aligned} h\to\bigvee_{i\in I}h_i &= \neg h\vee\bigvee h_i \\ &= \bigvee_{i\in I}(\neg h\vee h_i) \\ &= \bigvee_{i\in I}(h\to h_i) \end{aligned}$$

We can apply this equality with cartesian closedness to the setting of Proposition 3.1.40:

$$\begin{aligned} \bigwedge_{x\in X}\left(\mu_X(x)\to\bigvee_{g\in D/\mathbf{D}}\mu_{X_D}\left(F(g)(f(x))\right)\right) &= \bigwedge_{x\in X}\left(\bigvee_{g\in D/\mathbf{D}}\left(\mu_X(x)\to\mu_{X_D}\left(F(g)(f(x))\right)\right)\right) \\ &= \bigvee_{g\in D/\mathbf{D}}\left(\bigwedge_{x\in X}\left(\mu_X(x)\to\mu_{X_D}\left(F(g)(f(x))\right)\right)\right) \end{aligned}$$

which proves the thesis. □

The crucial property exploited in the proof of the previous corollary has been commutation of colimits and finite products, which is guaranteed by cartesian closedness of $\mathbf{H}$. In order to generalize Corollary 3.1.41 to other (regular) cardinals we need to introduce the notion of κ-continuity, which will guarantee commutation of suprema and infima (see [53, 54, 62, 105, 116] for further details).

Definition 3.1.42. Let $(P,\leq)$ be a poset and κ a regular cardinal, a *κ-ideal* I is a subset of P which is downward closed and κ-directed. We will denote by $\mathsf{Idl}_\kappa(P,\leq)$ the set of κ-ideals, which form a poset when ordered by inclusion.

Remark 3.1.43. If D is a κ-directed subset of $(P,\leq)$, then

$$\downarrow D := \{p\in P \mid p\leq d \text{ for some } d\in D\}$$

is a κ-ideal. Clearly it is downward closed. Moreover, if $\{p_i\}_{i\in I}\subseteq\downarrow D$ is a family with cardinality strictly less than κ, then for every $i\in I$ there exists $d_i\in D$ such that, for every $p_i\leq d_i$, but D is κ-directed and therefore there exists $d\in D$ which is a upper bound for $\{d_i\}_{i\in I}$ and thus also for $\{p_i\}_{i\in I}$.

Remark 3.1.44. The arbitrary intersection of a family $\{I_k\}_{k\in K}$ of ideals of a complete lattice $(P,\leq)$ is again an ideal. Let I be such intersection, if $q \in I$ and $p \leq q$, then $p \in I_k$ for every $k \in K$ and thus I is downward closed. On the other hand, if $\{p_i\}_{i\in I}$ is a family with cardinality less than κ contained in I, then for every $k \in K$ we have a $q_k \in I_k$ which is an upper bound for $\{p_i\}_{i\in I}$. Take

$$q := \bigwedge_{k\in K} q_k$$

then q is an upperbound for $\{p_i\}_{i\in I}$ too and it is in I because every I_k is downward closed.

Example 3.1.45. For every $p \in P$ and regular cardinal κ, the downward closure $\downarrow p$ of x is a κ-ideal. If $p \leq q$, then $\downarrow p \subseteq \downarrow q$, thus we have a monotone map $\downarrow\colon (P,\leq) \to (\mathsf{Idl}_\kappa(P,\leq),\subseteq)$.

Proposition 3.1.46. *Let $(P,\leq)$ be a poset, then the following are equivalent:*

1. $\downarrow\colon (P,\leq) \to (\mathsf{Idl}_\kappa(P,\leq),\subseteq)$ *has a left adjoint* sp*;*
2. *every κ-directed subset of P has a supremum.*

Proof. ($1 \Rightarrow 2$) Let D be a κ-directed subset of P, then its downward closure $\downarrow D$ is a κ-ideal by Remark 3.1.43. We claim that $\mathsf{sp}(\downarrow D)$ is the supremum for D. On one hand the unit of $\downarrow \dashv \mathsf{sp}$ yields

$$\downarrow D \subseteq \downarrow(\mathsf{sp}(\downarrow D))$$

so that $\mathsf{sp}(\downarrow D)$ is an upper bound for D. On the other hand, for every other $p \in P$ which is an upper bound we have $\downarrow D \subseteq \downarrow p$ and so, by adjointness $\mathsf{sp}(\downarrow D) \leq p$.
($2 \Rightarrow 1$) Since every ideal is κ-directed, we can define

$$\mathsf{sp}\colon (\mathsf{Idl}_\kappa(P,\leq),\subseteq) \to (P,\leq) \qquad I \mapsto \bigvee_{p\in I} p$$

Now, if $\mathsf{sp}(I) \leq q$ for some $q \in P$, then every element in I must be below q, showing $I \subseteq \downarrow q$. Vice versa, if $I \subseteq \downarrow q$ then q is an upper bound for I and therefore $\mathsf{sp}(I) \leq q$. $\square$

Definition 3.1.47. Given a regular cardinal κ, a complete lattice $(P,\leq)$ is *κ-continuous* if the function $\mathsf{sp}\colon (\mathsf{Idl}_\kappa(P,\leq),\subseteq) \to (P,\leq)$ has a left adjoint $\Downarrow$. A frame $\mathbf{H}$ is said to be *locally κ-compact* if it is κ-continuous when regarded as a lattice.

Example 3.1.48. The lattice $([0,1],\leq)$ is $\aleph_0$-continuous. To see this, for every $r \in [0,1]$, define

$$\Downarrow r := \begin{cases} \downarrow r \smallsetminus \{r\} & r \neq 0 \\ \{0\} & r = 0 \end{cases}$$

Clearly $\Downarrow r$ is downward closed, every finite set F contained in $\Downarrow r$ has an upper bound: this is tautological if $F = \emptyset$, while we can take the maximum of F if it is non empty. Notice that the supremum of $\Downarrow r$ is r itself: this is clear if $r = 0$, if $r \neq 0$, let s be the supremum of $\downarrow r$, clearly r is an upper bound for $\downarrow r$ and thus $s \leq r$, on the other hand, if $s < r$, then the density of $([0,1],\leq)$ entails the existence of $s < t < r$, but this is a contradiction. But now, given this observation, it is obvious that $\Downarrow r \subseteq I$ if and only if $r \leq \mathsf{sp}(I)$, showing that $\Downarrow \dashv \mathsf{sp}$.

Remark 3.1.49. The terminology of local κ-compactness comes from the fact that locally $\aleph_0$-compact frames are, up to isomorphism, the topologies of locally compact topological spaces [105].

We are now going to show that in a κ-continuous lattice directed suprema , i.e. suprema of directed sets, distribute over arbitrary infima.

Lemma 3.1.50. *Given a complete lattice $(P, \leq)$ and a regular cardinal κ, the following are equivalent:*

1. *$(P, \leq)$ is κ-continuous;*
2. *given a family $\{I_j\}_{j\in J}$ of κ-ideals, we have*

$$\bigvee_{x\in\prod_{j\in J} I_j}\left(\bigwedge_{j\in J}\pi_j(x)\right) = \bigwedge_{j\in J}\left(\bigvee_{y_j\in I_j} y_j\right)$$

where π_j denotes the projection $\prod_{j\in J} I_j \to I_j$.

Remark 3.1.51. Lemma 3.1.50, like Proposition 3.1.9, is an application of the classical Adjoint Functor Theorem for posets. For the sake of completeness, we will nonetheless provide a full proof of it.

Remark 3.1.52. Let us notice the following: for every fixed $x \in \prod_{j\in J} I_j$, let p_x be the infimum of the family $\{\pi_j(x)\}_{j\in J}$. By definition, $p_x \leq \pi_j(x)$ for every $j \in J$, so that p_x belongs to $\bigcap_{j\in J} I_J$. On the other hand, given $y \in \bigcap_{j\in J} I_J$, if we consider consider $x_y \in \prod_{j\in J} I_j$ defined by $y = \pi_j(x_y)$ for every $j \in J$, then y must coincide with p_{x_y}. Hence, the family $\{p_x\}_{x\in\prod_{j\in J} I_j}$ is cofinal in $\bigcap_{j\in J} I_j$ and therefore

$$\begin{aligned}\mathsf{sp}\left(\bigcap_{j\in J} I_j\right) &= \bigvee_{x\in\prod_{j\in J} I_j} p_x \\ &= \bigvee_{x\in\prod_{j\in J} I_j}\left(\bigwedge_{j\in J}\pi_j(x)\right)\end{aligned}$$

Proof. $(1 \Rightarrow 2)$ By hypothesis sp is a right adjoint, thus Remark 3.1.52 yields

$$\begin{aligned}\bigvee_{x\in\prod_{j\in J} I_j}\left(\bigwedge_{j\in J}\pi_j(x)\right) &= \mathsf{sp}\left(\bigcap_{j\in J} I_j\right) \\ &= \bigwedge_{j\in J}\mathsf{sp}(I_j) \\ &= \bigwedge_{j\in J}\left(\bigvee_{y_j\in I_j} y_j\right)\end{aligned}$$

$(2 \Rightarrow 1)$ Let p be an element of P, define

$$D_p := \{I \in \mathsf{Idl}_\kappa(P, \leq) \mid p \leq \mathsf{sp}(I)\}$$

Using Remark 3.1.44 we know that $\mathsf{Idl}_\kappa(P, \leq)$ is closed under arbitrary intersections, we can then put

$$\Downarrow(p) := \bigcap_{I\in D_p} I$$

Suppose that $\Downarrow(p) \subseteq J$ for some κ-ideal J. Since every κ-ideal is downward closed, it follows that

$$\begin{aligned} p &\leq \bigvee_{I \in D_p} \mathsf{sp}(I) \\ &= \mathsf{sp}\left(\bigcap_{I \in D_p} I\right) \\ &= \mathsf{sp}(\Downarrow(p)) \\ &\leq \mathsf{sp}(J) \end{aligned}$$

Vice versa, if $p \leq \mathsf{sp}(J)$ for some $J \in \mathsf{Idl}_\kappa(P, \leq)$ then $J \in D_p$, so, trivially, we have that $\Downarrow(p) \subseteq J$. □

Corollary 3.1.53. *Let $(P, \leq)$ be a κ-continuous lattice and $\{D_j\}_{j \in J}$ a family of κ-directed subsets of P, then*

$$\bigvee_{x \in \prod_{j \in J} D_j} \left(\bigwedge_{j \in J} \pi_j(x)\right) = \bigwedge_{j \in J} \left(\bigvee_{y_j \in D_j} y_j\right)$$

Proof. This follows at once from Remark 3.1.43 and Lemma 3.1.50 noticing that D_j is cofinal in $\downarrow D_j$. □

Corollary 3.1.54. *Let $\{p_{j,d}\}_{j \in J, d \in D}$ be a family of elements of a κ-continuous lattice $(P, \leq)$ such that $|J| < \kappa$ and, for every $j \in J$, the set $D_j := \{p_{j,d}\}_{d \in D}$ is κ-directed, then:*

$$\bigvee_{d \in D} \left(\bigwedge_{j \in J} p_{j,d}\right) = \bigwedge_{j \in J} \left(\bigvee_{d \in D} p_{j,d}\right)$$

Proof. As a first step notice that

$$\begin{aligned} \bigwedge_{j \in J} \left(\bigvee_{y_j \in D_j} y_j\right) &= \bigwedge_{j \in J} \mathsf{sp}(D_j) \\ &= \bigwedge_{j \in J} \left(\bigvee_{d \in D} p_{j,d}\right) \end{aligned}$$

Next, for every $d \in D$, put

$$p_d := \bigwedge_{j \in J} p_{j,d}$$

Now, for every $d \in D$, there is a unique $x_d \in \prod_{j \in J} D_j$ such that $p_d = \pi_j(x_d)$ showing that

$$\{p_d\}_{d \in D} \subseteq \left\{\bigwedge_{j \in J} \pi_j(x)\right\}_{x \in \prod_{j \in J} D_j}$$

We claim that this inclusion is cofinal. Let x be an element of $\prod_{j \in J} D_j$, the family $\{\pi_j(x)\}_{j \in J}$ has cardinality strictly less then κ and it is contained in D_j. Therefore, by Lemma 2.2.6, it has an upper bound $p_{j,d} \in D_j$. This shows that

$$\bigwedge_{j \in J} \pi_j(x) \leq p_d$$

This shows the desired cofinality.The thesis now follows from Corollary 3.1.53. □

Proposition 3.1.55. *Let* $\mathbf{H}$ *be a locally* κ*-compact frame, and let* X *be an object of* $\mathbf{Set}_\kappa$ *then:*

1. $(-)^{\Delta_{\mathbf{H}}(X)}$ *has rank* κ*;*
2. *if* $\mathbf{H}$ *is boolean then* $(-)^{(X,\mu_X)}$ *has rank* κ*.*

Proof. In view of Corollaries 2.2.8 and 3.1.54 it is enough to repeat verbatim the proof of Corollary 3.1.41.

1. The equality of Proposition 3.1.40 becomes

$$\bigvee_{g\in D/\mathbf{D}}\left(\bigwedge_{x\in X}\mu_{X_D}\left(F(g)(f(x))\right)\right)=\bigwedge_{x\in X}\left(\bigvee_{g\in D/\mathbf{D}}\mu_{X_D}\left(F(g)(f(x))\right)\right)$$

which holds since $\mathbf{H}$ is locally κ-compact.

2. Given a family $\{h_i\}_{i\in I}$ of elements of H, for every other h in it we have already noticed that

$$h\to\bigvee_{i\in I}h_i=\bigvee_{i\in I}(h\to h_i)$$

Thus, usinge the local κ-compactness of $\mathbf{H}$ we have

$$\begin{aligned}\bigwedge_{x\in X}\left(\mu_X(x)\to\bigvee_{g\in D/\mathbf{D}}\mu_{X_D}\left(F(g)(f(x))\right)\right)&=\bigwedge_{x\in X}\left(\bigvee_{g\in D/\mathbf{D}}(\mu_X(x)\to\mu_{X_D}\left(F(g)(f(x))\right))\right)\\&=\bigvee_{g\in D/\mathbf{D}}\left(\bigwedge_{x\in X}(\mu_X(x)\to\mu_{X_D}\left(F(g)(f(x))\right))\right)\end{aligned}$$

and the thesis follows from Proposition 3.1.40. □

3.2 Monads on Fuz(H)

In this section we will adapt the work done in Section 2.2 to the setting of fuzzy sets. Our main goal is to introduce new syntactic constructs, called *fuzzy algebraic theories*, and provide results similar to Corollaries 2.2.83 and 2.2.92, linking them to monads on $\mathbf{Fuz}(\mathbf{H})$.

3.2.1 Fuzzy algebraic theories

Let us start introducing the notion of fuzzy signature.

Definition 3.2.1. Let $\mathbf{Card}(\mathbf{H})$ be the class of all fuzzy sets whose underlying set is a cardinal. A *fuzzy signature* (or simply a signature) Σ is a triple $(O_\Sigma, C_\Sigma, \mathsf{ar}_\Sigma)$, where O_Σ is a class of *operations*, C_Σ a set of *constants* and ar_Σ is a function $O_\Sigma\to\mathbf{Card}(\mathbf{H})$ such that, for every (κ,μ_κ) in $\mathbf{Card}(\mathbf{H})$,

$$O_{\Sigma,(\kappa,\mu_\kappa)}:=\{o\in O_\Sigma\mid \mathsf{ar}_\Sigma(o)=(\kappa,\mu_\kappa)\}$$

is a set, called the set of *operations of arity* (κ,μ_κ). Given a regular cardinal κ, we will say that Σ is

- *κ-bounded* if, for every (λ, μ_λ) such that $\kappa \leq \lambda$, $O_{\Sigma,(\lambda,\mu_\lambda)}$ is empty;
- *strongly κ-bounded* if, for every $o \in O_\Sigma$, $\mathsf{ar}_\Sigma(o) = \Delta_{\mathbf{H}}(\mu)$ for some $\mu < \kappa$;
- *κ-accessible* if $(-)^{\mathsf{ar}_\Sigma(o)}$ has rank κ for every $o \in O_\Sigma$.

The category $\mathbf{FSign}_\kappa$ is defined as the category with κ-bounded fuzzy signatures as objects and in which a morphism $(f, g) \colon \Sigma_1 \to \Sigma_2$ is a pair of functions $f \colon O_{\Sigma_1} \to O_{\Sigma_2}$, $g \colon C_{\Sigma_1} \to C_{\Sigma_2}$ such that the following triangle commutes.

$$\begin{tikzcd} O_{\Sigma_1} \arrow[rr, "f"] \arrow[dr, "\mathsf{ar}_{\Sigma_1}"'] & & O_{\Sigma_2} \arrow[dl, "\mathsf{ar}_{\Sigma_2}"] \\ & \mathbf{Card}(\mathbf{H}) & \end{tikzcd}$$

Remark 3.2.2. Let Σ be κ-bounded, there is only a set of fuzzy sets whose underlying set has cardinality strictly less then κ, so, as in the case of algebraic theories, O_Σ is a set and $\mathbf{FSign}_\kappa$ is a category.

Remark 3.2.3. By definition and by Proposition 3.1.40 strongly κ-bounded and κ-accessible signatures are κ-bounded, thus they define two full subcategories $\mathbf{FSign}_{S,\kappa}$ and $\mathbf{FSign}_{A,\kappa}$ of $\mathbf{Sign}_\kappa$. We can point out some other relation between them.

- Point 1 of Corollary 3.1.41 entails that $\mathbf{FSign}_{S,\aleph_0}$ is a subcategory of $\mathbf{FSign}_{A,\aleph_0}$ while point 2 says that $\mathbf{FSign}_{A,\aleph_0} = \mathbf{FSign}_{\aleph_0}$ whenever $\mathbf{H}$ is a complete boolean algebra.
- If $\mathbf{H}$ is locally κ-compact, then, from Proposition 3.1.55 we obtain that, for every regular cardinal κ, $\mathbf{FSign}_{S,\kappa}$ is a subcategory of $\mathbf{FSign}_{A,\kappa}$ and that this last category coincides with $\mathbf{FSign}_\kappa$ if $\mathbf{H}$ is also boolean.

Remark 3.2.4. For every a fuzzy signature Σ we can construct an algebraic signature $\mathsf{cri}(\Sigma)$ putting

$$O_{\mathsf{cri}(\Sigma)} := O_\Sigma + C_\Sigma$$

and, denoting the obvious injections by $j_1 \colon O_\Sigma \to O_{\mathsf{cri}(\Sigma)}$ and $j_2 \colon C_\Sigma \to O_{\mathsf{cri}(\Sigma)}$:

$$\mathsf{ar}_{\mathsf{cri}(\Sigma)} \colon O_{\mathsf{cri}(\Sigma)} \to \mathbf{Card} \qquad x \mapsto \begin{cases} V_H(\mathsf{ar}_\Sigma(o)) & x = j_1(o) \\ 0 & x = j_2(c) \end{cases}$$

Given a regular cardinal κ, this construction extends to a functor $\mathsf{cri} \colon \mathbf{FSign}_\kappa \to \mathbf{Sign}_\kappa$: for every $(f, g) \colon O_{\Sigma_1} \to O_{\Sigma_2}$ we can define $\mathsf{cri}(f, g) : O_{\mathsf{cri}(\Sigma_1)} \to O_{\mathsf{cri}(\Sigma_2)}$ as $f + g \colon O_{\Sigma_1} + C_{\Sigma_1} \to O_{\Sigma_2} + C_{\Sigma_2}$. By construction, we get a morphism of $\mathbf{Sign}_\kappa$.

Example 3.2.5. The signature Σ_{FS} of *fuzzy semigroups* is given by

$$O_{\Sigma_{FS}} := \{\cdot\} \qquad C_{\Sigma_{FS}} = \emptyset$$

and in which the arity function is defined putting $\mathsf{ar}_{\Sigma_{FS}}(\cdot) = \Delta_{\mathbf{H}}(2)$.

Example 3.2.6. The signature Σ_{FG} of *fuzzy groups* is defined putting

$$O_{\Sigma_{FG}} := \left\{\cdot, (-)^{-1}\right\} \qquad C_{\Sigma_{FG}} := \{e\}$$

and in which

$$\mathsf{ar}_{\Sigma_{FG}}(\cdot) = \Delta_{\mathbf{H}}(2) \qquad \mathsf{ar}_{\Sigma_{FG}}\left((-)^{-1}\right) = \Delta_{\mathbf{H}}(1)$$

We are now ready to introduce Σ-algebras as in the previous chapter.

Definition 3.2.7. Let Σ be a fuzzy signature, a fuzzy Σ-*algebra* $\mathcal{A} = \left((A, \mu_A), \left\{o^{\mathcal{A}}\right\}_{o \in O_\Sigma}, \left\{c^{\mathcal{A}}\right\}_{c \in C_\Sigma}\right)$ is a triple where (A, μ_A) is a fuzzy set and, for every $o \in O_\Sigma$, $c \in C_\Sigma$,

$$o^{\mathcal{A}} \colon (A, \mu_A)^{\mathsf{ar}_\Sigma(o)} \to (A, \mu_A) \qquad c^{\mathcal{A}} \colon \nabla_{\mathbf{H}}(1) \to (A, \mu_A)$$

A Σ-*homomorphism* $f \colon \mathcal{A} \to \mathcal{B}$ is an arrow $f \colon (A, \mu_A) \to (B, \mu_B)$ such that, for every operation $o \in O_\Sigma$, the following diagrams commute

$$\begin{tikzcd} (A, \mu_A)^{\mathsf{ar}_\Sigma(o)} \arrow[r, "f^{\mathsf{ar}_\Sigma(o)}"] \arrow[d, "o^{\mathcal{A}}"'] & (B, \mu_B)^{\mathsf{ar}_\Sigma(o)} \arrow[d, "o^{\mathcal{B}}"] \\ (A, \mu_A) \arrow[r, "f"'] & (B, \mu_B) \end{tikzcd} \qquad \begin{tikzcd} & \nabla_{\mathbf{H}}(1) \arrow[dl, "c^{\mathcal{A}}"'] \arrow[dr, "c^{\mathcal{B}}"] & \\ (A, \mu_A) \arrow[rr, "f"'] & & (B, \mu_B) \end{tikzcd}$$

We will denote by Σ-**FAlg** the resulting category and by $V_\Sigma \colon \Sigma\text{-}\mathbf{FAlg} \to \mathbf{Fuz}(\mathbf{H})$ the forgetful functor.

Remark 3.2.8. Differently from the case of algebraic signatures, in our setting constants cannot be seen simply as operations of arity $(\emptyset, ?_H)$. For every(A, μ_A) we get

$$(A, \mu_A)^{(\emptyset, ?_H)} \simeq \Delta_{\mathbf{H}}(1)$$

Thus an operation of arity $(\emptyset, ?_H)$ must be interpreted as an arrow $\Delta_{\mathbf{H}}(1) \to (A, \mu_A)$, i.e. as an element of A with membership degree $\top$. However, limiting ourselves to this kind of constants would be too heavy a restriction for the expressivity of our formalism.

Take a fuzzy signature Σ and a Σ-algebra $\mathcal{A}$, we know that, for every $o \in O_\Sigma$

$$\begin{aligned} V_H\left((A, \mu_A)^{\mathsf{ar}\Sigma(o)}\right) &= \mathbf{Set}(V_H(\mathsf{ar}_\Sigma(o)), A) \\ &= \mathbf{Set}(\mathsf{ar}_{\mathsf{cri}(\Sigma)}(o), A) \\ &= A^{\mathsf{ar}_{\mathsf{cri}(\Sigma)}(o)} \end{aligned}$$

Thus we can define a $\mathsf{cri}(\Sigma)$-algebra $W_\Sigma(\mathcal{A})$ putting

$$o^{W_\Sigma(\mathcal{A})} := o^{\mathcal{A}} \qquad c^{W_\Sigma(\mathcal{A})} := c^{\mathcal{A}}$$

This can be immediately extended to a functor $W_\Sigma \colon \Sigma\text{-}\mathbf{FAlg} \to \mathsf{cri}(\Sigma)\text{-}\mathbf{Alg}$:

$$\begin{array}{ccc} \mathcal{A} & \longmapsto & W_\Sigma(\mathcal{A}) \\ f\big\downarrow & & \big\downarrow f \\ \mathcal{B} & \longmapsto & W_\Sigma(\mathcal{B}) \end{array}$$

Remark 3.2.9. It is worth to point out explicitly that a $\mathsf{cri}(\Sigma)$-homomorphism $f \colon W_\Sigma(\mathcal{A}) \to W_\Sigma(\mathcal{B})$ is the image of a Σ-homomorphism if and only if $f \colon (A, \mu_A) \to (B, \mu_B)$ is a morphism of $\mathbf{Fuz}(\mathbf{H})$.

Proposition 3.2.10. *The following hold true:*

1. *for every fuzzy signature Σ, the functor W_Σ has a right adjoint Δ_Σ;*
2. *for every strongly κ-bounded signature Σ, W_Σ has a left adjoint ∇_Σ;*
3. *for every morphism $(f,g)\colon \Sigma_1 \to \Sigma_2$ of $\mathbf{FSign}_\kappa$, there exists a functor $(f,g)^*\colon \Sigma_2\text{-}\mathbf{FAlg} \to \Sigma_1\text{-}\mathbf{FAlg}$ making the following square commute*

$$\begin{tikzcd}
\Sigma_2\text{-}\mathbf{FAlg} \arrow[r, "(f,g)^*"] \arrow[d, "W_{\Sigma_2}"'] & \Sigma_1\text{-}\mathbf{FAlg} \arrow[d, "W_{\Sigma_1}"] \\
\mathsf{cri}\,(\Sigma_2)\text{-}\mathbf{Alg} \arrow[r, "\mathsf{cri}(f,g)^*"'] & \mathsf{cri}\,(\Sigma_1)\text{-}\mathbf{Alg}
\end{tikzcd}$$

Proof. 1. Let $\mathcal{A} = \left(A, \{o^{\mathcal{A}}\}_{o\in O_{\mathsf{cri}(\Sigma)}}\right)$ be a $\mathsf{cri}\,(\Sigma)$-algebra, then for every $o \in O_\Sigma$ and $c \in C_\Sigma$ we have arrows of $\mathbf{Fuz}(\mathbf{H})$

$$o^{\mathcal{A}}\colon (\Delta_{\mathbf{H}}(A))^{\mathsf{ar}_\Sigma(o)} \to \Delta_{\mathbf{H}}(A) \qquad c^{\mathcal{A}}\colon \nabla_{\mathbf{H}}(1) \to \Delta_{\mathbf{H}}(A)$$

and we can define $\Delta_\Sigma(\mathcal{A})$ as the resulting fuzzy Σ-algebra. Notice that $W_\Sigma(\Delta_\Sigma(\mathcal{A})) = \mathcal{A}$ and $\mathsf{id}_{\mathcal{A}}$ has the universal property of a counit for $W_\Sigma \dashv \Delta_\Sigma$: given a $\mathsf{cri}\,(\Sigma)$-homomorphism $f\colon W_\Sigma(\mathcal{B}) \to \mathcal{A}$ we have for free that f is an arrow $V_\Sigma(\mathcal{B}) \to \Delta_{\mathbf{H}}(A)$ and thus it defines also a Σ-homomorphism $\mathcal{B} \to \Delta_{\mathbf{H}}(\mathcal{A})$.

2. Notice that, given two sets X and Y, we have that

$$(\nabla_{\mathbf{H}}(X))^{\Delta_{\mathbf{H}}(Y)} = \nabla_{\mathbf{H}}\left(X^Y\right)$$

Indeed, if $f\colon Y \to X$ is a function, then

$$\begin{aligned}
\mu_{Y^X}(Y) &= \bigwedge_{y\in Y} (\mu_Y(y) \to \mu_X(f(y))) \\
&= \bigwedge_{y\in Y} (\top \to \bot) \\
&= \bot
\end{aligned}$$

Thus, if Σ is strongly κ-bounded, given a $\mathsf{cri}\,(\Sigma)$-algebra $\mathcal{A} = \left(A, \{o^{\mathcal{A}}\}_{o\in O_{\mathsf{cri}(\Sigma)}}\right)$, we can construct a Σ-algebra structure $\nabla_\Sigma(\mathcal{A})$ on $\nabla_{\mathbf{H}}(A)$ simply using the arrows

$$o^{\mathcal{A}}\colon (\nabla_{\mathbf{H}}(A))^{\mathsf{ar}_\Sigma(o)} \to \nabla_{\mathbf{H}}(A) \qquad c^{\mathcal{A}}\colon \nabla_{\mathbf{H}}(1) \to \nabla_{\mathbf{H}}(A)$$

To see that in this way we get a left adjoint, consider $\mathsf{id}_{\mathcal{A}}\colon \mathcal{A} \to W_\Sigma(\nabla_\Sigma(\mathcal{A}))$ and suppose that a $\mathsf{cri}\,(\Sigma)$-homomorphism $f\colon \mathcal{A} \to W_\Sigma(\mathcal{B})$ is given, then f also defines a morphism of fuzzy sets $\nabla_{\mathbf{H}}(A) \to (B, \mu_B)$ and we can conclude.

3. This is done exactly as in Proposition 2.2.43. Given $\mathcal{A} = \left((A,\mu_A), \{o^{\mathcal{A}}\}_{o\in O_{\Sigma_2}}, \{c^{\mathcal{A}}\}_{c\in \mathbf{C}_{\Sigma_2}}\right)$, define $(f,g)^*(\mathcal{A})$ as the Σ_1-algebra on (A,μ_A) in which

$$o^{(f,g)^*(\mathcal{A})} := (f(o))^{\mathcal{A}} \qquad c^{(f,g)^*(\mathcal{A})} := (g(c))^{\mathcal{A}}$$

The action of $(f,g)^*$ on morphisms is the identity. □

We can also recover an analogous of Lemma 2.2.46.

Lemma 3.2.11. *Let Σ be a κ-accessible signature and $F\colon \mathbf{D} \to \Sigma$-$\mathbf{FAlg}$ be a functor with a κ-filtered domain, let also $((A,\mu_A),\{a_D\}_{D\in\mathbf{D}})$ be a colimiting cocone for $V_\Sigma \circ F$. Then there exists a unique $\mathcal{A}$ in Σ-$\mathbf{FAlg}$ such that $V_\Sigma(\mathcal{A}) = (A,\mu_A)$, and which makes every a_D a Σ-homomorphism $F(D) \to \mathcal{A}$. Moreover, the cocone $(\mathcal{A},\{a_D\}_{D\in\mathbf{D}})$ is colimiting for F.*

Proof. By definition of κ-accessible signature $\left((A,\mu_A)^{\mathsf{ar}_\Sigma(o)}, \left\{a_D^{\mathsf{ar}_\Sigma(o)}\right\}_{D\in\mathbf{D}}\right)$ is colimiting for the functor $(V_\Sigma(F(-)))^{\mathsf{ar}_\Sigma(o)}$. The proof now proceeds in the same way as the one of Lemma 2.2.46: given an arrow $f\colon D_1 \to D_2$ in $\mathbf{D}$, we have diagrams

$$\begin{tikzcd}
(V_\Sigma(F(D_1)))^{\mathsf{ar}_\Sigma(o)} \arrow[r, "o^{F(D_1)}"] \arrow[d, "F(f)^{\mathsf{ar}_\Sigma(o)}"'] & V_\Sigma(F(D_1)) \arrow[d, "F(f)"'] \arrow[dr, bend left, "a_{D_1}"] & \\
(V_\Sigma(F(D_2)))^{\mathsf{ar}_\Sigma(o)} \arrow[r, "o^{F(D_2)}"'] & V_\Sigma(F(D_2)) \arrow[r, bend right, "a_{D_2}"'] & (A,\mu_A)
\end{tikzcd}$$

and thus a unique arrow $o^{\mathcal{A}}\colon (A,\mu_A)^{\mathsf{ar}_\Sigma(o)} \to (A,\mu_A)$ such that

$$\begin{tikzcd}
(V_\Sigma(F(D)))^{\mathsf{ar}_\Sigma(o)} \arrow[r, "o^{F(D)}"] \arrow[d, "c_D^{\mathsf{ar}_\Sigma(o)}"'] & V_\Sigma(F(D)) \arrow[d, "c_D"] \\
(A,\mu_A)^{\mathsf{ar}_\Sigma} \arrow[r, "o^{\mathcal{A}}"'] & (A,\mu_A)
\end{tikzcd}$$

commutes. For a constant $c \in C_\Sigma$, we are forced to define $c^{\mathcal{A}}$ as $a_D \circ c^{F(D)}$ for any $D \in \mathbf{D}$. Notice that this definition does not depends on the choice of D: if D_1 and D_2 are objects of $\mathbf{D}$, then there exist arrows $f_1\colon D_1 \to D_3$ and $f_2\colon D_2 \to D_3$ and we have a diagram

$$\begin{tikzcd}
 & V_\Sigma(F(D_1)) \arrow[dr, "F(f_1)"] \arrow[drr, bend left, "a_{D_1}"] & & \\
\nabla_{\mathbf{H}}(1) \arrow[ur, "c^{F(D_1)}"] \arrow[rr, "c^{F(D_3)}"] \arrow[dr, "c^{F(D_3)}"'] & & V_\Sigma(F(D_3)) \arrow[r, "a_{D_2}"] & (A,\mu_A) \\
 & V_\Sigma(F(D_2)) \arrow[ur, "F(f_2)"'] \arrow[urr, bend right, "a_{D_2}"'] & &
\end{tikzcd}$$

and so

$$\begin{aligned}
a_{D_1} \circ c^{F(D_1)} &= a_{D_3} \circ F(f_1) \circ c^{F(D_1)} \\
&= a_{D_3} \circ c^{F(D_3)} \\
&= a_{D_3} \circ F(f_2) \circ c^{F(D_3)} \\
&= a_{D_2} \circ c^{F(D_2)}
\end{aligned}$$

Now let $\mathcal{A}$ be $\left((A, \mu_A), \{o^{\mathcal{A}}\}_{o \in O_\Sigma}, \{c^{\mathcal{A}}\}_{c \in C_\Sigma}\right)$. To show that $(\mathcal{A}, \{a_D\}_{D \in \mathbf{D}})$ is colimiting for F take another cocone $(\mathcal{B}, \{d_D\}_{\mathbf{D}})$, there is a unique $d\colon (A, \mu_A) \to (B, \mu_B)$, where $(B, \mu_B) = V_\Sigma(\mathcal{B})$, such that $d \circ a_D = d_D$, so it is enough to show that d is an arrow of Σ-**FAlg**. Computing we get

$$\begin{aligned} d \circ o^{\mathcal{A}} \circ a_D^{\mathsf{ar}_\Sigma(o)} &= d \circ a_D \circ o^{F(D)} \\ &= d_D \circ o^{F(D)} \\ &= o^{\mathcal{B}} \circ d_D^{\mathsf{ar}_\Sigma(o)} \\ &= o^{\mathcal{B}} \circ d^{\mathsf{ar}_\Sigma(o)} \circ a_D^{\mathsf{ar}_\Sigma(o)} \end{aligned} \qquad \begin{aligned} d \circ c^{\mathcal{A}} &= d \circ a_D \circ c^{F(D)} \\ &= d_D \circ c^{F(D)} \\ &= c^{\mathcal{B}} \end{aligned}$$

from which the thesis follows. □

Corollary 3.2.12. *Let κ be a regular cardinal and Σ a κ-accessible signature, then the following are true*

1. Σ-**FAlg** *has all κ-filtered colimits;*
2. V_Σ *has rank κ.*

The calculus of fuzzy algebraic sequents

We are now going to introduce two syntactic notions that will play the same role played by equations in the classical setting. Notice that the functor $\mathsf{cri}\colon \mathbf{FSign}_\kappa \to \mathbf{Sign}_\kappa$ allows us to speak of Σ-terms even if we have not yet built a left adjoint to V_Σ: this will be done in the next section.

Definition 3.2.13. Let Σ be a κ-bounded fuzzy signature, a Σ-*term* is simply a $\mathsf{cri}(\Sigma)$-term, i.e. an element of $T_{\mathsf{cri}(\Sigma)}(X)$ for some set X. We define the following sets:

- the set $\mathsf{Eq}(\Sigma)$ of Σ-*equations* coincides with the set of $\mathsf{cri}(\Sigma)$-equations, i.e.

$$\mathsf{Eq}(\Sigma) := \sum_{\lambda \in \kappa} T_{\mathsf{cri}(\Sigma)}(\lambda) \times T_{\mathsf{cri}(\Sigma)}(\lambda)$$

  We will still denote by $\lambda \mid t_1 \equiv t_2$ the image of the pair $(t_1, t_2) \in T_{\mathsf{cri}(\Sigma)}(\lambda) \times T_{\mathsf{cri}(\Sigma)}(\lambda)$ in $\mathsf{Eq}(\Sigma)$ and call λ the *context* of the equation;

- the set $\mathsf{MP}(\Sigma)$ of *membership propositions* is defined as

$$\mathsf{MP}(\Sigma) := \sum_{\lambda \in \kappa} H \times T_{\mathsf{cri}(\Sigma)}(\lambda)$$

  By $\lambda \mid \mathsf{m}(h, t)$ wi will denote the image in $\mathsf{MP}(\Sigma)$ of the pair $(h, t) \in H \times T_{\mathsf{cri}(\Sigma)}(\lambda)$ and we will again refer to λ as the *context* of the proposition;

- the set $\mathsf{Form}(\Sigma, \lambda)$ of Σ-*formulae in context* λ is

$$\mathsf{Form}(\Sigma, \lambda) := (T_{\mathsf{cri}(\Sigma)}(\lambda) \times T_{\mathsf{cri}(\Sigma)}(\lambda)) + (H \times T_{\mathsf{cri}(\Sigma)}(\lambda))$$

  while the set $\mathsf{Form}(\Sigma)$ of Σ-*formulae* is the coproduct $\sum_{\lambda \in \kappa} \mathsf{Form}(\Sigma, \lambda)$;

- finally, the set $\mathsf{Seq}(\Sigma)$ of Σ-*sequents* is

$$\mathsf{Seq}(\Sigma) := \sum_{\lambda \in \kappa} \mathcal{P}(\mathsf{Form}(\Sigma, \lambda)) \times \mathsf{Form}(\Sigma, \lambda)$$

  and we will write $\lambda \mid \Gamma \vdash \psi$ to denote the sequent given by the pair $(\Gamma, \psi) \in \mathcal{P}(\mathsf{Form}(\Sigma, \lambda)) \times \mathsf{Form}(\Sigma, \lambda)$, as before λ will be called *context*.

$$\frac{\phi \in \Gamma}{\lambda \mid \Gamma \vdash \phi}\ \textsc{A} \qquad \frac{\lambda \mid \Gamma \vdash \phi}{\lambda \mid \Gamma \cup \Delta \vdash \phi}\ \textsc{Weak} \qquad \frac{\{\lambda \mid \Gamma \vdash \phi\}_{\phi \in \Phi} \qquad \lambda \mid \Phi \vdash \psi}{\lambda \mid \Gamma \vdash \psi}\ \textsc{Cut}$$

$$\frac{}{\lambda \mid \Gamma \vdash t \equiv t}\ \textsc{Refl} \qquad \frac{\lambda \mid \Gamma \vdash t_1 \equiv t_2}{\lambda \mid \Gamma \vdash t_2 \equiv t_1}\ \textsc{Sym} \qquad \frac{\lambda \mid \Gamma \vdash t_1 \equiv t_2 \qquad \lambda \mid \Gamma \vdash t_2 \equiv t_3}{\lambda \mid \Gamma \vdash t_1 \equiv t_3}\ \textsc{Trans}$$

$$\frac{\sigma\colon \lambda_1 \to T_{\mathsf{cri}(\Sigma)}(\lambda_2) \qquad \lambda_1 \mid \Gamma \vdash \phi}{\lambda_2 \mid \Gamma[\sigma] \vdash \phi[\sigma]}\ \textsc{Subst} \qquad \frac{}{\lambda \mid \Gamma \vdash \mathsf{m}(\bot, t)}\ \textsc{Inf} \qquad \frac{\lambda \mid \Gamma \vdash \mathsf{m}(h, t)}{\lambda \mid \Gamma \vdash \mathsf{m}(h' \wedge h, t)}\ \textsc{Mon}$$

$$\frac{S \subseteq H \qquad \{\lambda \mid \Gamma \vdash \mathsf{m}(h, t)\}_{h \in S}}{\lambda \mid \Gamma \vdash \mathsf{m}(\mathsf{sup}(S), t)}\ \textsc{Sup} \qquad \frac{\lambda \mid \Gamma \vdash t \equiv s \qquad \lambda \mid \Gamma \vdash \mathsf{m}(h, t)}{\lambda \mid \Gamma \vdash \mathsf{m}(h, s)}\ \textsc{Fun}$$

$$\frac{o \in O_\Sigma \qquad \sigma\colon \mathsf{ar}_{\mathsf{cri}(\Sigma)}(j_1(o)) \to T_{\mathsf{cri}(\Sigma)}(\lambda) \qquad \{\lambda \mid \Gamma \vdash \mathsf{m}(h_\alpha, \sigma(\alpha))\}_{\alpha \in \mathsf{ar}_{\mathsf{cri}(\Sigma)}(j_1(o))}}{\lambda \mid \Gamma \vdash \mathsf{m}\left(\bigwedge_{\alpha \in \mathsf{ar}_{\mathsf{cri}(\Sigma)}(j_1(o))} \left(\mu_{\mathsf{ar}_\Sigma(o)}(\alpha) \to h_\alpha\right), j_1(o)(\sigma)\right)}\ \textsc{Exp}$$

$$\frac{o \in O_\Sigma \qquad \sigma_1, \sigma_2\colon \mathsf{ar}_{\mathsf{cri}(\Sigma)}(j_1(o)) \rightrightarrows T_{\mathsf{cri}(\Sigma)}(\lambda) \qquad \{\lambda \mid \Gamma \vdash \sigma_1(\alpha) \equiv \sigma_2(\alpha)\}_{\alpha \in \mathsf{ar}_{\mathsf{cri}(\Sigma)}(j_1(o))}}{\lambda \mid \Gamma \vdash j_1(o)(\sigma_1) \equiv j_1(o)(\sigma_2)}\ \textsc{Cong}$$

Figure 3.1: Derivation rules for the calculus of fuzzy algebraic sequents.

Remark 3.2.14. As will become clearer in the following, the intended meaning of a membership proposition $\lambda \mid \mathsf{m}(h, t)$ is "the membership degree of the term t is at least h".

Remark 3.2.15. Let σ be an arrow $\lambda_1 \to T_{\mathsf{cri}(\Sigma)}(\lambda_2)$, then we have a homomorphism $F_{\mathsf{cri}(\Sigma)}(\lambda_1) \to F_{\mathsf{cri}(\Sigma)}(\lambda_2)$. Considering $(\sigma_{\mathsf{cri}(\Sigma),*} \times \sigma_{\mathsf{cri}(\Sigma),*}) + (\mathsf{id}_H \times \sigma_{\mathsf{cri}(\Sigma),*})$ we get a function $\mathsf{Form}(\Sigma, \lambda_1) \to \mathsf{Form}(\Sigma, \lambda_2)$. We will denote by $\phi[\sigma]$ the image through it of $\phi \in \mathsf{Form}(\Sigma, \lambda_1)$. Similarly, we will denote by $\Gamma[\sigma]$ the image of $\Gamma \subseteq \mathsf{Form}(\Sigma, \lambda_1)$ under this arrow.

Notation. We will write $\lambda \mid \phi$ for $\lambda \mid \emptyset \vdash \phi$. As in Chapter 2 we will also use 0 to denote $\emptyset$ when it appears as a context.

Definition 3.2.16. Let S be a subset of $\mathsf{Seq}(\Sigma)$, its *deductive closure* $S^\vdash$ is the smallest subset of $\mathsf{Seq}(\Sigma)$ which contains S and it is closed under the rules of Fig. 3.1, i.e. if all the premises of a rule are in it, then the conclusion is. A sequent is *derivable* from S (or simply derivable if $S = \emptyset$) if it belongs to $S^\vdash$.

Remark 3.2.17. When Σ is strongly κ-accessible rule $\textsc{Exp}$ becomes

$$\frac{o \in O_\Sigma \qquad \sigma\colon \mathsf{ar}_{\mathsf{cri}(\Sigma)}(o) \to T_{\mathsf{cri}(\Sigma)}(\lambda) \qquad \{\lambda \mid \Gamma \vdash \mathsf{m}(h_\alpha, \sigma(\alpha))\}_{\alpha \in \mathsf{ar}_{\mathsf{cri}(\Sigma)}(o)}}{\lambda \mid \Gamma \vdash \mathsf{m}\left(\bigwedge_{\alpha \in \mathsf{ar}_{\mathsf{cri}(\Sigma)}(o)} h_\alpha, o(\sigma)\right)}\ \textsc{Exp}$$

We can now proceed as in the case of algebraic signatures.

Proposition 3.2.18. *Let Σ be a κ-bounded signature, then the following hold:*

1. *if S_1 and S_2 are subsets of $\mathsf{Seq}(\Sigma)$ and $S_1 \subseteq S_2$, then $S_1^\vdash \subseteq S_2^\vdash$;*

2. *for every* $S \subseteq \mathsf{Seq}(\Sigma)$, $(S^{\vdash})^{\vdash} = S^{\vdash}$.

Proof. 1. This follows at once since $S_2^{\vdash}$ contains S_2.

2. Clearly $S \subseteq S^{\vdash}$, so $S^{\vdash} \subseteq (S^{\vdash})^{\vdash}$. For the other inclusion it is enough to notice that, by definition, $S^{\vdash}$ is closed under the rules of Fig. 3.1. $\square$

Proposition 3.2.19. *There exists a functor* $\mathsf{Sqt}\colon \mathbf{FSign}_\kappa \to \mathbf{Set}$ *sending a signature* Σ *to the set of* Σ*-sequents.*

Proof. Let $(f,g)\colon \Sigma_1 \to \Sigma_2$ be a morphism in $\mathbf{FSign}_\kappa$, for every $\lambda \in \kappa$ Proposition 2.2.56 yields an arrow

$$\mathsf{tr}_{\mathsf{cri}(f,g),\lambda}\colon T_{\mathsf{cri}(\Sigma_1)}(\lambda) \times T_{\Sigma_1}(\lambda) \to T_{\mathsf{cri}(\Sigma_2)}(\lambda) \times T_{\Sigma_2}(\lambda)$$

On the other hand we also have the arrow $(\eta_{\mathsf{cri}(\Sigma_2),\lambda})_{\mathsf{cri}(\Sigma),*}\colon T_{\mathsf{cri}(\Sigma_1)}(\lambda) \to T_{\mathsf{cri}(\Sigma_2)}(\lambda)$, yielding

$$\mathsf{id}_H \times (\eta_{\mathsf{cri}(\Sigma_2),\lambda})_{\mathsf{cri}(\Sigma),*}\colon H \times T_{\mathsf{cri}(\Sigma_1)}(\lambda) \to H \times T_{\mathsf{cri}(\Sigma_2)}(\lambda)$$

These two arrows, in turn, define $\mathsf{tr}_{(f,g),\lambda}\colon \mathsf{Form}(\Sigma_1,\lambda) \to \mathsf{Form}(\Sigma_2,\lambda)$. We can now take as the image of (f,g) the arrow $\mathsf{tr}_{(f,g)}$ given by the sum of

$$\mathcal{P}(\mathsf{tr}_{(f,g),\lambda}) \times \mathsf{tr}_{(f,g),\lambda}\colon \mathcal{P}(\mathsf{Form}(\Sigma_1,\lambda)) \times \mathsf{Form}(\Sigma_1,\lambda) \to \mathcal{P}(\mathsf{Form}(\Sigma_2,\lambda)) \times \mathsf{Form}(\Sigma_2,\lambda)$$

The thesis now follows at once. $\square$

We need a little generalization of the previous construction to settle some technical points in the following. Let Σ_1 and Σ_2 be objects of $\mathbf{FSign}_\kappa$, let also λ_1 and λ_2 be elements of κ and, finally, let f be a function $T_{\mathsf{cri}(\Sigma_1)}(\lambda_1) \to T_{\mathsf{cri}(\Sigma_2)}(\lambda_2)$, then we can define

$$G_f\colon \mathsf{Form}(\Sigma_1,\lambda_1) \to \mathsf{Form}(\Sigma_2,\lambda_2) \qquad \phi \mapsto \begin{cases} f(t_1) \equiv f(t_2) & \phi \text{ is } t_1 \equiv t_2 \\ \mathsf{m}(h, f(t)) & \phi \text{ is } \mathsf{m}(h,t) \end{cases}$$

Given a set S of sequents in context λ_1, we will denote by S_f the sequent obtained applying G_f pointwise: a sequent is in S_f if and only if it is equal to $\lambda_2 \mid \{G_f(\psi)\}_{\psi\in\Gamma} \vdash G_f(\phi)$ for some element $\lambda_1 \mid \Gamma \vdash \phi$ of S.

Remark 3.2.20. Clearly, $\mathsf{tr}_{\mathsf{cri}(f,g),\lambda}$ coincides with $G_{(\eta_{\mathsf{cri}(\Sigma_2),\lambda})_{\mathsf{cri}(\Sigma),*}}$.

Lemma 3.2.21. *Given* Σ_1 *and* Σ_2 *in* $\mathbf{FSign}_\kappa$, $\lambda_1, \lambda_2 \in \kappa$ *and* $f\colon T_{\mathsf{cri}(\Sigma_1)}(\lambda_1) \to T_{\mathsf{cri}(\Sigma_2)}(\lambda_2)$, *for every set* S *the following are true:*

1. *if a sequent* $\lambda_1 \mid \Gamma \vdash \phi$ *is derivable from* S *using only rules A,* Weak*,* Cut*,* Refl*,* Sym*,* Trans*,* Inf*,* Mon*,* Sup*,* Fun*, then* $\lambda_2 \mid \{G_f(\psi)\}_{\psi\in\Gamma} \vdash G_f(\phi)$ *is derivable from* S_f*;*
2. *if for every* $o \in O_{\Sigma_1}$ *there exists* $o' \in O_{\Sigma_2}$ *such that* $\mathsf{ar}_{\Sigma_1}(o) = \mathsf{ar}\Sigma_2(o')$ *and the square*

$$\begin{tikzcd}
\left(T_{\mathsf{cri}(\Sigma_1)}(\lambda_1)\right)^{\mathsf{ar}_{\mathsf{cri}(\Sigma_1)}(j_1(o))} \arrow[r, "f^{\mathsf{ar}_{\mathsf{cri}(\Sigma_1)}(j_1(o))}"] \arrow[d, "(j_1(o))^{F_{\mathsf{cri}(\Sigma_1)}(\lambda_1)}"'] & \left(T_{\mathsf{cri}(\Sigma_2)}(\lambda_2)\right)^{\mathsf{ar}_{\mathsf{cri}(\Sigma_1)}(k_1(o'))} \arrow[d, "(k_1(o'))^{F_{\mathsf{cri}(\Sigma_2)}(\lambda_2)}"] \\
T_{\mathsf{cri}(\Sigma_1)}(\lambda_1) \arrow[r, "f"'] & T_{\mathsf{cri}(\Sigma_2)}(\lambda_2)
\end{tikzcd}$$

commutes, then the thesis of the previous point holds also adding Exp *and* Cong *to the list of used rules.*

Notation. In the previous lemma j_1 and k_1 denotes the inclusion $O_{\Sigma_1} \to O_{\mathsf{cri}(\Sigma_1)}$ and $O_{\Sigma_2} \to O_{\mathsf{cri}(\Sigma_2)}$ respectively. Notice that, with this notation, the square above makes sense because

$$\begin{aligned}\mathsf{ar}_{\mathsf{cri}(\Sigma_1)}(j_1(o)) &= V_{\mathbf{H}}(\mathsf{ar}_{\Sigma_1}(o)) \\ &= V_{\mathbf{H}}(\mathsf{ar}_{\Sigma_2}(o')) \\ &= \mathsf{ar}_{\mathsf{cri}(\Sigma_2)}(k_1(o'))\end{aligned}$$

Proof. 1. We proceed by induction on the derivation of $\lambda_1 \mid \Gamma \vdash \phi$ from S.

If $\lambda_1 \mid \Gamma \vdash \phi$ is in S there is nothing to show.

$\lambda_1 \mid \Gamma \vdash \phi$ is obtained applying rule A. Then ϕ is in Γ so that $G_f(\phi) \in \{G_f(\psi)\}_{\psi\in\Gamma}$ and an application of the same rule A yields the thesis.

$\lambda_1 \mid \Gamma \vdash \phi$ is obtained applying rule Weak. Thuse $\Gamma = \Gamma' \cup \Delta$ with $\lambda_1 \mid \Gamma' \vdash \phi$ derivable from S using only the listed rules. By the inductive hypothesis we can use again Weak to get

$$\frac{\lambda_2 \mid \{G_f(\psi)\}_{\psi\in\Gamma'} \vdash G_f(\phi)}{\lambda_2 \mid \{G_f(\psi)\}_{\psi\in\Gamma'} \cup \{G_f(\varphi)\}_{\varphi\in\Delta} \vdash G_f(\phi)}\ \textsc{Weak}$$

$\lambda_1 \mid \Gamma \vdash \phi$ is obtained applying rule Cut. Thus there exists $\lambda_1 \mid \Phi \vdash \phi$ satisfying the lemma such that, for every $\varphi \in \Phi$ the sequent $\lambda_1 \mid \Gamma \vdash \varphi$ satisfies the lemma too. The thesis now follows by induction applying

$$\frac{\{\lambda_2 \mid \{G_f(\psi)\}_{\psi\in\Gamma} \vdash G_f(\varphi)\}_{\varphi\in\Phi} \qquad \lambda_2 \mid \{G_f(\varphi)\}_{\varphi\in\Psi} \vdash G_f(\phi)}{\lambda_2 \mid \{G_f(\psi)\}_{\psi\in\Gamma} \vdash G_f(\phi)}\ \textsc{Cut}$$

$\lambda_1 \mid \Gamma \vdash \phi$ is obtained applying rule Refl. Then ϕ must by $t \equiv t$ for some $t \in T_{\mathsf{cri}(\Sigma_1)}(\lambda_1)$ and we can just apply again rule Refl to obtain $\lambda_2 \mid \{G_f(\psi)\}_{\psi\in\Gamma} \vdash f(t) \equiv f(t)$.

$\lambda_1 \mid \Gamma \vdash \phi$ is obtained applying rule Sym. Then ϕ must be $t_1 \equiv t_2$ for some $t_1, t_2 \in T_{\mathsf{cri}(\Sigma_1)}(\lambda_1)$ and $\lambda_1 \mid \Gamma \vdash t_1 \equiv t_2$ is derivable from S used only the given rules. We get the thesis considering

$$\frac{\lambda_2 \mid \{G_f(\psi)\}_{\psi\in\Gamma} \vdash f(t_1) \equiv f(t_2)}{\lambda_2 \mid \{G_f(\psi)\}_{\psi\in\Gamma} \vdash f(t_2) \equiv f(t_1)}\ \textsc{Sym}$$

$\lambda_1 \mid \Gamma \vdash \phi$ is obtained applying rule Trans. Then there exist $t_1, t_2, t_3 \in T_{\mathsf{cri}(\Sigma_1)}(\lambda_1)$ such that ϕ is $t_1 \equiv t_3$ and both $\lambda_1 \mid \Gamma \vdash t_1 \equiv t_2$, $\lambda_1 \mid \Gamma \vdash t_2 \equiv t_3$ both satisfies the hypotheses of our lemma. We conclude using again Trans.

$\lambda_1 \mid \Gamma \vdash \phi$ is obtained applying rule Inf. This is immediate.

$\lambda_1 \mid \Gamma \vdash \phi$ is obtained applying rule Mon. Then ϕ must be $\mathsf{m}(h' \wedge h, t)$ and, we can derive from S, using the admissible rules, the sequent $\lambda_1 \mid \Gamma \vdash \mathsf{m}(h, t)$, so that

$$\frac{\lambda_2 \mid \{G_f(\psi)\}_{\psi\in\Gamma} \vdash \mathsf{m}(h, f(t))}{\lambda_2 \mid \{G_f(\psi)\}_{\psi\in\Gamma} \vdash \mathsf{m}(h' \wedge h, f(t))}\ \textsc{Mon}$$

$\lambda_1 \mid \Gamma \vdash \phi$ is obtained applying rule SUP. As before, we must have a family $S \subseteq H$ such that for every $h \in S$ the sequent $\lambda_1 \mid \Gamma \vdash \mathsf{m}(h,t)$ satisfies our hypotheses, so that

$$\frac{S \subseteq H \qquad \{\lambda_2 \mid \Gamma \vdash \mathsf{m}(h, f(t))\}_{h\in S}}{\lambda_2 \mid \{G_f(\psi)\}_{\psi\in\Gamma} \vdash \mathsf{m}(\mathsf{sup}(S), f(t))}\ \textsc{Sup}$$

$\lambda_1 \mid \Gamma \vdash \phi$ is obtained applying rule FUN. This implies that we can derive, as always using the listed rules, the sequent $\lambda_1 \mid \Gamma \vdash t \equiv s$ and $\lambda_1 \mid \gamma \vdash \mathsf{m}(h,t)$, so that

$$\frac{\lambda_2 \mid \{G_f(\psi)\}_{\psi\in\Gamma} \vdash f(t) \equiv f(s) \qquad \lambda_2 \mid \{G_f(\psi)\}_{\psi\in\Gamma} \vdash \mathsf{m}(h, f(t))}{\lambda_2 \mid \{G_f(\psi)\}_{\psi\in\Gamma} \vdash \mathsf{m}(h, f(s))}\ \textsc{Fun}$$

2. Let us check the two new rules.

   $\lambda_1 \mid \Gamma \vdash \phi$ is obtained applying rule EXP. Then there must be an operation $o \in O_{\Sigma_1}$, a function $\sigma\colon \mathsf{ar}_{\mathsf{cri}(\Sigma_1)}(j_1(o)) \to T_{\mathsf{cri}(\Sigma_1)}(\lambda_1)$ and a family $\{\lambda_1 \mid \Gamma \vdash \mathsf{m}(h_\alpha, \sigma(\alpha))\}_{\alpha\in\mathsf{ar}_{\mathsf{cri}(\Sigma_1)}(j_1(o))}$ of sequents satisying our hypotheses. Since we have assumed that $f(j_1(o)(\sigma))$ and $k_1(o)(f\circ\sigma)$ coincide, the thesis follows applying again EXP to $o' \in O_{\Sigma_2}$, $f\circ\sigma\colon \mathsf{ar}_{\mathsf{cri}(\Sigma_2)}(k_1(o')) \to T_{\mathsf{cri}(\Sigma_2)}(\lambda_2)$ and to the family $\{\lambda_2 \mid \{G_f(\psi)\}_{\psi\in\Gamma} \vdash \mathsf{m}(h_\alpha, f(\sigma(\alpha)))\}_{\alpha\in\mathsf{ar}_{\mathsf{cri}(\Sigma_2)}(k_1(o'))}$.

   $\lambda_1 \mid \Gamma \vdash \phi$ is obtained applying rule CONG. The argument is similar as the one above: we must have $o \in O_{\Sigma_1}$, $\sigma_1, \sigma_2\colon \mathsf{ar}_{\mathsf{cri}(\Sigma_1)}(j_1(o)) \rightrightarrows T_{\mathsf{cri}(\Sigma_1)}(\lambda_1)$ and $\{\lambda_1 \mid \Gamma \vdash \sigma_1(\alpha) \equiv \sigma_2(\alpha)\}_{\alpha\in\mathsf{ar}_{\mathsf{cri}(\Sigma_1)}(j_1(o))}$, and we can conclude by the inductive hypothesis applying again rule Cong to o', $f\circ\sigma_1$, $f\circ\sigma_2$ and to the family $\{\lambda_2 \mid \{G_f(\psi)\}_{\psi\in\Gamma} \vdash f(\sigma_1(\alpha)) \equiv f(\sigma_2(\alpha))\}_{\alpha\in\mathsf{ar}_{\mathsf{cri}(\Sigma_2)}(k_1(o))}$. □

Corollary 3.2.22. *Let $(f,g)\colon \Sigma_1 \to \Sigma_2$ be a morphism of $\mathbf{FSign}_\kappa$. For every $S \subseteq \mathsf{Seq}(\Sigma_1)$, if a sequent is in $S^\vdash$, then its image under $\mathsf{tr}_{(f,g)}$ is in $(\mathsf{tr}_{(f,g)}(S))^\vdash$.*

Proof. Let $\lambda \mid \Gamma \vdash \phi$ be a sequent in $S^\vdash$. Notice that if a sequent is in S there is nothing to show. By Lemma 3.2.21 the only thing we need to show is that if a sequent is derived from S through an application of SUBST, then we can derive its image from $\mathsf{tr}_{(f,g)}(S)$. Suppose then that a sequent $\lambda_2 \mid \Gamma[\sigma] \vdash \phi[\sigma]$ is derived from S for some $\sigma\colon \lambda_1 \to T_{\mathsf{cri}(\Sigma_1)}(\lambda_2)$ and element $\lambda_1 \mid \Gamma \vdash \phi$ of $S^\vdash$. By the inductive hypothesis $\lambda_1 \mid \{\mathsf{tr}_{(f,g),\lambda_1}(\psi)\}_{\psi\in\Gamma} \vdash \mathsf{tr}_{(f,g),\lambda_1}(\phi)$ is in $(\mathsf{tr}_{(f,g)}(S))^\vdash$. Moreover, $(\eta_{\mathsf{cri}(\Sigma_2),\lambda_2})_{\mathsf{cri}(\Sigma_1),*}\circ\sigma$ is an arrow $\lambda_1 \to T_{\mathsf{cri}(\Sigma_2)}(\lambda_2)$ and therefore the sequent

$$\lambda_2 \mid \{\mathsf{tr}_{(f,g),\lambda_1}(\psi)[(\eta_{\mathsf{cri}(\Sigma_2),\lambda_2})_{\mathsf{cri}(\Sigma_1),*}\circ\sigma]\}_{\psi\in\Gamma} \vdash \mathsf{tr}_{(f,g),\lambda_1}(\phi)[(\eta_{\mathsf{cri}(\Sigma_2),\lambda_2})_{\mathsf{cri}(\Sigma_1),*}\circ\sigma]$$

is in $(\mathsf{tr}_{(f,g)}(S))^\vdash$. Now, the diagram

$$\begin{tikzcd}
 & & \lambda_1 \arrow[dll, "\eta_{\mathsf{cri}(\Sigma_1),\lambda_1}"'] \arrow[drr, "\eta_{\mathsf{cri}(\Sigma_1),\lambda_1}"] \arrow[ddll, "\sigma"] \arrow[ddrr, "\eta_{\mathsf{cri}(\Sigma_2),\lambda_1}"'] & & \\
T_{\mathsf{cri}(\Sigma_1)}(\lambda_1) \arrow[d, "\sigma_{\mathsf{cri}(\Sigma_1),*}"'] & & & & T_{\mathsf{cri}(\Sigma_1)}(\lambda_1) \arrow[d, "(\eta_{\mathsf{cri}(\Sigma_2),\lambda_2})_{\mathsf{cri}(\Sigma_1),*}"] \\
T_{\mathsf{cri}(\Sigma_1)}(\lambda_2) \arrow[drr, "(\eta_{\mathsf{cri}(\Sigma_2),\lambda_2})_{\mathsf{cri}(\Sigma_1),*}"] & & & & T_{\mathsf{cri}(\Sigma_2)}(\lambda_1) \arrow[dll, bend left, "((\eta_{\mathsf{cri}(\Sigma_2),\lambda_2})_{\mathsf{cri}(\Sigma_1),*}\circ\sigma)_{\mathsf{cri}(\Sigma_2),*}"] \\
\lambda_2 \arrow[u, "\eta_{\mathsf{cri}(\Sigma_1),\lambda_2}"] \arrow[rr, "\eta_{\mathsf{cri}(\Sigma_2),\lambda_2}"'] & & T_{\mathsf{cri}(\Sigma_2)}(\lambda_2) & &
\end{tikzcd}$$

shows that

$$((\eta_{\mathsf{cri}(\Sigma_2),\lambda_2})_{\mathsf{cri}(\Sigma_1),*} \circ \sigma)_{\mathsf{cri}(\Sigma_2),*} \circ (\eta_{\mathsf{cri}(\Sigma_2),\lambda_1})_{\mathsf{cri}(\Sigma_1),*} = (\eta_{\mathsf{cri}(\Sigma_2),\lambda_2})_{\mathsf{cri}(\Sigma_1),*} \circ \sigma$$

Therefore $\lambda_2 \mid \{\mathsf{tr}_{(f,g),\lambda_1}(\psi)[(\eta_{\mathsf{cri}(\Sigma_2),\lambda_2})_{\mathsf{cri}(\Sigma_1),*} \circ \sigma]\}_{\psi\in\Gamma} \vdash \mathsf{tr}_{(f,g),\lambda_1}(\phi)[(\eta_{\mathsf{cri}(\Sigma_2),\lambda_2})_{\mathsf{cri}(\Sigma_1),*} \circ \sigma]$ coincides with $\lambda_2 \mid \{\mathsf{tr}_{(f,g),\lambda_1}(\psi[\sigma])\}_{\psi\in\Gamma} \vdash \mathsf{tr}_{(f,g),\lambda_1}(\phi[\sigma])$ and we can conclude. □

Now let $\left((A,\mu_A), \left\{o^{\mathcal{A}}\right\}_{o\in O_\Sigma}, \left\{c^{\mathcal{A}}\right\}_{c\in C_\Sigma}\right)$ be a Σ-algebra, for every function $f\colon \lambda \to A$, we have a $\mathsf{cri}\,(\Sigma)$-homomorphism $f_{\mathsf{cri}(\Sigma,*)}\colon F_{\mathsf{cri}(\Sigma)}(\lambda) \to W_\Sigma(\mathcal{A})$ which, in particular, is a function $T_{\mathsf{cri}(\Sigma)}(\lambda) \to A$. So equipped, we are ready to define the notion of theory and introduce satisfability.

Definition 3.2.23. Let κ be a regular cardinal and Σ an object of $\mathbf{FSign}_\kappa$, a subset $\Lambda \subseteq \mathsf{Seq}(\Sigma)$ is a Σ-*theory* (or a theory) if $\Lambda = S^{\vdash}$ for some $S \subseteq \mathsf{Seq}(\Sigma)$, called a *set of axioms* for Λ.

Given a Σ-formula $\lambda \mid \phi$ with context λ and a Σ-algebra $\mathcal{A} = \left((A,\mu_A), \left\{o^{\mathcal{A}}\right\}_{o\in O_\Sigma}, \left\{c^{\mathcal{A}}\right\}_{c\in C_\Sigma}\right)$, we say that $\mathcal{A}$ *satisfies* ϕ *with respect to* $f\colon \lambda \to A$ and we will write $\mathcal{A} \vDash_f \phi$ if:

- $\lambda \mid \phi$ is the Σ-equation $\lambda \mid t_1 \equiv t_2$ and $f_{\mathsf{cri}(\Sigma,*)}(t_1) = f_{\mathsf{cri}(\Sigma,*)}(t_2)$;
- $\lambda \mid \phi$ is a membership proposition $\lambda \mid \mathsf{m}(h,t)$ and $h \leq \mu_A\left(f_{\mathsf{cri}(\Sigma,*)}(t)\right)$.

A sequent $\Gamma \vdash \phi$ with context λ is *satisfied* by $\mathcal{A}$ if, for every $f\colon \lambda \to A$, $\mathcal{A} \vDash_f \phi$ whenever $\mathcal{A} \vDash_f \psi$ for all $\psi \in \Gamma$. The category $\mathbf{Mod}(\Lambda)$ of *models* of a Σ-theory Λ is the full subcategory of Σ-$\mathbf{FAlg}$ given by algebras satisfying all the sequents in Λ. The restriction of $V_\Sigma\colon \Sigma\text{-}\mathbf{FAlg} \to \mathbf{Fuz}(\mathbf{H})$ to $\mathbf{Mod}(\Lambda)$ will be denoted by $V_\Lambda\colon \mathbf{Mod}(\Lambda) \to \mathbf{Fuz}(\mathbf{H})$.

First of all we shall prove that our semantics is sound for the rules of Fig. 3.1.

Lemma 3.2.24. *For every Σ-algebra $\mathcal{A} = \left((A,\mu_A), \left\{o^{\mathcal{A}}\right\}_{o\in O_\Sigma}, \left\{c^{\mathcal{A}}\right\}_{c\in C_\Sigma}\right)$, if $\mathcal{A}$ satisfies all the premises of a rule of the calculus of fuzzy algebraic sequents , then it satisfies also its conclusion.*

Proof. Let us proceed rule by rule.

A. This is tautological.

Weak. If $f\colon \lambda \to A$ is such that $\mathcal{A} \vDash_f \psi$ for every $\psi \in \Gamma \cup \Delta$ then, *a fortiori*, $\mathcal{A}$ satisfies any formula in Γ with respect to f and thus, by hypothesis $\mathcal{A} \vDash_f \phi$.

Cut. Let $f\colon \lambda \to A$ such that $\mathcal{A} \vDash_f \xi$ for every $\xi \in \Gamma$, then, since $\mathcal{A}$ satisfies $\lambda \mid \Gamma \vdash \phi$ for any $\phi \in \Phi$ we also have that it satisfies every element of Φ with respect to f and this implies $\mathcal{A} \vDash_f \psi$.

Refl. This follows from the reflexivity of equality.

Sym. This follows from the symmetry of equality.

Trans. This follows from the transitivity of equality.

Subst. As above, let us take a function $f\colon \lambda_1 \to A$ such that $\mathcal{A}$ satisfies every element $\psi[\sigma]$ of $\Gamma[\sigma]$ with respect to f. Now, we know that

$$(f_{\mathsf{cri}(\Sigma,*)} \circ \sigma)_{\mathsf{cri}(\Sigma,*)} = f_{\mathsf{cri}(\Sigma,*)} \circ \sigma_{\mathsf{cri}(\Sigma,*)}$$

and, by definition,

$$\psi[\sigma] = \begin{cases} \sigma_{\mathsf{cri}(\Sigma,*)}(t_1) \equiv \sigma_{\mathsf{cri}(\Sigma,*)}(t_2) & \psi \text{ is } t_1 \equiv t_2 \\ \mathsf{m}\big(h, \sigma_{\mathsf{cri}(\Sigma,*)}(t)\big) & \psi \text{ is } \mathsf{m}(h,t) \end{cases}$$

Thus $\mathcal{A} \vDash_f \psi[\sigma]$is equivalent to $\mathcal{A} \vDash_{f_{\mathsf{cri}(\Sigma,*)} \circ \sigma} \psi$. But then our hypothesis implies that $\mathcal{A} \vDash_{f_{\mathsf{cri}(\Sigma,*)} \circ \sigma} \phi$ and again this means that $\mathcal{A} \vDash_f \phi[\sigma]$.

Inf. This follows at once from the fact that $\perp$ is the bottom element of $\mathbf{H}$.

Mon. If $\mathcal{A}$ satisfies all the formulae in Γ with respect to some $f\colon \lambda \to A$ then $\mathcal{A} \vDash_f \lambda \mid \mathsf{m}(h, t)$, so that

$$h \leq \mu_A\big(f_{\mathsf{cri}(\Sigma,*)}(t)\big)$$

Therefore, for every other h' in $\mathbf{H}$ we also have $\mathcal{A} \vDash_f \lambda \mid \mathsf{m}(h' \wedge h, t)$ since the previous inequality entails

$$h' \wedge h \leq \mu_A\big(f_{\mathsf{cri}(\Sigma,*)}(t)\big)$$

Sup. As before if f is such that $\mathcal{A} \vDash_f \phi$ for every $\phi \in \Gamma$ then $\mathcal{A} \vDash_f \mathsf{m}(h, t)$ for all $h \in S$, implying that $\mu_A\big(f_{\mathsf{cri}(\Sigma,*)}(t)\big)$ is an upper bound for S.

Fun. This follows at once since μ_A is a function.

Exp. If $\mathcal{A} \vDash_f \phi$ for every $\phi \in \Gamma$, then $h_\alpha \leq \mu_A\big(f_{\mathsf{cri}(\Sigma,*)}\sigma(\alpha)\big)$ for every $\alpha \in \mathsf{ar}_{\mathsf{cri}(\Sigma)}(o)$. Since $f_{\mathsf{cri}(\Sigma,*)}$ is a homomorphism we have

$$\begin{aligned}
\bigwedge_{\alpha \in \mathsf{ar}_{\mathsf{cri}(\Sigma)}(j_1(o))} \big(\mu_{\mathsf{ar}_\Sigma(o)}(\alpha) \to h_\alpha\big) &\leq \bigwedge_{\alpha \in \mathsf{ar}_{\mathsf{cri}(\Sigma)}(j_1(o))} \big(\mu_{\mathsf{ar}_\Sigma(o)}(\alpha) \to f_{\mathsf{cri}(\Sigma,*)}(\sigma(\alpha))\big) \\
&= \mu_{A^{\mathsf{ar}_{\mathsf{cri}(\Sigma)}(j_1(o))}}\big(f_{\mathsf{cri}(\Sigma,*)} \circ \sigma\big) \\
&\leq \mu_A\left(o^{\mathcal{A}}(f_{\mathsf{cri}(\Sigma,*)} \circ \sigma)\right) \\
&= \mu_A\left(o^{\mathcal{A}}\left(f_{\mathsf{cri}(\Sigma,*)}^{\mathsf{ar}_{\mathsf{cri}(\Sigma)}(j_1(o))}(\sigma)\right)\right) \\
&= \mu_A(f_{\mathsf{cri}(\Sigma,*)}(o(\sigma)))
\end{aligned}$$

Cong. This follows at once since $o^{\mathcal{A}}$ is a function. □

We can provide some examples of theories.

Notation. We will stick with the convention used in Chapter 2: instead of using ordinals, variables will be denoted by x, y, z, eventually subscripted. We will use infix notation for operations of arity 2.

Example 3.2.25. The most basic examples of a fuzzy Σ-theory is the one generated by no axioms. Its models are all the Σ-algebras.

Example 3.2.26. Let Σ_{FS} be the signature of Example 3.2.5, we can define four Σ_{FS}-theories [98].

- The theory of *fuzzy semigroups* Λ_S is simply a translation of the theory of semigroups introduced in 2.2.39. More precisely is the one with the following axiom:

$$3 \mid (x \cdot y) \cdot z \equiv x \cdot (y \cdot z)$$

- The theory of *left ideals* Λ_{LI} is obtained adding to Λ_{FS} the axioms:

$$\{2 \mid \mathsf{m}(h, y) \vdash \mathsf{m}(h, x \cdot y)\}_{h \in H}$$

- Similarly the theory Λ_{RI} of *right ideals* is obtained using the axioms (again, one for every $h \in H$):

$$\{2 \mid \mathsf{m}(h, x) \vdash \mathsf{m}(h, x \cdot y)\}_{h \in H}$$

- We get the theory of *(bilateral) ideals* Λ_I adding to Λ_S both kind of previous axioms, i.e. all the sequents of the form

$$2 \mid \mathsf{m}(h, y) \vdash \mathsf{m}(h, x \cdot y) \qquad 2 \mid \mathsf{m}(h, x) \vdash \mathsf{m}(h, x \cdot y)$$

Example 3.2.27. Now let Σ_{FG} be the signature of Example 3.2.6, there are, at least, two interesting Σ_{FG}-theories appearing in the literature.

- We can translate the theory of gropus of Example 2.2.63 to get the theory Λ_{FG} of *fuzzy groups*. It is the theory with axioms given by

$$1 \mid x \cdot x^{-1} \equiv e \quad 1 \mid x^{-1} \cdot x \equiv e \quad 1 \mid e \cdot x \equiv x \quad 1 \mid x \cdot e \equiv x \quad 3 \mid (x \cdot y) \cdot z \equiv x \cdot (y \cdot z)$$

- The theory Λ_{NFG} of *normal fuzzy groups* is obtained adding to Λ_{FG} the axioms:

$$\{2 \mid \mathsf{m}(h, x) \vdash \mathsf{m}\big(h, y \cdot (x \cdot y^{-1})\big)\}_{h \in H}$$

Models for the theories Λ_{FG} and Λ_{NFG} are exactly the fuzzy groups and normal fuzzy groups described in [8, 9, 92, 111].

Proposition 3.2.28. *Given a morphism $(f, g)\colon \Sigma_1 \to \Sigma_2$ in $\mathbf{FSign}_\kappa$, then for every Σ_2-algebra $\mathcal{A}$, defined by $\Big((A, \mu_A), \{o^{\mathcal{A}}\}_{o \in O_{\Sigma_2}}, \{c^{\mathcal{A}}\}_{c \in C_{\Sigma_2}}\Big)$, the following hold true:*

1. *for every Σ_1-formula $\lambda \mid \phi$ and $h\colon \lambda \to A$, $(f, g)^*(\mathcal{A}) \vDash_h \phi$ if and only if $\mathcal{A} \vDash_h \mathsf{tr}_{(f,g),\lambda}(\phi)$;*
2. *$(f, g)^*(\mathcal{A})$ satisfies a sequent $\lambda \mid \Gamma \vdash \phi$ if and only if $\mathcal{A}$ satisfies $\lambda \mid \{\mathsf{tr}_{(f,g),\lambda}(\psi)\}_{\psi \in \Gamma} \vdash \mathsf{tr}_{(f,g),\lambda}(\phi)$;*
3. *if Λ_1 and Λ_2 are, respectively, a Σ_1-theory and a Σ_2-theory such that $\mathsf{tr}_{(f,g)}(\Lambda_1) \subseteq \Lambda_2$ and $\mathcal{A}$ is a model for Λ_2 then $(f, g)^*(\mathcal{A})$ belongs to $\mathbf{Mod}(\Lambda_1)$.*

Proof. 1. For every $k\colon \lambda \to A$, we have a $\mathsf{cri}(\Sigma_2)$-homomomorphism $k_{\mathsf{cri}(\Sigma_2,*)}\colon F_{\mathsf{cri}(\Sigma_2)}(\lambda) \to \mathcal{A}$ which is also a $\mathsf{cri}(\Sigma_1)$-homomorphism $\mathsf{cri}(f, g)^*\big(F_{\mathsf{cri}(\Sigma_2)}(\lambda)\big) \to \mathsf{cri}(f, g)^*(\mathcal{A})$. In particular, this means that $k_{\mathsf{cri}(\Sigma_2,*)} \circ \big(\eta_{\mathsf{cri}(\Sigma_2),\lambda}\big)_{\mathsf{cri}(\Sigma_1,*)}$ is the unique arrow of $\mathsf{cri}(\Sigma_1)$-$\mathbf{Alg}$ such that

$$k = k_{\mathsf{cri}(\Sigma_2),*} \circ \big(\eta_{\mathsf{cri}(\Sigma_2),\lambda}\big)_{\mathsf{cri}(\Sigma_1),*} \circ \eta_{\mathsf{cri}(\Sigma_1),\lambda}$$

Now the thesis follows at once noticing that, by construction

$$\mathsf{tr}_{(f,g),\lambda}(\phi) = \begin{cases} \big(\eta_{\mathsf{cri}(\Sigma_2),\lambda}\big)_{\mathsf{cri}(\Sigma_1,*)}(t_1) \equiv \big(\eta_{\mathsf{cri}(\Sigma_2),\lambda}\big)_{\mathsf{cri}(\Sigma_1,*)}(t_2) & \phi \text{ is } t_1 \equiv t_2 \\ \mathsf{m}\Big(h, \big(\eta_{\mathsf{cri}(\Sigma_2),\lambda}\big)_{\mathsf{cri}(\Sigma_1,*)}(t)\Big) & \phi \text{ is } \mathsf{m}(h, t) \end{cases}$$

2. Let us show the two implications.

($\Rightarrow$) Let $k\colon \lambda \to A$ such that $\mathcal{A} \vDash_k \mathsf{tr}_{(f,g),\lambda}(\psi)$ for every $\psi \in \Gamma$, by the previous point $(f, g)^*(\mathcal{A}) \vDash_k \psi$ and thus $\mathcal{A}$ also satisfies ϕ with respect to k. The thesis now follows applying again point 1.

($\Leftarrow$) The argument is pretty much the same as before. If $k\colon \lambda \to A$ is such that $(f, g)^*(\mathcal{A}) \vDash_k \psi$ for every element in Γ, then $\mathcal{A} \vDash_k \mathsf{tr}_{(f,g),\lambda}(\psi)$ and thus $(f, g)^*(\mathcal{A})$ also satisfies $\mathsf{tr}_{(f,g),\lambda}(\phi)$ with respect to k and this in turn entails the thesis.

3. This follows immediately from the previous two points. □

It is worth noticing that we do not have analogs for Lemma 2.2.64 and Corollary 2.2.65, as shown by the following example.

Example 3.2.29. Let $\mathbf{H}$ be $([0,1],\leq)$, Σ be the empty signature and Λ the theory with the axiom

$$2 \mid \mathsf{m}(1,x) \vdash x \equiv y$$

Since Σ is the empty signature, Σ-**FAlg** is simply **Fuz**(**H**) and $T_{\mathsf{cri}(\Sigma)}$ is $\mathsf{id}_{\mathbf{Fuz}(\mathbf{H})}$. For every $n \in \mathbb{N}$ we can define a constant function

$$\mu_n \colon 2 \to [0,1] \qquad t \mapsto \frac{n}{n+1}$$

and take as $\mathcal{A}_n$ simply $(2,\mu_n)$. By construction, there are no functions $2 \to 2$ such that $\mathcal{A}_n \vDash_f \mathsf{m}(1,x)$, so, for every $n \in \mathcal{N}$, $\mathcal{A}_n \in \mathbf{Mod}(\Lambda)$. Now, if $n \leq m$, id_2 defines an arrow $f_{n,m} \colon \mathcal{A}_n \to \mathcal{A}_m$ yielding a functor F from the $\aleph_0$-filtered category induced by $(\mathbb{N},\leq)$ into $\mathbf{Mod}(\Lambda)$. This functor F has a colimit in $\mathbf{Mod}(\Lambda)$: indeed, if $\big((C,\mu_C),\{c_n\}_{n\in\mathbb{N}}\big)$ is a cocone on F, then, for every $n \in \mathbb{N}$

$$\begin{aligned}\mu_C(c_0(0)) &= \mu_C(c_n(f_{0,n})(0))\\ &= \mu_C(c_n(0))\\ &\geq \mu_n(0)\\ &= \frac{n}{n+1}\end{aligned}$$

and therefore $\mu_C(c_0(0)) = 1$.Take now any other $c \in C$ and the function f sending 0 to $c_0(0)$ and 1 to c, by hypothesis $(C,\mu_C) \vDash_f \mathsf{m}(1,x)$, so that c must coincide with $c_0(0)$. This in turn shows that (C,μ_C) must be isomorphic to the terminal fuzzy set $(1,\delta_\top)$, which is a model for Λ and that F has a colimiting cocone given by $\Big((1,\delta_\top),\big\{!_{(2,\mu_n)}\big\}_{n\in\mathbb{N}}\Big)$.

On the other hand, if $\gamma_1 : 2 \to [0,1]$ is the function costant in 1, by Corollary 3.1.26, $V_\Lambda \circ F$ has $(2,\gamma_1)$ as colimit. $(2,\gamma_1)$ is not a model of Λ, hence V_Λ does not have rank $\aleph_0$.

The free model of a theory

In this section we are going to show that given a κ-bounded signature Σ and a Σ-theory Λ, the forgetful functor $V_\Lambda \colon \mathbf{Mod}(\Lambda) \to \mathbf{Fuz}(\mathbf{H})$, similarly to its **Set**-based analog U_Λ, has a left adjoint F_Λ.

Take a κ-bounded signature Σ and a set X. We can add the element of X to Σ defining another κ-bounded signature Σ_X as

$$O_{\Sigma_X} := O_\Sigma \qquad C_{\Sigma_X} := C_\Sigma + X \qquad \mathsf{ar}_{\Sigma_X} := \mathsf{ar}_\Sigma$$

Notation. Let us fix some notation to avoid confusion between the different roles of elements of X. Let $\iota_X : X \to C_\Sigma + X$ be the coprojection, for every set Y and $x \in X$ we have a function

$$(\iota_X(x))^{F_{\mathsf{cri}(\Sigma_X)}(Y)} \colon 1 \to T_{\mathsf{cri}(\Sigma_X)}(Y)$$

In particular, we will denote the element of $T_{\mathsf{cri}(\Sigma_X)}(\emptyset)$ picked out by $(\iota_X(x))^{F_{\mathsf{cri}(\Sigma_X)}(\emptyset)}$ with $\widehat{x}$. This allows us to define a function

$$\omega_X \colon X \to T_{\mathsf{cri}(\Sigma_X)}(\emptyset) \qquad x \mapsto \widehat{x}$$

Let ι_{C_Σ} be the coprojection $C_\Sigma \to C_{\Sigma_X}$, then we have a morphism of signatures $(\mathsf{id}_{O_\Sigma}, \iota_{C_\Sigma}) \colon \Sigma \to \Sigma_X$. Moreover, for every set X, we can promote $T_{\mathsf{cri}(\Sigma)}(X)$ to a $\mathsf{cri}\,(\Sigma_X)$-algebra $P_\Sigma(X)$ defining

$$x^{P_\Sigma(X)} \colon 1 \to T_{\mathsf{cri}(\Sigma)}(X) \qquad \emptyset \mapsto \eta_{\mathsf{cri}(\Sigma),X}(x)$$

On the other hand, $T_{\mathsf{cri}(\Sigma_X)}(\emptyset)$ carries a $\mathsf{cri}\,(\Sigma)$-algebra structure obtained by applying $\mathsf{cri}\,(\mathsf{id}_{O_\Sigma}, \iota_{C_\Sigma})^*$ to $F_{\mathsf{cri}(\Sigma_X)}(\emptyset)$. All these structures can be linked together by canonical morphisms as in the diagram below

$$\begin{tikzcd}
& & X \arrow[dll, bend right, "\eta_{\mathsf{cri}(\Sigma),X}"'] \arrow[dl, "\eta_{\mathsf{cri}(\Sigma_X),X}"'] \arrow[d, "\omega_X"] \arrow[dr, bend left, "\eta_{\mathsf{cri}(\Sigma),X}"] & \\
T_{\mathsf{cri}(\Sigma)}(X) \arrow[r, "\gamma_{1,X}"] \arrow[rr, bend left, "\gamma_{4,X}"] \arrow[rrr, bend right, "\mathsf{id}_{T_{\mathsf{cri}(\Sigma)}(X)}"'] & T_{\mathsf{cri}(\Sigma_X)}(X) \arrow[r, "\gamma_{2,X}"'] & T_{\mathsf{cri}(\Sigma_X)}(\emptyset) \arrow[r, "\gamma_{3,X}"] & T_{\mathsf{cri}(\Sigma)}(X)
\end{tikzcd}$$

where

$$\gamma_{1,X} = \left(\eta_{\mathsf{cri}(\Sigma_X),X}\right)_{\mathsf{cri}(\Sigma),*} \qquad \gamma_{2,X} = (\omega_X)_{\mathsf{cri}(\Sigma_X),*}$$
$$\gamma_{3,X} = \left(\eta_{\mathsf{cri}(\Sigma),X} \circ ?_X\right)_{\mathsf{cri}(\Sigma_X),*} \qquad \gamma_{4,X} = (\omega_X)_{\mathsf{cri}(\Sigma),*}$$

Notice that the last triangle commutes because $\gamma_{3,(X,\mu_X)}$ is a morphism of $\mathsf{cri}\,(\Sigma_X)$-algebras.

Finally, let us note that, for every function $f \colon X \to Y$, we can define an arrow $(\mathsf{id}_{O_\Sigma}, \mathsf{id}_{C_\Sigma} + f) \colon \Sigma_X \to \Sigma_Y$ in $\mathbf{FSign}_\kappa$. In particular, we can consider the $\mathsf{cri}\,(\Sigma_X)$-homomorphism $\gamma_f \colon F_{\mathsf{cri}(\Sigma_X)}(\emptyset) \to \mathsf{cri}\,(\mathsf{id}_{O_\Sigma}, \mathsf{id}_{C_\Sigma} + f)^* \left(F_{\mathsf{cri}(\Sigma_Y)}(\emptyset)\right)$ given by $\left(\eta_{\mathsf{cri}(\Sigma_Y),\emptyset}\right)_{\mathsf{cri}(\Sigma_X),*}$, moreover γ_f fits in the square:

$$\begin{tikzcd}
X \arrow[d, "\omega_{(X,\mu_X)}"'] \arrow[rr, bend left, "f"] & \emptyset \arrow[l, "?_X"'] \arrow[r, "?_Y"] \arrow[dl, "\eta_{\mathsf{cri}(\Sigma_X),\emptyset}"] \arrow[dr, "\eta_{\mathsf{cri}(\Sigma_Y),\emptyset}"'] & Y \arrow[d, "\omega_{(Y,\mu_Y)}"] \\
T_{\mathsf{cri}(\Sigma_X)}(\emptyset) \arrow[rr, "\gamma_f"'] & & T_{\mathsf{cri}(\Sigma_Y)}(\emptyset)
\end{tikzcd}$$

Lemma 3.2.30. *Given $\Sigma \in \mathbf{FSign}_\kappa$ for every set X, γ_3 is a $\mathsf{cri}\,(\Sigma)$- and $\mathsf{cri}\,(\Sigma_X)$-homomorphism with inverse $\gamma_{4,X}$. Moreover, for every $f \colon X \to Y$, the following diagram is commutative*

$$\begin{tikzcd}
T_{\mathsf{cri}(\Sigma_X)}(\emptyset) \arrow[r, "\gamma_{3,X}"] \arrow[d, "\gamma_f"'] & T_{\mathsf{cri}(\Sigma)}(X) \arrow[d, "T_{\mathsf{cri}(\Sigma)}(f)"] \\
T_{\mathsf{cri}(\Sigma_Y)}(\emptyset) \arrow[r, "\gamma_{3,Y}"'] & T_{\mathsf{cri}(\Sigma)}(Y)
\end{tikzcd}$$

Proof. We already know that $\gamma_{3,X} \circ \gamma_{4,X}$ is the identity on $T_{\mathsf{cri}(\Sigma)}(X)$. On the other hand, $F_{\mathsf{cri}(\Sigma_X)}(\emptyset)$ is the initial $\mathsf{cri}\,(\Sigma_X)$-algebra, thus $\gamma_{4,X} \circ \gamma_{3,X}$ must be the identity too. The same observation also shows the commutativity of the given square. $\square$

Next, we want to add all the structure of a fuzzy set (X, μ_X) to a Σ-theory.

Definition 3.2.31. Given a Σ-theory Λ, with $\Sigma \in \mathbf{Sign}_\kappa$, we define the Σ_X-theory $\Lambda_{(X,\mu_X)}$, by putting

$$\Lambda_{(X,\mu_X)} := \left(\mathsf{tr}_{(\mathsf{id}_{O_\Sigma}, \iota_{C_\Sigma})}(\Lambda) \cup \{0 \mid \mathsf{m}(\mu_X(x), \widehat{x})\}_{x\in X}\right)^{\vdash}$$

Remark 3.2.32. Every morphism $f\colon (X, \mu_X) \to (Y, \mu_Y)$ of $\mathbf{Fuz}(\mathbf{H})$ is, in particular, a function $f\colon X \to Y$, so that we can consider $(\mathsf{id}_{O_\Sigma}, \mathsf{id}_{C_\Sigma} + f)\colon \Sigma_X \to \Sigma_Y$ in $\mathbf{FSign}_\kappa$ as before. From the inequality

$$\mu_X(x) \leq \mu_Y(f(y))$$

and from Corollary 3.2.22, we can deduce that $\mathsf{tr}_{(\mathsf{id}_{O_\Sigma}, \mathsf{id}_{C_\Sigma}+f)}(\Lambda_{(X,\mu_X)})$ is a subset of $\Lambda_{(Y,\mu_Y)}$.

We are especially interested to the case in which $(X, \mu_X) = \nabla_{\mathbf{H}}(\lambda)$ for some cardinal $\lambda < \kappa$. In this case we can define the following auxiliary functions:

$$G_\lambda\colon \mathsf{Form}(\mathsf{cri}(\Sigma_\lambda), \lambda) \to \mathsf{Form}(\mathsf{cri}(\Sigma_\lambda), 0) \qquad \phi \mapsto \begin{cases} \gamma_{2,\lambda}(t_1) \equiv \gamma_{2,\lambda}(t_2) & \phi \text{ is } t_1 \equiv t_2 \\ \mathsf{m}(h, \gamma_{2,\lambda})(t) & \phi \text{ is } \mathsf{m}(h,t) \end{cases}$$

$$H_\lambda\colon \mathsf{Form}(\mathsf{cri}(\Sigma_\lambda), 0) \to \mathsf{Form}(\mathsf{cri}(\Sigma), \lambda) \qquad \phi \mapsto \begin{cases} \gamma_{3,\lambda}(t_1) \equiv \gamma_{3,\lambda}(t_2) & \phi \text{ is } t_1 \equiv t_2 \\ \mathsf{m}(h, \gamma_{3,\lambda})(t) & \phi \text{ is } \mathsf{m}(h,t) \end{cases}$$

$$K_\lambda\colon \mathsf{Form}(\mathsf{cri}(\Sigma), \lambda) \to \mathsf{Form}(\mathsf{cri}(\Sigma_\lambda), 0) \qquad \phi \mapsto \begin{cases} \gamma_{4,\lambda}(t_1) \equiv \gamma_{4,\lambda}(t_2) & \phi \text{ is } t_1 \equiv t_2 \\ \mathsf{m}(h, \gamma_{4,\lambda}(t)) & \phi \text{ is } \mathsf{m}(h,t) \end{cases}$$

Remark 3.2.33. By construction and by Proposition 2.1.11, we have identities

$$K_\lambda = G_\lambda \circ \mathsf{tr}_{(\mathsf{id}_{O_\Sigma}, \iota_{C_\Sigma}),\lambda} \qquad \mathsf{id}_{\mathsf{Form}(\mathsf{cri}(\Sigma),\lambda)} = H_\lambda \circ K_\lambda$$

We can also notice the commutativity of the diagram

$$\begin{tikzcd}
\emptyset \arrow[r, "?_\lambda"] \arrow[ddr, bend left, "\eta_{\mathsf{cri}(\Sigma_\lambda),\emptyset}"] \arrow[dd, "\eta_{\mathsf{cri}(\Sigma),\emptyset}"'] & \lambda \arrow[dr, bend left, "\omega_\lambda"] \arrow[dd, "\eta_{\mathsf{cri}(\Sigma),\lambda}"] & \\
& & T_{\mathsf{cri}(\Sigma_\lambda)}(\emptyset) \\
T_{\mathsf{cri}(\Sigma)}(\emptyset) \arrow[r, "T_{\mathsf{cri}(\Sigma)}(?_\lambda)"'] & T_{\mathsf{cri}(\Sigma)}(\lambda) \arrow[ur, bend right, "\gamma_{4,\lambda}"'] &
\end{tikzcd}$$

which shows that $\gamma_{4,\lambda} \circ T_{\mathsf{cri}(\Sigma)}(?_\lambda)$ coincides with $\mathsf{tr}_{(\mathsf{id}_{O_\Sigma}, \iota_{C_\Sigma}),0}$.

Proposition 3.2.34. *Let Σ be in $\mathbf{Sign}_\kappa$ and Λ a Σ-theory, then for every $\lambda \in \kappa$ the following are true:*

1. *if $\lambda \mid \Gamma \vdash \phi$ is in $\Lambda_{\nabla_{\mathbf{H}}(\lambda)}$ then $0 \mid \{G_\lambda(\psi)\}_{\psi\in\Gamma} \vdash G_\lambda(\phi)$ is in $\Lambda_{\nabla_{\mathbf{H}}(\lambda)}$ too;*
2. *if $0 \mid \Gamma \vdash \phi$ is in $\Lambda_{\nabla_{\mathbf{H}}(\lambda)}$ then $\lambda \mid \{H_\lambda(\psi)\}_{\psi\in\Gamma} \vdash H_\lambda(\phi)$ is in Λ;*
3. *if $\lambda \mid \Gamma \vdash \phi$ is in Λ then $0 \mid \{K_\lambda(\psi)\}_{\psi\in\Gamma} \vdash K_\lambda(\phi)$ belongs to $\Lambda_{\nabla_{\mathbf{H}}(\lambda)}$.*

Proof. 1. This follows at once applying rule Subst.

2. Let us start showing the thesis for the axioms for $\Lambda_{\nabla_{\mathbf{H}}(\lambda)}$.

   - $0 \mid \Gamma \vdash \phi$ is $0 \mid \left\{\mathsf{tr}_{(\mathsf{id}_{O_\Sigma}, \iota_{C_\Sigma}),0}(\psi')\right\}_{\psi' \in \Gamma'} \vdash \mathsf{tr}_{(\mathsf{id}_{O_\Sigma}, \iota_{C_\Sigma}),0}(\phi')$ for some element $0 \mid \Gamma' \vdash \phi'$ in Λ. Since, by Remark 3.2.33 $\mathsf{tr}_{(\mathsf{id}_{O_\Sigma}, \iota_{C_\Sigma}),0}$ is equal to $\gamma_{4,\lambda} \circ T_{\mathsf{cri}(\Sigma)}(?_\lambda)$ and $H_\lambda \circ K_\lambda$ is the identity, the sequent $\lambda \mid \{H_\lambda(\psi)\}_{\psi \in \Gamma} \vdash H_\lambda(\phi)$ must coincide with $0 \mid \Gamma'[\eta_{\mathsf{cri}(\Sigma),\lambda} \circ ?_\lambda] \vdash \phi'[\eta_{\mathsf{cri}(\Sigma),\lambda} \circ ?_\lambda]$ and the thesis follows applying rule Subst.
   - $0 \mid \Gamma \vdash \phi$ is $0 \mid \mathsf{m}(\bot, \widehat{\mu})$ for some $\mu \in \lambda$. Then, by construction, $\lambda \mid \{H_\lambda(\psi)\}_{\psi \in \Gamma} \vdash H_\lambda(\phi)$ is $\lambda \mid \mathsf{m}\big(\bot, \eta_{\mathsf{cri}(\Sigma),\lambda}(\mu)\big)$ which is in Λ by rule Inf.

   We can now proceed by induction on a derivation of $0 \mid \Gamma \vdash \phi$ from the axioms of $\Lambda_{\nabla_{\mathbf{H}}(\lambda)}$. By Lemma 3.2.21 the only case we have to deal with is the application of rule Subst. Suppose then that $0 \mid \Gamma \vdash \phi$ is obtained applying Subst, then there exists $\lambda_1 < \kappa$, a function $\sigma\colon \lambda_1 \to T_{\mathsf{cri}(\Sigma_\lambda)}(\emptyset)$ and a sequent $\lambda_1 \mid \Theta \vdash \varphi$ in $\Lambda_{\nabla_{\mathbf{H}}(\lambda)}$ such that $\Gamma = \Theta[\sigma]$ and $\phi = \varphi[\sigma]$. Now, if $\lambda_1 = 0$, σ must be $?_{T_{\mathsf{cri}(\Sigma_\lambda)}}$, so that $\sigma_{\mathsf{cri}(\Sigma_\lambda),*}$ must be the identity and there is nothing to show. Suppose then that λ_1 is not the empty set, so there is a function $f\colon \lambda \to \lambda_1$ which, in particular, defines a morphism $\nabla_{\mathbf{H}}(\lambda) \to \nabla_{\mathbf{H}}(\lambda_1)$ and, by Remark 3.2.32 an arrow $(\mathsf{id}_{O_\Sigma}, \mathsf{id}_{C_\Sigma} + f)\colon \Sigma_\lambda \to \Sigma_{\lambda_1}$. By the same Remark 3.2.32, we know that the sequent

   $$\lambda_1 \mid \left\{\mathsf{tr}_{(\mathsf{id}_{O_\Sigma}, \mathsf{id}_{C_\Sigma}+f),\lambda_1}(\alpha)\right\}_{\alpha \in \Theta} \vdash \mathsf{tr}_{(\mathsf{id}_{O_\Sigma}, \mathsf{id}_{C_\Sigma}+f),\lambda_1}(\varphi)$$

   is an element of $\Lambda_{\nabla_{\mathbf{H}}(\lambda_1)}$. Define

   $$\bar{\alpha} = \mathsf{tr}_{(\mathsf{id}_{O_\Sigma}, \mathsf{id}_{C_\Sigma}+f),\lambda_1}(\alpha) \qquad \bar{\varphi} = \mathsf{tr}_{(\mathsf{id}_{O_\Sigma}, \mathsf{id}_{C_\Sigma}+f),\lambda_1}(\varphi) \qquad \bar{\Theta} = \{\bar{\alpha}\}_{\alpha \in \Theta}$$

   Therefore the sequent $\lambda_1 \mid \bar{\Theta} \vdash \bar{\varphi}$ is in $\Lambda_{\nabla_{\mathbf{H}}(\lambda_1)}$. Point 1 and the inductive hypothesis entails that $\lambda_1 \mid \{H_{\lambda_1}(G_{\lambda_1}(\bar{\alpha}))\}_{\alpha \in \Theta} \vdash H_{\lambda_1}(G_{\lambda_1}(\bar{\varphi}))$ is in Λ, so that we get

   $$\frac{\gamma_{3,\lambda} \circ \sigma\colon \lambda_1 \to T_{\mathsf{cri}(\Sigma)}(\lambda) \qquad \lambda_1 \mid \{H_{\lambda_1}(G_{\lambda_1}(\bar{\alpha}))\}_{\alpha \in \Theta} \vdash H_{\lambda_1}(G_{\lambda_1}(\bar{\varphi}))}{\lambda \mid \{H_{\lambda_1}(G_{\lambda_1}(\bar{\alpha}))\,[\gamma_{3,\lambda} \circ \sigma]\}_{\alpha \in \Theta} \vdash H_{\lambda_1}(G_{\lambda_1}(\bar{\varphi}))\,[\gamma_{3,\lambda} \circ \sigma]}\ \textsc{Subst}$$

   Now, let γ be $\left(\eta_{\mathsf{cri}(\Sigma_{\lambda_1}),\lambda_1}\right)_{\mathsf{cri}(\Sigma_\lambda),*}$ so that, for any Σ_λ-formula β in context λ_1, we have

   $$\mathsf{tr}_{(\mathsf{id}_{O_\Sigma}, \mathsf{id}_{C_\Sigma}+f),\lambda_1}(\beta) = \begin{cases} \gamma(t_1) \equiv \gamma(t_2) & \beta \text{ is } t_1 \equiv t_2 \\ \mathsf{m}(h, \gamma(t)) & \beta \text{ is } \mathsf{m}(h,t) \end{cases}$$

   Then we have a diagram

   $$\begin{tikzcd}
   T_{\mathsf{cri}(\Sigma_\lambda)}(\emptyset) \arrow[dd, "\gamma_{3,\lambda}"'] & & T_{\mathsf{cri}(\Sigma_\lambda)}(\lambda_1) \arrow[ll, "\sigma_{\mathsf{cri}(\Sigma_\lambda),*}"'] \arrow[d, "\gamma"] \\
   & \lambda_1 \arrow[ul, "\sigma"'] \arrow[ur, bend left=15, "\eta_{\mathsf{cri}(\Sigma_\lambda),\lambda_1}"] \arrow[r, "\eta_{\mathsf{cri}(\Sigma_\lambda),\lambda_1}"] \arrow[dr, bend right=15, "\omega_{\lambda_1}"] \arrow[d, "\eta_{\mathsf{cri}(\Sigma),\lambda_1}"'] & T_{\mathsf{cri}(\Sigma_{\lambda_1})}(\lambda_1) \arrow[d, "\gamma_{2,\lambda_1}"] \\
   T_{\mathsf{cri}(\Sigma)}(\lambda) & T_{\mathsf{cri}(\Sigma)}(\lambda_1) \arrow[l, "(\gamma_{3,\lambda} \circ \sigma)_{\mathsf{cri}(\Sigma),*}"] & T_{\mathsf{cri}(\Sigma_\lambda)}(\emptyset) \arrow[l, "\gamma_{3,\lambda_1}"]
   \end{tikzcd}$$

showing that $\lambda \mid \left\{H_{\lambda_1}(G_{\lambda_1}(\bar{\alpha}))\left[\gamma_{3,\nabla_{\mathbf{H}}(\lambda)} \circ \sigma\right]\right\}_{\alpha\in\Theta} \vdash H_{\lambda_1}(G_{\lambda_1}(\bar{\varphi}))\left[\gamma_{3,\nabla_{\mathbf{H}}(\lambda)} \circ \sigma\right]$ is equal to the sequent $\lambda \mid \{H_\lambda(\psi)\}_{\psi\in\Gamma} \vdash H_\lambda(\phi)$ as desired.

3. By Remark 3.2.33 $K_\lambda = G_\lambda \circ \mathsf{tr}_{(\mathsf{id}_{O_\Sigma}, \iota_{C_\Sigma}),\lambda}$ and the thesis follows from point 1. □

Given a Σ-theory Λ, we can define a relation $\sim_{\Lambda_{(X,\mu_X)}}$ on $T_{\mathsf{cri}(\Sigma_X)}(\emptyset)$ putting $t_1 \sim_{\Lambda_{(X,\mu_X)}} t_2$ if and only if $0 \mid t_1 \equiv t_2$ belongs to $\Lambda_{(X,\mu_X)}$. Let us look more closely at the properties of $\sim_{\Lambda_{(X,\mu_X)}}$.

Proposition 3.2.35. *$\sim_{\Lambda_{(X,\mu_X)}}$ is a $\mathsf{cri}(\Sigma_X)$-congruence on $F_{\mathsf{cri}(\Sigma_X)}(\emptyset)$.*

Proof. By rules Refl, Sym and Trans we have that $\sim_{\Lambda_{(X,\mu_X)}}$ is an equivalence relation. On the other hand, given $o \in O_{\mathsf{cri}(\Sigma_X)}$ and $\sigma_1, \sigma_2 \colon \mathsf{ar}_{\mathsf{cri}(\Sigma_X)}(o) \rightrightarrows T_{\mathsf{cri}(\Sigma_X)}(\emptyset)$, if $\sigma_1(\alpha) \sim_{\Lambda_{(X,\mu_X)}} \sigma_2(\alpha)$, for every $\alpha \in \mathsf{ar}_{\mathsf{cri}(\Sigma_X)}(o)$ we know that $0 \mid \sigma_1(\alpha) \equiv \sigma_2(\alpha)$ belongs to $\Lambda_{(X,\mu_X)}$ and thus an application of Cong yields $o(\sigma_1) \sim_{\Lambda_{(X,\mu_X)}} o(\sigma_2)$. □

Let $\pi_{\Lambda_{(X,\mu_X)}} \colon T_{\mathsf{cri}(\Sigma_X)}(\emptyset) \to T_\Lambda(X,\mu_X)$ be the quotient map defined by $\sim_{\Lambda_{(X,\mu_X)}}$. By Proposition 3.2.35 and Lemma 2.2.68, for every $\mathsf{cri}(\Sigma_X)$-operation o of arity λ, we have a uniquely determined function $o_{\Lambda_{(X,\mu_X)}} \colon (T_\Lambda(X,\mu_X))^\lambda \to T_\Lambda(X,\mu_X)$ making $\pi_{\Lambda_{(X,\mu_X)}}$ a $\mathsf{cri}(\Sigma_X)$-homomorphism. Our next goal is to promote this algebra to an object of Σ-**FAlg**.

Lemma 3.2.36. *Let Σ be a κ-bounded fuzzy signature and Λ a Σ-theory, then the following hold true:*

1. *there exists a function $\mu_{\Lambda,(X,\mu_X)} \colon T_\Lambda(X,\mu_X) \to H$ such that for every $t \in T_{\mathsf{cri}(\Sigma_X)}(\emptyset)$, the sequent*
$$0 \mid \mathsf{m}\Big(\mu_{\Lambda,(X,\mu_X)}\left(\pi_{\Lambda_{(X,\mu_X)}}(t)\right), t\Big)$$
belongs to $\Lambda_{(X,\mu_X)}$;
2. *there exists a Σ_X-algebra $L_{\Lambda_{(X,\mu_X)}}$ on $\left(T_\Lambda(X,\mu_X), \mu_{\Lambda,(X,\mu_X)}\right)$ such that*
$$o^{L_{\Lambda(X,\mu_X)}} = j_1(o)_{\Lambda_{(X,\mu_X)}} \qquad c^{L_{\Lambda(X,\mu_X)}} = j_2(c)_{\Lambda_{(X,\mu_X)}}$$
where j_1 and j_2 are the inclusions of, respectively, O_{Σ_X} and C_{Σ_X} into $O_{\mathsf{cri}(\Sigma_X)}$;
3. *for every $\sigma \colon \lambda \to T_{\mathsf{cri}(\Sigma_X)}(\emptyset)$, $L_{\Lambda_{(X,\mu_X)}} \vDash_{\pi_{\Lambda_{(X,\mu_X)}}\circ\sigma} \phi$ if and only if $0 \mid \phi[\sigma]$ is in $\Lambda_{(X,\mu_X)}$;*
4. *$L_{\Lambda_{(X,\mu_X)}}$ is a model of $\Lambda_{(X,\mu_X)}$;*
5. *the Σ-algebra $F_\Lambda(X,\mu_X)$ obtained applying $(\mathsf{id}_{O_\Sigma}, \iota_{C_\Sigma})^*$ to $L_{\Lambda_{(X,\mu_X)}}$ is a model of Λ.*

Proof. 1. Let us start by defining a function
$$\mu'_{\Lambda,(X,\mu_X)} \colon T_{\mathsf{cri}(\Sigma_X)}(\emptyset) \to H \qquad t \mapsto \sup\left(\{h \in H \mid 0 \mid \mathsf{m}(h,t) \in \Lambda_{(X,\mu_X)}\}\right)$$
If t_1 and $t_2 \in T_{\mathsf{cri}(\Sigma_X)}(\emptyset)$ are such that $t_1 \sim_{\Lambda_{(X,\mu_X)}} t_2$ then both $0 \mid t_1 \equiv t_2$ and $0 \mid t_2 \equiv t_1$ belong to $\Lambda_{(X,\mu_X)}$ and thus we have derivations
$$\frac{0 \mid t_1 \equiv t_2 \qquad 0 \mid \mathsf{m}(h,t_1)}{0 \mid \mathsf{m}(h,t_2)}\ \textsc{Fun} \qquad \frac{0 \mid t_2 \equiv t_1 \qquad 0 \mid \mathsf{m}(h,t_2)}{0 \mid \mathsf{m}(h,t_1)}\ \textsc{Fun}$$
showing that
$$\{h \in H \mid 0 \mid \mathsf{m}(h,t_1) \in \Lambda_{(X,\mu_X)}\} = \{h \in H \mid 0 \mid \mathsf{m}(h,t_2) \in \Lambda_{(X,\mu_X)}\}$$

which implies $\mu'_{\Lambda,(X,\mu_X)}(t_1) = \mu'_{\Lambda,(X,\mu_X)}(t_2)$, therefore inducing $\mu_{\Lambda_{(X,\mu_X)}} : T_\Lambda(X, \mu_X) \to H$. Applying rule SUP, it follows that, for every $\mathsf{cri}(\Sigma_X)$-term $t \in T_{\mathsf{cri}(\Sigma_X)}(\emptyset)$, the membership proposition $0 \mid \mathsf{m}\Big(\mu'_{\Lambda,(X,\mu_X)}(t), t\Big)$ belongs to $\Lambda_{(X,\mu_X)}$ and we can conclude.

2. Let us split the cases between constants and operations.

   - $j_2(c)_{\Lambda_{(X,\mu_X)}}$ is an arrow $1 \to T_\Lambda(X, \mu_X)$ which automatically induces a morphism of fuzzy sets $j_2(c)_{\Lambda_{(X,\mu_X)}} : \nabla_{\mathbf{H}}(1) \to \Big(T_\Lambda(X, \mu_X), \mu_{\Lambda_{(X,\mu_X)}}\Big)$.
   - Now let o be an element of O_{Σ_X}, and recall that

$$\begin{aligned}\mathsf{ar}_{\mathsf{cri}(\Sigma_X)}(j_1(o)) &= V_{\mathbf{H}}\left(\mathsf{ar}_{\Sigma_X}(o)\right)\\ &= V_{\mathbf{H}}\left(\mathsf{ar}_{\Sigma}(o)\right)\end{aligned}$$

Hence, an element of $(T_\Lambda(X, \mu_X))^{\mathsf{ar}_{\Sigma_X}(o)}$ is just a function $\sigma \colon V_{\mathbf{H}}\left(\mathsf{ar}_\Sigma(o)\right) \to T_\Lambda(X, \mu_X)$. Now, for every $\tau \colon V_{\mathbf{H}}\left(\mathsf{ar}_\Sigma(o)\right) \to T_{\mathsf{cri}(\Sigma_X)}(\emptyset)$ we know, by the previous point, that the membership proposition $0 \mid \mathsf{m}\Big(\mu'_{\Lambda_{(X,\mu_X)}}(\tau(\alpha)), \tau(\alpha)\Big)$ is an element of $\Lambda_{(X,\mu_X)}$, thus we can apply rule EXP to get that the sequent

$$0 \mid \mathsf{m}\left(\bigwedge_{\alpha \in V_{\mathbf{H}}(\mathsf{ar}_\Sigma(o))} \Big(\mu_{\mathsf{ar}_\Sigma(o)}(\alpha) \to \mu'_{\Lambda,(X,\mu_X)}(\tau(\alpha))\Big), j_1(o)(\tau)\right)$$

is in $\Lambda_{(X,\mu_X)}$ too, implying that

$$\bigwedge_{\alpha \in V_{\mathbf{H}}(\mathsf{ar}_\Sigma(o))} \Big(\mu_{\mathsf{ar}_\Sigma(o)}(\alpha) \to \mu'_{\Lambda_{(X,\mu_X)}}(\tau(\alpha))\Big) \leq \mu'_{\Lambda,(X,\mu_X)}\left(j_1(o)(\tau)\right)$$

Take now $\sigma \colon V_{\mathbf{H}}\left(\mathsf{ar}_\Sigma(o)\right) \to T_\Lambda(X, \mu_X)$, assuming the axiom of choice, $\pi_{\Lambda_{(X,\mu_X)}}^{V_{\mathbf{H}}(\mathsf{ar}_\Sigma(o))}$ is surjective, therefore there exists another arrow $\tau \colon V_{\mathbf{H}}\left(\mathsf{ar}_\Sigma(o)\right) \to T_{\mathsf{cri}(\Sigma_X)}(\emptyset)$ such that

$$\pi_{\Lambda_{(X,\mu_X)}} \circ \tau = \sigma$$

Let μ be the membership degree of $\left(T_\Lambda(X, \mu_X), \mu_{\Lambda,(X,\mu_X)}\right)^{\mathsf{ar}_\Sigma(o)}$, then we have

$$\begin{aligned}\mu(\sigma) &= \bigwedge_{\alpha \in V_{\mathbf{H}}(\mathsf{ar}_\Sigma(o))} (\mu_{\mathsf{ar}_\Sigma(o)}(\alpha) \to \mu_{\Lambda,(X,\mu_X)}(\sigma(\alpha)))\\ &= \bigwedge_{\alpha \in V_{\mathbf{H}}(\mathsf{ar}_\Sigma(o))} (\mu_{\mathsf{ar}_\Sigma(o)}(\alpha) \to \mu_{\Lambda,(X,\mu_X)}((\pi_{\Lambda_{(X,\mu_X)}}(\tau(\alpha))))\\ &= \bigwedge_{\alpha \in V_{\mathbf{H}}(\mathsf{ar}_\Sigma(o))} (\mu_{\mathsf{ar}_\Sigma(o)}(\alpha) \to \mu'_{\Lambda,(X,\mu_X)}(\tau(\alpha)))\\ &\leq \mu'_{\Lambda,(X,\mu_X)}\left(j_1(o)(\tau)\right)\\ &= \mu_{\Lambda,(X,\mu_X)}\left(\pi_{\Lambda_{(X,\mu_X)}}\left(j_1(o)(\tau)\right)\right)\\ &= \mu_{\Lambda,(X,\mu_X)}\left(j_1(o)_{\Lambda_{(X,\mu_X)}}\left(\pi_{\Lambda_{(X,\mu_X)}} \circ \tau\right)\right)\\ &= \mu_{\Lambda_{(X,\mu_X)}}\left(j_1(o)_{\Lambda,(X,\mu_X)}\left(\sigma\right)\right)\end{aligned}$$

and we can conclude that $j_1(o)_{\Lambda_{(X,\mu_X)}}$ is indeed a morphism of **Fuz**(**H**).

3. Let us start by noticing that, since $\pi_{\Lambda_{(X,\mu_X)}}$ is a $\mathsf{cri}\,(\Sigma_X)$-homomorphism, we have that

$$\left(\pi_{\Lambda_{(X,\mu_X)}} \circ \sigma\right)_{\mathsf{cri}(\Sigma_X),*} = \pi_{\Lambda_{(X,\mu_X)}} \circ \sigma_{\mathsf{cri}(\Sigma_X),*}$$

Now we can split the cases.

- ϕ is $t_1 \equiv t_2$ for some $t_1, t_2 \in T_{\mathsf{cri}(\Sigma_X)}(\lambda)$. Then $L_{\Lambda_{(X,\mu_X)}} \vDash_{\pi_\Lambda \circ \sigma} \phi$ if and only if

$$\pi_{\Lambda_{(X,\mu_X)}}\big(\sigma_{\mathsf{cri}(\Sigma_X),*}(t_1)\big) = \pi_{\Lambda_{(X,\mu_X)}}\big(\sigma_{\mathsf{cri}(\Sigma_X),*}(t_2)\big)$$

  which, by construction is equivalent to the sequent

$$0 \mid \sigma_{\mathsf{cri}(\Sigma_X),*}(t_1) \equiv \sigma_{\mathsf{cri}(\Sigma_X),*}(t_2)$$

  being in $\Lambda_{(X,\mu_X)}$, but this is exactly the thesis.
- ϕ is $\mathsf{m}(h,t)$ for some $t \in T_{\mathsf{cri}(\Sigma_X)}(\lambda)$ and $h \in H$. Then $L_{\Lambda_{(X,\mu_X)}} \vDash_{\pi_\Lambda \circ \sigma} \phi$ if and only if

$$h \leq \mu'_{\Lambda,(X,\mu_X)}\big(\sigma_{\mathsf{cri}(\Sigma_X),*}(t)\big)$$

  which in turn is equivalent to $0 \mid \mathsf{m}\big(h, \sigma_{\mathsf{cri}(\Sigma_X),*}(t)\big) \in \Lambda_{(X,\mu_X)}$.

4. Take a sequent $\lambda \mid \Gamma \vdash \phi$ in $\Lambda_{(X,\mu_X)}$ and let $f \colon \lambda \to T_\Lambda(X,\mu_X)$ be a function such that $L_{\Lambda_{(X,\mu_X)}} \vDash \psi$ for every element $\psi \in \Gamma$. By the axiom of choice there exists a function $\sigma \colon \lambda \to T_{\mathsf{cri}(\Sigma_X)}$ such that $f = \pi_{\Lambda_{(X,\mu_X)}} \circ \sigma$, hence, applying the previous point, we get that $\{0 \mid \psi[\sigma]\}_{\psi\in\Gamma} \subseteq \Lambda_{(X,\mu_X)}$. Applying Cut and Subst we get the following derivation

$$\dfrac{\{0 \mid \psi[\sigma]\}_{\psi\in\Gamma} \qquad \dfrac{\sigma \colon \lambda \to T_{\mathsf{cri}(\Sigma_X)}(\emptyset) \qquad \lambda \mid \Gamma \vdash \phi}{0 \mid \Gamma[\sigma] \vdash \phi[\sigma]}\,\textsc{Subst}}{0 \mid \phi[\sigma]}\,\textsc{Cut}$$

   showing that $0 \mid \phi[\sigma]$ is an element of $\Lambda_{(X,\mu_X)}$. The previous point now yields the thesis.

5. This follows at once from the previous point, and Proposition 3.2.28 applied to $(\mathsf{id}_{O_\Sigma}, \iota_{C_\Sigma})$. $\square$

We can deduce a completeness result from the previous lemma.

Corollary 3.2.37. *Given a κ-bounded signature Σ, a sequent $\lambda \mid \phi$ is staisfied by all models of a Σ-theory Λ if and only if it belongs to Λ.*

Proof. ($\Rightarrow$) If $\lambda \mid \phi$ is satisfied by every model of Λ, then it is satisfied by $F_\Lambda(\nabla_{\mathbf{H}}(\lambda))$. The diagram

$$\begin{tikzcd}
 & T_{\mathsf{cri}(\Sigma)}(\lambda) \arrow[d, "\gamma_{1,\lambda}"] \arrow[rd, bend left, "\pi_{\Lambda_{\nabla_{\mathbf{H}}(\lambda)}}"] \arrow[dd, bend left=60, "\gamma_{4,\lambda}"] & \\
\lambda \arrow[ru, bend left, "\eta_{\mathsf{cri}(\Sigma),\lambda}"] \arrow[r, "\eta_{\mathsf{cri}(\Sigma_\lambda),\lambda}"] \arrow[rd, bend right, "\omega_\lambda"'] & T_{\mathsf{cri}(\Sigma_\lambda)}(\lambda) \arrow[r, "\left(\pi_{\Lambda_{\nabla_{\mathbf{H}}(\lambda)}} \circ \omega_\lambda\right)_{\mathsf{cri}(\Sigma_\lambda),*}"] \arrow[d, "\gamma_{2,\lambda}"] & T_\Lambda(\nabla_{\mathbf{H}}(\lambda)) \\
 & T_{\mathsf{cri}\left(\Sigma_{\nabla_{\mathbf{H}}(\lambda)}\right)}(\emptyset) \arrow[ru, bend right, "\pi_{\Lambda_{\nabla_{\mathbf{H}}(\lambda)}}"'] &
\end{tikzcd}$$

shows that $L_{\Lambda_{\nabla_{\mathbf{H}}(\lambda)}}$ satisfies $\lambda \mid \mathsf{tr}_{(\mathsf{id}_{O_\Sigma}, \iota_{C_\Sigma}),\lambda}(\phi)$ with respect to $\pi_{\Lambda_{\nabla_{\mathbf{H}}(\lambda)}} \circ \omega_\lambda$. Now, by Remark 3.2.33 $0 \mid \mathsf{tr}_{(\mathsf{id}_{O_\Sigma}, \iota_{C_\Sigma}),\lambda}(\phi)[\omega_\lambda]$ is just $0 \mid K_\lambda(\phi)$ and by the third point of Lemma 3.2.36 we know that it is an element of $\Lambda_{\nabla_{\mathbf{H}}(\lambda)}$. The thesis now follows from point 2 of Proposition 3.2.34 and from Remark 3.2.33.

($\Leftarrow$) This follows at once from Lemma 3.2.24. $\square$

We are now ready to show the main theorem of this section.

Theorem 3.2.38. *Let Σ be a κ-bounded signature and Λ a Σ-theory, the forgetful functor $V_\Lambda \colon$ $\mathbf{Mod}(\Lambda) \to \mathbf{Fuz}(\mathbf{H})$ has a left adjoint F_Λ.*

Proof. Let (X, μ_X) be a fuzzy set and define $\eta_{\Lambda,(X,\mu_X)}$ as $\pi_{\Lambda_{(X,\mu_X)}} \circ \omega_{(X,\mu_X)}$, so that, for every $x \in X$, $\eta_{\Lambda,(X,\mu_X)}(x)$ is the only element in the image $x^{L_{\Lambda_{(X,\mu_X)}}} \colon 1 \to T_\Lambda(X, \mu_X)$. By definition the sequent $0 \mid \mathsf{m}(\mu_X(x), \widehat{x})$ is in $\Lambda_{(X,\mu_X)}$, thus

$$\mu_X(x) \leq \mu_{\Lambda,(X,\mu_X)}\left(\eta_{\Lambda,(X,\mu_X)}(x)\right)$$

and we get a morphism $\eta_{\Lambda,(X,\mu_X)} \colon (X, \mu_X) \to (T_\Lambda(X, \mu_X), \mu_\Lambda)$. Take now a model $\mathcal{A}$ of Λ with $V_\Lambda(\mathcal{A}) = (A, \mu_A)$ and a morphism $f \colon (X, \mu_X) \to (A, \mu_A)$. We can use f to endow A with a Σ_X-algebra structure $\mathcal{A}_f$. This is easily done putting

$$o^{\mathcal{A}_f} := o^{\mathcal{A}} \qquad (\iota_{C_\Sigma}(c))^{\mathcal{A}_f} := \mathcal{A} \qquad (\iota_X(x))^{\mathcal{A}_f} := f(x)$$

where ι_{C_Σ} and ι_X are the coprojections. We want to show that $\mathcal{A}_f$ is a model for $\Lambda_{(X,\mu_X)}$.

On the one hand the unique arrow $(?_A)_{\mathsf{cri}(\Sigma_X),*} \colon T_{\mathsf{cri}(\Sigma_X)}(\emptyset) \to A$, induced by $?_A \colon \emptyset \to A$, must send the constant $\widehat{x}$ to $f(x)$. Now, since f is a morphism $(X, \mu_X) \to (A, \mu_A)$, it follows that

$$\mu_X(x) \leq \mu_A(f(x))$$

But this is the same as saying that $\mathcal{A}$ satisfies all the elements of $\{0 \mid \mathsf{m}(\mu_X(x), \widehat{x})\}_{x \in X}$.

On the other hand, notice that $(\mathsf{id}_{O_\Sigma}, \iota_{C_\Sigma})^*(\mathcal{A}_f) = \mathcal{A}$. Thus Proposition 3.2.28 entails that, for every sequent $\lambda \mid \Gamma \vdash \phi$ in Λ, $\mathcal{A}_f$ satisfies

$$\lambda \mid \left\{\mathsf{tr}_{(\mathsf{id}_{O_\Sigma}, \iota_{C_\Sigma})}(\psi)\right\}_{\psi \in \Gamma} \vdash \mathsf{tr}_{(\mathsf{id}_{O_\Sigma}, \iota_{C_\Sigma})}(\phi)$$

By Lemma 3.2.24 we conclude that $\mathcal{A}_f$ lies in $\mathbf{Mod}\big(\Lambda_{(X,\mu_X)}\big)$.

Now let t_1 and t_2 be elements of $T_{\mathsf{cri}(\Sigma_X)}(\emptyset)$ such that $t_1 \sim_{\Lambda_{(X,\mu_X)}} t_2$, then $0 \mid t_1 \equiv t_2$ belongs to $\Lambda_{(X,\mu_X)}$ and thus

$$(?_A)_{\mathsf{cri}(\Sigma_X),*}(t_1) = (?_A)_{\mathsf{cri}(\Sigma_X),*}(t_2)$$

Hence, there exists a unique $\mathsf{cri}(\Sigma_X)$-homomorphism $f_{\Lambda,*} \colon W_{\Sigma_X}\Big(L_{\Lambda_{(X,\mu_X)}}\Big) \to W_{\Sigma_X}(\mathcal{A}_f)$ such that the following diagram commutes

$$\begin{tikzcd}
T_{\mathsf{cri}(\Sigma_X)}(\emptyset) \arrow[r, "(?_A)_{\mathsf{cri}(\Sigma_X),*}"] \arrow[d, "\pi_{\Lambda_{(X,\mu_X)}}"'] & A \\
T_\Lambda(X, \mu_X) \arrow[ur, dotted, "f_{\Lambda,*}"'] &
\end{tikzcd}$$

On the other hand if $0 \mid \mathsf{m}(h,t)$ is in $\Lambda_{(X,\mu_X)}$ then

$$h \leq \mu_A\left((?_A)_{\mathsf{cri}(\Sigma_X),*}(t)\right)$$

and thus $f_{\Lambda,*}$ is actually a Σ_X-homomorphism $L_{\Lambda_{(X,\mu_X)}} \to \mathcal{A}_f$, hence, in particular, it is also a morphism $F_\Lambda(X,\mu_X) \to \mathcal{A}$ in $\mathbf{Mod}(\Lambda)$. Notice, moreover, that $f_{\Lambda,*}(\eta_{\Lambda,(X,\mu_X)}(x))$ must coincide with $f(x)$ since the following diagram commutes.

$$\begin{tikzcd}
1 \arrow[r, "x^{\mathcal{A}_f}"] \arrow[d, "x^{L_{\Lambda(X,\mu_X)}}"'] & A \\
T_\Lambda(X,\mu_X) \arrow[ur, bend right, "f_{\Lambda,*}"'] &
\end{tikzcd}$$

Now let $g\colon F_\Lambda(X,\mu_X) \to \mathcal{A}$ be another Σ-homomorphism such that $g \circ \eta_{\Lambda,(X,\mu_X)} = f$, this means that the following diagram commutes

$$\begin{tikzcd}
1 \arrow[r, "x^{\mathcal{A}_f}"] \arrow[d, "x^{L_{\Lambda(X,\mu_X)}}"'] & A \\
T_\Lambda(X,\mu_X) \arrow[ur, bend right, "g"'] &
\end{tikzcd}$$

i.e. that g is actually a $\mathsf{cri}\,(\Sigma_X)$-homomorphism. By the initiality of $T_{\mathsf{cri}(\Sigma_X)}(\emptyset)$, it follows that

$$g \circ \pi_{\Lambda_{(X,\mu_X)}} = (?_A)_{\mathsf{cri}(\Sigma_X),*}$$

and therefore $g = f_{\Lambda,*}$. $\square$

Notation. Given a Σ-theory Λ with Σ κ-bounded, we can define S_Λ as the composition $V_\Lambda \circ F_\Lambda$. In particular we will use the notation

$$S_\Lambda(X,\mu_X) = \left(T_\Lambda(X,\mu_X), \mu_{\Lambda,(X,\mu_X)}\right)$$

As before, when Λ is the theory without axioms, we will denote S_Λ and F_Λ by, respectively, S_Σ and F_Σ. Moreover we will use $\mu_{\Sigma,(X,\mu_X)}$ to denote the membership degree of $S_\Sigma(X,\mu_X)$.

Remark 3.2.39. Let Σ be a κ-bounded fuzzy signature, then we have a diagram

$$\begin{tikzcd}
\Sigma\text{-}\mathbf{FAlg} \arrow[r, "W_\Sigma"] \arrow[d, "V_\Sigma"'] & \mathsf{cri}\,(\Sigma)\text{-}\mathbf{Alg} \arrow[d, "U_{\mathsf{cri}(\Sigma)}"] \\
\mathbf{Fuz}(\mathbf{H}) \arrow[r, "V_{\mathbf{H}}"'] & \mathbf{Set}
\end{tikzcd}$$

By Corollary 3.1.26 and Proposition 3.2.10 $V_{\mathbf{H}}$ and W_Σ are left adjoints, thus there exists a natural isomorphism $\Theta\colon W_\Sigma \circ F_\Sigma \to F_{\mathsf{cri}(\Sigma)} \circ V_{\mathbf{H}}$. Let T_Σ be $W_\Sigma \circ F_\Sigma$, then the previous observation means that, for every (X,μ_X) in $\mathbf{Fuz}(\mathbf{H})$, there is an isomorphism of $\mathsf{cri}(\Sigma)$-algebras $\Theta_{(X,\mu_X)}\colon T_{\mathsf{cri}(\Sigma)}(X) \to T_\Sigma(X,\mu_X)$ which, moreover, fits in the triangle below:

$$\begin{tikzcd} & X \arrow[dl, "\eta_{\mathsf{cri}(\Sigma),X}"'] \arrow[dr, "V_{\mathbf{H}}(\eta_{\Sigma,(X,\mu_X)})"] & \\ T_{\mathsf{cri}(\Sigma)}(X) \arrow[rr, "\Theta_{(X,\mu_X)}"'] & & T_\Sigma(X,\mu_X) \end{tikzcd}$$

T_Σ sends $f\colon (X,\mu_X) \to (Y,\mu_Y)$ to $S_\Sigma(f)$, so we can add the following square to the triangle above.

$$\begin{tikzcd} T_{\mathsf{cri}(\Sigma)}(X) \arrow[r, "\Theta_{(X,\mu_X)}"] \arrow[d, "T_{\mathsf{cri}(\Sigma)}(f)"'] & T_\Sigma(X,\mu_X) \arrow[d, "S_\Sigma(f)"] \\ T_{\mathsf{cri}(\Sigma)}(Y) \arrow[r, "\Theta_{(Y,\mu_Y)}"'] & T_\Sigma(Y,\mu_Y) \end{tikzcd}$$

This last remark allows us to prove the following.

Proposition 3.2.40. *Given $\Sigma \in \mathbf{FSign}_\kappa$, for every Σ-theory Λ and fuzzy set (X,μ_X), there exists a unique natural transformation $\pi_\lambda\colon S_\Sigma \to S_\Lambda$ such that the triangle below commutes.*

$$\begin{tikzcd} & \mathsf{id}_{\mathbf{Fuz}(\mathbf{H})} \arrow[dl, "\eta_\Sigma"'] \arrow[dr, "\eta_\Lambda"] & \\ S_\Sigma \arrow[rr, "\pi_\Lambda"'] & & S_\Lambda \end{tikzcd}$$

Moreover, each component $\pi_{\Lambda,(X,\mu_X)}\colon S_\Sigma(X,\mu_X) \to S_\Lambda(X,\mu_X)$ defines a surjective Σ-homomorphism $F_\Sigma(X,\mu_X) \to F_\Lambda(X,\mu_X)$.

Proof. For every fuzzy set (X,μ_X), $F_\Lambda(X,\mu_X)$ is a Σ-algebra and we can define $\pi_{\Lambda,(X,\mu_X)}$ as the unique Σ-homomorphism fitting in the diagram

$$\begin{tikzcd} & (X,\mu_X) \arrow[dl, "\eta_{\Sigma,(X,\mu_X)}"'] \arrow[dr, "\eta_{\Lambda,(X,\mu_X)}"] & \\ S_\Sigma(X,\mu_X) \arrow[rr, dotted, "\pi_{\Lambda,(X,\mu_X)}"'] & & S_\Lambda(X,\mu_X) \end{tikzcd}$$

$\pi_{\Lambda,(X,\mu_X)}$ is a $\mathsf{cri}(\Sigma)$-homomorphism, therefore, using Lemma 3.2.30, we have

$$\pi_{\Lambda,(X,\mu_X)} = \pi_{\Lambda_{(X,\mu_X)}} \circ \gamma^{-1}_{3,X} \circ \Theta^{-1}_{(X,\mu_X)}$$

and this proves its surjectivity.

For naturality, take $f\colon (X,\mu_X) \to (Y,\mu_Y)$, then we can construct the diagram

$$
\begin{tikzcd}
S_\Sigma(X,\mu_X) \arrow[dd, "\pi_{\Lambda,(X,\mu_X)}"'] & (X,\mu_X) \arrow[l, "\eta_{\Sigma,(X,\mu_X)}"'] \arrow[r, "\eta_{\Sigma,(X,\mu_X)}"] \arrow[d, "f"] \arrow[ddl, bend left, "\eta_{\Lambda,(X,\mu_X)}"'] & S_\Sigma(X,\mu_X) \arrow[dd, "S_\Sigma(f)"] \\
 & (Y,\mu_Y) \arrow[d, "\eta_{\Lambda,(Y,\mu_Y)}"'] \arrow[dr, "\eta_{\Sigma,(Y,\mu_Y)}"] & \\
S_\Lambda(X,\mu_X) \arrow[r, "S_\Lambda(f)"'] & S_\Lambda(Y,\mu_Y) & S_\Sigma(Y,\mu_Y) \arrow[l, "\pi_{\Lambda,(Y,\mu_Y)}"]
\end{tikzcd}
$$

The thesis now follows since $S_\Sigma(f)$ and $S_\Lambda(f)$ are Σ-homomorphisms, respectively, $F_\Sigma(X,\mu_X) \to F_\Sigma(Y,\mu_Y)$ and $F_\Lambda(X,\mu_X) \to F_\Lambda(Y,\mu_Y)$. ☐

Given Corollary 3.2.12, the following result is now immediate.

Proposition 3.2.41. *For every κ-accessible signature Σ, the functor S_Σ has rank κ.*

Corollary 3.2.42. *Given a κ-accessible signature Σ, $(S_\Sigma, \mathsf{id}_{S_\Sigma \circ J_\kappa})$ is a left Kan extension of $S_\Sigma \circ J_\kappa$ along J_κ, where J_κ is the inclusion $\mathbf{Fuz}_\kappa(\mathbf{H}) \to \mathbf{Fuz}(\mathbf{H})$.*

Proof. Immediate from Theorem 3.1.39 and Proposition 3.2.41. ☐

We already know, by virtue of Example 3.2.29, that extending the previous result to arbitrary Σ-theories, to get a full analog of Corollary 2.2.65 is impossible. The next example, together with Theorem 3.1.39, show that the situation is even worse: given a Σ-theory Λ, with $\Sigma \in \mathbf{FSign}_\kappa$, $(S_\Lambda, \mathsf{id}_{S_\Lambda \circ J_\kappa})$ in general is not the left Kan extension of $S_\Lambda \circ J_\kappa$ along J_κ.

Example 3.2.43. Let $\mathbf{H}$ be $([0,1], \leq)$ and take Σ to be the signature with no operations nor constants. We can then consider the theory with the following set of axioms:

$$\{2 \mid \mathsf{m}(r,x) \vdash \mathsf{m}(r,y)\}_{r\in[0,1]} \cup \{2 \mid \mathsf{m}(1,x) \vdash x = y\}$$

A Σ-algebra is just a fuzzy set (X,μ_X) , while there are two kinds of models of Λ: $\Delta_{\mathbf{H}}(1)$ or fuzzy sets (X,μ_X) such that μ_X is constant at a value strictly smaller than 1. Given a fuzzy set (X,μ_X), let $s(X,\mu_X)$ be the supremum of the family $\{\mu_X(x)\}_{x\in X}$, and let $c_{s(X,\mu_X)}$ be the function $X \to H$ constant in $s(X,\mu_X)$ then:

$$S_\Lambda(X,\mu_X) = \begin{cases} \Delta_{\mathbf{H}}(1) & s(X,\mu_X) = 1 \\ \big(X, c_{s(X,\mu_X)}\big) & s(X,\mu_X) < 1 \end{cases}$$

To see this, notice that we have an $\eta_{\Lambda,(X,\mu_X)}\colon (X,\mu_X) \to S_\Lambda(X,\mu_X)$ which is the identity $(X,\mu_X) \to \big(X, c_{s(X,\mu_X)}\big)$ if $s(X,\mu_X) < 1$ or $!_{(X,\mu_X)}$, otherwise. If(Y,μ_Y) is a model of Λ and $f\colon (X,\mu_X) \to (Y,\mu_Y)$ a morphism of $\mathbf{Fuz}(\mathbf{H})$, then we have two cases:

- $s(X,\mu_X)$ is 1, then also $s(Y,\mu_Y)$ must be 1 , thus $S_\Lambda(X,\mu_X)$ and $((Y,\mu_Y),\delta_{y_0})$ are both $\Delta_{\mathbf{H}}(1)$ and the unique morphism between them is the identity;
- if $s(X,\mu_X) < 1$, the inequalities

$$\begin{aligned} \mu_X(x) \leq& \mu_Y(f(x)) \\ &= s(Y,\mu_Y) \end{aligned}$$

entails that $s(X,\mu_X) \leq s(Y,\mu_Y)$, therefore f itself defines a morphism $S_\Lambda(X,\mu_X) \to (Y,\mu_Y)$.

Given $f\colon (X,\mu_X)\to(Y,\mu_Y)$, the previous observations entail that

$$S_\Lambda(f)=\begin{cases} !_{(Y,\mu_Y)} & s(Y,\mu_Y)=1\\ f & s(Y,\mu_Y)<1\end{cases}$$

Now, take $(\mathbb{N},\mu_{\mathbb{N}})$ where

$$\mu_{\mathbb{N}}\colon \mathbb{N}\to[0,1] \qquad n\mapsto \frac{n}{n+1}$$

Then $S_\Lambda(\mathbb{N},\mu_{\mathbb{N}})$ is $(\Delta_{\mathbf{H}}(1)$, while, for any finite set $A\subseteq\mathbb{N}$, $S_\Lambda(A,\mu_{\mathbb{N}|A})$ is simply $(A,\mu_{\mathbb{N}|A})$. Moreover, given $A\subseteq B$, $S_\Lambda(i_{A,B})$ is again the inclusion $i_{A,B}$. By Lemma 2.2.89 and Theorem 3.1.39, we can now deduce that S_Λ is not the left Kan extension of its restriction to $\mathbf{Fuz}_{\aleph_0}(\mathbf{H})$ along $J_{\aleph_0}$.

3.2.2 Fuzzy algebraic theories and monads

In the previous section we have proved Theorem 3.2.38, showing that, for every given a κ-bounded signature Σ and a Σ-theory Λ, the forgetful functor $V_\Lambda\colon \mathbf{Mod}(\Lambda)\to\mathbf{Fuz}(\mathbf{H})$ has a left adjoint F_Λ. As in the case of ordinary algebraic theories, we can them appeal to Proposition 2.1.5 in order to equip the functor $S_\Lambda=V_\Lambda\circ F_\Lambda$ with a monad structure, getting $\mathbf{S}_\Lambda:=(S_\Lambda,\eta_\Lambda,\nu_\Lambda)$. While it is not true that V_Λ is monadic, we will show that this is true for a class of theories, called *basic*.

Our strategy will be the same as the one employed in Section 2.2.3, so let us start looking closely to the comparison functor $K_\Lambda\colon \mathbf{Mod}(\Lambda)\to\mathbf{EM}(\mathbf{S}_\Lambda)$.

Given $\mathcal{A}=\left(A,\{o^{\mathcal{A}}\}_{o\in O_\Sigma},\{c^{\mathcal{A}}\}_{c\in C_\Sigma}\right)$ in $\mathbf{Mod}(\Lambda)$, the component $\epsilon_{\Lambda,\mathcal{A}}$ of the counit of $F_\Lambda\dashv V_\Lambda$ is given by $(\mathsf{id}_A)_{\Lambda,*}\colon F_\Lambda((A,\mu_A))\to\mathcal{A}$. Thus, applying Propositions 2.1.5 and 2.1.14 we get:

- for every fuzzy set (X,μ_X), $\nu_{\Lambda,(X,\mu_X)}\colon S_\Lambda(S_\Lambda(X,\mu_X))\to S_\Lambda(X,\mu_X)$ is $\left(\mathsf{id}_{S_\Lambda(X)}\right)_{\Lambda,*}$, so that $\nu_{\Lambda,(X,\mu_X)}$ defines a Σ-homomorphism $F_\Lambda(S_\Lambda(X,\mu_X))\to F_\Lambda(X,\mu_X)$;
- the comparison functor $K_\Lambda\colon \mathbf{Mod}(\Lambda)\to\mathbf{EM}(\mathbf{S}_\Lambda)$ is defined by

$$\begin{array}{ccc}\mathcal{A} & \longmapsto & \left((A,\mu_A),(\mathsf{id}_{(A,\mu_A)})_{\Lambda,*}\right)\\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f}\\ \mathcal{B} & \longmapsto & \left((B,\mu_B),(\mathsf{id}_{(B,\mu_B)})_{\Lambda,*}\right)\end{array}$$

In order to construct an inverse to K_Λ, our first step is to mimic Definition 2.2.79

Definition 3.2.44. Let Λ be a Σ-theory, given an Eilenberg-Moore algebra $((X,\mu_X),\xi)$ for $\mathbf{S}_\Lambda$, its *associated* Σ*-algebra* $H_\Lambda(X,\xi)=\left((X,\mu_X),\{o^{H_\Lambda(X,\xi)}\}_{o\in O_\Sigma},\{c^{H_\Lambda(X,\xi)}\}_{c\in C_\Sigma}\right)$ is defined taking as $o^{H_\Lambda(X,\xi)}$ and $c^{H_\Lambda(X,\xi)}$ the compositions

$$(X,\mu_X)^{\mathsf{ar}_\Sigma(o)}\xrightarrow{\eta^{\mathsf{ar}_\Sigma(o)}_{\Lambda,(X,\mu_X)}}(S_\Lambda(X,\mu_X))^{\mathsf{ar}_\Sigma(o)}\xrightarrow{o^{F_\Lambda(X,\mu_X)}}S_\Lambda(X,\mu_X)\xrightarrow{\xi}(X,\mu_X)$$

$$\nabla_{\mathbf{H}}(1)\xrightarrow{c^{F_\Lambda(X,\mu_X)}}S_\Lambda(X,\mu_X)\xrightarrow{\xi}(X,\mu_X)$$

Proposition 3.2.45. *For every Σ-theory Λ, with $\Sigma \in \mathbf{FSign}_\kappa$, if $((X,\mu_X),\xi)$ is an Eilenberg-Moore algebra for $\mathbf{S}_\Lambda$, then the arrow ξ itself is a Σ-homomorphism $F_\Lambda(X,\mu_X) \to H_\Lambda((X,\mu_X),\xi)$. Moreover*

$$\xi = (\mathsf{id}_{(X,\mu_X)})_{\Lambda,*}$$

Proof. By definition, we have that

$$c^{H_\Lambda((X,\mu_X),\xi)} = \xi \circ c^{F_\Lambda(X,\mu_X)}$$

On the other hand, we have already proved that in the following diagram all the inner subdiagrams commute, so that the whole commutes too

$$\begin{tikzcd}
(X,\mu_X)^{\mathsf{ar}_\Sigma(o)} \arrow[r, "\eta_{\Lambda,(X,\mu_X)}^{\mathsf{ar}_\Sigma(o)}"] & (S_\Lambda(X,\mu_X))^{\mathsf{ar}_\Sigma(o)} \arrow[r, "o^{F_\Lambda(X,\mu_X)}"] & S_\Lambda(X,\mu_X) \arrow[ddr, bend left, "\xi"] & \\
(S_\Lambda(X,\mu_X))^{\mathsf{ar}_\Sigma(o)} \arrow[u, "\xi^{\mathsf{ar}_\Sigma(o)}"] \arrow[r, "\eta_{\Lambda,S_\Lambda(X,\mu_X)}^{\mathsf{ar}_\Sigma(o)}"] \arrow[dr, bend right, "\mathsf{id}_{S_\Lambda(X,\mu_X)}^{\mathsf{ar}_\Sigma(o)}"'] & (S_\Lambda(S_\Lambda(X,\mu_X)))^{\mathsf{ar}_\Sigma(o)} \arrow[u, "(S_\Lambda(\xi))^{\mathsf{ar}_\Sigma(o)}"'] \arrow[r, "o^{F_\Lambda(S_\Lambda(X,\mu_X))}"] \arrow[d, "\nu_{\Lambda,(X,\mu_X)}^{\mathsf{ar}_\Sigma(o)}"'] & S_\Lambda(S_\Lambda(X,\mu_X)) \arrow[u, "S_\Lambda(\xi)"'] \arrow[d, "\nu_{\Lambda,(X,\mu_X)}"'] & \\
& (S_\Lambda(X,\mu_X))^{\mathsf{ar}_\Sigma(o)} \arrow[r, "o^{F_\Lambda((X,\mu_X))}"'] & S_\Lambda(X,\mu_X) \arrow[r, "\xi"'] & (X,\mu_X)
\end{tikzcd}$$

The last part of the thesis follows at once from the identity $\xi \circ \eta_{\Lambda,(X,\mu_X)} = \mathsf{id}_{(X,\mu_X)}$. ☐

Example 3.2.46. Let $\mathbf{H}$ be the frame $(2,\leq)$ and consider the signature Σ with no operations and a constant c. We take now the Σ-theory Λ with axiom

$$2 \mid \mathsf{m}(1,c) \vdash x = y$$

We can compute explicitly S_Λ. We claim that

$$S_\Lambda(X,\mu_X) = (X,\mu_X) + \nabla_{\mathbf{H}}(1)$$

The coprojection $j_{\nabla_{\mathbf{H}}(1)} \colon \nabla_{\mathbf{H}}(1) \to (X,\mu_X) + \nabla_{\mathbf{H}}(1)$ equip this fuzzy set $S_\Lambda(X,\mu_X)$ with a Σ-algebra structure which is a model of Λ. The other coprojection gives us a morphism $\eta_{\Lambda,(X,\mu_X)} \colon (X,\mu_X) \to S_\Lambda(X,\mu_X)$ which has the universal property of the unit of $F_\Lambda \dashv V_\Lambda$. To see this, let $\mathcal{A} = ((A,\mu_A),c^{\mathcal{A}})$ be a model of Λ and $f \colon (X,\mu_X) \to (A,\mu_A)$ a morphism of $\mathbf{Fuz}(\mathbf{H})$. By the universal property of coproducts the unique Σ-homomorphism $f_{\Lambda,*} \colon (S_\Lambda(X,\mu_X), j_1) \to ((A,\mu_A),c^{\mathcal{A}})$ such that

$$f = f_{\Lambda,*} \circ \eta_{\Lambda,(X,\mu_X)}$$

is the one induced by f and $c^{\mathcal{A}}$. We can then conclude that, $\mathbf{S}_\Lambda$ is the exception monad of Example 2.1.3 with $\nabla_{\mathbf{H}}(1)$ as E. $S_\Lambda(S_\Lambda(X,\mu_X))$ is the coproduct of (X,μ_X) and two copies of $\nabla_{\mathbf{H}}(1)$, so that we have

$$\begin{tikzcd}
\nabla_{\mathbf{H}}(1) \arrow[r, "\mathsf{id}_{\nabla_{\mathbf{H}}(1)}"] \arrow[d, "j_{\nabla_{\mathbf{H}}(1),1}"'] & \nabla_{\mathbf{H}}(1) \arrow[d, "j_{\nabla_{\mathbf{H}}(1)}"] \\
(X,\mu_X) + \nabla_{\mathbf{H}}(1) + \nabla_{\mathbf{H}}(1) \arrow[r, "\nu_{\Lambda,(X,\mu_X)}"'] & (X,\mu_X) + \nabla_{\mathbf{H}}(1)
\end{tikzcd}$$

$$\begin{tikzcd}
\nabla_{\mathbf{H}}(1) \arrow[r, "\mathsf{id}_{\nabla_{\mathbf{H}}(1)}"] \arrow[d, "j_{\nabla_{\mathbf{H}}(1),2}"'] & \nabla_{\mathbf{H}}(1) \arrow[d, "j_{\nabla_{\mathbf{H}}(1)}"] \\
(X,\mu_X)+\nabla_{\mathbf{H}}(1)+\nabla_{\mathbf{H}}(1) \arrow[r, "\nu_{\Lambda,(X,\mu_X)}"'] & (X,\mu_X)+\nabla_{\mathbf{H}}(1)
\end{tikzcd}$$

where $j_{\nabla_{\mathbf{H}}(1),1}$ and $j_{\nabla_{\mathbf{H}}(1),2}$ are the two coprojections with domain $\nabla_{\mathbf{H}}(1)$. Considering the other coprojection $j_{(X,\mu_X)}\colon (X,\mu_X) \to S_\Lambda(S_\Lambda(X,\mu_X))$ we also have

$$\begin{tikzcd}
(X,\mu_X) \arrow[r, "\mathsf{id}_{(X,\mu_X)}"] \arrow[d, "j_{(X,\mu_X)}"'] & (X,\mu_X) \arrow[d, "\eta_{\Lambda,(X,\mu_X)}"] \\
(X,\mu_X)+\nabla_{\mathbf{H}}(1)+\nabla_{\mathbf{H}}(1) \arrow[r, "\nu_{\Lambda,(X,\mu_X)}"'] & (X,\mu_X)+\nabla_{\mathbf{H}}(1)
\end{tikzcd}$$

Now let $X = \{a, b\}$ be any set with two elements and c_X the function $X \to 2$ constant in 1. Then there are no Σ-algebra structures on (X, c_X) making it a model of Λ. On the other hand, we can define $\xi\colon S_\Lambda(X, c_X) \to (X, c_X)$ as the arrow induced by $\mathsf{id}_{(X,c_X)}$ and $\delta_a\colon \nabla_{\mathbf{H}}(1) \to (X, c_X)$. Clearly $\xi \circ \eta_{\Lambda,(X,c_X)}$ is the identity, while we have

$$\begin{aligned}
\xi \circ \nu_{\Lambda,(X,c_X)} \circ j_{(X,c_X)} &= \xi \circ \eta_{\Lambda,(X,c_X)} \circ \mathsf{id}_{(X,c_X)}\\
&= \xi \circ \eta_{\Lambda,(X,c_X)} \circ \xi \circ j_{(X,c_X)}\\
&= \xi \circ S_\Lambda(\xi) \circ j_{(X,c_X)}
\end{aligned}$$

$$\begin{aligned}
\xi \circ \nu_{\Lambda,(X,c_X)} \circ j_{\nabla_{\mathbf{H}}(1),1} &= \xi \circ j_{\nabla_{\mathbf{H}}(1)} \circ \mathsf{id}_{(X,c_X)} & \xi \circ \nu_{\Lambda,(X,c_X)} \circ j_{\nabla_{\mathbf{H}}(1),2} &= \xi \circ j_{\nabla_{\mathbf{H}}(1)} \circ \mathsf{id}_{(X,c_X)}\\
&= \xi \circ j_{\nabla_{\mathbf{H}}(1)} & &= \xi \circ j_{\nabla_{\mathbf{H}}(1)}\\
&= \xi \circ S_\Lambda(\xi) \circ j_{\nabla_{\mathbf{H}}(1),1} & &= \xi \circ S_\Lambda(\xi) \circ j_{\nabla_{\mathbf{H}}(1),2}
\end{aligned}$$

Therefore $((X, c_X), \xi)$ is an object of $\mathbf{EM}(\mathbf{S}_\Lambda)$ which cannot be in the essential image of the comparison functor $K_\Lambda\colon \mathbf{Mod}(\Lambda) \to \mathbf{EM}(\mathbf{S}_\Lambda)$ and which, moreover, is such that $H_\Lambda((X, c_X), \xi)$ is not in $\mathbf{Mod}(\Lambda)$.

The previous example shows that, in general $H_\Lambda((X, \mu_X), \xi)$ is not a model of Λ. We can nonetheless identify a class of theories such that this holds. As in [16, 91] the right class of theories is the one given by theories axiomatizable by axioms whose premises contains only variables.

Definition 3.2.47. Let Σ be a κ-bounded signature, a Σ-theory Λ is *basic* (or, using the terminology of [15], *simple*) if it has a set of axiom S such that, for any sequent $\lambda \mid \Gamma \vdash \phi$ in it, all the formulae in Γ contain only variables, i.e. elements in the image of $\eta_{\mathsf{cri}(\Sigma),\lambda}$.

Example 3.2.48. Fuzzy groups, fuzzy normal groups, fuzzy semigroups and left, right, bilateral ideals (Examples 3.2.27 and 3.2.26) are all examples of basic theories.

Lemma 3.2.49. *Let Σ be a κ-bounded signature. For every basic Σ-theory Λ, if $((X, \mu_X), \xi)$ is an object of $\mathbf{EM}(\mathbf{S}_\Lambda)$, then $H_\Lambda((X, \mu_X), \xi)$ is a model of Λ.*

Proof. Let S be a set of axiom for Λ such that for every sequent $\lambda \mid \Gamma \vdash \phi$ in it, each formula in Γ contains only variables. Let $f\colon \lambda \to X$ be a function, we can notice that, if $H_\Lambda((X, \mu_X), \xi) \vDash_f \Gamma$ then $F_\Lambda(X, \mu_X) \vDash_{\eta_{\Lambda,(X,\mu_X)} \circ f} \Gamma$ too. To see this, fix a formula ψ in Γ, and split the cases:

- if ψ is $x \equiv y$, let x and y be, respectively $\eta_{\mathsf{cri}(\Sigma),\lambda}(\alpha)$ and $\eta_{\mathsf{cri}(\Sigma),\lambda}(\beta)$ for some $\alpha, \beta \in \lambda$. By hypothesis

$$\begin{aligned} f(\alpha) &= f_{\mathsf{cri}(\Sigma),*}\left(\eta_{\mathsf{cri}(\Sigma),\lambda}(\alpha)\right) \\ &= f_{\mathsf{cri}(\Sigma),*}\left(\eta_{\mathsf{cri}(\Sigma),\lambda}(\beta)\right) \\ &= f(\beta) \end{aligned}$$

so that we also have

$$\begin{aligned} \left(\eta_{(X,\mu_X)} \circ f\right)_{\mathsf{cri}(\Sigma),*}(x) &= \left(\eta_{(X,\mu_X)} \circ f\right)_{\mathsf{cri}(\Sigma),*}\left(\eta_{\mathsf{cri}(\Sigma),\lambda}(\alpha)\right) \\ &= \eta_{(X,\mu_X)}(f(\alpha)) \\ &= \eta_{(X,\mu_X)}(f(\beta)) \\ &= \left(\eta_{(X,\mu_X)} \circ f\right)_{\mathsf{cri}(\Sigma),*}\left(\eta_{\mathsf{cri}(\Sigma),\lambda}(\beta)\right) \\ &= \left(\eta_{(X,\mu_X)} \circ f\right)_{\mathsf{cri}(\Sigma),*}(y) \end{aligned}$$

which is precisely what we claimed;

- if ψ is $\mathsf{m}(h, x)$ for some $h \in H$ and $x = \eta_{\mathsf{cri}(\Sigma),\lambda}(\alpha)$ for some $\alpha \in \lambda$, then

$$\begin{aligned} h &\leq \mu_X\left(f_{\mathsf{cri}(\Sigma),*}(x)\right) \\ &= \mu_X\left(f_{\mathsf{cri}(\Sigma),*}\left(\eta_{\mathsf{cri}(\Sigma),\lambda}(\alpha)\right)\right) \\ &= \mu_X(f(\alpha)) \\ &\leq \mu_{\Lambda,(X,\mu_X)}\left(\eta_{\Lambda,(X,\mu_X)}(f(\alpha))\right) \\ &= \mu_{\Lambda,(X,\mu_X)}\Big(\left(\eta_{(X,\mu_X)} \circ f\right)_{\mathsf{cri}(\Sigma),*}\left(\eta_{\mathsf{cri}(\Sigma),\lambda}(\alpha)\right)\Big) \\ &= \mu_{\Lambda,(X,\mu_X)}\Big(\left(\eta_{(X,\mu_X)} \circ f\right)_{\mathsf{cri}(\Sigma),*}(x)\Big) \end{aligned}$$

and we can conclude again.

Since $F_\Lambda(X, \mu_X)$ is a model for Λ, we can deduce from the previous observations that $F_\Lambda(X, \mu_X)$ satisfies ϕ with respect to $\eta_{\Lambda,(X,\mu_X)} \circ f$. Now, by Proposition 3.2.45, ξ is a Σ-homomorphism, thus, in particular, it is also a $\mathsf{cri}(\Sigma)$-homomorphism, then

$$\begin{aligned} \xi \circ \left(\eta_{\Lambda,(X,\mu_X)} \circ f\right)_{\mathsf{cri}(\Sigma,*)} &= \left(\xi \circ \eta_{\Lambda,(X,\mu_X)} \circ f\right)_{\mathsf{cri}(\Sigma),*} \\ &= \left(\mathsf{id}_{(X,\mu_X)} \circ f\right)_{\mathsf{cri}(\Sigma),*} \\ &= f_{\mathsf{cri}(\Sigma),*} \end{aligned}$$

We have again two cases.

- ϕ is $t \equiv s$, then

$$\begin{aligned} f_{\mathsf{cri}(\Sigma),*}(t) &= \xi\Big(\left(\eta_{\Lambda,(X,\mu_X)} \circ f\right)_{\mathsf{cri}(\Sigma),*}(t)\Big) \\ &= \xi\Big(\left(\eta_{\Lambda,(X,\mu_X)} \circ f\right)_{\mathsf{cri}(\Sigma),*}(s)\Big) \\ &= f_{\mathsf{cri}(\Sigma),*}(s) \end{aligned}$$

- ϕ is $\mathsf{m}(h,t)$, then

$$\begin{aligned} h &\leq \mu_{\Lambda,(X,\mu_X)}\Big(\left(\eta_{\Lambda,(X,\mu_X)} \circ f\right)_{\mathsf{cri}(\Sigma),*}(t)\Big) \\ &= \mu_X\Big(\xi\left(\eta_{\Lambda,(X,\mu_X)} \circ f\right)_{\mathsf{cri}(\Sigma),*}(t)\Big) \\ &= \mu_X(f_{\mathsf{cri}(\Sigma),*}(t)) \end{aligned}$$

In both cases we can conclude that $H_\Lambda((X,\mu_X),\xi) \vDash_f \phi$ and thus it belongs to $\mathbf{Mod}(\Lambda)$ $\square$

Consider now a morphism $f\colon (X,\xi_1) \to (Y,\xi_2)$ in $\mathbf{EM}(\mathbf{T}_\Lambda)$, then we have diagrams

$$\begin{tikzcd}
(X,\mu_X)^{\mathsf{ar}_\Sigma(o)} \arrow[r, "\eta^{\mathsf{ar}_\Sigma(o)}_{\Lambda,(X,\mu_X)}"] \arrow[d, "f^{\mathsf{ar}_\Sigma(o)}"'] & (S_\Lambda(X,\mu_X))^{\mathsf{ar}_\Sigma(o)} \arrow[r, "o^{F_\Lambda(X,\mu_X)}"] \arrow[d, "(S_\Lambda(f))^{\mathsf{ar}_\Sigma(o)}"'] & S_\Lambda(X,\mu_X) \arrow[r, "\xi_1"] \arrow[d, "S_\Lambda(f)"'] & (X,\mu_X) \arrow[d, "f"] \\
(Y,\mu_Y)^{\mathsf{ar}_\Sigma(o)} \arrow[r, "\eta^{\mathsf{ar}_\Sigma(o)}_{\Lambda,(Y,\mu_Y)}"'] & (S_\Lambda(Y,\mu_Y))^{\mathsf{ar}_\Sigma(o)} \arrow[r, "o^{F_\Lambda(Y,\mu_Y)}"'] & S_\Lambda(Y,\mu_Y) \arrow[r, "\xi_2"'] & (Y,\mu_Y)
\end{tikzcd}$$

$$\begin{tikzcd}
 & S_\Lambda(X,\mu_X) \arrow[r, "\xi_1"] \arrow[d, "S_\Lambda(f)"'] & (X,\mu_X) \arrow[d, "f"] \\
\nabla_{\mathbf{H}}(1) \arrow[ur, bend left, "c^{F_\Lambda(X,\mu_X)}"] \arrow[dr, bend right, "c^{F_\Lambda(Y,\mu_Y)}"'] & & \\
 & S_\Lambda(Y,\mu_Y) \arrow[r, "\xi_2"] & (Y,\mu_Y)
\end{tikzcd}$$

made by commutative rectangles and triangles, therefore f is a Σ-homomorphism $H_\Lambda(X,\xi_1) \to H_\Lambda(Y,\xi_2)$. This, in turn allows us to define a functor $H_\Lambda\colon \mathbf{EM}(\mathbf{T}_\Lambda) \to \mathbf{Mod}(\Lambda)$

$$\begin{tikzcd}
((X,\mu_X),\xi_1) \arrow[r, mapsto] \arrow[d, "f"'] & H_\Lambda((X,\mu_X),\xi_1) \arrow[d, "f"] \\
((Y,\mu_Y),\xi_2) \arrow[r, mapsto] & H_\Lambda((Y,\mu_Y),\xi_2)
\end{tikzcd}$$

Theorem 3.2.50. *For every $\Sigma \in \mathbf{FSign}_\kappa$ and basic Σ-theory Λ, the functor $K_\Lambda\colon \mathbf{Mod}(\Lambda) \to \mathbf{EM}(\mathbf{S}_\Lambda)$ has $H_\Lambda\colon \mathbf{EM}(\mathbf{S}_\Lambda) \to \mathbf{Mod}(\Lambda)$ as an inverse.*

Proof. H_Λ and K_Λ both act on arrows as the identity, hence it is enough to show that they are mutually inverse on objects.

On one hand, if $((X,\mu_X),\xi)$ be an Eilenberg-Moore algebra for $\mathbf{S}_\Lambda$, by construction we have

$$K_\Lambda(H_\Lambda((X,\mu_X),\xi)) = (X, (\mathsf{id}_{(X,\mu_X)})_{\Lambda,*})$$

Proposition 3.2.45 entails $\xi = (\mathsf{id}_{(X,\mu_X)})_{\Lambda,*}$ so that $K_\Lambda \circ H_\Lambda = \mathsf{id}_{\mathbf{EM}(\mathbf{S}_\Lambda)}$.

On the other hand, if $\mathcal{A} = \left(A, \left\{o^{\mathcal{A}}\right\}_{o\in O_\Sigma}, \left\{c^{\mathcal{A}}\right\}_{c\in C_\Sigma}\right)$ is a model of Λ, then we have a diagram

$$\begin{tikzcd}
(A,\mu_A)^{\mathsf{ar}_\Sigma(o)} \arrow[r, "\eta^{\mathsf{ar}_\Sigma(o)}_{\Lambda,(A,\mu_A)}"] \arrow[dr, bend right, "\mathsf{id}^{\mathsf{ar}_\Sigma(o)}_{(A,\mu_A)}"'] & (S_\Lambda(A,\mu_A))^{\mathsf{ar}_\Sigma(o)} \arrow[r, "o^{F_\Lambda(A,\mu_A)}"] \arrow[d, "(\mathsf{id}_A)^{\mathsf{ar}_\Sigma(o)}_{\Lambda,*}"] & S_\Lambda(A,\mu_A) \arrow[d, "(\mathsf{id}_{(A,\mu_A)})_{\Lambda,*}"] \\
 & (A,\mu_A)^{\mathsf{ar}_\Sigma(o)} \arrow[r, "o^{\mathcal{A}}"'] & (A,\mu_A)
\end{tikzcd}$$

which is commutative since $K_\Lambda(\mathcal{A})$ is an object of $\mathbf{EM}(\mathbf{T}_\Lambda)$ and $(\mathsf{id}_A)_{\Lambda,*}$ is a Σ-homomorphism. In particular this shows that $o^{\mathcal{A}} = o^{H_\Lambda(K_\Lambda(\mathcal{A}))}$. Now it is enough to notice that we have another diagram

$$\begin{tikzcd} & \nabla_{\mathbf{H}}(1) \arrow[dl, bend right, "c^{F_\Lambda(A,\mu_A)}"'] \arrow[dr, bend left, "c^{\mathcal{A}}"] & \\ S_\Lambda(A,\mu_A) \arrow[rr, "(\mathsf{id}_{(A,\mu_A)})_{\Lambda,*}"'] & & (A,\mu_A) \end{tikzcd}$$

to conclude that $H_\Lambda \circ K_\Lambda = \mathsf{id}_{\mathbf{Mod}(\Lambda)}$. □

Corollary 3.2.51. *Let Σ be a κ-bounded signature and Λ a Σ-theory, then V_Λ is strictly monadic.*

Let $I_\Lambda : \mathbf{Mod}(\Lambda) \to \Sigma\text{-}\mathbf{FAlg}$ be the inclusion of models of Λ into the category of Σ-algebras. By Corollary 3.2.51 we know that there is a functor $F\colon \mathbf{EM}(\mathbf{S}_\Lambda) \to \mathbf{EM}(\mathbf{S}_\Sigma)$ fitting in the diagram below

$$\begin{tikzcd} \mathbf{Mod}(\Lambda) \arrow[rr, "I_\Lambda"] \arrow[dd, "K_\Lambda"'] \arrow[dr, "V_\Lambda"] & & \Sigma\text{-}\mathbf{FAlg} \arrow[dd, "K_\Sigma"] \arrow[dl, "V_\Sigma"'] \\ & \mathbf{Fuz}(\mathbf{H}) & \\ \mathbf{EM}(\mathbf{S}_\Lambda) \arrow[rr, dotted, "F"'] \arrow[ur, "U_{\mathbf{S}_\Lambda}"] & & \mathbf{EM}(\mathbf{S}_\Sigma) \arrow[ul, "U_{\mathbf{T}_\Sigma}"'] \end{tikzcd}$$

Moreover, for every $\mathcal{A} \in \mathbf{Mod}(\Lambda)$, $(\mathsf{id}_{(A,\mu_A)})_{\Lambda,*} \circ \pi_{\Lambda,A}$ is the unique Σ-homomorphism which makes the following diagram commutative

$$\begin{tikzcd} & (A,\mu_A) \arrow[dl, bend right, "\eta_{\Sigma,(A,\mu_A)}"'] \arrow[d, "\eta_{\Lambda,(A,\mu_A)}"] \arrow[dr, bend left, "\mathsf{id}_{(A,\mu_A)}"] & \\ S_\Sigma((A,\mu_A)) \arrow[r, "\pi_{\Lambda,(A,\mu_A)}"'] & S_\Lambda((A,\mu_A)) \arrow[r, "(\mathsf{id}_{(A,\mu_A)})_*"'] & (A,\mu_A) \end{tikzcd}$$

Applying this argument to $I_\Lambda(H_\Lambda(X,\xi))$, and using Proposition 2.2.80 we get that F is given by

$$\begin{array}{ccc} ((X,\mu_X),\xi_1) & \longmapsto & ((X,\mu_X),\xi_1 \circ \pi_{\Lambda,(X,\mu_X)}) \\ f\big\downarrow & & \big\downarrow f \\ ((Y,\mu_Y),\xi_2) & \longmapsto & ((Y,\mu_Y),\xi_2 \circ \pi_{\Lambda,(Y,\mu_Y)}) \end{array}$$

If we apply Proposition 2.1.24, the previous observations now yield the following result.

Proposition 3.2.52. *Given $\Sigma \in \mathbf{Sign}_\kappa$ and a Σ-theory Λ, there exists a morphism of monads $\pi_\Lambda\colon \mathbf{S}_\Sigma \to \mathbf{S}_\Lambda$ whose component at (X,μ_X) is given by $\pi_{\Lambda,(X,\mu_X)}$.*

3.3 Two HSP theorems for fuzzy algebraic theories

In this section we prove two results for our calculus analogous to the classic HSP theorem [25], applying the abstract machinery developed by Milius and Urbat [95] to our case.

3.3.1 Milius and Urbat's theorem

Let us start recalling the tools introduced in [95], adapted to our situation.

Definition 3.3.1. An object X of a category $\mathbf{X}$ is *projective with respect to an arrow* $f\colon Z \to Y$ if for any $h\colon X \to Y$ there exists a $k\colon X \to Z$ such that the following diagram commutes

$$\begin{tikzcd} & Z \arrow[d, "f"] \\ X \arrow[ur, dotted, bend left, "k"] \arrow[r, "h"'] & Y \end{tikzcd}$$

Let $(\mathcal{E}, \mathcal{M})$ be a proper factorization system on $\mathbf{X}$. For every subclass $\mathcal{X}$ of objects of $\mathbf{X}$, we define $\mathcal{E}_{\mathcal{X}}$ as the class of $e \in \mathcal{E}$ such that for every $X \in \mathcal{X}$, X is projective with respect to e.

An *MU-structure* is a triple $(\mathbf{X}, (\mathcal{E}, \mathcal{M}), \mathcal{X})$ where $\mathbf{X}$ is a category, $(\mathcal{E}, \mathcal{M})$ a proper factorization system on it and $\mathcal{X}$ a class of objects of $\mathbf{X}$ such that

1. $\mathbf{X}$ has all (small) products and it is $\mathcal{E}$-cowellpowered;
2. for every object X of $\mathbf{X}$ there exists $e\colon Y \to X$ in $\mathcal{E}_{\mathcal{X}}$ with $Y \in \mathcal{X}$.

A full subcategory $\mathbf{Y}$ of $\mathbf{X}$ will be called a *variety* if it is closed under $\mathcal{E}_{\mathcal{X}}$-quotients, $\mathcal{M}$-subobjects and small products, i.e. if:

- if $Y \in \mathbf{Y}$, then for every $[e] \in \mathcal{E}_{\mathcal{X}}\text{-}\mathsf{Quot}(Y)$, $\mathsf{cod}(e)$ belongs to $\mathbf{Y}$;
- if $Y \in \mathbf{Y}$, then for every $[m] \in \mathcal{M}\text{-}\mathsf{Sub}(Y)$, $\mathsf{dom}(e)$ belongs to $\mathbf{Y}$;
- if I is a set and $\{Y_i\}_{i\in I}$ a family of objects of $\mathbf{Y}$, then their product in $\mathbf{X}$ belongs to $\mathbf{Y}$, too.

Remark 3.3.2. Notice that if $\mathcal{X}$ and $\mathcal{Y}$ are two subclasses of objects of $\mathbf{X}$ with $\mathcal{X} \subseteq \mathcal{Y}$, then $\mathcal{E}_{\mathcal{Y}} \subseteq \mathcal{E}_{\mathcal{X}}$.

Let us prove some properties of $\mathcal{E}_{\mathcal{X}}$.

Proposition 3.3.3. *Let $(\mathbf{X}, (\mathcal{E}, \mathcal{M}), \mathcal{X})$ be an MU-structure, then the following hold:*

1. *if $f\colon Z \to Y$ is an isomorphism, then $f \in \mathcal{E}_{\mathcal{X}}$;*
2. *if $f\colon X \to Y$ and $g\colon Y \to Z$ belong to $\mathcal{E}_{\mathcal{X}}$, then $g \circ f \in \mathcal{E}_{\mathcal{X}}$ too;*
3. *given $f\colon X \to Y$ and $g\colon Y \to Z$, if $g \circ f \in \mathcal{E}_{\mathcal{X}}$ then $g \in \mathcal{E}_{\mathcal{X}}$.*

Proof. 1. By point 1 of Definition 2.1.40, $f \in \mathcal{E}$. On the other hand, if $X \in \mathcal{X}$ and h is an arrow $X \to Y$, then the following diagram witnesses $f \in \mathcal{E}_{\mathcal{X}}$.

$$\begin{tikzcd} & Z \arrow[d, "f"] \\ X \arrow[ur, bend left, "f^{-1}\circ h"] \arrow[r, "h"'] & Y \end{tikzcd}$$

2. By point 2 of Definition 2.1.40, $g \circ f$ is an element of $\mathcal{E}$, so we are left with projectivity. Let $h\colon A \to Z$ be an arrow with domain in $\mathcal{X}$ and consider the following diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[d, "g"] \\ A \arrow[u, dotted, "k_1"] \arrow[ur, dotted, "k_2"] \arrow[r, "h"'] & Z \end{tikzcd}$$

k_2 exists applying projectivity of g to h and k_1 exists applying projectivity of f to k_2. We have

$$\begin{aligned}(g \circ f) \circ k_1 &= g \circ (f \circ k_1)\\ &= g \circ k_2\\ &= h\end{aligned}$$

3. By point 3 of Corollary 2.1.42 we know that $g \in \mathcal{E}$, so let $h\colon A \to Z$ be an arrow with domain $\mathcal{X}$, since $g \circ f \in \mathcal{E}_{\mathcal{X}}$ we get the solid part of the following diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[d, "g"] \\ A \arrow[u, "k_1"] \arrow[ur, dotted, "k_2"] \arrow[r, "h"'] & Z \end{tikzcd}$$

   Now let k_2 be $f \circ k_1$, computing we get

$$\begin{aligned}g \circ k_2 &= g \circ (f \circ k_1)\\ &= (g \circ f) \circ k_1\\ &= h\end{aligned}$$

   from which the thesis follows at once □

Definition 3.3.4 ([17]). Let $(\mathbf{X}, (\mathcal{E}, \mathcal{M}), \mathcal{X})$ be an MU-structure, an *$\mathcal{X}$-equation* is an arrow $e \in \mathcal{E}\text{-}\mathsf{Quot}(X)$ with domain X in $\mathcal{X}$. We say that an object Y of $\mathbf{X}$ *satisfies* a $\mathcal{X}$-equation $e\colon X \to Z$, if for every $h\colon X \to Y$ there exists $q\colon Z \to Y$ such that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "e"'] & Y \\ Z \arrow[ur, dotted, bend right, "q"'] & \end{tikzcd}$$

Given a class E of $\mathcal{X}$-equations, we define $\mathcal{V}(E)$ as the full subcategory of $\mathbf{X}$ given by objects that satisfy e for every $e \in E$. A full subcategory $\mathbf{Y}$ is *$\mathcal{X}$-equationally presentable* if there exists a class E of $\mathcal{X}$-equations such that $\mathbf{Y} = \mathcal{V}(E)$.

Remark 3.3.5. The definition of $\mathcal{X}$-equations and all the machinery involved is given in [95] in more general terms. However, when applied to the two MU-structures on $\mathbf{Fuz}(\mathbf{H})$ in which we are interested, Milius and Urbat's definition reduces to ours (cfr. their Remark 3.4 in [95]).

We can now notice that $\mathcal{X}$-equationally presentable subcategories are varieties.

Lemma 3.3.6. *Let* $\mathbf{Y}$ *be a* $\mathbf{X}$*-equationally presentable subcategory of* $\mathbf{X}$*. Then* $\mathbf{Y}$ *is a variety.*

Proof. Let $\mathbf{Y}$ be $\mathcal{V}(E)$ for some class E of $\mathcal{X}$-equations, we have to prove the three closure properties.

- $\mathcal{E}_{\mathcal{X}}$-quotients. Let $q\colon Y \to Q$ be an arrow in $\mathcal{E}_{\mathcal{X}}$ with $Y \in \mathbf{Y}$ and fix a $\mathcal{X}$-equation $e\colon X \to Z$ in E. Let $h\colon X \to Q$ be another arrow, since $q \in \mathcal{E}_{\mathcal{X}}$ we get the dotted $k\colon X \to Y$ in the diagram

$$\begin{tikzcd} Z \arrow[r, dotted, "k'"] & Y \arrow[d, "q"] \\ X \arrow[u, "e"] \arrow[ur, dotted, "k"] \arrow[r, "h"'] & Q \end{tikzcd}$$

  On the other hand, $Y \in \mathcal{V}(E)$so there also exists the other dotted arrow $k'\colon Z \to Y$ ÿ and the thesis now follows.

- $\mathcal{M}$-subobjects. Let Y be an object of $\mathbf{Y}$ and $m\colon M \to Y$ an arrow in $\mathcal{M}$. As before fix an element $e\colon X \to Z$ of E and an arrow $h\colon X \to M$. Since $Y \in \mathcal{V}(E)$ there exists $k\colon Z \to Y$ making the solid part of the following diagram commutative

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "e"'] & M \arrow[d, "m"] \\ Z \arrow[ur, dotted, "k'"] \arrow[r, "k"'] & Y \end{tikzcd}$$

  Now, e is in $\mathcal{E}$ and $(\mathcal{E}, \mathcal{M})$ is a factorization system, so there is $k'\colon Z \to M$ witnessing $M \in \mathcal{V}(E)$.

- Small products. Let $\{Y_i\}_{i \in I}$ be a small family of objects in $\mathbf{Y}$ and let $e\colon X \to Z$ be a given element of E. For every arrow $h\colon X \to \prod_{i \in I} Y_i$, we get the solid part of the following diagram

$$\begin{tikzcd} & X \arrow[dl, bend right, "e"'] \arrow[d, "h"] & \\ Z \arrow[r, dotted, "q"] \arrow[rr, dotted, bend right, "q_i"'] & \prod\limits_{i \in I} Y_i \arrow[r, "\pi_i"] & Y_i \end{tikzcd}$$

  Since Y_i is an object of $\mathbf{Y} = \mathcal{V}(E)$, we get the existence of the dotted $q_i\colon E \to Y_i$ such that

$$q_i \circ e = \pi_i \circ h$$

  Let q be the induced arrow into the product, then, for every $i \in I$:

$$\begin{aligned} \pi_i \circ q \circ e &= q_i \circ e \\ &= \pi_i \circ h \end{aligned}$$

  and thus $q \circ e = h$ as desired. □

Definition 3.3.7. Let $(\mathbf{X}, (\mathcal{E}, \mathcal{M}), \mathcal{X})$ be an MU-structure, a X an object of $\mathcal{X}$. An *$\mathcal{X}$-equation over X* is a class $\mathfrak{I}_X \subseteq X/\mathcal{E}$ of $\mathcal{X}$-equations with the same domain such that:

1. there is a *minimum* $e_X \in \mathfrak{I}_X$ such that $e_X \leq e'$ for every other $e' \in \mathfrak{I}_X$;
2. for every $e\colon X \to Z$ in $\mathfrak{I}_X$, if $q\colon Z \to V$ is in $\mathcal{E}_{\mathcal{X}}$, then $q \circ e \in \mathfrak{I}_X$.

An object Y *satisfies* $\mathfrak{I}_X$ if, for every $h\colon X \to A$ there is $e\colon X \to Z$ in $\mathfrak{I}_X$ and $q\colon Z \to A$ such that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "e"'] & Y \\ Z \arrow[ur, bend right, "q"'] & \end{tikzcd}$$

A $\mathcal{X}$*-equational theory* $\mathfrak{I}$ is a family $\{\mathfrak{I}_X\}_{X\in\mathcal{X}}$ of $\mathcal{X}$-equations over objects of $\mathcal{X}$ such that:

1. (*substitution invariance*) for every arrow $h\colon X \to Y$ between objects of $\mathcal{X}$ and $e\colon Y \to Z$ in $\mathfrak{I}_Y$, if $m_{e\circ h} \circ e_{e\circ h}$ is a $(\mathcal{E}, \mathcal{M})$-factorization of $e \circ h$, then $e_{e\circ h}$ is in $\mathfrak{I}_{\mathcal{X}}$;
2. ($\mathcal{E}_{\mathcal{X}}$*-completeness*) for every $e\colon Y \to Z$ in $\mathfrak{I}_Y$, there exists another $e'\colon X \to Z$ in $\mathfrak{I}_X$ which belongs also to $\mathcal{E}_{\mathcal{X}}$.

An object Y *satisfies* $\mathfrak{I}$ if it satisfies all its elements $\mathfrak{I}_X$. We will denote by $\mathcal{V}_*(\mathfrak{I})$ the full subcategory of $\mathbf{X}$ given by the objects satisfying $\mathfrak{I}$.

Proposition 3.3.8. *Let $\mathfrak{I}_X$ be an equation over an object X with minimum e_X, then an object Y satisfies $\mathfrak{I}_X$ if an only if it belongs to $\mathcal{V}(\{e_X\})$.*

Proof. ($\Rightarrow$) Let $h\colon X \to Y$ be an arrow, by hypothesis there exists $e \in \mathfrak{I}_X$ and q such that $q \circ e = h$. Since $e_X \leq e$, then there is a k such that $k \circ e_X = e$ and the thesis now follows taking $q \circ k$.

($\Leftarrow$) This is tautological since $e_X \in \mathfrak{I}_X$. $\square$

Corollary 3.3.9. *Let $\mathfrak{I} = \{\mathfrak{I}_X\}_{X\in\mathcal{X}}$ be an equational theory, and define $E_{\mathfrak{I}}$ to be the collection of the minima of all the $\mathfrak{I}_X$, then*

$$\mathcal{V}_*(\mathfrak{I}) = \mathcal{V}(E_{\mathfrak{I}})$$

In particular, this implies that $\mathcal{V}_*(\mathfrak{I})$ is a variety.

$\mathcal{X}$-equational theories are useful, because we can provide a simple criterion criterion to establish if an object satisfies a given $\mathfrak{I}$.

Proposition 3.3.10. *Given an MU-structure $(\mathbf{X}, (\mathcal{E}, \mathcal{M}), \mathcal{X})$ and an $\mathcal{X}$-equational theory $\mathfrak{I}$, an object Y belongs to $\mathcal{V}_*(\mathfrak{I})$ if and only if there exists $X \in \mathcal{X}$ and $e \in \mathfrak{I}_X$ with codomain Y.*

Proof. ($\Rightarrow$) By point 2 of Definition 3.3.1 there is $e\colon X \to Y$ in $\mathcal{E}_{\mathcal{X}}$, with $X \in \mathcal{X}$. By hypothesis Y satisfies $\mathfrak{I}$, thus it satisfies $\mathfrak{I}_X$ and so there is $e'\colon X \to Z$ in $\mathfrak{I}_X$ and $q\colon Z \to Y$ fitting in the diagram

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "e'"'] & Y \\ Z \arrow[ur, bend right, "q"'] & \end{tikzcd}$$

By the third point of Proposition 3.3.3, $q \in \mathcal{E}_{\mathcal{X}}$ and the thesis now follows since $\mathfrak{I}_X$ is closed under composition with elements of $\mathcal{E}_{\mathcal{X}}$.

($\Leftarrow$) Let $e\colon X \to Y$ be an element of $\mathfrak{I}_X$ with codomain Y, by $\mathcal{E}_{\mathcal{X}}$ completeness there is another $e'\colon Y' \to Y$ in $\mathfrak{I}_{Y'}$ which is also in $\mathcal{E}_{\mathcal{X}}$. Take now any other $Z \in \mathcal{X}$ and suppose that an arrow $X \to Y$ is given.

Since $e' \in \mathcal{E}_{\mathcal{X}}$ we get a $k\colon Z \to Y'$ which makes the following diagram commute

$$\begin{tikzcd}
Z \arrow[rr, dotted, "k"] \arrow[dr, "h"'] & & Y' \arrow[dl, "e'"] \\
& Y &
\end{tikzcd}$$

If we factor $e' \circ k$ as $m_{e' \circ k} \circ e_{e' \circ k}$, by substitution invariance we have that $e_{e' \circ k} \in \mathfrak{I}_Z$, getting

$$\begin{tikzcd}
Z \arrow[r, "k"] \arrow[d, "e_{e' \circ k}"'] \arrow[dr, "h"] & Y' \arrow[d, "e'"] \\
Z' \arrow[r, "m_{e' \circ k}"'] & Y
\end{tikzcd}$$

But now, this diagram witnesses that Y satisfies $\mathfrak{I}_Z$ and the thesis now follows. $\square$

Take now a variety $\mathbf{Y}$, then for every $X \in \mathcal{X}$ we can define $\mathcal{I}(\mathbf{Y})_X$ putting

$$\mathcal{I}(\mathbf{Y})_X := \{e \in X/\mathcal{E} \mid \mathsf{cod}(e) \in \mathbf{Y}\}$$

The following proposition guarantees us that in this way we get an $\mathcal{X}$-equational theory.

Proposition 3.3.11. *Let $(\mathbf{X}, (\mathcal{E}, \mathcal{M}), \mathcal{X})$ be an MU-structure, then for every variety $\mathbf{Y}$, the family*

$$\mathcal{I}(\mathbf{Y}) := \{\mathcal{I}(\mathbf{Y})_X\}_{X \in \mathcal{X}}$$

is an $\mathcal{X}$-equational theory.

Proof. First of all we have to show that, for every $X \in \mathcal{X}$, $\mathcal{I}(\mathbf{Y})_X$ is an $\mathcal{X}$-equation over X.

1. By definition of MU-structure ,$\mathbf{X}$ is $\mathcal{E}$-cowellpowered. Thus there exists a set $\{e_i\}_{i \in I} \subseteq \mathcal{I}(\mathbf{Y})_X$ such that, for every $e \in \mathcal{I}(\mathbf{Y})_X$, $e \equiv e_i$ for some $i \in I$. Let X_i be the codomain of e_i, we have a diagram

$$\begin{tikzcd}
& X \arrow[dl, "e_f"'] \arrow[d, "f"] \arrow[dr, "e_i"] & \\
Y \arrow[r, "m_f"'] & \prod_{i \in I} X_i \arrow[r, "\pi_i"'] & X_I
\end{tikzcd}$$

   where f is the arrow induced by $\{e_i\}_{i \in I}$ and e_f, m_f an $(\mathcal{E}, \mathcal{M})$-factorization of it. e_f belongs to $\mathcal{I}(\mathbf{Y})_X$ since $\mathbf{Y}$ is a variety and, by construction, $e_f \leq e_i$ for every $i \in I$. The thesis now follows since any element of $\mathcal{I}(\mathbf{Y})_X$ is equivalent to one of $\{e_i\}_{i \in I}$.

2. Let $e\colon X \to Z$ be in $\mathcal{I}(\mathbf{Y})_X$, if $q\colon Z \to Z'$ is in $\mathcal{E}_{\mathcal{X}}$ then Z' belongs to $\mathbf{Y}$ and thus $q \circ e \in \mathcal{I}(\mathbf{Y})_X$.

Next, we have to show that $\mathcal{I}(\mathbf{Y})$ enjoys the substitution invariance and $\mathcal{E}_{\mathcal{X}}$-completeness properties.

1. Let $h\colon X \to Y$ be an arrow between two objects of $\mathcal{X}$ and let $e \in \mathcal{I}(\mathbf{Y})_Y$. Factoring $e \circ h$ we get a diagram

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "e_{e\circ h}"'] & Y \arrow[d, "e"] \\ Z' \arrow[r, "m_{e\circ h}"'] & Z \end{tikzcd}$$

Z is in $\mathbf{Y}$ so, since $\mathbf{Y}$ is a variety, Z' is in $\mathbf{Y}$ too and thus $e_{e\circ h}$ belongs to $\mathcal{I}(\mathbf{Y})_X$.

2. Let $e\colon Y \to Z$ be an element of $\mathfrak{I}_Y$, by definition of MU-structure there exists $e'\colon X \to Z$ in $\mathcal{E}_{\mathcal{X}}$ which, by definition, is in $\mathcal{I}(\mathbf{Y})_X$ and we are done. □

Lemma 3.3.12. *Given an MU-structure* $(\mathbf{X}, (\mathcal{E}, \mathcal{M}), \mathcal{X})$, *the following hold true:*

1. *for every variety* $\mathbf{Y}$, $\mathcal{V}_*(\mathcal{I}(\mathbf{Y})) = \mathbf{Y}$;
2. *for every* $\mathcal{X}$*-equational theory* $\mathfrak{I}$, $\mathcal{I}(\mathcal{V}_*(\mathfrak{I})) = \mathfrak{I}$.

Proof. 1. Let us show the two inclusions.

($\subseteq$) Let Y be an object of $\mathcal{V}_*(\mathcal{I}(\mathbf{Y}))$, by Proposition 3.3.10 there exists $X \in \mathcal{X}$ and $e \in \mathcal{I}(\mathbf{Y})_X$ with codomain Y and thus $Y \in \mathbf{Y}$ by definition of $\mathcal{I}(\mathbf{Y})_X$.

($\supseteq$) By definition of MU-structure, for every $Y \in \mathbf{Y}$ there exists $e\colon X \to Y$ in $\mathcal{E}_{\mathcal{X}}$ with domain in $\mathcal{X}$. Hence $e \in \mathcal{I}(\mathbf{Y})_X$ and Proposition 3.3.10 yields $Y \in \mathcal{V}_*(\mathcal{I}(\mathbf{Y}))$.

2. As in the previous point, we are going to show the two inclusions

($\subseteq$) Given $e\colon X \to Y$ in $\mathcal{I}(\mathcal{V}_*(\mathfrak{I}))_X$, we know that $Y \in \mathcal{V}_*(\mathfrak{I})$. By Proposition 3.3.10 there exists $X' \in \mathcal{X}$ and $e'\colon X' \to Y$ in $\mathfrak{I}_{X'}$. By $\mathcal{E}_{\mathcal{X}}$-completeness we get another $e''\colon X'' \to Y$ in $\mathfrak{I}_{X''}$ which, moreover, is in $\mathcal{E}_{\mathcal{X}}$. Take the diagram

$$\begin{tikzcd} & X'' \arrow[d, "e''"] \\ X \arrow[ur, dotted, bend left, "h"] \arrow[r, "e"'] & Y \end{tikzcd}$$

The existence of the dotted h is guaranteed by the projectivity of X with respect to e''. We can factor $e'' \circ h$ to get a square

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "e_{e''\circ h}"'] \arrow[dr, "e"] & X'' \arrow[d, "e''"] \\ Z \arrow[r, "m_{e''\circ h}"'] & Y \end{tikzcd}$$

By the third point of Proposition 3.3.3, $m_{e''\circ h} \in \mathcal{E}_{\mathcal{X}}$. By substitution invariance $e_{e''\circ h}$ is an object of $\mathfrak{I}_X$, which is closed under composition with arrows in $\mathcal{E}_{\mathcal{X}}$, therefore $e \in \mathfrak{I}_X$ too.

($\supseteq$) Take $e\colon X \to Y$ in $\mathfrak{I}_X$, thus $Y \in \mathcal{V}_*(\mathfrak{I}_X)$ by Proposition 3.3.10 and so $e \in \mathcal{I}(\mathcal{V}_*(\mathfrak{I}))_X$. □

Corollary 3.3.13 ([95, Th. 3.16])**.** *A full subcategory* $\mathbf{Y}$ *of* $\mathbf{X}$ *is* $\mathcal{X}$*-equationally presentable if and only if it is a variety.*

Proof. ($\Rightarrow$) This is the content of Lemma 3.3.6.

($\Leftarrow$) By Lemma 3.3.12 we know that $\mathbf{Y} = \mathcal{V}_*(\mathcal{I}(\mathbf{Y}))$, therefore Corollary 3.3.9 yields the thesis. □

3.3.2 Application to fuzzy algebras

We now want to apply the machinery developed in the previous section to fuzzy Σ-algebras for some κ-bounded signature Σ. In order to do so we are going to define two MU-structures on Σ-**FAlg**.

Lemma 3.3.14. *For any κ-bounded signature Σ, there exists a proper factorization system $(\mathcal{E}_\Sigma, \mathcal{M}_\Sigma)$ on Σ-**FAlg**, where $e \in \mathcal{E}_\Sigma$ if and only if $V_\Sigma(e)$ is an epimorphism and $m \in \mathcal{M}_\Sigma$ if and only if $V_\Sigma(m)$ is a regular monomorphism.*

Proof. This follows from Theorem 2.1.46, Remark 3.1.27, and Corollaries 3.1.31 and 3.2.51 □

Remark 3.3.15. Notice that, by Proposition 2.1.30 and Corollary 3.2.51, $\mathcal{M}_\Sigma$ is exactly the class of regular monos in Σ-**FAlg**.

Next, we define the following two classes of Σ-algebras putting

$$\mathcal{X}_0 := \{F_\Sigma(\nabla_{\mathbf{H}}(X)) \mid X \in \mathbf{Set}\} \quad \mathcal{X}_{\mathsf{M}} := \{F_\Sigma(X, \mu_X) \mid (X, \mu_X) \in \mathbf{Fuz}(\mathbf{H}) \text{ and } |\mathsf{supp}(X, \mu_X)| < \kappa\}$$

The following lemma assures us that in this way we get two MU-structures.

Lemma 3.3.16. *With the definitions given above, the following hold true*

1. $(\mathcal{E}_\Sigma)_{\mathcal{X}_0} = \mathcal{E}_\Sigma$;
2. $(\mathcal{E}_\Sigma)_{\mathcal{X}_{\mathsf{M}}} = \{e \in \mathcal{E}_\Sigma \mid V_\Sigma(e)$ *is split*$\}$;
3. $(\Sigma\text{-}\mathbf{FAlg}, (\mathcal{E}_\Sigma, \mathcal{M}_\Sigma), \mathcal{X}_0)$ *and* $(\Sigma\text{-}\mathbf{FAlg}, (\mathcal{E}_\Sigma, \mathcal{M}_\Sigma), \mathcal{X}_{\mathsf{M}})$ *are MU-structures.*

Proof. 1. It is enough to show that every arrow in $\mathcal{E}_\Sigma$ is in $(\mathcal{E}_\Sigma)_{\mathcal{X}_0}$. Let $e\colon \mathcal{A} \to \mathcal{B}$ be an arrow in $\mathcal{E}_\Sigma$ and let $h\colon F_\Sigma(\nabla_{\mathbf{H}}(X)) \to \mathcal{B}$ be any morphism of Σ-**FAlg**. By definition e is surjective. So, if $(A, \mu_A) = V_\Sigma(\mathcal{A})$, for any $x \in X$ there exists $k(x) \in A$ such that

$$e(k(x)) = h\left(\eta_{\Sigma, \nabla_{\mathbf{H}}(X)}(x)\right)$$

This defines a function $k\colon X \to A$ where $\mathcal{A}$ is the algebra $\left((A, \mu_A), \{o^{\mathcal{A}}\}_{o \in O_\Sigma}, \{c^{\mathcal{A}}_{c \in C_\Sigma}\}\right)$. k is also a morphism $k\colon \nabla_{\mathbf{H}}(X) \to (A, \mu_A)$, therefore, by adjointness, we get a Σ-homomorphism $k_{\Sigma,*}\colon F_\Sigma(\nabla_{\mathbf{H}}(X)) \to \mathcal{A}$, and computing, we have

$$\begin{aligned}(e \circ k_{\Sigma,*}) \circ \eta_{\Sigma, \nabla_{\mathbf{H}}(X)} &= e \circ (k_{\Sigma,*} \circ \eta_{\Sigma, \nabla_{\mathbf{H}}(X)}) \\ &= e \circ k \\ &= h \circ \eta_{\Sigma, \nabla_{\mathbf{H}}(X)}\end{aligned}$$

Hence, we can deduce that $e \circ k_{\Sigma,*} = h$.

2. Let us show the two inclusions.

($\subseteq$) Take an element $e\colon \mathcal{A} \to \mathcal{B}$ in $(\mathcal{E}_\Sigma)_{\mathcal{X}_M}$, and consider the component in $\mathcal{B}$ of the counit $\epsilon\colon F_\Sigma \circ V_\Sigma \to \mathsf{id}_{\Sigma\text{-}\mathbf{FAlg}}$ of the adjunction $F_\Sigma \dashv V_\Sigma$. If (A, μ_A) and (B, μ_B) are, respectively, $V_\Sigma(\mathcal{A})$ and $V_\Sigma(\mathcal{B})$, we get a diagram:

$$\begin{tikzcd} & \mathcal{A} \arrow[d, "e"] \\ F_\Sigma(B, \mu_B) \arrow[ur, dotted, bend left, "k"] \arrow[r, "\epsilon_{\mathcal{B}}"'] & \mathcal{B} \end{tikzcd}$$

where the dotted κ exists since $e \in (\mathcal{E}_\Sigma)_{\mathcal{X}_\mathsf{M}}$. Now the thesis follows noticing that

$$\begin{aligned} e \circ k \circ \eta_{\Sigma,(B,\mu_B)} &= \epsilon_{\mathcal{B}} \circ \eta_{\Sigma,(B,\mu_B)} \\ &= \mathsf{id}_{(B,\mu_B)} \end{aligned}$$

($\supseteq$) Now let $e\colon \mathcal{A} \to \mathcal{B}$ be such that $V_\Sigma(e)$ is split and let s be a section of it. Given an arrow $h\colon F_\Sigma(X,\mu_X) \to \mathcal{B}$ we can consider define k as the composition

$$(X,\mu_X) \xrightarrow{\eta_{\Sigma,(X,\mu_X)}} S_\Sigma(X,\mu_X) \xrightarrow{\quad h \quad} (B,\mu_B) \xrightarrow{\quad s \quad} (A,\mu_A)$$

where, as usual, (A,μ_A) and (B,μ_B) are $V_\Sigma(\mathcal{A})$ and $V_\Sigma(\mathcal{B})$. By adjointness we get a Σ-homomorphism $k_{\Sigma,*}\colon F_\Sigma(X,\mu_X) \to \mathcal{A}$ and

$$\begin{aligned} (e \circ k_{\Sigma,*}) \circ \eta_{\Sigma,(X,\mu_X)} &= e \circ \big(k_{\Sigma,*} \circ \eta_{\Sigma,(X,\mu_X)}\big) \\ &= e \circ \big(s \circ h \circ \eta_{\Sigma(X,\mu_X)}\big) \\ &= (e \circ s) \circ \big(h \circ \eta_{\Sigma,(X,\mu_X)}\big) \\ &= \mathsf{id}_{(B,\mu_B)} \circ h \circ \eta_{\Sigma,(X,\mu_X)} \\ &= h \circ \eta_{\Sigma,(X,\mu_X)} \end{aligned}$$

so $k_{\Sigma,*}$ is the desired lifting.

3. Let us prove all the conditions of Definition 3.3.1.

   (a) Σ-**FAlg** has all products by Proposition 2.1.30 and Corollaries 3.1.26 and 3.2.51. Moreover, Σ-**FAlg** is also $\mathcal{E}_\Sigma$-cowellpowered: $V_\mathbf{H} \circ V_\Sigma\colon \Sigma\text{-}\mathbf{FAlg} \to \mathbf{Set}$ is faithful, it sends $e \in \mathcal{A}/\mathcal{E}_\Sigma$ to a surjective arrow with domain A and **Set** is cowellpowered with respect to surjective functions.

   (b) Let $\mathcal{A}$ be an object of Σ-**FAlg** and take (A,μ_A) to be $V_\Sigma(\mathcal{A})$. We can consider two arrows:

   $$\mathsf{id}_A\colon \nabla_\mathbf{H}(A) \to (A,\mu_A) \qquad \mathsf{id}_{(A,\mu_A)}\colon (A,\mu_A) \to (A,\mu_A)$$

   which induce

   $$e_0\colon F_\Sigma(\nabla_\mathbf{H}(A)) \to \mathcal{A} \qquad e_\mathsf{M}\colon F_\Sigma(A,\mu_A) \to \mathcal{A}$$

   Now, by construction we have the following two equalities

   $$e_0 \circ \eta_{\Sigma,\nabla_\mathbf{H}(A)} = \mathsf{id}_A \qquad e_\mathsf{M} \circ \eta_{\Sigma,\nabla_\mathbf{H}(A)} = \mathsf{id}_\mathcal{A}$$

   showing that e_0 is surjective and e_M is split. □

Remark 3.3.17. We will say that an arrow in $(\mathcal{E}_\Sigma)_{\mathcal{X}_\mathsf{M}}$ is a *split $\mathcal{E}_\Sigma$-quotient*. Notice that such a morphism is not a split epimorphism in Σ-**FAlg**.

We want now to relate formulae of our sequent calculus to $\mathcal{X}_0$- and $\mathcal{X}_\mathsf{M}$-equations. Recall that, for every $(X,\mu_X) \in \mathbf{Fuz}(\mathbf{H})$, Remark 3.2.39 entails the existence of a $\mathsf{cri}(\Sigma)$-isomorphism $\Theta_{(X,\mu_X)}\colon T_{\mathsf{cri}(\Sigma)}(X) \to T_\Sigma(X,\mu_X)$. Moreover, fix a bijection $j\colon |X| \to X$ and take $R_{(X,\mu_X)}$ to be $j^{-1}(\mathsf{supp}(X,\mu_X))$. Finally, define the function $\Xi_{(X,\mu_X)}\colon T_{\mathsf{cri}(\Sigma)}(|X|) \to T_\Sigma(X,\mu_X)$ as the composition

$$T_{\mathsf{cri}(\Sigma)}(|X|) \xrightarrow{T_{\mathsf{cri}(\Sigma)}(j)} T_{\mathsf{cri}(\Sigma)}(X) \xrightarrow{\Theta_{(X,\mu_X)}} T_\Sigma(X,\mu_X)$$

Definition 3.3.18. Let Σ be a κ_1-bounded signature and $e\colon F_\Sigma(X,\mu_X) \to \mathcal{B}$ an $\mathcal{X}_{\mathsf{M}}$-equation. Let also κ the smallest regular cardinal greater or equal than $\mathsf{sup}\,(\kappa_1, |X|)$, so that, in particular, Σ is κ-bounded. We define $\Gamma_{(X,\mu_X)}$ as

$$\Gamma_{(X,\mu_X)} := \{\mathsf{m}\big(\mu_X(j(\alpha)), \eta_{\mathsf{cri}(\Sigma),|X|}(\alpha)\big)\}_{\alpha \in R_{(X,\mu_X)}}$$

A sequent $|X| \mid \Gamma_{(X,\mu_X)} \vdash \phi$ will be called a *e-sequent* if

- ϕ is $t_1 \equiv t_2$ and $e\left(\Xi_{(X,\mu_X)}(t_1)\right) = e\left(\Xi_{(X,\mu_X)}(t_2)\right)$;
- ϕ is $\mathsf{m}(h,t)$ and $h \leq \mu_B\left(e\left(\Xi_{(X,\mu_X)}(t)\right)\right)$.

We define Λ_e as the theory generated by all the e-sequents.

Lemma 3.3.19. *Let Σ be a κ-bounded signature and $e\colon T_\Sigma(X,\mu_X) \to \mathcal{B}$ an $\mathcal{X}_{\mathsf{M}}$-equation such that $|X| < \kappa$. Then* $\mathbf{Mod}(\Lambda_e) = \mathcal{V}(\{e\})$.

Proof. ($\subseteq$) Let $\mathcal{C}$ be a model of Λ_e and $h\colon F_\Sigma(X,\mu_X) \to \mathcal{C}$ a Σ-homomorphism. Let s_1 and s_2 be elements of $T_\Sigma(X,\mu_X)$ such that

$$e(s_1) = e(s_2)$$

By Remark 3.2.39, we also have $t_1, t_2 \in T_{\mathsf{cri}(\Sigma)}(|X|)$ such that

$$\Xi_{(X,\mu_X)}(t_1) = s_1 \qquad \Xi_{(X,\mu_X)}(t_2) = s_2$$

In **Set** we can form a diagram

$$\begin{tikzcd}
 & T_{\mathsf{cri}(\Sigma)}(|X|) \arrow[d, "T_{\mathsf{cri}(\Sigma)}(j)"] \arrow[rdd, dotted, bend left, "(h\circ\eta_{\Sigma,X}\circ j)_{\mathsf{cri}(\Sigma),*}"] & \\
|X| \arrow[ur, bend left, "\eta_{\mathsf{cri}(\Sigma),|X|}"] \arrow[d, "j"'] & T_{\mathsf{cri}(\Sigma)}(X) \arrow[d, "\Theta_{(X,\mu_X)}"] & \\
X \arrow[ur, bend left, "\eta_{\mathsf{cri}(\Sigma),X}"] \arrow[r, "\eta_{\Sigma,X}"'] & T_\Sigma(X,\mu_X) \arrow[r, "h"'] & C
\end{tikzcd}$$

which shows that

$$h \circ \Xi_{(X,\mu_X)} = (h \circ \eta_{\Sigma,X} \circ j)_{\mathsf{cri}(\Sigma),*}$$

Notice that, for every $\alpha \in |X|$ we have

$$\begin{aligned}
\mu_X\left(j(\alpha)\right) &\leq \mu_C(h(\eta_{\Sigma,X}\left(j\left(\alpha\right)\right))) \\
&= \mu_C\Big((h \circ \eta_{\Sigma,X} \circ j)_{\mathsf{cri}(\Sigma),*}\left(\eta_{\mathsf{cri}(\Sigma),|X|}(\alpha)\right)\Big)
\end{aligned}$$

Since by hypothesis $\mathcal{C}$ is a model of Λ_e, we get

$$\begin{aligned}
h(s_1) &= h\left(\Xi_{(X,\mu_X)}(t_1)\right) \\
&= (h \circ \eta_{\Sigma,X} \circ j)_{\mathsf{cri}(\Sigma),*}(t_1) \\
&= (h \circ \eta_{\Sigma,X} \circ j)_{\mathsf{cri}(\Sigma),*}(t_2) \\
&= h\left(\Xi_{(X,\mu_X)}(t_2)\right) \\
&= h(s_2)
\end{aligned}$$

By Proposition 2.2.67 we get a $\mathsf{cri}\,(\Sigma)$-homomorphism g making the following diagram commutative

$$\begin{tikzcd}
T_\Sigma(X,\mu_X) \arrow[r, "h"] \arrow[d, "e"'] & C \\
B \arrow[ur, dotted, bend right, "g"'] &
\end{tikzcd}$$

Now let b be an element of B, since e is surjective there exists $t \in T_{\mathsf{cri}(\Sigma)}(|X|)$ such that

$$e\left(\Xi_{(X,\mu_X)}(t)\right) = b$$

Using again that $\mathcal{C}$ is a model of Λ_e, we obtain

$$\begin{aligned}
\mu_B(b) &= \mu_B\left(e\left(\Xi_{(X,\mu_X)}(t)\right)\right) \\
&\leq \mu_C\left(\left(h \circ \eta_{\Sigma,X} \circ j\right)_{\mathsf{cri}(\Sigma),*}(t)\right) \\
&= \mu_C\left(h\left(\Xi_{(X,\mu_X)}(t)\right)\right) \\
&= \mu_C\left(g\left(e\left(\Xi_{(X,\mu_X)}(t)\right)\right)\right) \\
&= \mu_C(g(b))
\end{aligned}$$

So, by Remark 3.2.4 g is a Σ-homomorphism and we can conclude.

($\supseteq$) Now let $\mathcal{C}$ be an object in $\mathcal{V}(\{e\})$ and $|X| \mid \Gamma_{(X,\mu_X)} \vdash \phi$ an e-sequent. Given a function $f\colon |X| \to C$ such that

$$\mu_X(j(\alpha)) \leq \mu_C(f(\alpha))$$

This implies that $g := f \circ j^{-1}$ is a morphism $(X,\mu_X) \to (C,\mu_C)$ of $\mathbf{Fuz}(\mathbf{H})$ inducing a Σ-homomorphism $g_\circ\colon F_\Sigma(X,\mu_X) \to \mathcal{C}$. Notice that we have a diagram

$$\begin{tikzcd}
 & & & T_{\mathsf{cri}(\Sigma)}(|X|) \arrow[dl, "T_{\mathsf{cri}(\Sigma)}(j)"'] \arrow[ddl, bend left, "\Xi_{(X,\mu_X)}"] \arrow[dddl, bend left, "f_{\mathsf{cri}(\Sigma),*}"] \\
|X| \arrow[urrr, bend left, "\eta_{\mathsf{cri}(\Sigma),|X|}"] \arrow[dr, "j"] \arrow[ddrr, bend right, "f"'] & & T_{\mathsf{cri}(\Sigma)}(X) \arrow[d, "\Theta_{(X,\mu_X)}"'] \arrow[dd, bend left=60, "g_{\mathsf{cri}(\Sigma),*}"] & \\
 & X \arrow[ur, bend left, "\eta_{\mathsf{cri}(\Sigma),X}"] \arrow[r, "\eta_{\Sigma,X}"] \arrow[dr, "g"'] & T_\Sigma(X,\mu_X) \arrow[d, "g_{\Sigma,*}"'] & \\
 & & C &
\end{tikzcd}$$

Since $\mathcal{C}$ is in $\mathcal{V}(\{e\})$ we also have a $k\colon \mathcal{B} \to \mathcal{C}$ such that $g_{\Sigma,*} = k \circ e$. Let us split the cases.

- ϕ is $t_1 \equiv t_2$ for some $t_1, t_2 \in T_{\mathsf{cri}(\Sigma)}(|X|)$. Then we have

$$\begin{aligned}
f_{\mathsf{cri}(\Sigma),*}(t_1) &= g_{\Sigma,*}\left(\Xi_{(X,\mu_X)}(t_1)\right) \\
&= k\left(e\left(\Xi_{(X,\mu_X)}(t_1)\right)\right) \\
&= k\left(e\left(\Xi_{(X,\mu_X)}(t_2)\right)\right) \\
&= g_{\Sigma,*}\left(\Xi_{(X,\mu_X)}(t_2)\right) \\
&= f_{\mathsf{cri}(\Sigma),*}(t_2)
\end{aligned}$$

- ϕ is $\mathsf{m}(h,t)$ for some $h \in H$ and $t \in T_{\mathsf{cri}(\Sigma)}(|X|)$. Computing we get

$$\begin{aligned} h &\leq \mu_B\left(e\left(\Xi_{(X,\mu_X)}(t)\right)\right) \\ &\leq \mu_C\left(k\left(e\left(\Xi_{(X,\mu_X)}(t)\right)\right)\right) \\ &= \mu_C\left(g_{\Sigma,*}\left(\Xi_{(X,\mu_X)}(t)\right)\right) \\ &= \mu_C\left(f_{\mathsf{cri}(\Sigma),*}(t)\right) \end{aligned}$$

and we can conclude. □

Remark 3.3.20. We can refine the previous construction a little. Let Σ be a signature, (X,μ_X) a fuzzy set and κ a regular cardinal such that Σ is κ-bounded and $|\mathsf{supp}(X,\mu_X)| < \kappa$. Take also an $\mathcal{X}_{\mathsf{M}}$-equation $e\colon F_\Sigma(X,\mu_X) \to \mathcal{B}$. Since Σ is λ-bounded for every regular λ greater than $|X|$ we can still consider an e-sequent $|X| \mid \Gamma_{(X,\mu_X)} \vdash \phi$. Notice also that every term in ϕ is the image of some other term $t \in T_{\mathsf{cri}}(\Sigma)(\alpha)$ for some $|\mathsf{supp}(X,\mu_X)| \leq \alpha < \kappa$. Fix an injection $\iota\colon |\mathsf{supp}(X,\mu_X)| \to \alpha$ and a bijection $h\colon R_{(X,\mu_X)} \to |\mathsf{supp}(X,\mu_X)|$, if $i\colon R_{(X,\mu_X)} \to |X|$ is the inclusion we can find $f_\phi\colon |X| \to \alpha$ fitting in the following diagram

$$\begin{tikzcd} R_{(X,\mu_X)} \arrow[r, "h"] \arrow[d, "i"'] & {|\mathsf{supp}(X,\mu_X)|} \arrow[d, "\iota"] \\ {|X|} \arrow[r, dotted, "f_\phi"'] & \alpha \end{tikzcd}$$

Let us now define $\sigma_\phi\colon |X| \to T_{\mathsf{cri}(\Sigma)}(\alpha)$ as the composition

$$|X| \xrightarrow{f_\phi} \alpha \xrightarrow{\eta_{\mathsf{cri}(\Sigma),\alpha}} T_{\mathsf{cri}(\Sigma)}(\alpha)$$

Define Λ'_e as the theory which has as axioms the sequents of type

$$\mu \mid \Gamma_{(X,\mu_X)}[\sigma_\phi] \vdash \phi[\sigma_\phi]$$

whenever $|X| \mid \Gamma_{(X,\mu_X)} \vdash \phi$ is an e-sequent. We claim that $\mathbf{Mod}(\Lambda_e) = \mathbf{Mod}(\Lambda'_e)$.

($\subseteq$) This follows since Λ'_e is contained in Λ_e: by definition all the axioms of the former are derivable from the ones of the latter by an application of rule Subst.

($\supseteq$) Let $\mathcal{A}$ be a model for Λ'_e and $|X| \mid \Gamma_{(X,\mu_X)} \vdash \phi$ an e-sequent. Let also $g\colon |X| \to A$ be a function such that, for every $\beta \in |X|$

$$\mu_X(j(\beta)) \leq \mu_A(g(\beta))$$

Given such g, we can always find $\overline{g}\colon \alpha \to A$ such that

$$g = \overline{g} \circ f_\phi$$

We can then consider the following commutative diagram

$$\begin{tikzcd}
{|\mathsf{supp}(X,\mu_X)|} \arrow[r, "h^{-1}"] \arrow[dr, "\iota"'] & R_{(X,\mu_X)} \arrow[r, "i"] & {|X|} \arrow[dl, "f_\phi"'] \arrow[d, "\eta_{\mathsf{cri}(\Sigma),|X|}"] \arrow[dr, bend left, "g"] & \\
& \alpha \arrow[rr, bend right, "\overline{g}"] \arrow[drr, bend right, "\eta_{\mathsf{cri}(\Sigma),\alpha}"'] & T_{\mathsf{cri}(\Sigma)}(|X|) \arrow[r, "g_{\mathsf{cri}(\Sigma),*}"] \arrow[dr, "T_{\mathsf{cri}(\Sigma)}(f_\phi)"'] & A \\
& & & T_{\mathsf{cri}(\Sigma)}(\alpha) \arrow[u, "\overline{g}_{\mathsf{cri}(\Sigma),*}"']
\end{tikzcd}$$

By construction $\mathcal{A}$ satisfies all elements of $\Gamma_{(X,\mu_X)}[\sigma_\phi]$ with respect to $\overline{g}$ and we can conclude.

This, together with Lemma 3.3.19, shows that $\mathcal{V}(\{e\})$ is the category of models of a theory, which has a set of axioms whose contexts are all less or equal than κ.

We want now to go in the other direction: which kinds of sequents allow us to recover an $\mathcal{X}_{\mathsf{M}}$- or an $\mathcal{X}_0$-equation? The answer is provided by the following definition.

Definition 3.3.21. Let Σ be a κ-bounded signature, a sequent $\lambda \mid \Gamma \vdash \phi$ is said to be

- *unconditional* ([95, App. B.5]) if Γ is the empty set;
- *of type* M if $\Gamma = \{\mathsf{m}(h_i, \eta_{\mathsf{cri}(\Sigma),\lambda}(x_i))\}_{i\in I}$ for some family of variables $\{x_i\}_{i\in I}$ and $\{h_i\}_{i\in I} \subseteq H$.

A Σ-theory Λ is said to be *unconditional* (of *type* M) if it has a set of axioms made by unconditional sequents (sequents of type M).

Lemma 3.3.22. *Let $\lambda \mid \Gamma \vdash \phi$ be a sequent of type* M *and $\Lambda_{\Gamma,\phi}$ the theory with it as a single axiom. Then there exists a $\mathcal{X}_{\mathsf{M}}$-equation $e_{\Gamma,\phi}$ such that*

$$\mathbf{Mod}(\Lambda^{\Gamma,\phi}) = \mathcal{V}\left(\{e_{\Gamma,\phi}\}\right)$$

Moreover, if $\Gamma = \emptyset$, then $e_{\emptyset,\phi}$ is an $\mathcal{X}_0$-equation.

Proof. Let α be an element of λ, we can define

$$\mu_\lambda(\alpha) := \mathsf{sup}\left(\{h \in H \mid \mathsf{m}(h, \eta_{\mathsf{cri}(\Sigma),\lambda}(\alpha)) \in \Gamma\}\right)$$

In this way we get a fuzy set (λ, μ_λ). Applying F_Λ we get the following diagram in $\mathbf{Fuz}(\mathbf{H})$.

$$\begin{tikzcd}
& (\lambda,\mu_\lambda) \arrow[dl, "\eta_{\Sigma,(\lambda,\mu_\lambda)}"'] \arrow[dr, "\eta_{\Lambda^{\Gamma,\phi},(\lambda,\mu_\lambda)}"] & \\
S_\Sigma(\lambda,\mu_\lambda) \arrow[rr, "\pi_{\Lambda^{\Gamma,\phi},(\lambda,\mu_\lambda)}"'] & & S_{\Lambda^{\Gamma,\phi}}(\lambda,\mu_\lambda)
\end{tikzcd}$$

So that we can take $\pi_{\Lambda^{\Gamma,\phi},(\lambda,\mu_\lambda)}$ as $e_{\Gamma,\phi}$.

$(\subseteq)$ Let $\mathcal{A}$ be an algebra satisfying $\lambda \mid \Gamma \vdash \phi$ and $h\colon F_\Sigma(\lambda,\mu_\lambda) \to \mathcal{A}$ a Σ-homomorphism. We can apply freenes of $F_{\Lambda^{\Gamma,\phi}}(\lambda,\mu_\lambda)$ to $h \circ \eta_{\Sigma,(\lambda,\mu_\lambda)}$ to get the dotted k in the following diagram, proving the thesis.

$$
\begin{tikzcd}
(\lambda,\mu_\lambda) \arrow[r, "\eta_{\Sigma,(\lambda,\mu_\lambda)}"] \arrow[d, "\eta_{\Lambda^{\Gamma,\phi},(\lambda,\mu_\lambda)}"'] & S_\Sigma(\lambda,\mu_\lambda) \arrow[dl, "e_{\Gamma,\phi}"'] \arrow[d, "h"] \\
S_{\Lambda^{\Gamma,\phi}}(\lambda,\mu_\lambda) \arrow[r, dotted, "k"'] & (A,\mu_A)
\end{tikzcd}
$$

($\supseteq$) If $f\colon \lambda \to A$ is an arrow such that $\mathcal{A} \vDash_f \psi$ for every $\psi \in \Gamma$, then f itself defines an arrow $f\colon (\lambda,\mu_\lambda) \to (A,\mu_A)$. By hypothesis, $\mathcal{A}$ is in $\mathcal{V}(\{e_{\Gamma,\phi}\})$, thus we get a $k\colon S_{\Lambda^{\Gamma,\phi}}(\lambda,\mu_\lambda) \to \mathcal{A}$ as in the following diagram.

$$
\begin{tikzcd}
(\lambda,\mu_\lambda) \arrow[r, "\eta_{\Sigma,(\lambda,\mu_\lambda)}"] \arrow[d, "\eta_{\Lambda^{\Gamma,\phi},(\lambda,\mu_\lambda)}"'] \arrow[dr, "f"] & S_\Sigma(\lambda,\mu_\lambda) \arrow[dl, "e_{\Gamma,\phi}"] \arrow[d, "f_\circ"] \\
S_{\Lambda^{\Gamma,\phi}}(\lambda,\mu_\lambda) \arrow[r, dotted, "k"'] & (A,\mu_A)
\end{tikzcd}
$$

Moreover, recall that, by Remark 3.2.39, we have a $\mathsf{cri}(\Sigma)$-isomorphism $\Theta_{(\lambda,\mu_\lambda)}\colon T_{\mathsf{cri}(\Sigma)}(\lambda) \to T_\Sigma(\lambda,\mu_\lambda)$

$$
\begin{tikzcd}
T_{\mathsf{cri}(\Sigma)}(\lambda) \arrow[rr, "\Theta_{(\lambda,\mu_\lambda)}"] \arrow[dr, "f_{\mathsf{cri}(\Sigma),*}"'] & & T_\Sigma(\lambda,\mu_\lambda) \arrow[dl, "f_{\Sigma,*}"] \\
& A &
\end{tikzcd}
$$

Now, notice that $S_{\Lambda^{\Gamma,\phi}}(\lambda,\mu_\lambda)$ satisfies all the formulae in Γ with respect to $\eta_{\Lambda^{\Gamma,\phi},(\lambda,\mu_\lambda)}$. Thus it also satisfies ϕ with respect to it. In particular, since, by construction, $e_{\Gamma,\phi} = \left(\eta_{\Lambda^{\Gamma,\phi},(\lambda,\mu_\lambda)}\right)_{f_{\Sigma,*}}$ this implies the following two things:

- if ϕ is $t_1 \equiv t_2$ then $e_{\Gamma,\phi}(t_1) = e_{\Gamma,\phi}(t_2)$;
- if ϕ is $\mathsf{m}(h,t)$ then $h \leq \mu_{\Lambda^{\Gamma,\phi},(\lambda,\mu_\lambda)}(e_{\Gamma,\phi}(t))$.

From these two observations the thesis follows at once

To prove the second half of the thesis just notice that μ_λ is constant at $\perp$ whenever Γ is empty. □

Corollary 3.3.23. *If Λ is a theory of type M, then there is a class E of $\mathcal{X}_{\mathsf{M}}$-equations such that*

$$\mathbf{Mod}(\Lambda) = \mathcal{V}(E)$$

If, moreover, Λ is unconditional then every element of E can be taken to be a $\mathcal{X}_0$-equation.

Putting together Lemmas 3.3.19 and 3.3.22 with Corollary 3.3.13 we get the following result.

Theorem 3.3.24. *Let Σ be a κ-bounded fuzzy signature and let $\mathbf{Y}$ be a full subcategory of Σ-**FAlg**, then the following hold true:*

1. *$\mathbf{Y}$ is closed under $\mathcal{E}_\Sigma$-quotients, (small) products and regular monomorphisms if and only if there exists a class of type M theories $\{\Lambda_i\}_{i\in I}$ such that $\mathcal{A} \in \mathbf{Y}$ if and only if $\mathcal{A} \in \mathbf{Mod}(\Lambda_i)$ for all $i \in I$;*
2. *$\mathbf{Y}$ is closed under split $\mathcal{E}_\Sigma$-quotients, (small) products and regular monomorphisms if and only if there exists a class of unconditional theories $\{\Lambda_i\}_{i\in I}$ such that $\mathcal{A} \in \mathbf{Y}$ if and only if $\mathcal{A} \in \mathbf{Mod}(\Lambda_i)$ for all $i \in I$.*

Proof.
1. ($\Rightarrow$) By Corollary 3.3.13 there is a class of E of $\mathcal{X}_{\mathsf{M}}$-equations such that $\mathbf{Y} = \mathcal{V}(E)$. The thesis follows from Lemma 3.3.19.

   ($\Leftarrow$) This follows from the first half of Corollary 3.3.23 and from Corollary 3.3.13.

2. ($\Rightarrow$) We proceed as in the previous case: by Corollary 3.3.13 there is a class of E of $\mathcal{X}_0$-equations such that $\mathbf{Y} = \mathcal{V}(E)$, Lemma 3.3.19 yields the thesis.

   ($\Leftarrow$) This follows from the second half of Corollary 3.3.23 and from Corollary 3.3.13. □

If Σ is κ-bounded, then it is λ-bounded for every regular λ greater than κ, so we can write down sequents with arbitrarily large contexts and the theorem above makes sense even if E is a proper class. But, due to the way in which we have defined Σ-theories, we cannot put together all the Λ_e's to form a unique theory: for us, in fact, the sequents of a theory all have contexts bounded by a regular cardinal. Luckily, for unconditional theories, this issue disappears.

Corollary 3.3.25. *Let Σ be a κ-bounded fuzzy signature and let $\mathbf{Y}$ be a full subcategory of Σ-$\mathbf{FAlg}$, $\mathbf{Y}$ is closed under $\mathcal{E}_\Sigma$-quotients, (small) products and regular monomorphisms if and only if there exists an unconditional theory Λ such that $\mathbf{Y} = \mathbf{Mod}(\Lambda)$.*

Proof. ($\Rightarrow$) By Corollary 3.3.13 there exists a class E of $\mathcal{X}_0$-equations such that $\mathbf{Y} = \mathcal{V}(E)$. For every $e \in E$, using Remark 3.3.20 we can find a theory Λ_e such that $\mathcal{V}(e) = \mathbf{Mod}(\Lambda_e)$ and Λ_e is axiomatized only by sequents with a context smaller then κ. The thesis now follows taking the theory generated by all the axioms.

($\Leftarrow$)This follows from Corollary 3.3.23. □

CHAPTER 4
Conclusions for Part I

The first part of this thesis has explored the topic of *algebraic theories*, both in their classical form and in a new version, taylored for the category $\mathbf{Fuz}(\mathbf{H})$ of *fuzzy sets*.

In Chapter 2, we reviewed both the categorical and syntactical approaches to this subject, and demonstrated how they are related by restating and proving the well-known results of Linton and Lawvere [76, 78]. In particular, we discussed the notion of *monads* and analyze the related categories of *Eilenberg-Moore algebras*, showing how to compute limits and colimits in them. We then turned our attention to monads on the category $\mathbf{Set}$ of sets and functions, with a focus on those that preserve *κ-filtered colimits*. These monads are determined by their restriction on the subcategory of sets with cardinality less than κ: if a monad preserves such colimits, then it must be a left Kan extension of its restriction.

We focused on this class of monads because they correspond precisely to algebraic theories. Given a set of operations with arities bounded by some cardinal κ, and a set of equations, we demonstrates how a monad can be constructed such that its category of Eilenberg-Moore algebras is isomorphic to the category of models of these equations. Such monad is defined constructing for any set, the *free model* over it and this, in turn, allows us to deduce a completeness theorem for the calculus of equations.

Finally, we ended Chapter 2 showing that the construction associating a monad to an algebraic theory, which can be thought as a functor assigning the semantics to a given syntax, is part of an adjunction. Specifically, given a monad $\mathbf{T}$, with rank, we were able to extract from it an algebraic theory whose category of models is isomorphic to $\mathbf{EM}(\mathbf{T})$.

In the next chapter, Chapter 3, we have moved from the category $\mathbf{Set}$ to $\mathbf{Fuz}(\mathbf{H})$, the category of fuzzy sets. Fuzzy sets are pairs that consist of a set and a function into a given frame $\mathbf{H}$. Such function expresses the *membership degree* of an element in the whole set.

To capture the equational aspects of fuzzy sets, we have introduced a *fuzzy sequent calculus*. While classical equations capture equalities, the membership function's information is captured using syntactic items called *membership propositions* of the form $\mathsf{m}(h, t)$, which can be interpreted as "the membership degree of term t is at least h". We have then introduced the concept of *fuzzy algebras* to provide a sound and complete semantics for this calculus. Completeness here means that a formula is satisfied by all models of a given theory if and only if it is derivable from the theory using the rules of our calculus.

As in the classical context, there is a notion of *free model* of a theory Λ and thus an associated monad $\mathbf{S}_\Lambda$ on the category $\mathbf{Fuz}(\mathbf{H})$. In general Eilenberg-Moore algebras for such a monad are not equivalent to models of Λ. However we have shown that this equivalence holds if Λ is *basic*.

Unfortunately, the correspondence between fuzzy algebraic theories and monads does not hold in the same way as it does for classical ones. We plan to investigate this phenomenon further in future work. One possible approach would be to apply the work of Nishizawa and Power [100] to $\mathbf{Fuz}(\mathbf{H})$, where $\mathbf{H}$ is a κ-algebraic frame and determine if our notion of algebraic theory is related with their notion of $\mathbf{Fuz}(\mathbf{H})$-*Lawvere theory*. Another approach could involve characterizing the monads that arise from a fuzzy algebraic theory.

Finally, using the results provided in [95] we have proved that, given a signature Σ, subcategories of

Σ-**FAlg** which are closed under products, regular monomorphisms and epimorphic images correspond precisely to categories of models for *unconditional theories*, i.e. theories axiomatised by sequents without premises. Moreover, using the same results, we have also proved that the categories of models of *theories of type* M, i.e. those whose axioms' premises contain only membership propositions involving variables, are exactly those subcategories closed under products, strong monomorphisms and split epimorphisms.

Our category $\mathbf{Fuz}(\mathbf{H})$ of fuzzy sets has crisp arrows and crisp equality: arrows are ordinary functions between the underlying sets and equalities can be judged to be either true or false. A way to further "fuzzifying" concepts is to use the topos of $\mathbf{H}$-*sets* over the frame $\mathbf{H}$ introduced in [47]: this is equivalent to the topos of sheaves over $\mathbf{H}$ and contains $\mathbf{Fuz}(\mathbf{H})$ as a (non full) subcategory. By construction, equalities and functions are "fuzzy". It would be interesting to study an application of our approach to this context. A promising feature is that, in an $\mathbf{H}$-set, the membership degree function is built-in as simply the equality relation, so it would not be necessary to distinguish between equations and membership propositions. Even more generally, we can replace $\mathbf{H}$ with an arbitrary quantale $\mathbf{Q} := (Q, \leq)$ and consider the category of sets endowed with a "$\mathbf{Q}$-valued equivalence relation" [27].

PART II
$\mathcal{M}, \mathcal{N}$-ADHESIVE CATEGORIES

On the axioms of $\mathcal{M},\mathcal{N}$-adhesivity

CHAPTER 5

Contents

The introduction of *adhesive categories* marked a watershed moment for the algebraic approaches to the rewriting of graph-like structures [42, 73]. Until then, key results of the approaches on e.g. parallelism and confluence had to be proven over and over again for each different formalism at hand, despite the obvious similarity of the procedure. Differently from previous solutions to such problems, as the one witnessed by the *butterfly lemma* for graph rewriting [39, Lemma 3.9.1], the introduction of adhesive categories provided such a disparate set of formalisms with a common abstract framework where many of these general results could be recast and uniformly proved once and for all.

Despite the elegance and effectiveness of the framework, proving that a given category satisfies the conditions for being adhesive can be a daunting task. For this reason, we look for simpler general criteria implying adhesivity for a class of categories. Similar criteria have already been provided for the core framework of adhesive categories; e.g. every elementary topos is adhesive [74], and a category is (quasi)adhesive if and only if can be suitably embedded in a topos [52, 67]. This covers many useful categories such as sets, graphs, and so on. On the other hand, there are many categories of interest which are not (quasi)adhesive, such as directed graphs, posets, and many of their subcategories. In these cases we can try to prove the more general $\mathcal{M},\mathcal{N}$-adhesivity [60, 104] for suitable classes $\mathcal{M}$ and $\mathcal{N}$. However, so far this has been achieved only by means of *ad hoc* arguments. To this end, one of the results of this chapter is a new criterion for $\mathcal{M},\mathcal{N}$-adhesivity, based on the verification of some properties of functors connecting the category of interest to a family of suitable adhesive categories. This criterion allows us to prove in a uniform and systematic way some previous results about the adhesivity of categories built by products, exponents, and the comma construction. Moreover, this result will be extensively exploited in Chapter 6 in order to show the $\mathcal{M},\mathcal{N}$ of a host of categories of graphs and hypergraphs.

The next result presented here regards the relationship between $\mathcal{M},\mathcal{N}$-adhesivity and the existence of binary suprema in the poset of subobjects of a given object X. It is well known [67] that in a quasiadhesive category any two regular subobjects (i.e. subobjects represented by a regular mono) have a join which is again a regular subobject. Vice versa it is also known [52] that if regular monos are *adhesive*, then the existence of a regular join for any pair of regular subobjects entails quasiadhesivity. Generalizing the approach of [52] we will show that, if $\mathcal{M}$ and $\mathcal{N}$ are nice enough, $\mathcal{M},\mathcal{N}$-adhesivity entails the existence of suprema for some pairs of subobjects and that, vice versa, the existence of these suprema together with every arrow in $\mathcal{M}$ being $\mathcal{N}$*-adhesive* is enough to guarantee $\mathcal{M},\mathcal{N}$-adhesivity.

The framework of $\mathcal{N}$-adhesive morphisms, in turn, allows us to generalize also the embedding results provided in [52, 72]: every (quasi)adhesive category can be embedded in a Grothendieck topos via a functor preserving pullbacks and pushouts along (regular) monomorphisms. Under some hypotheses on the classes $\mathcal{M}$ and $\mathcal{N}$ we will prove that an $\mathcal{M},\mathcal{N}$-adhesive category admits a full and faithful functor into a Grothendieck topos which preserves pullbacks and $\mathcal{M},\mathcal{N}$-pushouts.

The first section of this chapter is based on the material present in [36]. The remaining part of the chapter is entirely new and, at the moment, a paper about these new results is submitted to *Theoretical Computer Science* for publication.

Synopsis In Section 5.1 after recalling the definition of Van Kampen square and of $\mathcal{M},\mathcal{N}$-adhesive category, we prove a new criterion for $\mathcal{M},\mathcal{N}$-adhesivity. Section 5.2 is devoted to study the relationship between $\mathcal{M},\mathcal{N}$-adhesivity and the existence of suprema in the poset of subobjects. Using the results of this section, in Section 5.3 we will provide a new proof of the adhesivity of elementary toposes and show that, under some hypotheses on $\mathcal{M}$ and $\mathcal{N}$, every $\mathcal{M},\mathcal{N}$-adhesive category can be embedded in a Grothendieck topos via a functor preserving pullbacks and $\mathcal{M},\mathcal{N}$-pushouts.

5.1 $\mathcal{M},\mathcal{N}$-adhesive categories

In this section we recall some definitions and results about $\mathcal{M},\mathcal{N}$-adhesive categories and provide a new criterion to prove this property. Intuitively, an adhesive category is one in which pushouts of monomorphisms exist and behave more or less as they do in a topos [73, 74] (see also Section 5.3).

5.1.1 The Van Kampen condition

The key property that $\mathcal{M},\mathcal{N}$-adhesive categories enjoy is given by the so-called *Van Kampen condition* [33, 67, 73]. We will recall it and examine some of its consequences. We will end this section with the definition of $\mathcal{M},\mathcal{N}$-adhesivity and some of its variants.

Definition 5.1.1. Let **X** be a category and consider the two diagrams below

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "m"'] & B \arrow[d, "n"] \\
C \arrow[r, "g"'] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
& A' \arrow[rr, "f'"] \arrow[dl, "m'"'] \arrow[dd, "a", near end] & & B' \arrow[dl, "n'"'] \arrow[dd, "b"] \\
C' \arrow[rr, "g'", near start, crossing over] \arrow[dd, "c"'] & & D' & \\
& A \arrow[rr, "f", near end] \arrow[dl, "m"] & & B \arrow[dl, "n"] \\
C \arrow[rr, "g"'] & & D \arrow[from=uu, "d", near start, crossing over] &
\end{tikzcd}
$$

We say that the left square is a *Van Kampen square* if:

1. it is a pushout square;
2. whenever the right cube has pullbacks as back and left faces, then its top face is a pushout if and only if the front and right faces are pullbacks.

Pushout squares which enjoy the "if" half of this condition are called *stable*.

Let us make two rather technical remarks.

Remark 5.1.2. Take $m\colon X \to Y$ and $n\colon X \to Z$ to be two arrows and consider two pushout squares

$$
\begin{tikzcd}
X \arrow[r, "n"] \arrow[d, "m"'] & Z \arrow[d, "q_1"] \\
Y \arrow[r, "q_2"'] & Q
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "n"] \arrow[d, "m"'] & Z \arrow[d, "p_1"] \\
Y \arrow[r, "p_2"'] & P
\end{tikzcd}
$$

and let ϕ be the canonical isomorphism $Q \to P$. Take a cube in which the left and back faces are pullbacks

$$
\begin{tikzcd}
& X' \arrow[rr, "n'"] \arrow[dl, "m'"'] \arrow[dd, "x", near end] & & Z' \arrow[dl, "p_1'"'] \arrow[dd, "z"] \\
Y' \arrow[rr, "g", near start, crossing over] \arrow[dd, "y"'] & & P' & \\
& X \arrow[rr, "n", near end] \arrow[dl, "m"] & & Z \arrow[dl, "p_1"] \\
Y \arrow[rr, "p_2"'] & & P \arrow[from=uu, "p", near start, crossing over] &
\end{tikzcd}
$$

We can add ϕ^{-1} to get a second cube on the first pushout square.

$$
\begin{tikzcd}
& X' \arrow[rr, "n'"] \arrow[dl, "m"'] \arrow[ddd, "x"] & & Z' \arrow[dl, "p_1'"'] \arrow[ddd, "z"] \\
Y' \arrow[rr, "g"] \arrow[dddd, "y"'] & & P' \arrow[d, "p"] & \\
& & P & \\
& X \arrow[rr, "n"] \arrow[dl, "m"] & & Z \arrow[dl, "q_1"] \arrow[ul, bend right, "p_1"'] \\
Y \arrow[rr, "q_2"'] \arrow[uurr, bend left, "p_2"] & & Q \arrow[from=uu, "\phi^{-1}"', crossing over] &
\end{tikzcd}
$$

Now, we can notice the following facts.

- If all the vertical faces in the first cube are pullbacks then, since ϕ is an isomorphism the ones in the second cube are pullbacks too. Thus if the square

$$\begin{tikzcd} X \arrow[r, "n"] \arrow[d, "m"'] & Z \arrow[d, "q_1"] \\ Y \arrow[r, "q_2"'] & Q \end{tikzcd}$$

is a stable pushout, also the other one is so.

- If the top face of the cubes is a pushout and the first square is Van Kampen, then all the vertical faces in the first cube are pullbacks, and this, using again the fact that ϕ is an isomorphism, entails that the second square is Van Kampen too.

Summing up, if a stable (Van Kampen) pushout square of m along n exists, then every other pushout square of m along n is stable (Van Kampen).

Remark 5.1.3. Take a pushout square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "m"'] & B \arrow[d, "n"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

and an arrow $d\colon D' \to D$. Suppose that two cubes are given, in which all the vertical faces are pullbacks.

$$\begin{tikzcd} & A' \arrow[rr, "g'"] \arrow[dl, "m'"'] \arrow[dd, "a"] & & B' \arrow[dl, "n'"'] \arrow[dd, "b"] \\ C'' \arrow[rr, "f''"] \arrow[dd, "c'"'] & & D' \arrow[dd, "d"] & \\ & A \arrow[rr, "g"] \arrow[dl, "m"] & & B \arrow[dl, "n"] \\ C \arrow[rr, "f"'] & & D & \end{tikzcd} \qquad \begin{tikzcd} & A'' \arrow[rr, "g''"] \arrow[dl, "m''"'] \arrow[dd, "a'"] & & B'' \arrow[dl, "n''"'] \arrow[dd, "b'"] \\ C' \arrow[rr, "f'"] \arrow[dd, "c"'] & & D' \arrow[dd, "d"] & \\ & A \arrow[rr, "g"] \arrow[dl, "m"] & & B \arrow[dl, "n"] \\ C \arrow[rr, "f"'] & & D & \end{tikzcd}$$

The top faces fit together in the following diagram

$$\begin{tikzcd} A'' \arrow[rrr, "g''"] \arrow[ddd, "m''"'] & & & B'' \arrow[ddd, "n''"] \\ & A' \arrow[r, "g'"] \arrow[d, "m'"'] \arrow[ul, "\phi_1"'] & B' \arrow[d, "n'"] \arrow[ur, "\phi_2"] & \\ & C' \arrow[r, "f'"'] \arrow[dl, "\phi_3"] & D' \arrow[dr, "\mathsf{id}_{D'}"] & \\ C'' \arrow[rrr, "f''"'] & & & D' \end{tikzcd}$$

in which ϕ_1, ϕ_2 and ϕ_3 are canonical isomorphism between pullbacks. It is now clear that the inner square is a pushout if and only if the outer one is a pushout too. This means that to prove the stabilty of a pushout square, it is enough to verify it for a cube with chosen pullbacks as vertical faces.

Before proceeding further, we must recall a classical result about pullbacks.

Lemma 5.1.4. *Let* $\mathbf{X}$ *be a category, and consider the following diagram in which the right square is a pullback.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "a"'] & Y \arrow[r, "g"] \arrow[d, "b"] & Z \arrow[d, "c"] \\ A \arrow[r, "h"'] & B \arrow[r, "k"'] & C \end{tikzcd}$$

Then the whole rectangle is a pullback if and only if the left square is one.

Proof. ($\Rightarrow$) Let $q_1 \colon Q \to Y$ and $q_2 \colon Q \to A$ be two arrows such that $b \circ q_1 = h \circ q_2$, if we copute we get

$$\begin{aligned} c \circ g \circ q_1 &= k \circ b \circ q_1 \\ &= k \circ h \circ q_2 \end{aligned}$$

and applying the pullback property of the whole rectangle we get the dotted l in the following diagram

$$\begin{tikzcd} Q \arrow[r, dotted, "l"'] \arrow[rrr, bend left, "g \circ q_1"] \arrow[dr, bend right, "q_2"'] & X \arrow[r, "f"'] \arrow[d, "a"'] & Y \arrow[r, "g"'] \arrow[d, "b"] & Z \arrow[d, "c"] \\ & A \arrow[r, "h"'] & B \arrow[r, "k"'] & C \end{tikzcd}$$

All we have to prove is that $f \circ l = q_1$. On the one hand we have for free that

$$g \circ f \circ l = g \circ q_1$$

On the other hand

$$\begin{aligned} b \circ f \circ l &= h \circ a \circ l \\ &= h \circ q_2 \\ &= b \circ q_1 \end{aligned}$$

and we can conclude since the right square in the original diagram is a pullback.

For uniqueness: if $l' \colon Q \to X$ is such that

$$f \circ l' = q_1 \qquad a \circ l' = q_2$$

then

$$g \circ f \circ l' = g \circ q_1$$

and we can conclude applying the pullback property of the outer rectangle.

($\Leftarrow$) Take two arrows $q_1 \colon Q \to Z$ and $q_2 \colon Q \to A$ such that

$$c \circ q_1 = k \circ h \circ q_2$$

We can apply the pullback property of the right square to get the dotted $q \colon Q \to Y$ in the following

$$\begin{tikzcd}
Q \arrow[dr, dotted, "l"] \arrow[drr, dotted, "q"] \arrow[drrr, bend left, "q_1"] \arrow[ddr, bend right, "q_2"'] & & & \\
& X \arrow[r, "f"] \arrow[d, "a"'] & Y \arrow[r, "g"] \arrow[d, "b"'] & Z \arrow[d, "c"] \\
& A \arrow[r, "h"'] & B \arrow[r, "k"'] & C
\end{tikzcd}$$

Now, by construction we have

$$b \circ q = h \circ q_2$$

and thus, since the left square is a pullback, we get also a unique $l \colon Q \to X$ such that

$$f \circ l = q \qquad a \circ l = q_2$$

but then we clearly have

$$\begin{aligned} g \circ f \circ l &= g \circ q \\ &= q_1 \end{aligned}$$

We are left with uniqueness. Let $l' \colon Q \to X$ be another arrow such that

$$q_1 = g \circ f \circ l' \qquad q_2 = a \circ l'$$

But then we must also have

$$\begin{aligned} b \circ f \circ l' &= h \circ a \circ l' \\ &= h \circ q_2 \\ &= b \circ q \end{aligned}$$

which implies $f \circ l' = q$, from which $l = l'$ follows. $\square$

Corollary 5.1.5. *Let* $\mathbf{X}$ *be a category and suppose that the solid part of the following cube is given*

$$\begin{tikzcd}
& Y' \arrow[rr, "g'"] \arrow[dl, dotted, "q'"'] \arrow[dd, "y"] & & Z' \arrow[dl, "r'"'] \arrow[dd, "z"] \\
B' \arrow[rr, "k'"] \arrow[dd, "b"'] & & C' \arrow[dd, "c"] & \\
& Y \arrow[rr, "g"] \arrow[dl, "q"] & & Z \arrow[dl, "r"] \\
B \arrow[rr, "k"'] & & C &
\end{tikzcd}$$

If the front face is a pullback then there is a unique $q' : Y' \to B'$ filling the diagram. If, moreover, the other two vertical faces are also pullbacks, then the following square is a pullback too.

$$\begin{tikzcd}
Y' \arrow[r, "q'"] \arrow[d, "y"'] & B' \arrow[d, "b"] \\
Y \arrow[r, "q"'] & B
\end{tikzcd}$$

Proof. Let us compute:

$$\begin{aligned}
c \circ r' \circ g' &= r \circ z \circ g' \\
&= r \circ g \circ y \\
&= k \circ q \circ y
\end{aligned}$$

Since the front face is a pullback, this guarantees the existence of q'. The second half of the thesis follows applying Lemma 5.1.4 to the following rectangle.

$$\begin{tikzcd}
Y' \arrow[r, "q'"'] \arrow[d, "y"'] \arrow[rr, bend left, "r' \circ g'"] & B' \arrow[r, "k'"'] \arrow[d, "b"'] & C' \arrow[d, "c"] \\
Y \arrow[r, "q"] \arrow[rr, bend right, "r \circ g"'] & B \arrow[r, "k"] & C
\end{tikzcd}$$

□

We can dualize Lemma 5.1.4 to get half of the following.

Lemma 5.1.6. *Let* **X** *be a category, and consider the following diagram in which the left square is a pushout.*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "p"'] & Y \arrow[r, "g"] \arrow[d, "q"] & Z \arrow[d, "r"] \\
A \arrow[r, "h"'] & B \arrow[r, "k"'] & C
\end{tikzcd}$$

Then the whole rectangle is a pushout if and only if the right square is one.

Moreover, if **X** *has pullbacks and the left square is stable, then stability of the whole rectangle is equivalent to that of the right square.*

Proof. The first half follows from Lemma 5.1.4 by duality. Let us show the second one.
($\Rightarrow$) Take a cube

$$\begin{tikzcd}
& Y' \arrow[rr, "g'"] \arrow[dl, "q'"'] \arrow[dd, "y"' near end] & & Z' \arrow[dl, "r'"'] \arrow[dd, "z"] \\
B' \arrow[rr, crossing over, "k'" near start] \arrow[dd, "b"'] & & C' & \\
& Y \arrow[rr, "g" near end] \arrow[dl, "q"] & & Z \arrow[dl, "r"] \\
B \arrow[rr, "k"'] & & C \arrow[from=uu, crossing over, "c"' near start] &
\end{tikzcd}$$

in which all the vertical faces are pullbacks. Pulling back y along f and b along h we get the solid part of another cube

$$\begin{tikzcd}
& X' \arrow[rr, "f'"] \arrow[dl, dotted, "p'"'] \arrow[dd, "x"' near end] & & Y' \arrow[rr, "g'"] \arrow[dl, "q'"'] \arrow[dd, "y"' near end] & & Z' \arrow[dl, "r'"'] \arrow[dd, "z"] \\
A' \arrow[rr, "h'" near start] \arrow[dd, "a"'] & & B' \arrow[rr, "k'" near start] \arrow[dd, "b" near start] & & C' \arrow[dd, "c" near start] & \\
& X \arrow[rr, "f" near start] \arrow[dl, "p"] & & Y \arrow[rr, "g" near end] \arrow[dl, "q"] & & Z \arrow[dl, "r"] \\
A \arrow[rr, "h"'] & & B \arrow[rr, "k"'] & & C &
\end{tikzcd}$$

and Corollary 5.1.5 shows that the dotted $p' \colon X' \to A'$ exists and that the new square is again a pullback. By Lemma 5.1.4 the whole composite cube has pullbacks as vertical faces and thus the top one is a pushout. Now the thesis follows from the first half of this lemma.

$(\Leftarrow)$ Take the following cube with pullbacks as vertical faces

$$\begin{tikzcd}
& X' \arrow[rrrr, "s"] \arrow[dl, "p'"'] \arrow[dd, "x"' near end] & & & & Z' \arrow[dl, "r'"'] \arrow[dd, "z"] \\
A' \arrow[rrrr, "t" near start] \arrow[dd, "a"'] & & & & C' \arrow[dd, "c" near start] & \\
& X \arrow[rr, "f"] \arrow[dl, "p"] & & Y \arrow[rr, "g" near end] \arrow[dl, "q"] & & Z \arrow[dl, "r"] \\
A \arrow[rr, "h"'] & & B \arrow[rr, "k"'] & & C &
\end{tikzcd}$$

Since $\mathbf{X}$ has pullbacks, we can construct the solid part of the cube

$$\begin{tikzcd}
& Y' \arrow[rr, "g'"] \arrow[dl, dotted, "q'"'] \arrow[dd, "y"' near end] & & Z' \arrow[dl, "r'"'] \arrow[dd, "z"] \\
B' \arrow[rr, "k'" near start] \arrow[dd, "b"'] & & C' \arrow[dd, "c" near start] & \\
& Y \arrow[rr, "g" near end] \arrow[dl, "q"] & & Z \arrow[dl, "r"] \\
B \arrow[rr, "k"'] & & C &
\end{tikzcd}$$

in which the three vertical faces are pullbacks. By Corollary 5.1.5 we also get the dotted q' and a cube with pullbacks as vertical faces. By hypothesis this cube has a stable pushout as bottom face. Thus its top face is a pushout, too. Now,

$$z \circ s = g \circ f \circ x \qquad c \circ t = k \circ h \circ a$$

Thus there exists $h' \colon A' \to B'$ and $f' \colon X' \to Y'$ such that

$$t = k' \circ h' \quad b \circ h' = h \circ a \quad s = g' \circ f' \quad y \circ f' = f \circ x$$

Moreover,

$$
\begin{aligned}
k' \circ q' \circ f' &= r' \circ g' \circ f' & \qquad b \circ q' \circ f' &= q \circ y \circ f' \\
&= r' \circ s & &= q \circ f \circ x \\
&= t \circ p' & &= h \circ p \circ x \\
&= k' \circ h' \circ p' & &= h \circ a \circ p' \\
& & &= b \circ h' \circ p'
\end{aligned}
$$

Therefore we have a diagram

$$
\begin{tikzcd}
& X' \arrow[rr, "f'"] \arrow[dl, "p'"'] \arrow[dd, "x" near end] & & Y' \arrow[rr, "g'"] \arrow[dl, "q'"'] \arrow[dd, "y" near end] & & Z' \arrow[dl, "r'"'] \arrow[dd, "z"] \\
A' \arrow[rr, "h'" near start] \arrow[dd, "a"'] & & B' \arrow[rr, "k'" near start] \arrow[dd, "b" near start] & & C' \arrow[dd, "c" near start] & \\
& X \arrow[rr, "f" near start] \arrow[dl, "p"] & & Y \arrow[rr, "g" near end] \arrow[dl, "q"] & & Z \arrow[dl, "r"] \\
A \arrow[rr, "h"'] & & B \arrow[rr, "k"'] & & C &
\end{tikzcd}
$$

Applying Lemma 5.1.4 to the rectangles

$$
\begin{tikzcd}
X' \arrow[r, "f'"'] \arrow[rr, bend left, "s"] \arrow[d, "x"'] & Y' \arrow[r, "g'"'] \arrow[d, "y"'] & Z' \arrow[d, "z"] \\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
A' \arrow[r, "h'"'] \arrow[rr, bend left, "s"] \arrow[d, "a"'] & B' \arrow[r, "k'"'] \arrow[d, "b"'] & C' \arrow[d, "c"] \\
A \arrow[r, "h"'] & B \arrow[r, "k"'] & C
\end{tikzcd}
$$

we get that all the faces of the left cube are pullbacks, and so both halves of the top face are pushouts. □

We can now prove another property of Van Kampen squares.

Proposition 5.1.7. *Let $m\colon A \to C$ be a monomorphism in a category $\mathbf{X}$. Then every Van Kampen square*

$$
\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "m"'] & B \arrow[d, "n"] \\
C \arrow[r, "f"'] & D
\end{tikzcd}
$$

is also a pullback square and n is a monomorphism.

Proof. Take the following cube:

$$
\begin{tikzcd}
& A \arrow[rr, "g"] \arrow[dl, "\mathsf{id}_A"'] \arrow[dd, "\mathsf{id}_A"' near end] & & B \arrow[dl, "\mathsf{id}_B"'] \arrow[dd, "\mathsf{id}_B"] \\
A \arrow[rr, "g" near end] \arrow[dd, "m"'] & & B \arrow[dd, "n" near start] & \\
& A \arrow[rr, "g" near end] \arrow[dl, "m"] & & B \arrow[dl, "n"] \\
C \arrow[rr, "f"'] & & D &
\end{tikzcd}
$$

By construction the top face of the cube is a pushout and the back one a pullback. The left face is a pullback because m is mono, thus the Van Kampen property yields that the front and the right faces are pullbacks too and the thesis follows. □

Finally, we can show a kind of left cancellation property for pullbacks.

Lemma 5.1.8. *Let* $\mathbf{X}$ *be a category with pullbacks, given the following diagrams:*

$$
\begin{tikzcd}
Y \arrow[r, "f_2"] \arrow[d, "f_1"'] & X_2 \arrow[d, "r_2"] \\
X_1 \arrow[r, "r_1"'] & R
\end{tikzcd}
\qquad
\begin{tikzcd}
Z_1 \arrow[r, "z_1"] \arrow[d, "x_1"'] & W \arrow[r, "w"] \arrow[d, "r"'] & Q' \arrow[d, "q"] \\
X_1 \arrow[r, "r_1"'] & R \arrow[r, "s"'] & Q
\end{tikzcd}
\qquad
\begin{tikzcd}
Z_2 \arrow[r, "z_2"] \arrow[d, "x_2"'] & W \arrow[r, "w"] \arrow[d, "r"'] & Q' \arrow[d, "q"] \\
X_2 \arrow[r, "r_2"'] & R \arrow[r, "s"'] & Q
\end{tikzcd}
$$

if the first square is a stable pushout and the whole rectangles and their left halves are pullbacks, then their common right half is a pullback too.

Proof. Pulling back q along s we get a square

$$
\begin{tikzcd}
U \arrow[r, "u"] \arrow[d, "h"'] & Q' \arrow[d, "q"] \\
R \arrow[r, "s"'] & S
\end{tikzcd}
$$

Notice that

$$q \circ w \circ z_1 = s \circ r_1 \circ x_1 \qquad q \circ w \circ z_2 = s \circ r_2 \circ x_2$$

Thus we get $u_1 \colon Z_1 \to U$ and $u_2 \colon Z_2 \to U$ fitting in the rectangles

$$
\begin{tikzcd}
Z_1 \arrow[r, "u_1"'] \arrow[rr, bend left, "w \circ z_1"] \arrow[d, "x_1"'] & U \arrow[r, "u"'] \arrow[d, "h"'] & Q' \arrow[d, "q"] \\
X_1 \arrow[r, "r_1"'] & R \arrow[r, "s"'] & Q
\end{tikzcd}
\qquad
\begin{tikzcd}
Z_2 \arrow[r, "u_2"'] \arrow[rr, bend left, "w \circ z_2"] \arrow[d, "x_2"'] & U \arrow[r, "u"'] \arrow[d, "h"'] & Q' \arrow[d, "q"] \\
X_2 \arrow[r, "r_2"'] & R \arrow[r, "s"'] & Q
\end{tikzcd}
$$

which, by hypothesis and Lemma 5.1.4 have left halves which are pullbacks. Now,

$$s \circ r_1 \circ f_1 = s \circ r_2 \circ f_2$$

Pulling back q along this arrow we get another square

$$
\begin{tikzcd}
Z_0' \arrow[r, "t"] \arrow[d, "y"'] & Q' \arrow[d, "q"] \\
R \arrow[r, "s \circ r_1 \circ f_1"'] & S
\end{tikzcd}
$$

In particular, we obtain the dotted $b_1\colon Z'_0 \to Z_1$ and $b_2\colon Z'_0 \to Z_2$ in

$$
\begin{tikzcd}
Z'_0 \arrow[r, dotted, "b_1"'] \arrow[rrr, bend left, "t"] \arrow[d, "y"'] & Z_1 \arrow[r, "u_1"'] \arrow[d, "x_1"'] & U \arrow[r, "u"'] \arrow[d, "h"'] & Q' \arrow[d, "q"] \\
Y \arrow[r, "f_1"'] & X_1 \arrow[r, "r_1"'] & R \arrow[r, "s"'] & Q
\end{tikzcd}
\qquad
\begin{tikzcd}
Z'_0 \arrow[r, dotted, "b_2"'] \arrow[rrr, bend left, "t"] \arrow[d, "y"'] & Z_2 \arrow[r, "u_2"'] \arrow[d, "x_2"'] & U \arrow[r, "u"'] \arrow[d, "h"'] & Q' \arrow[d, "q"] \\
Y \arrow[r, "f_2"'] & X_2 \arrow[r, "r_2"'] & R \arrow[r, "s"'] & Q
\end{tikzcd}
$$

in which, using again Lemma 5.1.4, all of the squares on the bottom rows are pullbacks.

We are going to construct another row above these two rectangles. By hypothesis

$$q \circ w = s \circ r$$

Thus there exists a unique $g\colon W \to U$ such that

$$r = h \circ g \qquad w = u \circ g$$

Moreover, we also have that

$$
\begin{aligned}
h \circ g \circ z_1 &= r \circ z_1 & \qquad h \circ g \circ z_2 &= r \circ z_2 \\
&= r_1 \circ x_1 & &= r_2 \circ x_2 \\
&= h \circ u_1 & &= h \circ u_2
\end{aligned}
$$

and

$$
\begin{aligned}
u \circ g \circ z_1 &= w \circ z_1 & \qquad u \circ g \circ z_1 &= w \circ z_2 \\
&= u \circ u_1 & &= u \circ u_2
\end{aligned}
$$

which together show that

$$g \circ z_1 = u_1 \qquad g \circ z_2 = u_2$$

Summing up, we can depict all the arrows we have constucted so far in the following diagrams

$$
\begin{tikzcd}
Z'_0 \arrow[r, "b_1"] \arrow[d, "\mathsf{id}_{Z'}"'] & Z_1 \arrow[r, "z_1"] \arrow[d, "\mathsf{id}_{Z_1}"'] & W \arrow[r, "w"] \arrow[d, "g"'] & Q' \arrow[d, "\mathsf{id}_{Q'}"] \\
Z'_0 \arrow[r, "b_1"] \arrow[d, "y"'] & Z_1 \arrow[r, "u_1"] \arrow[d, "x_1"'] & U \arrow[r, "u"] \arrow[d, "h"'] & Q' \arrow[d, "q"] \\
Y \arrow[r, "f_1"'] & X_1 \arrow[r, "r_1"'] & R \arrow[r, "s"'] & Q
\end{tikzcd}
\qquad
\begin{tikzcd}
Z'_0 \arrow[r, "b_2"] \arrow[d, "\mathsf{id}_{Z'}"'] & Z_2 \arrow[r, "z_2"] \arrow[d, "\mathsf{id}_{Z_2}"'] & W \arrow[r, "w"] \arrow[d, "g"'] & Q' \arrow[d, "\mathsf{id}_{Q'}"] \\
Z'_0 \arrow[r, "b_2"] \arrow[d, "y"'] & Z_2 \arrow[r, "u_2"] \arrow[d, "x_2"'] & U \arrow[r, "u"] \arrow[d, "h"'] & Q' \arrow[d, "q"] \\
Y \arrow[r, "f_2"'] & X_2 \arrow[r, "r_2"'] & R \arrow[r, "s"'] & Q
\end{tikzcd}
$$

If we show that g is an isomorphism we are done. Consider the cubes

$$
\begin{tikzcd}
& Z'_0 \arrow[rr, "b_2"] \arrow[dl, "b_1"'] \arrow[dd, "y"'] & & Z_2 \arrow[dl, "z_2"'] \arrow[dd, "x_2"] \\
Z_1 \arrow[rr, "z_1"] \arrow[dd, "x_1"'] & & W \arrow[dd, "r"] & \\
& Y \arrow[rr, "f_2"] \arrow[dl, "f_1"'] & & X_2 \arrow[dl, "r_2"] \\
X_1 \arrow[rr, "r_1"'] & & R &
\end{tikzcd}
\qquad
\begin{tikzcd}
& Z'_0 \arrow[rr, "b_2"] \arrow[dl, "b_1"'] \arrow[dd, "y"'] & & Z_2 \arrow[dl, "u_2"'] \arrow[dd, "F_j(b)"] \\
Z_1 \arrow[rr, "u_1"] \arrow[dd, "x_1"'] & & U \arrow[dd, "h"] & \\
& Y \arrow[rr, "f_2"] \arrow[dl, "f_1"'] & & X_2 \arrow[dl, "r_2"] \\
X_1 \arrow[rr, "r_1"'] & & R &
\end{tikzcd}
$$

in which the vertical faces are pullbacks. Since the bottom face is a stable pushout we can deduce that

$$\begin{tikzcd} Z_0' \arrow[r, "b_2"] \arrow[d, "b_1"'] & Z_2 \arrow[d, "z_2"] \\ Z_1 \arrow[r, "z_1"'] & W \end{tikzcd} \qquad \begin{tikzcd} Z_0' \arrow[r, "b_2"] \arrow[d, "b_1"'] & Z_2 \arrow[d, "u_2"] \\ Z_1 \arrow[r, "u_1"'] & U \end{tikzcd}$$

are pushout squares too. The arrow g fits in the following diagram

$$\begin{tikzcd} Z_0' \arrow[r, "b_2"] \arrow[d, "b_1"'] & Z_2 \arrow[d, "z_2"] \arrow[ddr, bend left, "u_2"] & \\ Z_1 \arrow[r, "z_1"'] \arrow[drr, bend right, "u_1"'] & W \arrow[dr, "g"] & \\ & & U \end{tikzcd}$$

and thus it is an isomorphism. □

5.1.2 Definition of $\mathcal{M},\mathcal{N}$-adhesivity

In this section we will define the notion of $\mathcal{M},\mathcal{N}$-adhesivity and explore some of the consequence of such a property. Let us start fixing some terminology.

Definition 5.1.9. Let $\mathbf{X}$ be a category and $\mathcal{A}$, $\mathcal{B}$ two classes of arrows, we say that $\mathcal{A}$ is

- *stable under pushouts (pullbacks)* if for every pushout (pullbacks) square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "m"'] & B \arrow[d, "n"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

  if $m \in \mathcal{A}$ ($n \in \mathcal{A}$) then $n \in \mathcal{A}$ ($m \in \mathcal{A}$);
- *closed under composition* if $g, f \in \mathcal{A}$ implies $g \circ f \in \mathcal{A}$ whenever g and f are composable;
- *closed under $\mathcal{B}$-decomposition* if $g \circ f \in \mathcal{A}$ and $g \in \mathcal{B}$ implies $f \in \mathcal{A}$;
- *closed under decomposition* if it is closed under $\mathcal{A}$-decomposition.

Remark 5.1.10. Clearly, "decomposition" corresponds to "left cancellation", but we prefer to stick to the name commonly used in literature (see e.g. [60]).

Example 5.1.11. In every category $\mathbf{X}$, split monomorphism (i.e. those arrows which have a left inverse) are stable under pushouts. Indeed, take a square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "m"'] & B \arrow[d, "n"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

with m a split monomorphism. Let $r\colon C \to A$ be a left inverse of m, then

$$\begin{aligned} f \circ r \circ m &= f \circ \mathsf{id}_A \\ &= f \\ &= \mathsf{id}_B \circ f \end{aligned}$$

This equality in turn entails the existence of a unique $t\colon D \to B$ fitting in the following diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "m"'] & B \arrow[d, "n"] \arrow[ddr, bend left, "\mathsf{id}_B"] & \\ C \arrow[r, "g"'] \arrow[dr, bend right, "r"'] & D \arrow[dr, dotted, "t"] & \\ & A \arrow[r, "f"'] & B \end{tikzcd}$$

Lemma 5.1.12. *Let $\mathcal{M}$ be a class of monos in a category $\mathbf{X}$ which is stable under pullbacks and contains all isomorphisms. If pushouts along arrows in $\mathcal{M}$ exist and are Van Kampen and every split mono is contained in $\mathcal{M}$, then $\mathcal{M}$ is closed under pushouts.*

Proof. Take two pushout squares

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "m"'] & Z \arrow[d, "n"] \\ Y \arrow[r, "g"'] & Q \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "m"] \arrow[d, "m"'] & Y \arrow[d, "p"] \\ Y \arrow[r, "q"'] & P \end{tikzcd}$$

with $m \in \mathcal{M}$. p and q are split monomorphisms: indeed by the universal property of pushouts there exists the dotted arrow $t\colon P \to Y$ in the following diagram

$$\begin{tikzcd} X \arrow[r, "m"] \arrow[d, "m"'] & Y \arrow[d, "p"] \arrow[ddr, bend left, "\mathsf{id}_Y"] & \\ Y \arrow[r, "q"'] \arrow[drr, bend right, "\mathsf{id}_Y"'] & P \arrow[dr, dotted, "t"] & \\ & & Y \end{tikzcd}$$

By hypothesis p and q are in $\mathcal{M}$, we can then consider the following cube, in which the top, left, front and back faces are pushouts.

$$\begin{tikzcd} & X \arrow[rr, "f"] \arrow[dl, "m"'] \arrow[dd, "m" near end] & & Z \arrow[dl, "n"'] \arrow[dd, "n"] \\ Y \arrow[rr, "g" near start, crossing over] \arrow[dd, "p"'] & & Q & \\ & Y \arrow[rr, "g" near end] \arrow[dl, "q"] & & Q \arrow[dl, "s"] \\ P \arrow[rr, "h"'] & & A \arrow[from=uu, "r" near start, crossing over] & \end{tikzcd}$$

Notice that the right face commutes too: the following rectangles are pushouts by Lemma 5.1.6

$$\begin{tikzcd} X \arrow[r, "m"] \arrow[d, "f"'] & Y \arrow[r, "p"] \arrow[d, "g"] & P \arrow[d, "h"] \\ Z \arrow[r, "n"'] & Q \arrow[r, "r"'] & A \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "m"] \arrow[d, "f"'] & Y \arrow[r, "q"] \arrow[d, "g"] & P \arrow[d, "h"] \\ Z \arrow[r, "n"'] & Q \arrow[r, "s"'] & A \end{tikzcd}$$

and, by construction,

$$p \circ m = q \circ m$$

and this entails that

$$s \circ n = r \circ n$$

By hypothesis all the square beside the right one are Van Kampen, thus, by Proposition 5.1.7 are also pullbacks. Since the bottom and top squares are pushouts this entails that the front faces are pullbacks. Now, r is split mono by Example 5.1.11, thus it is in $\mathcal{M}$, but this now entails that n is in $\mathcal{M}$ too. □

We are now ready to give the definition of $\mathcal{M},\mathcal{N}$-adhesive category

Definition 5.1.13 ([60, 104])**.** Let $\mathbf{X}$ be a category, $\mathcal{M} \subseteq \mathcal{M}(\mathbf{X})$ and $\mathcal{N} \subseteq \mathcal{A}(\mathbf{X})$, we say that the pair $(\mathcal{M},\mathcal{N})$ is a *preadhesive structure* on $\mathbf{X}$ if the following conditions hold.

1. $\mathcal{M}$ and $\mathcal{N}$ contain all isomorphisms and are closed under composition and decomposition;
2. $\mathcal{N}$ is closed under $\mathcal{M}$-decomposition;
3. $\mathcal{M}$ and $\mathcal{N}$ are stable under pullbacks and pushouts.

Given a preadhesive structure $(\mathcal{M},\mathcal{N})$, we say that $\mathbf{X}$ is $\mathcal{M},\mathcal{N}$*-adhesive* if

1. for every $m\colon X \to Y$ in $\mathcal{M}$ and $g\colon Z \to Y$, a pullback square

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "n"'] & X \arrow[d, "m"] \\ Z \arrow[r, "g"'] & Y \end{tikzcd}$$

   exists, such pullbacks will be called $\mathcal{M}$*-pullbacks*;
2. for every $m\colon X \to Y$ in $\mathcal{M}$ and $n\colon X \to Z$ in $\mathcal{N}$, a pushout square

$$\begin{tikzcd} X \arrow[r, "n"] \arrow[d, "m"'] & Z \arrow[d, "q"] \\ Y \arrow[r, "p"'] & Q \end{tikzcd}$$

   exists, such pushouts will be called $\mathcal{M},\mathcal{N}$*-pushouts*;
3. $\mathcal{M},\mathcal{N}$-pushouts are Van Kampen squares.

Remark 5.1.14. Our notion of $\mathcal{M},\mathcal{N}$-adhesivity is slightly different from the one of [60]: in that paper, $\mathcal{M},\mathcal{N}$-pushouts are required to satisfy a Van Kampen condition which is weaker then ours. More precisely, in [60] a pushout square

$$\begin{tikzcd} A \arrow[r, "n"] \arrow[d, "m"'] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

is *Van Kampen square* if, for every cube as the one below, with b, c and d in $\mathcal{M}$ and pullbacks as back and left faces, then its top face is a pushout if and only if the front and right faces are pullbacks.

$$\begin{tikzcd} & A' \arrow[rr, "g'"] \arrow[dl, "m'"'] \arrow[dd, "a"'] & & B' \arrow[dl, "n'"'] \arrow[dd, "b"] \\ C' \arrow[rr, "f'"] \arrow[dd, "c"'] & & D' \arrow[dd, "d"] & \\ & A \arrow[rr, "g"] \arrow[dl, "m"] & & B \arrow[dl, "n"] \\ C \arrow[rr, "f"'] & & D & \end{tikzcd}$$

Remark 5.1.15. A list of examples of $\mathcal{M},\mathcal{N}$-adhesive categories will be provided in Chapter 6.

Proposition 5.1.7 yields at once the following fact.

Proposition 5.1.16. *If* $\mathbf{X}$ *is an* $\mathcal{M},\mathcal{N}$*-adhesive category, then* $\mathcal{M},\mathcal{N}$*-pushouts are also pullback squares.*

Relation with $\mathcal{M}$-adhesivity

We will end this section proving that, under suitable hypothesis, $\mathcal{M},\mathcal{N}$-adhesivity subsumes $\mathcal{M}$-adhesivity as defined in [13].

Definition 5.1.17. Let $\mathbf{X}$ be a category, a *stable system of monos* is a class $\mathcal{M}$ of monomorphisms closed under composition, containing all isomorphisms and stable under pullbacks.

Lemma 5.1.18. *Let a stable system of monos* $\mathcal{M}$ *on a category* $\mathbf{X}$ *and let also* $f\colon X \to Y$ *be an arrow in* $\mathbf{X}$*. For every mono* $m\colon Y \to Z$*, if* $m \circ f \in \mathcal{M}$ *then* $f \in \mathcal{M}$*.*

Proof. Take the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathsf{id}_X"'] & Y \arrow[r, "\mathsf{id}_Y"] \arrow[d, "\mathsf{id}_Y"'] & Y \arrow[d, "m"] \\ X \arrow[r, "f"'] & Y \arrow[r, "m"'] & Z \end{tikzcd}$$

Since m is mono the right square is a pullback, the thesis now follows from Lemma 5.1.4. □

Definition 5.1.19 ([13])**.** Let $\mathcal{M}$ be stable system of monos on a category $\mathbf{X}$. $\mathbf{X}$ is $\mathcal{M}$*-adhesive* if

1. it has $\mathcal{M}$-pullbacks;

2. for every $m\colon X \to Y$ in $\mathcal{M}$ and for any arrow $f\colon X \to Z$, a pushout square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "m"'] & Z \arrow[d, "n"] \\ Y \arrow[r, "g"'] & Q \end{tikzcd}$$

   exists and it is and it is a Van Kampen square.

Remark 5.1.20. We will stick to the notion of $\mathcal{M}$-adhesivity as defined in [13], as noted in Remark 5.1.14, other authors have introduced weaker notions of $\mathcal{M}$-adhesivity, where the Van Kampen condition is required to hold only for some cubes; see, e.g. [22, 42, 43, 45, 118], where our $\mathcal{M}$-adhesive categories are called *adhesive HLR categories*.

On the other hand, in [13] no requirement about the existence of pullbacks or $\mathcal{M}$-pullbacks is made, while in [52, 67, 73] adhesive and quasiadhesive categories are required to have all pullbacks. Mimicking the definition of $(\mathcal{M},\mathcal{N})$-adhesivity, for us an $\mathcal{M}$-adhesive category must have $\mathcal{M}$-pullbacks, .

Proposition 5.1.21. *Let* $\mathbf{X}$ *be an* $\mathcal{M}$*-adhesive category and suppose that every split mono is in* $\mathcal{M}$*, then* $\mathcal{M}$ *is stable under pushouts.*

Proof. This follows at once from Lemma 5.1.12. □

Example 5.1.22. The first, and fundamental, example is when $\mathcal{M}$ is the class of all monomorphisms: in this case $\mathcal{M}$-adhesivity is simply called *adhesivity*.

One would weaken the previous example using regular monos instead of ordinary monomorphisms. The problem is that $\mathcal{R}(\mathbf{X})$ is not in general closed under composition (see Example 2.1.56). This problem is solved by the following proposition.

Proposition 5.1.23. *Let* $\mathbf{X}$ *be a category with* $\mathcal{R}(\mathbf{X})$*-pullbacks, then the following are equivalent:*

1. $\mathcal{R}(\mathbf{X})$ *is a stable system of monos and* $\mathbf{X}$ *is* $\mathcal{R}(\mathbf{X})$*-adhesive;*
2. *pushouts along regular monos exists and are Van Kampen.*

Proof. $(1 \Rightarrow 2)$ This is tautological.

$(2 \Rightarrow 1)$ We only have to show that show that $\mathcal{R}(\mathbf{X})$ is closed under composition. If $m\colon X \to Y$ has a left inverse r then m is the equalizer of id_Y and $m \circ r$. On the one hand we have

$$\begin{aligned} m \circ r \circ m &= m \circ \mathsf{id}_X \\ &= m \end{aligned}$$

On the other hand if $z\colon Z \to Y$ is such that

$$m \circ r \circ z = z$$

then $r \circ z$ is the unique arrow $Z \to X$ satisfying the previous equation. Thus $\mathcal{R}(\mathbf{X})$ contains every split mono, and, by Lemma 5.1.12, we can deduce that it is also stable under pushouts. Now, if $m\colon X \to Y$

and $n\colon Y \to Z$ are in $\mathcal{R}(\mathbf{X})$, the previous observation allows are to construct the following diagram, in which all squares are pushouts along regular monos:

$$\begin{tikzcd}
X \arrow[r, "m"] \arrow[d, "m"'] & Y \arrow[r, "n"] \arrow[d, "p"] & Z \arrow[d, "t"] \\
Y \arrow[r, "q"] \arrow[d, "n"'] & P \arrow[r, "a"] \arrow[d, "u"] & A \arrow[d, "v"] \\
Z \arrow[r, "s"'] & B \arrow[r, "w"'] & C
\end{tikzcd}$$

By Proposition 5.1.7 all the inner squares are also pullbacks, by Lemma 5.1.4 the outer square is a pullback too, but this entails that $n \circ m$ is the equalizer of $v \circ t$ and $w \circ s$. □

Remark 5.1.24. A category with pullbacks and pushouts along regular monos and in which such pushouts are Van Kampen is what in the literature is usually called a *quasiadhesive category*, a notable exception is [52], in which *rm-adhesive* is used.

Lemma 5.1.25. *Let $\mathcal{M}$ be a stable system of monos in a category $\mathbf{X}$ which is also stable under pushouts, then the following are equivalent:*

1. *$\mathbf{X}$ is $\mathcal{M}$-adhesive;*
2. *$\mathbf{X}$ is $(\mathcal{M}, \mathcal{A}(\mathbf{X}))$-adhesive.*

Proof. ($1 \Rightarrow 2$) Since the axioms of $(\mathcal{M}, \mathcal{A}(\mathbf{X}))$-adhesivity are exactly those of $\mathcal{M}$-adhesivity, the only thing to verify is that $(\mathcal{M}, \mathcal{A}(\mathbf{X}))$ is a preadhesive structure (Definition 5.1.13).

1. Closure under composition and decomposition of $\mathcal{A}(\mathbf{X})$ doesn't need to be proved, and surely it contains all isomorphisms. Closure under decomposition of $\mathcal{M}$ follows from Lemma 5.1.18.
2. This is obvious.
3. $\mathcal{A}(\mathbf{X})$ is clearly stable under pullbacks and pushouts, while stability of $\mathcal{M}$ is one of the hypotheses.

($2 \Rightarrow 1$) This is clear. □

We can apply Proposition 5.1.21 to obtain the following corollary at once.

Corollary 5.1.26. *Let $\mathcal{M}$ be a stable system of monos in a category $\mathbf{X}$ and suppose that it contains all split monomorphisms., then the following are equivalent:*

1. *$\mathbf{X}$ is $\mathcal{M}$-adhesive;*
2. *$\mathbf{X}$ is $(\mathcal{M}, \mathcal{A}(\mathbf{X}))$-adhesive.*

If we specialize the previous results to the classes of monos and regular monos we get the following.

Corollary 5.1.27. *A category $\mathbf{X}$ is adhesive if an only if it is $(\mathcal{M}(\mathbf{X}), \mathcal{A}(\mathbf{X}))$-adhesive and it is quasiadhesive if and only if it is $(\mathcal{R}(\mathbf{X}), \mathcal{A}(\mathbf{X}))$-adhesive).*

5.1.3 A criterion for $\mathcal{M}, \mathcal{N}$-adhesivity

In this section we present a criterion which allows us to deduce $\mathcal{M}, \mathcal{N}$-adhesivity from the existence of a family of functors with sufficiently nice properties. We will start adapting Definition A.1.1.

Definition 5.1.28. Let $G\colon \mathbf{D} \to \mathbf{X}$ be a diagram and J a setz. Given a family $F = \{F_j\}_{j\in J}$ of functors $F_j\colon \mathbf{X} \to \mathbf{Y}_j$ we say that it:

1. *jointly preserves (co)limits* of G if given a (co)limiting (co)cone $(L, \{l_D\}_{D\in\mathbf{D}})$ for G, for every $j \in J$, the (co)cone $(F_j(L), \{F_j(l_D)\}_{D\in\mathbf{D}})$ is (co)limiting for $F_j \circ G$;
2. *jointly reflects (co)limits* of G if a (co)cone $(L, \{l_D\}_{D\in\mathbf{D}})$ is (co)limiting for G whenever for every $j \in J$, $(F_j(L), \{F_j(l_D)\}_{D\in\mathbf{D}})$ is (co)limiting for $F_j \circ G$;
3. *jointly creates (co)limits* of G if G has a (co)limit in $\mathbf{X}$ whenever $F_j \circ G$ has one for every $j \in J$, and F jointly preserves and reflects (co)limits along G.

Remark 5.1.29. Joint preservation, reflection or creation of (co)limits of for a family of functors $F_j\colon \mathbf{X} \to \mathbf{Y}_j$ is equivalent to the usual preservation, reflection or creation of (co)limits for the functor

$$\mathbf{X} \to \prod_{j\in J} \mathbf{Y}_j$$

induced by the family $F = \{F_j\}_{j\in J}$.

Remark 5.1.30. We can unpack a bit the definition of jointly creation of limits. If $G\colon \mathbf{D} \to \mathbf{Y}$ is a functor and $F = \{F_j\}_{j\in J}$ a family of functors creating limits of G. Suppose that, for every $j \in J$, a limiting cone $(L_j, \{l_{D,j}\}_{D\in\mathbf{D}})$ for $F_j \circ G$ is given. Then in $\mathbf{X}$ there exists a cone $(L, \{l_D\}_{D\in\mathbf{D}})$ which is limiting for G and, moreover, there exists a unique isomorphism $\phi_j\colon F_j(L) \to L_j$ fitting in the following diagram

$$\begin{tikzcd}
F_j(L) \arrow[rr, "\phi_j"] \arrow[dr, "F_j(l_D)"'] & & L_j \arrow[dl, "l_{D,k}"] \\
& F_j(G(D)) &
\end{tikzcd}$$

Theorem 5.1.31. *Let $(\mathcal{M}, \mathcal{N})$ be a preadhesive structure on a category $\mathbf{X}$, and let F be a non-empty family of functors $F_j\colon \mathbf{X} \to \mathbf{Y}_j$ such that for every $j \in J$, $\mathbf{Y}_j$ is $\mathcal{M}_j, \mathcal{N}_j$-adhesive. Then the followings are true:*

1. *if every F_j preserves pullbacks, $F_j(\mathcal{M}) \subseteq \mathcal{M}_j$ and $F_j(\mathcal{N}) \subseteq \mathcal{N}_j$ for every $j \in J$, F jointly preserves $\mathcal{M}, \mathcal{N}$-pushouts, and jointly reflects pushout squares*

$$\begin{tikzcd}
F_j(A) \arrow[r, "F_j(f)"] \arrow[d, "F_j(m)"'] & F_j(B) \arrow[d, "F_j(n)"] \\
F_j(C) \arrow[r, "F_j(g)"'] & F_j(D)
\end{tikzcd}$$

with $m, n \in \mathcal{M}$ and $f \in \mathcal{N}$, then $\mathcal{M}, \mathcal{N}$-pushouts in $\mathbf{X}$ are stable. Moreover, if, in addition, F jointly reflects $\mathcal{M}$-pullbacks and $\mathcal{N}$-pullbacks, then $\mathcal{M}, \mathcal{N}$-pushouts are Van Kampen squares;

2. *if F satisfies the assumptions of the previous points and jointly creates both $\mathcal{M}$-pullbacks and $\mathcal{N}$-pullbacks, then $\mathbf{X}$ is $\mathcal{M},\mathcal{N}$-adhesive;*
3. *if F jointly creates all pushouts and all pullbacks, then $\mathbf{X}$ is $\mathcal{M}_F,\mathcal{N}_F$-adhesive, where*

$$\mathcal{M}_F := \{m \in \mathbf{X} \mid F_j(m) \in \mathcal{M}_j \text{ for every } j \in J\} \quad \mathcal{N}_F := \{n \in \mathbf{X} \mid F_j(n) \in \mathcal{N}_j \text{ for every } j \in J\}$$

Proof. 1. Take a cube in which the bottom face is an $\mathcal{M},\mathcal{N}$-pushout and all the vertical faces are pullbacks, as the one below on the left. Applying each $F_j \in F$ we get another cube in $\mathbf{Y}_j$ as the one below on the right.

$$\begin{tikzcd}
& A' \arrow[rr, "f'"] \arrow[dl, "m'"'] \arrow[dd, "a"] & & B' \arrow[dl, "n'"'] \arrow[dd, "b"] \\
C' \arrow[rr, "g'"] \arrow[dd, "c"'] & & D' \arrow[dd, "d"] & \\
& A \arrow[rr, "f"] \arrow[dl, "m"] & & B \arrow[dl, "n"] \\
C \arrow[rr, "g"'] & & D &
\end{tikzcd}
\qquad
\begin{tikzcd}
& F_j(A') \arrow[rr, "F_j(f')"] \arrow[dl, "F_j(m')"'] \arrow[dd, "F_j(a)"] & & F_j(B') \arrow[dl, "F_j(n')"'] \arrow[dd, "F_j(b)"] \\
F_j(C') \arrow[rr, "F_j(g')"] \arrow[dd, "F_j(c)"'] & & F_j(D') \arrow[dd, "F_j(d)"] & \\
& F_j(A) \arrow[rr, "F_j(f)"] \arrow[dl, "F_j(m)"] & & F_j(B) \arrow[dl, "F_j(n)"] \\
F_j(C) \arrow[rr, "F_j(g)"'] & & F_j(D) &
\end{tikzcd}$$

By hypothesis the bottom face of the right cube is an $\mathcal{M}_j,\mathcal{N}_j$-pushout and the vertical faces are pullbacks, thus the top face of it is a pushout. Now $m', n' \in \mathcal{M}$ and $f' \in \mathcal{N}$ since they are the pullbacks of m, n and f, respectively, therefore the thesis follows from the hypothesis on F.

Suppose now that F jointly reflects $\mathcal{M}$-pullbacks and $\mathcal{N}$-pullbacks. We have to show that the front faces of the first cube above are pullbacks if the top one is a pushout. In the second cube, the bottom and top face are $\mathcal{M}_j,\mathcal{N}_j$-pushouts and the back faces are pullbacks, thus the front faces are pullbacks too by $\mathcal{M}_j,\mathcal{N}_j$-adhesivity. Now, notice that $f \in \mathcal{M}$ and $g \in \mathcal{N}$ (since $\mathcal{M}$ and $\mathcal{N}$ are closed under pushouts). Since F jointly reflects pullbacks along arrows in $\mathcal{M}$ or in $\mathcal{N}$ we get the thesis.

2. The first thing to check is that $\mathcal{M}_F$ is a class of monos. Let $m\colon X \to Y$ be an arrow in $\mathcal{M}$, by hypothesis, for every $j \in J$, $F_j(m)$ is a mono in $\mathbf{X}_j$, thus we have a pullback square

$$\begin{tikzcd}
F_j(X) \arrow[r, "\mathsf{id}_{F_j(X)}"] \arrow[d, "\mathsf{id}_{F_j(X)}"'] & F_j(X) \arrow[d, "F_j(m)"] \\
F_j(X) \arrow[r, "F_j(m)"'] & F_j(Y)
\end{tikzcd}$$

Since F jointly creates pullbacks we can deduce that the following square

$$\begin{tikzcd}
X \arrow[r, "\mathsf{id}_X"] \arrow[d, "\mathsf{id}_X"'] & X \arrow[d, "m"] \\
X \arrow[r, "m"'] & Y
\end{tikzcd}$$

is a pullback in $\mathbf{X}$ and this implies m being a monomorphism.

Next, we have to show the three properties defining $\mathcal{M},\mathcal{N}$-adhesivity.

Existence of $\mathcal{M}$-pullbacks. Let $m\colon B \to D$ be an arrow in $\mathcal{M}$ and $g\colon C \to D$ any other arrow. Take $j \in J$, since $\mathbf{Y}_j$ is $\mathcal{M}_j,\mathcal{N}_j$-adhesive and $F_j(m) \in \mathcal{M}_j$, we get a pullback square

$$\begin{tikzcd} P_j \arrow[r, "p_j"] \arrow[d, "q_j"'] & F_j(B) \arrow[d, "F_j(m)"] \\ F_j(C) \arrow[r, "F_j(g)"'] & F_j(D) \end{tikzcd}$$

Since F jointly creates $\mathcal{M}$-pullbacks we can conclude.

Existence of $\mathcal{M},\mathcal{N}$-pushouts. if $m\colon A \to C$ is in $\mathcal{M}$ and $n\colon A \to B$ in $\mathcal{N}$, we get an $\mathcal{M}_j,\mathcal{N}_j$-pushout square

$$\begin{tikzcd} F_j(A) \arrow[r, "F_j(n)"] \arrow[d, "F_j(m)"'] & F_j(B) \arrow[d, "p_j"] \\ F_j(C) \arrow[r, "q_j"'] & Q_j \end{tikzcd}$$

in each $\mathbf{Y}_j$ and we can conclude because F jointly creates $\mathcal{M},\mathcal{N}$-pushouts.

$\mathcal{M},\mathcal{N}$-pushouts are Van Kampen square. This follows at once from the second half of point 1.

3. By the previous point it is enough to show that $(\mathcal{M}_F,\mathcal{N}_F)$ is a preadhesive structure.

1. If $f \in \mathbf{X}$ is an isomorphism then so is $F_j(f)$ for every $F_j \in F$. Thus $F_j(f)$ belongs to $\mathcal{M}_j$ and $\mathcal{N}_j$ for every $j \in J$, which implies that f is in $\mathcal{M}_F$ and in $\mathcal{N}_F$. The parts regarding composition and decomposition follow immediately by functoriality of each $F_j \in F$.

2. Suppose that $g \circ f \in \mathcal{N}_F$, with $g \in \mathcal{M}_F$. Then for every $j \in F$,

$$F_j(g \circ f) = F_j(g) \circ F_j(f)$$

is in $\mathcal{N}_j$ and $F_j(g) \in \mathcal{M}_j$, thus $F_j(f) \in \mathcal{N}_j$ and so $f \in \mathcal{N}_F$.

3. Take a pullback square with $n \in \mathcal{M}_F$ $(\mathcal{N}_F)$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "m"'] & B \arrow[d, "n"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

then applying any $F_j \in F$ we get that $F_j(m)$ is the pullback of $F_j(n)$ along $F_j(g)$, since $F_j(n)$ is in $\mathcal{M}_j$ (in $\mathcal{N}_j$), which implies that $F_j(m) \in \mathcal{M}_j$ $(\mathcal{N}_j)$.

For pushouts the argument is the same: given a pushout square with $m \in \mathcal{M}_F$ $(\mathcal{N}_F)$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "m"'] & B \arrow[d, "n"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

then $F_j(n) \in \mathcal{M}_j$ ($\mathcal{N}_j$) since it is the pushout of $F_j(m)$ and the thesis follows. $\square$

Applying the previous theorem to the families given by, respectively, projections, evaluations and the inclusion we get immediately the following three corollaries (cf. [42, Thm. 4.15]).

Corollary 5.1.32. *Let $\{\mathbf{X}_i\}_{i\in I}$ be a non-empty family of categories such that each $\mathbf{X}_i$ is $\mathcal{M}_i,\mathcal{N}_i$-adhesive. Then the product category $\prod_{i\in I}\mathbf{X}_i$ is $\prod_{i\in I}\mathcal{M}_i,\prod_{i\in I}\mathcal{N}_i$-adhesive, where*

$$\prod_{i\in I}\mathcal{M}_i := \left\{ m\in\mathcal{A}\left(\prod_{i\in I}\mathbf{X}_i\right) \mid \pi_i(m)\in\mathcal{M}_i \textit{ for every } i\in I\right\}$$
$$\prod_{i\in I}\mathcal{N}_i := \left\{ n\in\mathcal{A}\left(\prod_{i\in I}\mathbf{X}_i\right) \mid \pi_i(n)\in\mathcal{N}_i \textit{ for every } i\in I\right\}$$

where $\pi_i\colon \prod_{i\in I}\mathbf{X}_i\to\mathbf{X}_i$ is the projection functor.

Proof. Limits and colimits in $\prod_{i\in I}X_i$ are computed componentwise. Thus, $\{\pi_i\}_{i\in I}$ jointly creates all limits and colimits, and the thesis follows from point 3 of Theorem 5.1.31. $\square$

Corollary 5.1.33. *Let $\mathbf{X}$ be an $\mathcal{M},\mathcal{N}$-adhesive category. Then for every category $\mathbf{Y}$, the category of functors $\mathbf{X}^{\mathbf{Y}}$ is $\mathcal{M}^{\mathbf{Y}},\mathcal{N}^{\mathbf{Y}}$-adhesive, where*

$$\mathcal{M}^{\mathbf{Y}} := \left\{\eta\in\mathcal{A}\left(\mathbf{X}^{\mathbf{Y}}\right) \mid \eta_Y\in\mathcal{M} \textit{ for every object } Y \textit{ of } \mathbf{Y}\right\}$$
$$\mathcal{N}^{\mathbf{Y}} := \left\{\eta\in\mathcal{A}\left(\mathbf{X}^{\mathbf{Y}}\right) \mid \eta_Y\in\mathcal{N} \textit{ for every object } Y \textit{ of } \mathbf{Y}\right\}$$

Proof. This is proved as in the case of products since in a functor category limits and colimits are, again, computed componentwise. $\square$

Corollary 5.1.34. *Let $\mathbf{X}$ be a full subcategory of an $\mathcal{M},\mathcal{N}$-adhesive category $\mathbf{Y}$. Let also $(\mathcal{M}',\mathcal{N}')$ be a preadhesive structure on $\mathbf{X}$ such that $\mathcal{M}'\subseteq\mathcal{M}$ and $\mathcal{N}'\subseteq\mathcal{N}$. Suppose that $\mathbf{X}$ is closed in $\mathbf{Y}$ under pullbacks and $\mathcal{M}',\mathcal{N}'$-pushouts. Then $\mathbf{X}$ is $\mathcal{M}',\mathcal{N}'$-adhesive.*

Proof. A full and faithful functor reflects limits and colimits, and the hypotheses entail that the inclusion functor creates pullbacks and $\mathcal{M}',\mathcal{N}'$-pushouts. $\square$

Application to comma categories

In this section we will show how to apply Theorem 5.1.31 to the comma construction in order to guarantee some adhesivity properties under suitable hypotheses. Our starting point is the following result relating limits and colimits in the comma category $L{\downarrow}R$ with those preserved by $L\colon\mathbf{A}\to\mathbf{X}$ or $R\colon\mathbf{B}\to X$.

Lemma 5.1.35. *Let $L\colon\mathbf{A}\to\mathbf{X}$ and $R\colon\mathbf{B}\to\mathbf{X}$ be functors and $F\colon\mathbf{D}\to L{\downarrow}R$ be a diagram such that L preserves colimits along $U_L\circ F$. Then the family $\{U_L,U_R\}$ (see Appendix A.2) jointly creates colimits of F.*

Proof. Suppose that $U_L\circ F$ and $U_R\circ F$ have colimiting cocones $(A,\{a_D\}_{D\in\mathbf{D}})$ and $(B,\{b_D\}_{D\in\mathbf{D}})$ respectively. By hypothesis $(L(A),\{L(a_D)\}_{D\in\mathbf{D}})$ is colimiting for $L\circ U_L\circ F$. Now, if we define

$$F(D) := (A_D, B_D, f_D)$$

then we have arrows $R(a_i) \circ f_D \colon L(A_D) \to R(B)$ that forms a cocone on $L \circ U_L \circ F$: if $d \colon D \to D'$ is an arrow in $\mathbf{D}$ then $F(d)$ is an arrow in $L{\downarrow}R$ and so

$$\begin{aligned} R(b_{D'}) \circ f_{D'} \circ L(U_L(F(d))) &= R(b_{D'}) \circ R(U_R(F(d))) \circ f_D \\ &= R(b_{D'} \circ U_R(F(d))) \circ f_D \\ &= R(b_D) \circ f_D \end{aligned}$$

Thus there exists $f \colon L(A) \to R(B)$ such that

$$\begin{tikzcd} L(A_D) \arrow[r, "L(a_D)"] \arrow[d, "f_D"'] & L(A) \arrow[d, "f"] \\ R(B_D) \arrow[r, "R(b_D)"'] & R(B) \end{tikzcd}$$

Notice that f is the unique arrow in $\mathbf{X}$ wich makes (a_D, b_D) an arrow $(A_D, B_D, f_D) \to (A, B, f)$ of $L{\downarrow}R$. If we show that $\big((A, B, f), \{(a_D, b_D)\}_{D\in\mathbf{D}}\big)$ is colimiting for F we are done.

First of all, let us show that it is a cocone. Given $d \colon D \to D'$ in $\mathbf{D}$ we have:

$$\begin{aligned} (a_{D'}, b_{D'}) \circ F(d) &= (a_{D'}, b_{D'}) \circ (U_L(F(d)), U_R(F(d))) \\ &= (a_{D'} \circ U_L(F(d)), b_{D'} \circ U_R(F(d))) \\ &= (a_D, b_D) \end{aligned}$$

For the colimiting property, let $\big((X, Y, g), \{(x_D, y_D)\}_{D\in\mathbf{D}}\big)$ be another cocone on F. In particular $\big(X, \{x_D\}_{D\in\mathbf{D}}\big)$ and $\big(Y, \{y_D\}_{D\in\mathbf{D}}\big)$ are cocones on $U_L \circ F$ and $U_R \circ F$ respectively, so we have uniquely determined arrows $x \colon A \to X$ and $y \colon B \to Y$ such that

$$x \circ a_D = x_D \qquad y \circ b_D = y_D$$

Let us show that (x, y) is an arrow of $L{\downarrow}R$. Given $D \in \mathbf{D}$ we have

$$\begin{aligned} R(y) \circ f \circ L(a_D) &= R(y) \circ R(b_D) \circ f_D \\ &= R(y \circ b_D) \circ f_D \\ &= R(y_D) \circ f_D \\ &= g \circ L(x_D) \\ &= g \circ L(x \circ a_D) \\ &= g \circ L(x) \circ L(a_D) \end{aligned}$$

from which it follows that the following diagram commutes.

$$\begin{tikzcd} L(A) \arrow[r, "L(x)"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ R(B) \arrow[r, "R(y)"'] & Y \end{tikzcd}$$

This shows that $\big((A, B, f), \{(a_D, b_D)\}_{D\in\mathbf{D}}\big)$ is colimiting for F and the thesis follows. $\square$

Proposition A.2.2 and Lemma 5.1.35 now yields the following.

Corollary 5.1.36. *The family $\{U_L, U_R\}$ jointly creates limits along every diagram $F\colon \mathbf{D} \to L{\downarrow}R$ such that R preserves the limit of $U_R \circ I$.*

We can use Corollary 5.1.36 to characterize monos in comma categories.

Corollary 5.1.37. *If R preserves pullbacks then an arrow (h,k) in $L{\downarrow}R$ is mono if and only if both h and k are monomorphisms.*

Proof. ($\Rightarrow$) If $(h,k)\colon (A,B,f) \to (A',B',g)$ is a mono then the following square is a pullback in $L{\downarrow}R$

$$\begin{tikzcd}
(A,B,f) \arrow[r, "\mathsf{id}_{(A,B,f)}"] \arrow[d, "\mathsf{id}_{(A,B,f)}"'] & (A,B,f) \arrow[d, "(h,k)"] \\
(A,B,f) \arrow[r, "(h,k)"'] & (A',B',g)
\end{tikzcd}$$

Using Corollary 5.1.36 we deduce that the following two squares are pullbacks in $\mathbf{A}$ and $\mathbf{B}$.

$$\begin{tikzcd}
A \arrow[r, "\mathsf{id}_A"] \arrow[d, "\mathsf{id}_A"'] & A \arrow[d, "h"] \\
A \arrow[r, "h"'] & A'
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "\mathsf{id}_B"] \arrow[d, "\mathsf{id}_B"'] & B \arrow[d, "k"] \\
B \arrow[r, "k"'] & B'
\end{tikzcd}$$

From which it follows that h and k are monos.

($\Leftarrow$) Since h and k are monos then we have two pullback squares

$$\begin{tikzcd}
A \arrow[r, "\mathsf{id}_A"] \arrow[d, "\mathsf{id}_A"'] & A \arrow[d, "h"] \\
A \arrow[r, "h"'] & A'
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "\mathsf{id}_B"] \arrow[d, "\mathsf{id}_B"'] & B \arrow[d, "k"] \\
B \arrow[r, "k"'] & B'
\end{tikzcd}$$

By Corollary 5.1.36 this implies that

$$\begin{tikzcd}
(A,B,f) \arrow[r, "\mathsf{id}_{(A,B,f)}"] \arrow[d, "\mathsf{id}_{(A,B,f)}"'] & (A,B,f) \arrow[d, "(h,k)"] \\
(A,B,f) \arrow[r, "(h,k)"'] & (A',B',g)
\end{tikzcd}$$

is a pullback in $L{\downarrow}R$ and we are done. □

Applying Theorem 5.1.31 and Corollary 5.1.36 we get at once the following result.

Theorem 5.1.38 ([22])**.** *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be respectively* $\mathcal{M}, \mathcal{N}$*-adhesive and* $\mathcal{M}', \mathcal{N}'$*-adhesive categories,* $L\colon \mathbf{A} \to \mathbf{X}$ *a functor that preserves* $\mathcal{M}, \mathcal{N}$*-pushouts, and* $R\colon \mathbf{B} \to \mathbf{X}$ *a pullback preserving one. Then* $L{\downarrow}R$ *is* $\mathcal{M}{\downarrow}\mathcal{M}', \mathcal{N}{\downarrow}\mathcal{N}'$*-adhesive, where*

$$\mathcal{M}{\downarrow}\mathcal{M}' := \{(h,k) \in \mathcal{A}(L{\downarrow}R) \mid h \in \mathcal{M}, k \in \mathcal{M}'\}$$
$$\mathcal{N}{\downarrow}\mathcal{N} := \{(h,k) \in \mathcal{A}(L{\downarrow}R) \mid h \in \mathcal{N}, k \in \mathcal{N}'\}$$

Take now L to be $\mathsf{id}_{\mathbf{X}}$ and $\delta_X : \mathbf{1} \to \mathbf{X}$ the functor which picks an object X. It is now obvious to notice that δ_X preserves all pullbacks, (actually all connected limits [35, 103]) thus, applying Theorem 5.1.38 (and Proposition A.3.5) we get the following.

Corollary 5.1.39. *Let* $\mathbf{X}$ *be* $\mathcal{M}, \mathcal{N}$*-adhesive, then for every object* $X \in \mathbf{X}$*, the slice category* $\mathbf{X}/X$ *is* $\mathcal{M}/X, \mathcal{N}/X$*-adhesive, where*

$$\mathcal{M}/X := \{m \in \mathcal{A}(\mathbf{X}/X) \mid m \in \mathcal{M}\}$$
$$\mathcal{N}/X := \{n \in \mathcal{A}(\mathbf{X}/X) \mid n \in \mathcal{N}\}$$

5.2 $\mathcal{M}, \mathcal{N}$-unions and $\mathcal{M}, \mathcal{N}$-adhesivity

Johnstone, Lack and Sobociński [67] and Garner [52] have provided a criterion to establish quasiadhesivity, involving the closure of regular monos under unions. The aim of this section is to adapt their results to the setting of $\mathcal{M}, \mathcal{N}$-adhesivity.

5.2.1 $\mathcal{N}$-(pre)adhesive morphisms

The first step that we need to take is to generalize the notion of *(pre)adhesive morphism* provided in [52].

Definition 5.2.1. Given a class $\mathcal{N}$ of arrows of a category $\mathbf{X}$, we say that $\mathcal{N}$ is a *matching class* if

1. it contains all isomorphisms;
2. is closed under composition and decomposition;
3. is stable under pullbacks and pushouts.

Given a matching class $\mathcal{N}$, a morphism $m\colon X \to Y$ in $\mathbf{X}$ is $\mathcal{N}$*-preadhesive* if for every $n\colon X \to Z$ in $\mathcal{N}$, a stable pushout square

$$\begin{tikzcd} X \arrow[r, "n"] \arrow[d, "m"'] & Z \arrow[d, "p"] \\ Y \arrow[r, "q"'] & W \end{tikzcd}$$

exists and it is also a pullback of p along q. m will be called $\mathcal{N}$*-adhesive* if for every pullback square as the one below, n is $\mathcal{N}$-preadhesive.

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "n"'] & X \arrow[d, "m"] \\ W \arrow[r, "f"'] & Y \end{tikzcd}$$

We will denote by $\mathcal{N}_{\mathsf{pa}}$ and by $\mathcal{N}_{\mathsf{a}}$ the classes of, respectively, $\mathcal{N}$-preadhesive and $\mathcal{N}$-adhesive morphisms.

Notation. Instead of "$\mathcal{A}(\mathbf{X})$-(pre)adhesive" we will use "(pre)adhesive".

Example 5.2.2. If $\mathbf{X}$ is an $\mathcal{M},\mathcal{N}$-adhesive category then $\mathcal{N}$ is a matching class. Moreover, $\mathcal{M},\mathcal{N}$-pushouts are Van Kampen squares, so every $m \in \mathcal{M}$ is preadhesive. Since $\mathcal{M}$ is closed under pullback this implies that every arrow in $\mathcal{M}$ is also adhesive.

The following proposition collects some useful facts about $\mathcal{N}$-(pre)adhesive morphisms.

Proposition 5.2.3. *Let $\mathcal{N}$ be a matching class on a category $\mathbf{X}$, then the following hold true:*

1. *if m is $\mathcal{N}$-adhesive then it is $\mathcal{N}$-preadhesive;*
2. *every isomorphism is $\mathcal{N}$-adhesive;*
3. *if $n \in \mathcal{N}$ is $\mathcal{N}$-preadhesive then it is a regular mono;*
4. *the class $\mathcal{N}_{\mathsf{pa}}$ is closed under composition;*
5. *$\mathcal{N}_{\mathsf{a}}$ is stable under pullbacks;*
6. *if $\mathbf{X}$ has pullbacks along $\mathcal{N}$-adhesive arrows, then $\mathcal{N}_{\mathsf{a}}$ is closed under composition.*

Proof. 1. This follows at once noticing that the following square is a pullback.

$$\begin{tikzcd} X \arrow[r, "\mathsf{id}_X"] \arrow[d, "m"'] & X \arrow[d, "m"] \\ Y \arrow[r, "\mathsf{id}_Y"'] & Y \end{tikzcd}$$

2. Isomorphisms are closed under pullbacks, thus it is enough to show that every isomorphism $m\colon X \to Y$ is $\mathcal{N}$-preadhesive. Let $n\colon X \to Z$ be an element of $\mathcal{N}$, we have a pushout square

$$\begin{tikzcd} X \arrow[rr, "n"] \arrow[d, "m"'] & & Z \arrow[d, "\mathsf{id}_Z"] \\ Y \arrow[r, "m^{-1}"'] & X \arrow[r, "n"'] & Z \end{tikzcd}$$

Given $f\colon W \to Z$ and $g\colon W \to Y$ such that

$$f = n \circ m^{-1} \circ g$$

we can notice that $m^{-1} \circ f$ is the unique arrow such that

$$g = m \circ \left(m^{-1} \circ g\right)$$

and from the commutativity of the following diagram we can deduce that the pushout square above is also a pullback.

$$\begin{tikzcd} W \arrow[r, "g"'] \arrow[rrrr, bend left, "f"] \arrow[rr, bend right, "g"'] & Y \arrow[r, "m^{-1}"] & X \arrow[rr, "n"] \arrow[d, "m"] & & Z \arrow[d, "\mathsf{id}_Z"] \\ & & Y \arrow[r, "m^{-1}"'] & X \arrow[r, "n"'] & Z \end{tikzcd}$$

For stability, take $f\colon Z' \to Z$ such that the following pullback square exists

$$\begin{tikzcd}
P \arrow[r, "h"] \arrow[d, "g"'] & Z' \arrow[d, "f"] \\
X \arrow[r, "n"'] & Z
\end{tikzcd}$$

Then by Lemma 5.1.4 in the following cube all the vertical faces are pullbacks

$$\begin{tikzcd}
 & P \arrow[rrr, "h"] \arrow[dl, "\mathsf{id}_P"'] \arrow[dd, "g"] & & & Z' \arrow[dl, "\mathsf{id}_{Z'}"'] \arrow[dd, "f"] \\
P \arrow[rr, "\mathsf{id}_P"] \arrow[dd, "g"'] & & P \arrow[r, "h"] \arrow[ddd, "g"] & Z' \arrow[ddd, "f"] & \\
 & X \arrow[dl, "\mathsf{id}_X"'] \arrow[d, "\mathsf{id}_X"] & & & \\
X \arrow[d, "m"'] & X \arrow[rrr, "n"] \arrow[dl, "m"] & & & Z \arrow[dl, "\mathsf{id}_Z"] \\
Y \arrow[rr, "m^{-1}"'] & & X \arrow[r, "n"'] & Z &
\end{tikzcd}$$

and we can conclude from Remarks 5.1.2 and 5.1.3 that the pushouts of m along n are stable.

3. Since n is in $\mathcal{N}$ and $\mathcal{N}$-preadhesive we can consider its pushout along itself

$$\begin{tikzcd}
X \arrow[r, "n"] \arrow[d, "n"'] & Y \arrow[d, "f"] \\
Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

which, is also a pullback. Thus n is the equalizer of $f, g\colon Y \rightrightarrows Z$.

4. Let $n\colon X \to Z$ be an element of $\mathcal{N}$, and $m\colon X \to Y$, $k\colon Y \to Z$ two $\mathcal{N}$-preadhesive morphisms, since $\mathcal{N}$ is stable under pushouts, we get the following two pushout squares, which are also pullbacks

$$\begin{tikzcd}
X \arrow[r, "n"] \arrow[d, "m"'] & Z \arrow[d, "p_1"] \\
Y \arrow[r, "p_2"'] & P
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r, "p_2"] \arrow[d, "k"'] & P \arrow[d, "q_1"] \\
Z \arrow[r, "q_2"'] & Q
\end{tikzcd}$$

By Lemmas 5.1.4 and 5.1.6, pasting them together gives us a pushout square for n along $m' \circ m$

which is also a pullback. For stability, Take an arrow $p\colon P' \to P$, we have a cube

$$
\begin{tikzcd}
 & & X' \arrow[rrr, "n'"] \arrow[dl, "m'"'] \arrow[ddd, "x"] & & & Z' \arrow[dl, "p_1'"] \arrow[ddd, "z"] \\
 & Y' \arrow[rrr, "p_2'"] \arrow[dl, "k'"'] \arrow[ddd, "y"] & & & P' \arrow[dl, "q_1'"] \arrow[ddd, "p"] & \\
Z' \arrow[rrr, "q_2'"] \arrow[ddd, "z"'] & & & Q' \arrow[ddd, "q"] & & \\
 & & X \arrow[rrr, "n"] \arrow[dl, "m"'] & & & Z \arrow[dl, "p_1"] \\
 & Y \arrow[rrr, "p_2"] \arrow[dl, "k"'] & & & P \arrow[dl, "q_1"] & \\
Z \arrow[rrr, "q_2"'] & & & Q & &
\end{tikzcd}
$$

in which all the vertical squares are pullbacks. Thus the two halves of the top face are pushouts and by Lemma 5.1.6 also the whole top face is one. The thesis follows from Remark 5.1.3

5. Let $m\colon X \to Y$ be $\mathcal{N}$-adhesive, and consider the the following rectangle in which both squares are pullbacks

$$
\begin{tikzcd}
A \arrow[r, "p"] \arrow[d, "q"'] & Z \arrow[r, "g"] \arrow[d, "n"'] & X \arrow[d, "m"] \\
B \arrow[r, "r"'] & W \arrow[r, "f"'] & Y
\end{tikzcd}
$$

By Lemma 5.1.4 the outer rectangle is a pullback and thus q is $\mathcal{N}$-preadhesive, proving that n is $\mathcal{N}$-adhesive.

6. Let $m_1\colon X \to Y$ and $m_2\colon Y \to Z$ be $\mathcal{N}$-adhesive arrows, then for every $n\colon N \to Z$ in $\mathcal{N}$ we can consider the following diagram, in which the squares are pullbacks

$$
\begin{tikzcd}
Q \arrow[r, "q_1"] \arrow[d, "q_2"'] & P \arrow[r, "p_1"] \arrow[d, "p_2"'] & N \arrow[d, "n"] \\
X \arrow[r, "m_1"'] & Y \arrow[r, "m_2"'] & Z
\end{tikzcd}
$$

By Lemma 5.1.4 the whole rectangle is a pullback and both p_1 and q_1 are $\mathcal{N}$-preadhesive, therefore the thesis follows from point 4. □

Corollary 5.2.4. *In any category* $\mathbf{X}$, $\mathcal{A}(\mathbf{X})_{\mathsf{a}} \subseteq \mathcal{R}(\mathbf{X})$.

Corollary 5.2.5. *Let $\mathcal{N}$ be a matching class on a category $\mathbf{X}$ with pullbacks, then:*

1. *$\mathcal{N}_{\mathsf{a}} \cap \mathcal{M}(\mathbf{X})$ is a stable system of monos;*
2. *if $\mathcal{N}_{\mathsf{a}} \cap \mathcal{M}(\mathbf{X})$ is stable under pushouts, then $(\mathcal{N}_{\mathsf{a}} \cap \mathcal{M}(\mathbf{X}), \mathcal{N})$ is a preadhesive structure*

Proof.
1. By point 2 of Proposition 5.2.3 every isomorphism is in $\mathcal{N}_{\mathsf{a}} \cap \mathcal{M}(\mathbf{X})$, stability under pullbacks follows from point 5 while closure under composition is entailed by point 6.
2. This follows at once from the previous point and Lemma 5.1.18. □

In general we cannot guarantee closure of $\mathcal{N}_{\mathfrak{a}}$ under all pushouts, nonetheless we can still establish some result along this line.

Lemma 5.2.6. *Let $\mathcal{N}$ be a matching class in a category* $\mathbf{X}$ *with pullbacks and consider the following pushout*

$$\begin{tikzcd} X \arrow[r, "n"] \arrow[d, "m_1"'] & Z \arrow[d, "m_2"] \\ Y \arrow[r, "g"'] & W \end{tikzcd}$$

with $n \in \mathcal{N}$. If m_1 is mono and $\mathcal{N}$-adhesive, then:

1. *m_2 is mono;*
2. *m_2 is $\mathcal{N}$-preadhesive;*
3. *m_2 is $\mathcal{N}$-adhesive.*

Proof. 1. Since $\mathbf{X}$ has pullbacks, we have a diagram

$$\begin{tikzcd} X \arrow[dr, dotted, "h"] \arrow[drr, bend left, "n"] \arrow[ddr, bend right, "n"'] & & \\ & P \arrow[r, "p"] \arrow[d, "q"'] & Z \arrow[d, "m_2"] \\ & Z \arrow[r, "m_2"'] & W \end{tikzcd}$$

in which the square is a pullback, so that the dotted h exists because of its universal property. We can then build a cube

$$\begin{tikzcd}[column sep=small, row sep=small] & X \arrow[rr, "h"] \arrow[dl, "\mathrm{id}_X"'] \arrow[dd, "\mathrm{id}_X" near end] & & P \arrow[dl, "p"'] \arrow[dd, "q"] \\ X \arrow[rr, "n" near start] \arrow[dd, "m_1"'] & & Z \arrow[dd, "m_2" near start] & \\ & X \arrow[rr, "n" near end] \arrow[dl, "m_1"] & & Z \arrow[dl, "m_2"] \\ Y \arrow[rr, "g"'] & & W & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small, row sep=small] & X \arrow[rr, "h"] \arrow[dl, "\mathrm{id}_X"'] \arrow[dd, "\mathrm{id}_X" near end] & & P \arrow[dl, "q"'] \arrow[dd, "p"] \\ X \arrow[rr, "n" near start] \arrow[dd, "m_1"'] & & Z \arrow[dd, "m_2" near start] & \\ & X \arrow[rr, "n" near end] \arrow[dl, "m_1"] & & Z \arrow[dl, "m_2"] \\ Y \arrow[rr, "g"'] & & W & \end{tikzcd}$$

By point 1 of Proposition 5.2.3 the bottom and front faces are stable pushouts and pullbacks because m_1 is $\mathcal{N}$-adhesive, and the left squares are pullbacks by hypothesis. Lemma 5.1.4 entails that the rectangles

$$\begin{tikzcd} X \arrow[r, "h"'] \arrow[rr, bend left, "n"] \arrow[d, "\mathrm{id}_X"'] & P \arrow[r, "p"'] \arrow[d, "q"'] & Z \arrow[d, "m_2"] \\ X \arrow[r, "n"] \arrow[rr, bend right, "g \circ m_1"'] & Z \arrow[r, "m_2"] & W \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "h"'] \arrow[rr, bend left, "n"] \arrow[d, "\mathrm{id}_X"'] & P \arrow[r, "q"'] \arrow[d, "p"'] & Z \arrow[d, "m_2"] \\ X \arrow[r, "n"] \arrow[rr, bend right, "g \circ m_1"'] & Z \arrow[r, "m_2"] & W \end{tikzcd}$$

are pullbacks, thus the same lemma shows that also the back faces of the two cubes are pullbacks too. By stability of the bottom faces it follows that

$$
\begin{tikzcd}
X \arrow[r, "h"] \arrow[d, "\mathrm{id}_X"'] & P \arrow[d, "p"] \\
X \arrow[r, "n"'] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "h"] \arrow[d, "\mathrm{id}_X"'] & P \arrow[d, "q"] \\
X \arrow[r, "n"'] & Z
\end{tikzcd}
$$

are pushouts and thus p and q are isomorphisms.

2. Let $k\colon Z \to Q$ be another arrow in $\mathcal{N}$ and consider the diagram

$$
\begin{tikzcd}
X \arrow[r, "n"] \arrow[d, "m_1"'] & Z \arrow[r, "k"] \arrow[d, "m_2"] & Q \arrow[d, "f"] \\
Y \arrow[r, "g"] \arrow[rr, bend right, "t"'] & W \arrow[r, dotted, "s"] & S
\end{tikzcd}
$$

in which the left square and the external rectangle are stable pushouts and pullbacks. Since

$$f \circ k \circ n = t \circ m_1$$

the universal property of the left square yields the dotted s. By Lemma 5.1.6 the square so obtained is a stable pushout. Thus we are left with showing that it is a pullback. Given the solid part of the diagram

$$
\begin{tikzcd}
L \arrow[dr, dotted, "l_2"] \arrow[drrr, bend left, "l_1"] \arrow[ddr, bend right, "l_2"'] & & & \\
& Z \arrow[rr, "k"] \arrow[d, "\mathrm{id}_Z"] & & Q \arrow[d, "f"] \\
& Z \arrow[r, "m_2"'] & W \arrow[r, "s"'] & S
\end{tikzcd}
$$

we have

$$
\begin{aligned}
f \circ l_1 &= s \circ m_2 \circ l_2 \\
&= f \circ k \circ m_2
\end{aligned}
$$

By the previous point, f is mono, and thus the following rectangle is a pullback

$$
\begin{tikzcd}
Z \arrow[rr, "k"] \arrow[d, "\mathrm{id}_Z"'] & & Q \arrow[d, "f"] \\
Z \arrow[r, "m_2"'] & W \arrow[r, "s"'] & S
\end{tikzcd}
$$

The thesis now follows applying the previous point and Lemma 5.1.8 to the following diagrams.

$$
\begin{tikzcd}
X \arrow[r, "n"] \arrow[d, "m_1"'] & Z \arrow[d, "m_2"] \\
Y \arrow[r, "g"'] & W
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "n"] \arrow[d, "m_1"'] & Z \arrow[r, "k"] \arrow[d, "m_2"'] & Q \arrow[d, "f"] \\
Y \arrow[r, "g"'] & W \arrow[r, "s"'] & S
\end{tikzcd}
\qquad
\begin{tikzcd}
Z \arrow[r, "\mathsf{id}_Z"] \arrow[d, "\mathsf{id}_Z"'] & Z \arrow[r, "k"] \arrow[d, "m_2"'] & Q \arrow[d, "f"] \\
Z \arrow[r, "m_2"'] & W \arrow[r, "s"'] & S
\end{tikzcd}
$$

3. Take an arrow $w\colon W' \to W$ and consider the following cube, in which the solid faces are pullbacks

$$
\begin{tikzcd}
& X' \arrow[rr, "n'"] \arrow[dl, dotted, "m'_1"'] \arrow[dd, "x"' near start] & & Z' \arrow[dl, "m'_2"'] \arrow[dd, "z"] \\
Y' \arrow[rr, "g'" near end, crossing over] \arrow[dd, "y"'] & & W' & \\
& X \arrow[rr, "n" near end] \arrow[dl, "m_1"] & & Z \arrow[dl, "m_2"] \\
Y \arrow[rr, "g"'] & & W \arrow[from=uu, "w"' near start, crossing over] &
\end{tikzcd}
$$

By Corollary 5.1.5 the arrow $m'_1\colon X' \to Y'$ exists and the added face is a pullback. Since the bottom face is a stable pushout then the top face is a pushout too. By point 5 of Proposition 5.2.3, m'_1 is $\mathcal{N}$-adhesive and, since $\mathcal{N}$ is matching, n' is in $\mathcal{N}$. The previous point of this lemma implies that m'_2 is $\mathcal{N}$-preadhesive and we can conclude. □

Corollary 5.2.7. *If $\mathbf{X}$ is a category with pullbacks then $(\mathcal{A}(\mathbf{X})_{\mathsf{a}}, \mathcal{A}(\mathbf{X}))$ is a preadhesive structure.*

Proof. By Corollary 5.2.4 we know that $\mathcal{A}(\mathbf{X})_{\mathsf{a}} \subseteq \mathcal{M}(\mathbf{X})$, by Lemma 5.2.6 this implies that $\mathcal{A}(\mathbf{X})_{\mathsf{a}}$ is stable under pushout and we can conclude appealing to Corollary 5.2.5. □

Finally, $\mathcal{N}$-adhesivity allows us to compute suprema of certain pairs of subobjects.

Proposition 5.2.8. *Let $\mathcal{N}$ be a matching class in a category $\mathbf{X}$ with pullbacks. Given an $\mathcal{N}$-adhesive mono $m\colon M \to X$ and another mono $n\colon N \to X$ in $\mathcal{N}$, consider the diagram*

$$
\begin{tikzcd}
P \arrow[r, "p_1"] \arrow[d, "p_2"'] & M \arrow[d, "u_2"] \arrow[ddr, bend left, "m"] & \\
N \arrow[r, "u_1"'] \arrow[drr, bend right, "n"'] & U \arrow[dr, dotted, "u"] & \\
& & X
\end{tikzcd}
$$

in which the outer boundary form a pullback and the inner square a pushout. Then the dotted arrow $u\colon U \to X$ is a monomorphism and, in $(\mathsf{Sub}(X), \leq)$

$$[u] = [m] \vee [n]$$

Remark 5.2.9. Notice that the p_2 and p_1 are both monos, moreover, p_2 is $\mathcal{N}$-preadhesive while $p_1 \in \mathcal{N}$, as the pullback of n. Thus the inner pushout exists.

Proof. Consider the following two pullback squares

$$
\begin{tikzcd}
Q \arrow[r, "q_1"] \arrow[d, "q_2"'] & N \arrow[d, "n"] \\
U \arrow[r, "u"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
W \arrow[r, "w_1"] \arrow[d, "w_2"'] & M \arrow[d, "m"] \\
U \arrow[r, "u"'] & X
\end{tikzcd}
$$

By construction we have the following equalities

$$
\begin{aligned}
n &= u \circ u_1 & m &= u \circ u_2 \\
u \circ u_2 \circ p_1 &= m \circ p_1 & u \circ u_1 \circ p_1 &= n \circ p_2 \\
&= n \circ p_2 & &= m \circ p_1
\end{aligned}
$$

which give us the arrows $f_1 \colon N \to Q$, $f_2 \colon P \to Q$, $g_1 \colon M \to W$, $g_2 \colon P \to W$ making the following diagrams commute

$$
\begin{tikzcd}
N \arrow[r, "f_1"'] \arrow[rr, bend left, "\mathsf{id}_N"] \arrow[d, "\mathsf{id}_N"'] & Q \arrow[r, "q_1"'] \arrow[d, "q_2"'] & N \arrow[d, "n"] \\
N \arrow[r, "u_1"] \arrow[rr, bend right, "n"'] & U \arrow[r, "u"] & X
\end{tikzcd}
\quad
\begin{tikzcd}
P \arrow[r, "f_2"'] \arrow[rr, bend left, "p_2"] \arrow[d, "p_1"'] & Q \arrow[r, "q_1"'] \arrow[d, "q_2"'] & N \arrow[d, "n"] \\
M \arrow[r, "u_2"] \arrow[rr, bend right, "m"'] & U \arrow[r, "u"] & X
\end{tikzcd}
\quad
\begin{tikzcd}
M \arrow[r, "g_1"'] \arrow[rr, bend left, "\mathsf{id}_M"] \arrow[d, "\mathsf{id}_M"'] & W \arrow[r, "w_1"'] \arrow[d, "w_2"'] & M \arrow[d, "m"] \\
M \arrow[r, "u_2"] \arrow[rr, bend right, "m"'] & U \arrow[r, "u"] & X
\end{tikzcd}
\quad
\begin{tikzcd}
P \arrow[r, "g_2"'] \arrow[rr, bend left, "p_1"] \arrow[d, "p_2"'] & W \arrow[r, "w_1"'] \arrow[d, "w_2"'] & M \arrow[d, "m"] \\
N \arrow[r, "u_1"] \arrow[rr, bend right, "n"'] & U \arrow[r, "u"] & X
\end{tikzcd}
$$

Their outer edges are pullbacks, thus in the following cubes, the vertical faces are pullbacks

$$
\begin{tikzcd}
& P \arrow[rr, "\mathsf{id}_P"] \arrow[dl, "p_2"'] \arrow[dd, "\mathsf{id}_P"', near start] & & P \arrow[dl, "f_2"'] \arrow[dd, "p_1"] \\
N \arrow[rr, "f_1", near start] \arrow[dd, "\mathsf{id}_N"'] & & Q \arrow[dd, "q_2"', near start] & \\
& P \arrow[rr, "p_1", near end] \arrow[dl, "p_2"] & & M \arrow[dl, "u_2"] \\
N \arrow[rr, "u_1"'] & & U &
\end{tikzcd}
\qquad
\begin{tikzcd}
& P \arrow[rr, "p_1"] \arrow[dl, "\mathsf{id}_P"'] \arrow[dd, "\mathsf{id}_P"', near start] & & M \arrow[dl, "g_1"'] \arrow[dd, "\mathsf{id}_M"] \\
P \arrow[rr, "g_2", near start] \arrow[dd, "p_2"'] & & W \arrow[dd, "w_2"', near start] & \\
& P \arrow[rr, "p_1", near end] \arrow[dl, "p_2"] & & M \arrow[dl, "u_2"] \\
N \arrow[rr, "u_1"'] & & U &
\end{tikzcd}
$$

p_2 is $\mathcal{N}$-preadhesive, so the top faces are pushouts and therefore f_1, and g_1 are isomorphisms with inverses given by q_1 and w_1. But then, since

$$
u_1 = q_2 \circ f_1 \qquad u_2 = w_2 \circ g_1
$$

we can further deduce that the squares below are both pullbacks.

$$
\begin{tikzcd}
N \arrow[r, "\mathsf{id}_N"] \arrow[d, "u_1"'] & N \arrow[d, "n"] \\
U \arrow[r, "u"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[r, "\mathsf{id}_M"] \arrow[d, "u_2"'] & M \arrow[d, "m"] \\
U \arrow[r, "u"'] & X
\end{tikzcd}
$$

We have three diagrams

$$
\begin{tikzcd}
P \arrow[r, "p_1"] \arrow[d, "p_2"'] & M \arrow[d, "m"] \\
N \arrow[r, "n"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
N \arrow[r, "u_1"] \arrow[d, "\mathsf{id}_N"'] & U \arrow[r, "\mathsf{id}_U"] \arrow[d, "\mathsf{id}_U"'] & U \arrow[d, "u"] \\
N \arrow[r, "u_1"] \arrow[rr, bend right, "n"'] & U \arrow[r, "u"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[r, "u_2"] \arrow[d, "\mathsf{id}_M"'] & U \arrow[r, "\mathsf{id}_U"] \arrow[d, "\mathsf{id}_U"'] & U \arrow[d, "u"] \\
M \arrow[r, "u_2"] \arrow[rr, bend right, "m"'] & U \arrow[r, "u"] & X
\end{tikzcd}
$$

and we have just proved that the rectangles are pullbacks. Thus we can apply Lemma 5.1.8 to deduce that

$$
\begin{tikzcd}
U \arrow[r, "\mathsf{id}_U"] \arrow[d, "\mathsf{id}_U"'] & U \arrow[d, "u"] \\
U \arrow[r, "u"'] & X
\end{tikzcd}
$$

is a pullback, but this means exactly that u is a mono.

For the second half: suppose that $k\colon K \to X$ is an upper bound for m and n, thus there exists $k_1\colon M \to K$ and $k_2\colon N \to K$ such that

$$m = k \circ k_1 \qquad n = k \circ k_2$$

But then

$$
\begin{aligned}
k \circ k_1 \circ p_1 &= m \circ p_1 \\
&= n \circ p_2 \\
&= k \circ k_2 \circ p_2
\end{aligned}
$$

Since k is mono, this implies that there exists a unique $h\colon U \to K$ such that

$$k_2 = h \circ u_1 \qquad k_1 = h \circ u_1$$

and we have

$$
\begin{aligned}
k \circ h \circ u_1 &= k \circ k_2 & \qquad k \circ h \circ u_2 &= k \circ k_1 \\
&= n & &= m \\
&= u \circ u_1 & &= u \circ u_2
\end{aligned}
$$

showing that $u = k \circ h$, i.e. $u \leq k$. □

5.2.2 From $\mathcal{M},\mathcal{N}$-unions to $\mathcal{M},\mathcal{N}$-adhesivity

Given a preadhesive structure $(\mathcal{M},\mathcal{N})$ and suppose that $\mathcal{M} \subseteq \mathcal{N}_\mathsf{a}$, in this section we will show how to deduce $\mathcal{M},\mathcal{N}$-adhesivity from the closure of $\mathcal{M}$ under some kind of unions.

Definition 5.2.10. Let $(\mathcal{M},\mathcal{N})$ be a preadhesive structure. A monomorphism $u\colon U \to X$ is an $\mathcal{M},\mathcal{N}$-*union* if there exist $m \in \mathcal{M}$ and $n \in \mathcal{M} \cap \mathcal{N}$ such that, in the poset $(\mathsf{Sub}(X), \leq)$,

$$[u] = [m] \vee [n]$$

We will say that $\mathcal{M}$ is *closed under $\mathcal{M},\mathcal{N}$-unions*, if it contains all such monos.

We will need a technical lemma involving kernel pairs. Take a category **X** with pullbacks endowed with a preadhesive structure $(\mathcal{M},\mathcal{N})$, and take also the following $\mathcal{M},\mathcal{N}$-pushout square

$$\begin{tikzcd} X \arrow[r, "n"] \arrow[d, "m"'] & Z \arrow[d, "p"] \\ Y \arrow[r, "q"'] & W \end{tikzcd}$$

Pulling back n and q along themselves, we get two diagrams

$$\begin{tikzcd} X \arrow[dr, dotted, "\gamma_n"] \arrow[drr, bend left, "\mathsf{id}_X"] \arrow[ddr, bend right, "\mathsf{id}_X"'] & & \\ & K_n \arrow[r, "x_1"] \arrow[d, "x_2"'] & X \arrow[d, "n"] \\ & X \arrow[r, "n"'] & Z \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[dr, dotted, "\gamma_q"] \arrow[drr, bend left, "\mathsf{id}_Y"] \arrow[ddr, bend right, "\mathsf{id}_Y"'] & & \\ & K_q \arrow[r, "y_1"] \arrow[d, "y_2"'] & Y \arrow[d, "q"] \\ & Y \arrow[r, "q"'] & W \end{tikzcd}$$

with the dotted arrows $\gamma_n \colon Z \to K_n$ and $\gamma_q \colon Y \to K_q$. Moreover, we have

$$\begin{aligned} q \circ m \circ x_1 &= p \circ n \circ x_1 \\ &= p \circ n \circ x_2 \\ &= q \circ m \circ x_2 \end{aligned}$$

Thus we have an arrow $k \colon K_n \to K_q$ as in the following squares.

$$\begin{tikzcd} K_n \arrow[r, "x_1"] \arrow[d, dotted, "k"'] & X \arrow[d, "m"] \\ K_q \arrow[r, "y_1"'] & Y \end{tikzcd} \qquad \begin{tikzcd} K_n \arrow[r, "x_2"] \arrow[d, dotted, "k"'] & X \arrow[d, "m"] \\ K_q \arrow[r, "y_2"'] & Y \end{tikzcd}$$

We can also construct another commutative square. From the following chains of equalities

$$\begin{aligned} y_1 \circ \gamma_q \circ m &= \mathsf{id}_Y \circ m \\ &= m \\ &= m \circ \mathsf{id}_X \\ &= m \circ x_1 \circ \gamma_n \\ &= y_1 \circ \circ k \circ \gamma_n \end{aligned} \qquad \begin{aligned} y_2 \circ \gamma_q \circ m &= \mathsf{id}_Y \circ m \\ &= m \\ &= m \circ \mathsf{id}_X \\ &= m \circ x_2 \circ \gamma_n \\ &= y_2 \circ \circ k \circ \gamma_n \end{aligned}$$

we can deduce the commutativity of the square below.

$$\begin{tikzcd} X \arrow[r, "\gamma_n"] \arrow[d, "m"'] & K_n \arrow[d, "k"] \\ Y \arrow[r, "\gamma_q"'] & K_q \end{tikzcd}$$

Lemma 5.2.11. *Let $(\mathcal{M},\mathcal{N})$ be a preadhesive structure on a category $\mathbf{X}$ with pullbacks such that $\mathcal{M} \subseteq \mathcal{N}_\mathfrak{a}$, $\mathcal{M} \cap \mathcal{N}$ contains every split mono and $\mathcal{M}$ is closed under $\mathcal{M},\mathcal{N}$-unions. Then given an $\mathcal{M},\mathcal{N}$-pushout square*

$$\begin{tikzcd}
X \arrow[r, "n"] \arrow[d, "m"'] & Z \arrow[d, "p"] \\
Y \arrow[r, "q"'] & W
\end{tikzcd}$$

all the squares in the following diagrams, constructed as above, are stable pushouts and pullbacks.

$$\begin{tikzcd}
X \arrow[r, "\gamma_n"] \arrow[d, "m"'] & K_n \arrow[r, "x_1"] \arrow[d, "k"'] & X \arrow[r, "n"] \arrow[d, "m"'] & Z \arrow[d, "p"] \\
Y \arrow[r, "\gamma_q"'] & K_q \arrow[r, "y_1"'] & Y \arrow[r, "q"'] & W
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "\gamma_n"] \arrow[d, "m"'] & K_n \arrow[r, "x_2"] \arrow[d, "k"'] & X \arrow[r, "n"] \arrow[d, "m"'] & Z \arrow[d, "p"] \\
Y \arrow[r, "\gamma_q"'] & K_q \arrow[r, "y_2"'] & Y \arrow[r, "q"'] & W
\end{tikzcd}$$

Proof. The rightmost square in both diagrams is a pushout by hypothesis, since it is an $\mathcal{M},\mathcal{N}$-pushout and m is $\mathcal{N}$-adhesive. Now, by Lemma 5.1.4 the rectangles

$$\begin{tikzcd}
K_n \arrow[r, "x_2"] \arrow[d, "x_1"'] & X \arrow[r, "m"] \arrow[d, "n"'] & Y \arrow[d, "q"] \\
X \arrow[r, "n"'] & Y \arrow[r, "p"'] & W
\end{tikzcd}
\qquad
\begin{tikzcd}
K_n \arrow[r, "x_1"] \arrow[d, "x_2"'] & X \arrow[r, "m"] \arrow[d, "n"'] & Y \arrow[d, "q"] \\
X \arrow[r, "n"'] & Y \arrow[r, "p"'] & W
\end{tikzcd}$$

are pullbacks. But then also the following rectangles are pullbacks.

$$\begin{tikzcd}
K_n \arrow[r, "k"'] \arrow[rr, bend left, "m \circ x_2"] \arrow[d, "x_1"'] & K_q \arrow[r, "y_2"'] \arrow[d, "y_1"'] & Y \arrow[d, "q"] \\
X \arrow[r, "m"] \arrow[rr, bend right, "p \circ n"'] & Y \arrow[r, "q"] & W
\end{tikzcd}
\qquad
\begin{tikzcd}
K_n \arrow[r, "k"'] \arrow[rr, bend left, "m \circ x_1"] \arrow[d, "x_2"'] & K_q \arrow[r, "y_1"'] \arrow[d, "y_2"'] & Y \arrow[d, "q"] \\
X \arrow[r, "m"] \arrow[rr, bend right, "p \circ n"'] & Y \arrow[r, "q"] & W
\end{tikzcd}$$

Therefore their left halves, which are the central squares of the original diagrams, are pullbacks, too. In particular this shows that k belongs to $\mathcal{M}$, and thus, it is $\mathcal{N}$-adhesive. We can now consider the following two cubes in which all faces are pullbacks

$$\begin{tikzcd}
& K_n \arrow[rr, "x_1"] \arrow[dl, "k"'] \arrow[dd, "x_2"', near end] & & X \arrow[dl, "m"'] \arrow[dd, "n"] \\
K_q \arrow[rr, "y_1", near start, crossing over] \arrow[dd, "y_2"'] & & Y & \\
& X \arrow[rr, "n", near end] \arrow[dl, "m"'] & & Z \arrow[dl, "p"] \\
Y \arrow[rr, "q"'] & & W \arrow[from=uu, "q"', near start, crossing over] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& K_n \arrow[rr, "x_2"] \arrow[dl, "k"'] \arrow[dd, "x_1"', near end] & & X \arrow[dl, "m"'] \arrow[dd, "n"] \\
K_q \arrow[rr, "y_2", near start, crossing over] \arrow[dd, "y_1"'] & & Y & \\
& X \arrow[rr, "n", near end] \arrow[dl, "m"'] & & Z \arrow[dl, "p"] \\
Y \arrow[rr, "q"'] & & W \arrow[from=uu, "q"', near start, crossing over] &
\end{tikzcd}$$

which prove that the two central squares in the original diagram are also pushouts.

We are left with the last square. We can deduce that it is a pullback applying Lemma 5.1.4 to the rectangle

$$\begin{tikzcd}
X \arrow[r, "\gamma_n"'] \arrow[d, "m"'] \arrow[rr, bend left, "\mathsf{id}_X"] & K_n \arrow[r, "x_1"'] \arrow[d, "k"'] & X \arrow[d, "m"] \\
Y \arrow[r, "\gamma_q"] \arrow[rr, bend right, "\mathsf{id}_X"'] & K_q \arrow[r, "y_1"] & Y
\end{tikzcd}$$

By construction, γ_n is a split mono, thus it is in $\mathcal{N}$. By hypothesis, $m \in \mathcal{M}$ is $\mathcal{N}$-adhesive, and we can build the following diagram in which the inner square is a pushout.

$$\begin{tikzcd}
X \arrow[r, "\gamma_n"] \arrow[d, "m"'] & K_n \arrow[d, "p_1"] \arrow[ddr, bend left, "k"] & \\
Y \arrow[r, "p_2"'] \arrow[drr, bend right, "\gamma_q"'] & E \arrow[dr, dotted, "e"] & \\
 & & K_q
\end{tikzcd}$$

We already know that the outer edges form a pullback square. The arrow γ_q is in $\mathcal{N}$ because it is a split mono, and k is $\mathcal{N}$-adhesive,. Thus, by Proposition 5.2.8, we get a mono $e\colon E \to K_q$ filling the diagram and such that

$$[e] = [k] \vee [\gamma_q]$$

Since γ_q is also in $\mathcal{M}$, e is an $\mathcal{M},\mathcal{N}$-union, and thus, it belongs to $\mathcal{M}$. Now, by construction we have

$$\begin{aligned}
\mathsf{id}_Y \circ m &= m \\
&= m \circ \mathsf{id}_X \\
&= m \circ x_1 \circ \gamma_n
\end{aligned}$$

thus there exists an $h\colon E \to Y$ filling the diagram

$$\begin{tikzcd}
X \arrow[r, "\gamma_n"'] \arrow[d, "m"'] \arrow[rr, bend left, "\mathsf{id}_X"] & K_n \arrow[r, "x_1"'] \arrow[d, "p_1"] & X \arrow[d, "m"] \\
Y \arrow[r, "p_2"] \arrow[rr, bend right, "\mathsf{id}_Y"'] & E \arrow[r, dotted, "h"] & Y
\end{tikzcd}$$

In this diagram the left square and the whole rectangle are pushouts. Thus by Lemma 5.1.6 the right square is a pushout too. Now, $x_1 \in \mathcal{N}$ as it is the pullback of n, and thus, h belongs to $\mathcal{N}$ too. On the other hand we have already proved that in the diagram

$$\begin{tikzcd}
K_1 \arrow[r, "p_1"'] \arrow[rr, bend left, "k"] \arrow[d, "x_1"'] & E \arrow[r, "e"'] \arrow[d, "h"] & K_q \arrow[d, "y_1"] \\
X \arrow[r, "m"] \arrow[rr, bend right, "m"'] & Y \arrow[r, "\mathsf{id}_Y"] & Y
\end{tikzcd}$$

the whole rectangle is a pushout. Hence, using again Lemma 5.1.6, it follows that its right half

$$\begin{tikzcd}
E \arrow[r, "e"] \arrow[d, "h"'] & K_q \arrow[d, "y_1"] \\
Y \arrow[r, "\mathsf{id}_Y"'] & Y
\end{tikzcd}$$

is a pushout too. By hypothesis, e is $\mathcal{N}$-adhesive, and thus, the previous square is also a pullback, showing that e is an isomorphism.

We are left with stability: $n \in \mathcal{N}$ by hypothesis, γ_n is in $\mathcal{N}$ because it is a split mono and x_1 and x_2 belongs to $\mathcal{N}$ as they are pullbacks of n. Since we have proved that m and k are in $\mathcal{M}$ we know that they are $\mathcal{N}$-adhesive and we can conclude. ☐

We are now going to prove that if $\mathcal{M}$ is composed of $\mathcal{N}$-adhesive morphism then three quarters of the Van Kampen condition are satisfied. In order to do so we need the following technical lemma.

Lemma 5.2.12. *Let* $\mathbf{X}$ *be a category with pullbacks and consider the following cube in which the left, back, bottom and top faces are pullbacks.*

$$\begin{tikzcd}
& X' \arrow[rr, "n'"] \arrow[dl, "m'"'] \arrow[dd, "x" near end] & & Z' \arrow[dl, "p'"'] \arrow[dd, "z"] \\
Y' \arrow[rr, "q'" near start, crossing over] \arrow[dd, "y"'] & & W' & \\
& X \arrow[rr, "n" near end] \arrow[dl, "m"] & & Z \arrow[dl, "p"] \\
Y \arrow[rr, "q"'] & & W \arrow[from=uu, crossing over, "w" near start] &
\end{tikzcd}$$

Suppose that p and p' are monos and that the top face is a stable pushout. Then the right face is a pullback.

Proof. Since p is a mono, by Lemma 5.1.4, the rectangle

$$\begin{tikzcd}
Z' \arrow[r, "z"] \arrow[d, "\mathsf{id}_{Z'}"'] & Z \arrow[r, "\mathsf{id}_Z"] \arrow[d, "\mathsf{id}_Z"'] & Z \arrow[d, "p"] \\
Z' \arrow[r, "z"'] & Z \arrow[r, "p"'] & W
\end{tikzcd}$$

is a pullback. Take now the following three diagrams

$$
\begin{tikzcd}
X' \arrow[r, "n'"] \arrow[d, "m'"'] & Z' \arrow[d, "p'"] \\
Y' \arrow[r, "q'"'] & W'
\end{tikzcd}
\qquad
\begin{tikzcd}
X' \arrow[r, "n'"'] \arrow[d, "m'"'] \arrow[rr, bend left, "n\circ x"] & Z' \arrow[r, "z"'] \arrow[d, "p'"'] & Z \arrow[d, "p"] \\
Z' \arrow[r, "q'"] \arrow[rr, bend right, "q\circ y"'] & W' \arrow[r, "w"] & W
\end{tikzcd}
\qquad
\begin{tikzcd}
Z' \arrow[r, "\mathsf{id}_{Z'}"'] \arrow[d, "\mathsf{id}_{Z'}"'] \arrow[rr, bend left, "z"] & Z' \arrow[r, "z"'] \arrow[d, "p'"'] & Z \arrow[d, "p"] \\
Z' \arrow[r, "p'"] \arrow[rr, bend right, "p\circ z"'] & W' \arrow[r, "w"] & W
\end{tikzcd}
$$

By hypothesis the first square is a stable pushout and the left half of the first rectangle is a pullback. Since also the bottom face is a pullback by hypothesis, it follows that the whole first rectangle is a pullback too. By the previous observation, the whole second rectangle is a pullback and, since p' is a mono, its first half is a pullback square. We can then apply Lemma 5.1.8 to get the thesis. $\square$

Corollary 5.2.13. *Let $(\mathcal{M},\mathcal{N})$ be a preadhesive structure on a category $\mathbf{X}$ with pullbacks, and suppose that every arrow in $\mathcal{M}$ is $\mathcal{N}$-adhesive. For every $m \in \mathcal{M}$, $n \in \mathcal{N}$ and cube*

$$
\begin{tikzcd}
& X' \arrow[rr, "n'"] \arrow[dl, "m'"'] \arrow[dd, "x"] & & Z' \arrow[dl, "p'"'] \arrow[dd, "z"] \\
Y' \arrow[rr, "q'"] \arrow[dd, "y"'] & & W' \arrow[dd, "w"] & \\
& X \arrow[rr, "n"] \arrow[dl, "m"] & & Z \arrow[dl, "p"] \\
Y \arrow[rr, "q"'] & & W &
\end{tikzcd}
$$

if the top and bottom faces are pushouts and the left and back ones are pullbacks, then the right face is a pullback.

Proof. $\mathcal{M}$ and $\mathcal{N}$ are closed under pullbacks, thus the top face is an $\mathcal{M},\mathcal{N}$-pushout, and so it is stable because m' is $\mathcal{N}$-adhesive. Since m' and m are $\mathcal{N}$-adhesive, the top and bottom faces are also pullbacks. The arrows p and p' are in $\mathcal{M}$ as they are the pushouts of, respectively, m and m'. Thus they are monomorphisms and the thesis now follows from Lemma 5.2.12. $\square$

We are now ready to prove the main theorem of this section.

Theorem 5.2.14. *Let $(\mathcal{M},\mathcal{N})$ be a preadhesive structure on a category $\mathbf{X}$ with pullbacks and suppose that every split mono is in $\mathcal{M}\cap\mathcal{N}$, $\mathcal{M}\subseteq\mathcal{N}_{\mathsf{a}}$ and $\mathcal{M}$ is closed under $\mathcal{M},\mathcal{N}$-unions. Then $\mathbf{X}$ is $\mathcal{M},\mathcal{N}$-adhesive.*

Proof. Every elements of $\mathcal{M}$ is adhesive. Thus we already know that for any $n \in \mathcal{N}$ and every $\in \mathcal{M}$ a stable pushout square

$$
\begin{tikzcd}
X \arrow[r, "n"] \arrow[d, "m"'] & Z \arrow[d, "p"] \\
Y \arrow[r, "q"'] & W
\end{tikzcd}
$$

exists. Since $\mathbf{X}$ has all pullbacks by hypothesis, all that we have to show is the remaining half of the Van Kampen condition. Take a cube in which the top and bottom faces are pushout and the left and back ones are pullbacks

$$\begin{tikzcd}
& X' \arrow[rr, "n'"] \arrow[dl, "m'"'] \arrow[dd, "x"] & & Z' \arrow[dl, "p'"'] \arrow[dd, "z"] \\
Y' \arrow[rr, "q'", near start] \arrow[dd, "y"'] & & W' \arrow[dd, "w", near start] & \\
& X \arrow[rr, "n", near end] \arrow[dl, "m"] & & Z \arrow[dl, "p"] \\
Y \arrow[rr, "q"'] & & W &
\end{tikzcd}$$

Then m' and n' belong to, respectively, $\mathcal{M}$ and $\mathcal{N}$. Thus the top face is a stable pushout square, which is also a pullback. By Corollary 5.2.13 we already know that the right face is a pullback, let us prove that the other one is a pullback, too.

By Lemma 5.2.11, in the following diagrams all squares are stable pushouts and pullbacks.

$$\begin{tikzcd}
X \arrow[r, "\gamma_n"] \arrow[d, "m"'] & K_n \arrow[r, "x_1"] \arrow[d, "k"'] & X \arrow[r, "n"] \arrow[d, "m"'] & Z \arrow[d, "p"] \\
Y \arrow[r, "\gamma_q"'] & K_q \arrow[r, "y_1"'] & Y \arrow[r, "q"'] & W
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "\gamma_n"] \arrow[d, "m"'] & K_n \arrow[r, "x_2"] \arrow[d, "k"'] & X \arrow[r, "n"] \arrow[d, "m"'] & Z \arrow[d, "p"] \\
Y \arrow[r, "\gamma_q"'] & K_q \arrow[r, "y_2"'] & Y \arrow[r, "q"'] & W
\end{tikzcd}$$

$$\begin{tikzcd}
X' \arrow[r, "\gamma_{n'}"] \arrow[d, "m'"'] & K_{n'} \arrow[r, "x'_1"] \arrow[d, "k'"'] & X' \arrow[r, "n'"] \arrow[d, "m'"'] & Z' \arrow[d, "p'"] \\
Y' \arrow[r, "\gamma_{q'}"'] & K_{q'} \arrow[r, "y'_1"'] & Y \arrow[r, "q'"'] & W'
\end{tikzcd}
\qquad
\begin{tikzcd}
X' \arrow[r, "\gamma_{n'}"] \arrow[d, "m'"'] & K_{n'} \arrow[r, "x'_2"] \arrow[d, "k'"'] & X' \arrow[r, "n'"] \arrow[d, "m'"'] & Z' \arrow[d, "p'"] \\
Y \arrow[r, "\gamma_{q'}"'] & K_{q'} \arrow[r, "y'_2"'] & Y \arrow[r, "q'"'] & W
\end{tikzcd}$$

By Corollary 5.1.5, there exist $t_1\colon K_{n'} \to K_n$ and $t_2\colon K_{q'} \to K_q$ fitting in the following diagrams

$$\begin{tikzcd}
& K_{n'} \arrow[rr, "x'_1"] \arrow[dl, dotted, "t_1"'] \arrow[dd, "x'_2"', near end] & & X' \arrow[dl, "x"'] \arrow[dd, "n'"] \\
K_n \arrow[rr, "x_1", near start] \arrow[dd, "x_2"'] & & X \arrow[dd, "n", near start] & \\
& X' \arrow[rr, "n'", near end] \arrow[dl, "x"'] & & Z' \arrow[dl, "z"] \\
X \arrow[rr, "n"'] & & Z &
\end{tikzcd}
\qquad
\begin{tikzcd}
& K_{q'} \arrow[rr, "y'_1"] \arrow[dl, dotted, "t_2"'] \arrow[dd, "y'_2"', near end] & & Y' \arrow[dl, "y"'] \arrow[dd, "q'"] \\
K_q \arrow[rr, "y_1", near start] \arrow[dd, "y_2"'] & & Y \arrow[dd, "q"', near start] & \\
& Y' \arrow[rr, "q'", near end] \arrow[dl, "y"'] & & W' \arrow[dl, "w"] \\
Y \arrow[rr, "q"'] & & W &
\end{tikzcd}$$

and the left face of the first cube is a pullback square. We compute to obtain

$$
\begin{aligned}
x_1 \circ t_1 \circ \gamma_{n'} &= x \circ x_1' \circ \gamma_{n'} & y_1 \circ t_2 \circ \gamma_{q'} &= y \circ y_1' \circ \gamma_{q'} & y_1 \circ t_2 \circ k' &= y \circ y_1' \circ k' \\
&= x \circ \mathsf{id}_{X'} & &= y \circ \mathsf{id}_{Y'} & &= y \circ m' \circ x_1' \\
&= \mathsf{id}_X \circ x & &= \mathsf{id}_Y \circ y & &= m \circ x \circ x_1' \\
&= x_1 \circ \gamma_n \circ x & &= y_1 \circ \gamma_n \circ y & &= m \circ x_1 \circ t_1 \\
& & & & &= y_1 \circ k \circ t_1
\end{aligned}
$$

$$
\begin{aligned}
x_2 \circ t_1 \circ \gamma_{n'} &= x \circ x_2' \circ \gamma_{n'} & y_2 \circ t_2 \circ \gamma_{q'} &= y \circ y_2' \circ \gamma_{q'} & y_2 \circ t_2 \circ k' &= y \circ y_2' \circ k' \\
&= x \circ \mathsf{id}_{X'} & &= y \circ \mathsf{id}_{Y'} & &= y \circ m' \circ x_2' \\
&= \mathsf{id}_X \circ x & &= \mathsf{id}_Y \circ y & &= m \circ x \circ x_2' \\
&= x_2 \circ \gamma_n \circ x & &= y_2 \circ \gamma_n \circ y & &= m \circ x_2 \circ t_1 \\
& & & & &= y_2 \circ k \circ t_1
\end{aligned}
$$

Therefore the following three squares commute

$$
\begin{tikzcd}
X' \arrow[r, "\gamma_{n'}"] \arrow[d, "x"'] & K_{n'} \arrow[d, "t_1"] \\
X \arrow[r, "\gamma_n"'] & K_n
\end{tikzcd}
\qquad
\begin{tikzcd}
Y' \arrow[r, "\gamma_{q'}"] \arrow[d, "y"'] & K_{q'} \arrow[d, "t_2"] \\
Y \arrow[r, "\gamma_q"'] & K_q
\end{tikzcd}
\qquad
\begin{tikzcd}
K_{n'} \arrow[r, "k'"] \arrow[d, "t_1"'] & K_{q'} \arrow[d, "t_2"] \\
K_n \arrow[r, "k"'] & K_q
\end{tikzcd}
$$

The first one of the squares above is a pullback: this follows applying Lemma 5.1.4 to the rectangle below.

$$
\begin{tikzcd}
X' \arrow[r, "\gamma_{n'}"'] \arrow[d, "x"'] \arrow[rr, bend left, "\mathsf{id}_{X'}"] & K_{n'} \arrow[r, "x_2'"'] \arrow[d, "t_1"'] & X' \arrow[d, "x"] \\
X \arrow[r, "\gamma_n"] \arrow[rr, bend right, "\mathsf{id}_X"'] & K_n \arrow[r, "x_2"] & X
\end{tikzcd}
$$

We can then use these arrows t_1 and t_2 to construct the following cube

$$
\begin{tikzcd}
& X' \arrow[rr, "m'"] \arrow[dl, "\gamma_{n'}"'] \arrow[dd, "x"'] & & Y' \arrow[dl, "\gamma_{q'}"'] \arrow[dd, "y"] \\
K_{n'} \arrow[rr, "k'"] \arrow[dd, "t_1"'] & & K_{q'} \arrow[dd, "t_2"] & \\
& X \arrow[rr, "m"] \arrow[dl, "\gamma_n"'] & & Y \arrow[dl, "\gamma_q"] \\
K_n \arrow[rr, "k"'] & & K_q &
\end{tikzcd}
$$

which has pullbacks as left and back faces and stable pushouts as top and bottom ones. The morphisms γ_q and $\gamma_{q'}$ are split monos, thus by Lemma 5.2.12 the right face is a pullback. Switching γ_n and m we get another cube

$$
\begin{tikzcd}
& X' \arrow[rr, "\gamma_{n'}"] \arrow[dl, "m'"'] \arrow[dd, "x" near end] & & K_{n'} \arrow[dl, "k'"'] \arrow[dd, "t_1"] \\
Y' \arrow[rr, "\gamma_{q'}" near start, crossing over] \arrow[dd, "y"'] & & K_{q'} & \\
& X \arrow[rr, "\gamma_n" near end] \arrow[dl, "m"] & & K_n \arrow[dl, "k"] \\
Y \arrow[rr, "\gamma_q"'] & & K_q \arrow[from=uu, "t_2" near start, crossing over] &
\end{tikzcd}
$$

to which we can apply Corollary 5.2.13 to get again that the right face is a pullback. Now, by Lemma 5.1.4, the following rectangle is a pullback

$$
\begin{tikzcd}
K_{n'} \arrow[r, "x_1'"] \arrow[d, "t_1"'] & X' \arrow[r, "m'"] \arrow[d, "x"] & Y' \arrow[d, "y"] \\
K_n \arrow[r, "x_1"'] & X \arrow[r, "m"'] & Y
\end{tikzcd}
$$

Thus we can apply Lemma 5.1.8 to the diagrams

$$
\begin{tikzcd}
X \arrow[r, "\gamma_n"] \arrow[d, "m"'] & K_n \arrow[d, "k"] \\
Y \arrow[r, "\gamma_q"'] & K_q
\end{tikzcd}
\qquad
\begin{tikzcd}
Y' \arrow[r, "\gamma_{q'}"'] \arrow[d, "y"'] \arrow[rr, bend left, "\mathsf{id}_{Y'}"] & K_{q'} \arrow[r, "y_2'"'] \arrow[d, "t_2"] & Y' \arrow[d, "y"] \\
Y \arrow[r, "\gamma_q"] \arrow[rr, bend right, "\mathsf{id}_Y"'] & K_q \arrow[r, "y_2"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
K_{n'} \arrow[r, "k'"'] \arrow[d, "t_1"'] \arrow[rr, bend left, "m' \circ x_1'"] & K_{q'} \arrow[r, "y_2'"'] \arrow[d, "q'"] & Y' \arrow[d, "y"] \\
K_n \arrow[r, "k"] \arrow[rr, bend right, "m \circ x_1"'] & K_q \arrow[r, "y_2"] & Y
\end{tikzcd}
$$

to deduce that the square below is a pullback, too.

$$
\begin{tikzcd}
K_{q'} \arrow[r, "y_2'"] \arrow[d, "t_2"'] & Y' \arrow[d, "y"] \\
K_q \arrow[r, "y_2"'] & Y
\end{tikzcd}
$$

This in turn also entails that the following rectangle is a pullback.

$$
\begin{tikzcd}
K_{q'} \arrow[r, "t_2"] \arrow[d, "y_2'"'] & K_q \arrow[r, "y_1"] \arrow[d, "y_2"] & Y \arrow[d, "q"] \\
Y' \arrow[r, "y"'] & Y \arrow[r, "q"'] & W
\end{tikzcd}
$$

We can now notice that the diagrams

$$
\begin{tikzcd}
X' \arrow[r, "n'"] \arrow[d, "m'"'] & Z' \arrow[d, "p'"] \\
Y' \arrow[r, "q'"'] & W'
\end{tikzcd}
\qquad
\begin{tikzcd}
K_{q'} \arrow[r, "y_1'"'] \arrow[d, "y_2'"'] \arrow[rr, bend left, "y_1 \circ t_2"] & Y' \arrow[r, "y"'] \arrow[d, "q'"'] & Y \arrow[d, "q"] \\
Y' \arrow[r, "q'"] \arrow[rr, bend right, "q \circ y"'] & W' \arrow[r, "w"] & W
\end{tikzcd}
\qquad
\begin{tikzcd}
X' \arrow[r, "m'"'] \arrow[d, "n'"'] \arrow[rr, bend left, "m \circ x"] & Y' \arrow[r, "y"'] \arrow[d, "q'"'] & Y \arrow[d, "q"] \\
Z' \arrow[r, "p'"] \arrow[rr, bend right, "p \circ z"'] & W' \arrow[r, "w"] & W
\end{tikzcd}
$$

satisfy the hypothesis of Lemma 5.1.8, and this yields the thesis. □

The previous theorem yields at once the following two corollaries

Corollary 5.2.15. *Let* $\mathbf{X}$ *be a category with pullbacks, then*

1. *if* $\mathcal{M}(\mathbf{X}) \subseteq \mathcal{A}(\mathbf{X})_{\mathfrak{a}}$ *then* $\mathbf{X}$ *is adhesive;*
2. *if* $\mathcal{R}(\mathbf{X}) \subseteq \mathcal{A}(\mathbf{X})_{\mathfrak{a}}$ *and it is closed under binary joins then* $\mathbf{X}$ *is quasiadhesive.*

Proof. 1. By Corollary 5.2.7 $(\mathcal{A}(\mathbf{X})_{\mathfrak{a}}, \mathcal{A}(\mathbf{X})_{\mathfrak{a}})$ is a preadhesive structure, which, by Corollary 5.2.4, coincides with $(\mathcal{M}(\mathbf{X}), \mathcal{A}(\mathbf{X}))$. The thesis now follows from Corollary 5.1.27 and Theorem 5.2.14.

2. As before, Corollaries 5.2.4 and 5.2.7 entails that $(\mathcal{R}(\mathbf{X}), \mathcal{A}(\mathbf{X}))$ is a preadhesive structure on $\mathbf{X}$ to which we can apply Theorem 5.2.14 and get the thesis appealing to Corollary 5.1.27. □

Corollary 5.2.16. *Let* $\mathcal{M}$ *be a stable system of monos in a category* $\mathbf{X}$ *with pullbacks. Suppose that* $\mathcal{M}$ *is stable under pushouts, it contains all split monos, it is closed under binary joins and* $\mathcal{M} \subseteq \mathcal{A}(\mathbf{X})_{\mathfrak{a}}$*. Then* $\mathbf{X}$ *is an* $\mathcal{M}$*-adhesive category.*

Proof. This follows at once from Corollary 5.1.26 and Theorem 5.2.14. □

Remark 5.2.17. In Corollaries 5.2.15 and 5.2.16, closure under joins means that, given $m\colon M \to X$, $n\colon N \to X$ in $\mathcal{R}(\mathbf{X})$ or in $\mathcal{M}$, any representative of $[m] \vee [n]$, which exists by virtue of Proposition 5.2.8, is again in $\mathcal{R}(\mathbf{X})$ or in $\mathcal{M}$.

5.2.3 From $\mathcal{M},\mathcal{N}$-adhesivity to $\mathcal{M},\mathcal{N}$-unions

In the previous section we deduced $\mathcal{M},\mathcal{N}$-adhesivity from the closure of $\mathcal{M}$ under some kinds of unions. In this section we will go in the opposite direction.

Definition 5.2.18. Let $f\colon X \to Y$ be an arrow in a category $\mathbf{X}$ such that the pushout square below exists.

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "f"'] & Y \arrow[d, "y_1"] \\
Y \arrow[r, "y_2"'] & Q_f
\end{tikzcd}
$$

The *codiagonal* $\upsilon_f \colon Q_f \to Y$ is the unique arrow fitting in the following diagram.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "f"'] & Y \arrow[d, "y_1"] \arrow[ddr, bend left, "\mathsf{id}_Y"] & \\
Y \arrow[r, "y_2"'] \arrow[drr, bend right, "\mathsf{id}_Y"'] & Q_f \arrow[dr, dotted, "\upsilon_f"] & \\
 & & Y
\end{tikzcd}$$

Given a preadhesive structure $(\mathcal{M},\mathcal{N})$, an $\mathcal{M},\mathcal{N}$*-codiagonal* is the codiagonal of an arrow $n \in \mathcal{M} \cap \mathcal{N}$.

Let us list some useful properties of codiagonals.

Lemma 5.2.19. *Let* $f \colon X \to Y$ *be a morphism in a category* $\mathbf{X}$ *and suppose that* f *admits a codiagonal* $\upsilon_f \colon Q_f \to Y$*, then the following hold true:*

1. υ_f *is the coequalizer of the pair of coprojections* $y_1, y_2 \colon Y \rightrightarrows Q_f$*;*
2. *if a pullback of* y_1 *along* y_2 *exists, then the pair* $y_1, y_2 \colon Y \rightrightarrows Q_f$ *has an equalizer* $e \colon E \to Y$ *and, moreover, the following square is a pullback*

$$\begin{tikzcd}
E \arrow[r, "e"] \arrow[d, "e"'] & X \arrow[d, "y_1"] \\
X \arrow[r, "y_2"'] & Q_f
\end{tikzcd}$$

Proof. 1. Let $z \colon Q_f \to Z$ be such that

$$z \circ y_1 = z \circ y_2$$

Then

$$\begin{aligned} z \circ y_1 \circ \upsilon_f \circ y_1 &= z \circ y_1 \circ \mathsf{id}_Y \\ &= z \circ y_1 \end{aligned} \qquad \begin{aligned} z \circ y_1 \circ \upsilon \circ y_2 &= z \circ y_1 \circ \mathsf{id}_Y \\ &= z \circ y_1 \\ &= z \circ y_2 \end{aligned}$$

and we can consider the following commutative diagram

$$\begin{tikzcd}
Y \arrow[r, shift left, "y_1"] \arrow[r, shift right, "y_2"'] & Q_f \arrow[r, "z"] \arrow[d, "\upsilon_f"'] & Z \\
 & Y \arrow[r, "y_1"'] & Q_f \arrow[u, "z"']
\end{tikzcd}$$

Uniqueness follows from the fact that υ_f is a split epi.

2. First of all we can notice that in every square, not necessarily a pullback one, as the one in the diagram below, the existence of the codiagonal implies $p_1 = p_2$

$$\begin{tikzcd} P \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, "y_1"] \arrow[ddr, bend left, "\mathsf{id}_X"] & \\ X \arrow[r, "y_2"'] \arrow[drr, bend right, "\mathsf{id}_X"'] & Q_f \arrow[dr, "v_f"] & \\ & & X \end{tikzcd}$$

By hypothesis, y_1 has a pullback along y_2 as in the following diagram

$$\begin{tikzcd} E \arrow[r, "e"] \arrow[d, "e"'] & X \arrow[d, "y_1"] \\ X \arrow[r, "y_2"'] & Q_f \end{tikzcd}$$

Thus, if $z\colon Z \to X$ is an arrows such that

$$y_1 \circ z = y_2 \circ z$$

then the universal property of pullback yields a unique $g\colon Z \to E$ such that $z = e \circ g$. $\square$

The following lemma is a generalization of [52, Prop. 4.4].

Lemma 5.2.20. *Let $(\mathcal{M},\mathcal{N})$ be a preadhesive structure on a category $\mathbf{X}$ with pullbacks and $u\colon U \to X$ an $\mathcal{M},\mathcal{N}$-union. Suppose that $\mathcal{M} \subseteq \mathcal{N}_{\mathfrak{a}}$, that $\mathcal{M} \cap \mathcal{N}$ contains all split monomorphisms and that $\mathcal{N}$ contains all $\mathcal{M},\mathcal{N}$-codiagonals. Then:*

1. *u admits pushouts along itself (i.e. it has a* cokernel pair*);*
2. *there exists an epi $e_u\colon M \to E_u$ and an element $m_u\colon E_u \to X$ of $\mathcal{M} \cap \mathcal{N}$ such that $u = m_u \circ e_u$.*

Remark 5.2.21. Notice that, if $\mathcal{M} \subseteq \mathcal{N}_{\mathfrak{a}}$, then for every $n \in \mathcal{M} \cap \mathcal{N}$ a pushout square

$$\begin{tikzcd} N \arrow[r, "n"] \arrow[d, "n"'] & X \arrow[d, "n_1"] \\ X \arrow[r, "n_2"'] & Q_n \end{tikzcd}$$

of n along itself exists, and thus there also exists the codiagonal v_n.

Proof. 1. Let $m\colon M \to X$ in $\mathcal{M}$ and $n\colon N \to X$ in $\mathcal{M} \cap \mathcal{N}$ be arrows such that

$$[u] = [m] \vee [n]$$

By Proposition 5.2.8, we can consider the following diagram in which the outer edges form a pullback and the inner square is a pushout.

$$\begin{tikzcd}
P \arrow[r, "p_1"] \arrow[d, "p_2"'] & M \arrow[d, "u_2"] \arrow[ddr, bend left, "m"] & \\
N \arrow[r, "u_1"'] \arrow[drr, bend right, "n"'] & U \arrow[dr, "u"] & \\
& & X
\end{tikzcd}$$

Pulling back m along v_n, we obtain a pullback square

$$\begin{tikzcd}
T \arrow[r, "t_1"] \arrow[d, "t_2"'] & M \arrow[d, "m"] \\
Q_n \arrow[r, "v_n"'] & X
\end{tikzcd}$$

Now, we have identities

$$\begin{aligned}
m \circ \mathsf{id}_M &= m & \qquad m \circ \mathsf{id}_M &= m \\
&= \mathsf{id}_X \circ m & &= \mathsf{id}_X \circ m \\
&= v_n \circ n_1 \circ m & &= v_n \circ n_2 \circ m
\end{aligned}$$

and thus there exist $l_1, l_2 \colon M \rightrightarrows T$ as in the following diagram

$$\begin{tikzcd}
M \arrow[d, "m"'] \arrow[dr, dotted, "l_1"] \arrow[drr, bend left, "\mathsf{id}_M"] & & \\
X \arrow[dr, bend right, "n_1"'] & T \arrow[r, "t_1"] \arrow[d, "t_2"'] & M \arrow[d, "m"] \\
& Q_n \arrow[r, "v_n"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[d, "m"'] \arrow[dr, dotted, "l_2"] \arrow[drr, bend left, "\mathsf{id}_M"] & & \\
X \arrow[dr, bend right, "n_2"'] & T \arrow[r, "t_1"] \arrow[d, "t_2"'] & M \arrow[d, "m"] \\
& Q_n \arrow[r, "v_n"'] & X
\end{tikzcd}$$

By Lemma 5.1.4, the following are pullback squares

$$\begin{tikzcd}
M \arrow[r, "l_1"] \arrow[d, "m"'] & T \arrow[d, "t_2"] \\
X \arrow[r, "n_1"'] & Q_n
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[r, "l_2"] \arrow[d, "m"'] & T \arrow[d, "t_2"] \\
X \arrow[r, "n_2"'] & Q_n
\end{tikzcd}$$

Therefore, since n is $\mathcal{N}$-adhesive, the top face of the following cube is a pushout.

$$\begin{tikzcd}
& P \arrow[rr, "p_1"] \arrow[dl, "p_1"'] \arrow[dd, "p_2"' near start] & & M \arrow[dl, "l_1"'] \arrow[dd, "m"] \\
M \arrow[rr, "l_2" near start, crossing over] \arrow[dd, "m"'] & & T & \\
& N \arrow[rr, "n" near end] \arrow[dl, "n"'] & & X \arrow[dl, "n_1"] \\
X \arrow[rr, "n_2"'] & & Q_n \arrow[from=uu, "t_2" near start, crossing over] &
\end{tikzcd}$$

Now, t_1 is the pullback of an $\mathcal{M},\mathcal{N}$-codiagonal. Thus, it is in $\mathcal{N}$, while t_2 is in $\mathcal{N}_{\mathfrak{a}}$ since it is the pullback of m. Therefore the pushout square below exists.

$$\begin{tikzcd}
T \arrow[r, "t_1"] \arrow[d, "t_2"'] & M \arrow[d, "q_1"] \\
Q_n \arrow[r, "q_2"'] & Q
\end{tikzcd}$$

Suppose now that the solid part of the next diagram is given

$$\begin{tikzcd}
U \arrow[rr, "u"] \arrow[dd, "u"'] & & X \arrow[dl, "z_1"'] \arrow[d, "n_1"] \\
& Z & Q_n \arrow[d, "q_2"] \\
X \arrow[ur, "z_2"] \arrow[r, "n_2"'] & Q_n \arrow[r, "q_2"'] & Q \arrow[ul, dotted, "z"']
\end{tikzcd}$$

Precomposing with u_1 and u_2 we get the following identities

$$\begin{aligned}
z_1 \circ m &= z_1 \circ u \circ u_2 & \qquad z_1 \circ n &= z_1 \circ u \circ u_1 \\
&= z_2 \circ u \circ u_2 & &= z_2 \circ u \circ u_1 \\
&= z_2 \circ m & &= z_2 \circ n
\end{aligned}$$

The second chain of the equalities above allows us to deduce the existence of the dotted $w\colon Q_n \to Z$.

$$\begin{tikzcd}
N \arrow[r, "n"] \arrow[d, "n"'] & X \arrow[d, "n_1"] \arrow[ddr, bend left, "z_1"] & \\
X \arrow[r, "n_2"'] \arrow[drr, bend right, "z_2"'] & Q_n \arrow[dr, dotted, "w"] & \\
& & Z
\end{tikzcd}$$

We compute to obtain

$$\begin{aligned} w \circ t_2 \circ l_2 &= w \circ n_2 \circ m \\ &= z_2 \circ m \\ &= z_1 \circ m \\ &= w \circ n_1 \circ m \\ &= w \circ t_2 \circ l_1 \end{aligned}$$

By construction and by our previous observations, t_1 is a codiagonal for p_1. Thus the first point of Lemma 5.2.19 implies the existence of a unique $k\colon M \to Z$ making the following diagram commutative

$$\begin{tikzcd} T \arrow[r, "t_1"] \arrow[d, "t_2"'] & M \arrow[d, "q_1"] \arrow[ddr, bend left, "k"] & \\ Q_n \arrow[r, "q_2"'] \arrow[drr, bend right, "w"'] & Q \arrow[dr, dotted, "z"] & \\ & & Z \end{tikzcd}$$

which, in turn, implies the existence of the dotted z. Computing further we have

$$\begin{aligned} z_1 &= w \circ n_1 & z_2 &= w \circ n_2 \\ &= z \circ q_2 \circ n_1 & &= z \circ q_2 \circ n_2 \end{aligned}$$

Moreover, if $z'\colon Q \to Z$ is such that

$$z_1 = z' \circ q_2 \circ n_1 \qquad z_2 = z' \circ q_2 \circ n_2$$

then we also have

$$\begin{aligned} z' \circ q_2 \circ n_1 &= z_1 & z' \circ q_2 \circ n_2 &= z_2 \\ &= w \circ n_1 & &= w \circ n_2 \end{aligned}$$

which shows that $w = z' \circ q_2$. On the other hand

$$\begin{aligned} z' \circ q_1 \circ t_1 &= z' \circ q_2 \circ t_2 \\ &= w \circ t_2 \end{aligned}$$

and so we also have that $z' \circ q_1 = k$, allowing us to conclude that $z = z'$. We can now deduce that the following square is a pushout

$$\begin{tikzcd} U \arrow[r, "u"] \arrow[d, "u"'] & X \arrow[d, "q_2 \circ n_1"] \\ X \arrow[r, "q_2 \circ n_2"'] & Q \end{tikzcd}$$

2. By the previous point u has pushout along itself. Therefore there exists a codiagonal $v_u\colon Q \to U$. In particular, $q_2 \circ n_1$ and $q_2 \circ n_2$ are split monos and thus elements of $\mathcal{M} \cap \mathcal{N}$. By the second point

of Lemma 5.2.19, they have an equalizer $m_u \colon E_u \to X$ which, since $\mathcal{M}$ and $\mathcal{N}$ are stable under pullback, is also an element of $\mathcal{M} \cap \mathcal{N}$. Since, by construction we have

$$q_2 \circ n_1 \circ u = q_2 \circ n_2 \circ u$$

we also get an arrow $e_u \colon U \to E_u$ such that $u = m_u \circ e_u$. To show that this arrow is epi, start with the equalities

$$\begin{aligned} m &= u \circ u_2 & \qquad n &= u \circ u_2 \\ &= m_u \circ e_u \circ u_2 & &= m_u \circ e_u \circ u_1 \end{aligned}$$

Since $\mathcal{M}$ and $\mathcal{N}$ are closed under decomposition and $\mathcal{M}$-decomposition we can deduce that $e_u \circ u_2$ belongs to $\mathcal{M}$ and that $e_u \circ u_1$ is an element of $\mathcal{M} \cap \mathcal{N}$.

Now let $b \colon B \to E_u$ be another mono such that

$$b \circ b_1 = e_u \circ u_1 \qquad b \circ b_2 = e_u \circ u_2$$

for some $b_1 \colon N \to B$ and $b_2 \colon M \to B$. Then

$$\begin{aligned} b \circ b_1 \circ p_2 &= e_u \circ u_1 \circ p_2 \\ &= e_u \circ u_2 \circ p_1 \\ &= b \circ b_2 \circ p_1 \end{aligned}$$

which, since b is a mono, entails

$$b_1 \circ p_2 = b_2 \circ p_1$$

Thus there exists $\hat{b} \colon U \to B$ such that

$$b_1 = \hat{b} \circ u_1 \qquad b_2 = \hat{b} \circ u_2$$

By computing further we get

$$\begin{aligned} b \circ \hat{b} \circ u_1 &= b \circ b_1 & \qquad b \circ \hat{b} \circ u_2 &= b \circ b_2 \\ &= e_u \circ u_1 & &= e_u \circ u_2 \end{aligned}$$

which shows that $[e_u] \leq [b]$, implying that e_u is a union of $e_u \circ u_2$ and $e_u \circ u_1$. By the previous point and point 2 of Lemma 5.2.19, there exist a diagram in which the outer edges form a pushout, the inner square is a pullback and c is the equalizer of c_1 and c_2.

$$\begin{tikzcd}
U \arrow[r, dotted, "e"'] \arrow[rr, bend left, "e_u"] \arrow[dr, bend right, "e_u"'] & C \arrow[r, "c"'] \arrow[d, "c"'] & E_u \arrow[d, "c_2"] \\
& E_u \arrow[r, "c_1"'] & \hat{Q}
\end{tikzcd}$$

The existence of $e \colon U \to C$ can then be inferred from the universal property of pullbacks. If we show that c is invertible, then we are done. Notice that c_1 and c_2 are in $\mathcal{M} \cap \mathcal{N}$ since they are split monos. Thus $c \in \mathcal{M} \cap \mathcal{N}$ too. Suppose that the solid part of the following diagram is given.

$$\begin{tikzcd}
C \arrow[r, "c"] \arrow[d, "c"'] & E_u \arrow[r, "m_u"] & X \arrow[d, "n_1"] \arrow[dl, "z_1"'] \\
E_u \arrow[d, "c"'] & Z & Q_n \arrow[d, "q_2"] \\
X \arrow[r, "n_2"'] \arrow[ur, "z_2"] & Q_n \arrow[r, "q_2"'] & Q \arrow[ul, dotted, "z"]
\end{tikzcd}$$

Then we have

$$\begin{aligned}
z_1 \circ u &= z_1 \circ m_u \circ e_u \\
&= z_1 \circ m_u \circ c \circ e \\
&= z_2 \circ m_u \circ c \circ e \\
&= z_2 \circ m_u \circ e_u \\
&= z_2 \circ u
\end{aligned}$$

and thus there exists $z\colon Q \to Z$ such that

$$z_1 = z \circ q_2 \circ n_1 \qquad z_2 = z \circ q_2 \circ n_2$$

Uniqueness of such a z follows at once since $q_2 \circ n_1$ and $q_2 \circ n_2$ are the coprojections of a pushout. Thus we can conclude that the square below is a pushout.

$$\begin{tikzcd}
U \arrow[r, "m_u \circ c"] \arrow[d, "m_u \circ c"'] & X \arrow[d, "q_2 \circ n_1"] \\
X \arrow[r, "q_2 \circ n_2"'] & Q
\end{tikzcd}$$

Now, $\mathcal{M}$ and $\mathcal{N}$ are closed under composition. Thus $m_u \circ c$ is in $\mathcal{M} \cap \mathcal{N}$ and, since $\mathcal{M} \subseteq \mathcal{N}_{\mathsf{a}}$, it follows from the third point of Proposition 5.2.3 that $m_u \circ c$ is a regular mono. The dual of Proposition 2.1.50 thus entails that $m_u \circ c$ is the equalizer of $q_2 \circ n_1$ and $q_2 \circ n_2$, and therefore c must be an isomorphism. □

We are now ready to prove the main theorem of this section (see [67, Thm. 19]).

Theorem 5.2.22. *Let* $\mathbf{X}$ *be an* $\mathcal{M},\mathcal{N}$*-adhesive category with pullbacks. If* $\mathcal{M} \cap \mathcal{N}$ *contains all split monomorphisms and* $\mathcal{N}$ *contains all* $\mathcal{M},\mathcal{N}$*-codiagonals, then* $\mathcal{M}$ *is closed under* $\mathcal{M},\mathcal{N}$*-unions.*

Proof. Let $u\colon U \to X$ be the $\mathcal{M},\mathcal{N}$-union of $m\colon M \to X$ in $\mathcal{M}$ and $n\colon N \to X$ in $\mathcal{M} \cap \mathcal{N}$. By Example 5.2.2 and Proposition 5.2.8, we know that these arrows fit in a diagram

$$\begin{tikzcd}
P \arrow[r, "p_1"] \arrow[d, "p_2"'] & M \arrow[d, "u_2"] \arrow[ddr, bend left, "m"] & \\
N \arrow[r, "u_1"'] \arrow[drr, bend right, "n"'] & U \arrow[dr, "u"] & \\
 & & X
\end{tikzcd}$$

in which the outer edges form a pullback and the inner square is a pushout. Notice that $p_2 \in \mathcal{M}$ and $p_1 \in \mathcal{N}$. Thus, by Proposition 5.1.7, the inner square is also a pullback. By Lemma 5.2.20, we also know that $u = m_u \circ e_u$ for some epi $e_u \colon Y \to E_u$ and $m_u \colon E_u \to X$ in $\mathcal{M} \cap \mathcal{N}$. As we have noticed before, the decomposition properties of $\mathcal{M}$ and $\mathcal{N}$ imply that $e_u \circ u_2 \in \mathcal{M}$ and $e_u \circ u_1 \in \mathcal{M} \cap \mathcal{N}$. Our strategy to prove the theorem consists in showing that e_u is an isomorphism.

First of all notice that e_u is a mono because $u = m_u \circ e_u$. Thus in the following diagram every square is a pullback and, applying Lemma 5.1.4, we can deduce that the composite square is a pullback too.

$$\begin{tikzcd}
P \arrow[r, "p_1"] \arrow[d, "p_2"'] & M \arrow[r, "\mathsf{id}_M"] \arrow[d, "u_2"'] & M \arrow[d, "u_2"] \\
N \arrow[r, "u_1"'] \arrow[d, "\mathsf{id}_U"'] & U \arrow[r, "\mathsf{id}_U"'] \arrow[d, "\mathsf{id}_U"'] & U \arrow[d, "e_u"] \\
N \arrow[r, "u_2"'] & U \arrow[r, "e_u"'] & E_u
\end{tikzcd}$$

.

Next, since the arrow n is in $\mathcal{M}$, p_1 is in $\mathcal{M}$ as it is its pullback and $u_1 \in \mathcal{M}$ since it is the pushout of p_1. We can then build the following two pushout squares, which, by Proposition 5.1.16, are also pullbacks.

$$\begin{tikzcd}
N \arrow[r, "e_u \circ u_1"] \arrow[d, "e_u \circ u_1"'] & E_u \arrow[d, "e_1"] \\
E_u \arrow[r, "e_2"'] & Q_{e_u \circ u_1}
\end{tikzcd}
\qquad
\begin{tikzcd}
N \arrow[r, "e_u \circ u_1"] \arrow[d, "u_1"'] & E_u \arrow[d, "a_1"] \\
U \arrow[r, "a_2"'] & A
\end{tikzcd}$$

Notice that he solid part of the following diagram is commutative. Thus the dotted arrow a exists and, by Lemma 5.1.6, the bottom rectangle is a pushout.

$$\begin{tikzcd}
N \arrow[r, "u_1"] \arrow[d, "u_1"'] & U \arrow[r, "e_u"] & E_u \arrow[d, "a_1"'] \arrow[dd, bend left, "e_1"] \\
U \arrow[rr, "a_2"] \arrow[d, "e_u"'] & & A \arrow[d, dotted, "a"'] \\
E_u \arrow[rr, "e_2"'] & & Q_{e_u \circ u_1}
\end{tikzcd}$$

Moreover, since $u_1 \in \mathcal{M}$ and $e_u \circ u_1$ is in $\mathcal{N}$, the upper half of the square above is also a pullback.

Now, e_2 is the pushout of $e_u \circ u_1$. Thus it is in $\mathcal{M}$, and so it is a mono. This, together with Lemma 5.1.4, entails that the following rectangle is a pullback.

$$\begin{tikzcd}
U \arrow[r, "e_u"] \arrow[d, "\mathsf{id}_U"'] & E_u \arrow[r, "\mathsf{id}_{E_u}"] \arrow[d, "\mathsf{id}_{E_u}"'] & E_u \arrow[d, "e_2"] \\
U \arrow[r, "e_u"'] & E_u \arrow[r, "e_2"'] & Q_{e_u \circ u_1}
\end{tikzcd}$$

The arrow a_2 is in $\mathcal{M}$ as it is the pushout of $e_u \circ u_1$. Thus we can apply Lemma 5.1.8 to the diagrams

$$\begin{tikzcd}
N \arrow[r, "e_u \circ u_1"] \arrow[d, "u_1"'] & E_u \arrow[d, "a_1"] \\
U \arrow[r, "a_2"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
N \arrow[r, "u_1"] \arrow[d, "e_u \circ u_1"'] & U \arrow[r, "e_u"] \arrow[d, "a_2"'] & E_u \arrow[d, "e_2"] \\
E_u \arrow[r, "a_1"'] & A \arrow[r, "a"'] & Q_{e_u \circ u_1}
\end{tikzcd}
\qquad
\begin{tikzcd}
U \arrow[r, "\mathsf{id}_U"] \arrow[d, "\mathsf{id}_U"'] & U \arrow[r, "e_u"] \arrow[d, "a_2"'] & E_u \arrow[d, "e_2"] \\
U \arrow[r, "a_2"] \arrow[rr, bend right, "e_2 \circ e_u"'] & A \arrow[r, "a"] & Q_{e_u \circ u_1}
\end{tikzcd}$$

to get that also the following square is a pullback.

$$\begin{tikzcd}
U \arrow[r, "a_2"] \arrow[d, "e_u"'] & A \arrow[d, "a"] \\
E_u \arrow[r, "e_2"'] & Q_{e_u \circ u_1}
\end{tikzcd}$$

On the other hand, the arrow $p_1 \colon P \to M$ is in $\mathcal{M} \cap \mathcal{N}$ as it is the pullback of n. Thus we can consider the following pushout of p_1 along itself.

$$\begin{tikzcd}
P \arrow[r, "p_1"] \arrow[d, "p_1"'] & M \arrow[d, "m_1"] \\
M \arrow[r, "m_2"'] & Q_{p_1}
\end{tikzcd}$$

We can then construct the solid part of the rightmost rectangle in the diagram below, inducing the dotted $b \colon Q_{p_1} \to A$. Notice that the first rectangle is a pushout by Lemma 5.1.6 so that the right half of the second diagram also is a pushout, again because of Lemma 5.1.6, and since b belongs to $\mathcal{M}$.

$$\begin{tikzcd}
P \arrow[r, "p_2"] \arrow[d, "p_1"'] & N \arrow[r, "u_1"] \arrow[d, "u_1"'] & U \arrow[r, "e_u"] & E_u \arrow[d, "a_1"] \\
M \arrow[r, "u_2"'] & U \arrow[rr, "a_2"'] & & A
\end{tikzcd}
\qquad
\begin{tikzcd}
P \arrow[r, "p_1"'] \arrow[d, "p_1"'] \arrow[rrr, bend left, "e_u \circ u_1 \circ p_2"] & M \arrow[r, "u_2"'] \arrow[d, "m_1"'] & U \arrow[r, "e_u"'] & E_u \arrow[d, "a_1"] \\
M \arrow[r, "m_2"] \arrow[rrr, bend right, "a_2 \circ u_2"'] & Q_{p_1} \arrow[rr, dotted, "b"] & & A
\end{tikzcd}$$

We can compose with the codiagonal $\upsilon_{p_1} \colon Q_{p_1} \to M$ to obtain the solid diagram

$$\begin{tikzcd}
M \arrow[r, "u_2"] \arrow[d, "m_1"] \arrow[dd, bend right, "\mathsf{id}_M"'] & U \arrow[r, "e_u"] & E_u \arrow[d, "a_1"'] \arrow[dd, bend left, "\mathsf{id}_{E_u}"] \\
Q_{p_1} \arrow[rr, "b"] \arrow[d, "\upsilon_{p_1}"] & & A \arrow[d, dotted, "r"'] \\
M \arrow[r, "u_2"'] & U \arrow[r, "e_u"'] & E_u
\end{tikzcd}$$

Since the upper half of the square above is a pushout, the dotted $r\colon A \to Q_{e_u \circ u_1}$ exists. Moreover, since the outer edges make a pushout square, the lower half is a pushout too, by Lemma 5.1.6. The codiagonal v_{p_1} belongs to $\mathcal{N}$, therefore Proposition 5.1.16 allows us to conclude that the bottom rectangle of the previous diagram is also a pullback.

We can now notice that for every $z_1\colon Z \to M$ and $z_2\colon Z \to E_u$ such that

$$m \circ z_1 = m_u \circ z_2$$

we have the following chain of equalities

$$\begin{aligned} m_u \circ e_u \circ u_2 \circ z_1 &= u \circ u_2 \circ z_1 \\ &= m \circ z_1 \\ &= m_u \circ z_2 \end{aligned}$$

which, since m_u is mono, entails

$$z_2 = e_u \circ u_2 \circ z_1$$

This, in turn, can be rephrased by saying that the square below is a pullback

$$\begin{tikzcd} M \arrow[r, "\mathsf{id}_M"] \arrow[d, "e_u \circ u_2"'] & M \arrow[d, "m"] \\ E_u \arrow[r, "m_u"'] & X \end{tikzcd}$$

In particular, we can now apply Lemma 5.1.8 to the following $\mathcal{M},\mathcal{N}$-pushout square

$$\begin{tikzcd} N \arrow[r, "e_u \circ u_1"] \arrow[d, "e_u \circ u_1"'] & E_u \arrow[d, "e_1"] \\ E_u \arrow[r, "e_2"'] & Q_{e_u \circ u_1} \end{tikzcd}$$

and to the pullback rectangles

$$\begin{tikzcd} M \arrow[r, "m_2"'] \arrow[rrr, bend left, "\mathsf{id}_M"] \arrow[d, "e_u \circ u_2"'] & Q_{p_1} \arrow[rr, "v_{p_1}"'] \arrow[d, "a \circ b"'] & & M \arrow[d, "m"] \\ E_u \arrow[r, "e_2"] \arrow[rrr, bend right, "m_u"'] & Q_{e_u \circ u_1} \arrow[r, "v_{e_u \circ u_1}"] & E_u \arrow[r, "m_u"] & X \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "m_1"'] \arrow[rrr, bend left, "\mathsf{id}_M"] \arrow[d, "e_u \circ u_2"'] & Q_{p_1} \arrow[rr, "v_{p_1}"'] \arrow[d, "a \circ b"'] & & M \arrow[d, "m"] \\ E_u \arrow[r, "e_1"] \arrow[rrr, bend right, "m_u"'] & Q_{e_u \circ u_1} \arrow[r, "v_{e_u \circ u_1}"] & E_u \arrow[r, "m_u"] & X \end{tikzcd}$$

to show that the outer rectangle in the diagram below is a pullback, so that, in particular, $a \circ b \in \mathcal{M}$. We can also apply Lemma 5.1.4 to deduce that the left half of the rectangle is a pullback, too.

$$\begin{tikzcd} Q_{p_1} \arrow[r, "v_{p_1}"] \arrow[d, "a \circ b"'] & M \arrow[r, "\mathsf{id}_M"] \arrow[d, "e_u \circ u_2"'] & M \arrow[d, "m"] \\ Q_{e_u \circ u_1} \arrow[r, "v_{e_u \circ u_1}"'] & E_u \arrow[r, "m_u"'] & X \end{tikzcd}$$

We compute to obtain

$$\begin{aligned} \upsilon_{e_u \circ u_1} \circ a \circ b &= e_u \circ u_2 \circ \upsilon_{p_1} \\ &= r \circ b \end{aligned} \qquad \begin{aligned} \upsilon_{e_u \circ u_1} \circ a \circ a_1 &= \upsilon_{e_u \circ u_1} \circ e_1 \\ &= \mathsf{id}_{E_u} \\ &= r \circ a_1 \end{aligned}$$

Therefore $r = \upsilon_{e_u \circ u_1} \circ a$. We can then apply Lemma 5.1.4 to the following rectangle, showing that its left half is a pullback

$$\begin{tikzcd}
Q_{p_1} \arrow[r, "\mathsf{id}_{Q_{p_1}}"] \arrow[d, "b"'] & Q_{p_1} \arrow[r, "\upsilon_{p_1}"] \arrow[d, "a \circ b"'] & M \arrow[d, "e_u \circ u_2"] \\
A \arrow[r, "a"] \arrow[rr, bend right, "r"'] & Q_{e_u \circ u_1} \arrow[r, "\upsilon_{e_u \circ u_1}"] & E_u
\end{tikzcd}$$

Suppose now that the solid part of the diagram below is given

$$\begin{tikzcd}
Q_{p_1} \arrow[r, "b"] \arrow[d, "\upsilon_{p_1}"'] & A \arrow[r, "a"] & Q_{e_u \circ u_1} \arrow[d, "\upsilon_{e_u \circ u_1}"'] \arrow[ddr, bend left, "z_1"] & \\
M \arrow[r, "u_2"'] \arrow[drrr, bend right, "z_2"'] & U \arrow[r, "e_u"'] & E_u \arrow[dr, dotted, "z"] & \\
& & & Z
\end{tikzcd}$$

We want to show that the inner rectangle is a pushout. Uniqueness of the dotted $z\colon E_u \to Z$ is guaranteed by the fact that $\upsilon_{e_u \circ u_1}$ is an epimorphism. So it is enough to construct an arrow fitting in the diagram.

First of all we can notice that

$$z_1 \circ e_1 \circ e_u \circ u_1 = z_1 \circ e_2 \circ e_u \circ u_1$$

while we also have

$$\begin{aligned} z_1 \circ e_1 \circ e_u \circ u_2 &= z_1 \circ a \circ a_1 \circ e_u \circ u_2 \\ &= z_1 \circ a \circ b \circ m_1 \\ &= z_2 \circ \upsilon_{p_1} \circ m_1 \\ &= z_2 \circ \mathsf{id}_M \\ &= z_2 \circ \upsilon_{p_1} \circ m_2 \\ &= z_1 \circ a \circ b \circ m_2 \\ &= z_1 \circ a \circ a_2 \circ u_2 \\ &= z_1 \circ e_2 \circ e_u \circ u_2 \end{aligned}$$

which implies that

$$z_1 \circ e_1 \circ e_u = z_1 \circ e_2 \circ e_u$$

which, since e_u is an epimorphism, allows us to conclude that

$$z_1 \circ e_1 = z_1 \circ e_2$$

So equipped, we can now compute:

$$\begin{aligned} z_1 \circ e_1 \circ v_{e_u \circ u_1} \circ e_1 &= z_1 \circ e_1 \circ \mathsf{id}_{E_u} \\ &= z_1 \circ e_1 \end{aligned} \qquad \begin{aligned} z_1 \circ e_1 \circ v_{e_u \circ u_1} \circ e_2 &= z_1 \circ e_1 \circ \mathsf{id}_{E_u} \\ &= z_1 \circ e_1 \\ &= z_1 \circ e_2 \end{aligned}$$

showing

$$z_1 = z_1 \circ e_1 \circ v_{e_u \circ u_1}$$

Moreover, computing again we obtain

$$\begin{aligned} z_2 \circ v_{p_1} &= z_1 \circ a \circ b \\ &= z_1 \circ \mathsf{id}_{E_u} \circ a \circ b \\ &= z_1 \circ e_1 \circ v_{e_u \circ u_1} \circ a \circ b \\ &= z_1 \circ e_1 \circ e_u \circ u_2 \circ v_{p_1} \end{aligned}$$

and v_{p_1} is an epimorphism, thus

$$z_2 = z_1 \circ e_1 \circ e_u \circ u_2$$

Summing up, $z_1 \circ e_1$ fills our original diagram, thus its inner rectangle is indeed a pushout.

We are now ready to collect all our arrows in the following cube

$$\begin{tikzcd}
 & & Q_{p_1} \arrow[rrr, "v_{p_1}"] \arrow[ddll, "b"'] \arrow[ddd, "\mathsf{id}_{Q_{p_1}}"] & & & M \arrow[dl, "u_2"] \arrow[ddd, "\mathsf{id}_M"] \\
 & & & & U \arrow[dl, "e_u"] \arrow[ddd, "\mathsf{id}_U"] & \\
A \arrow[rrr, "r"] \arrow[ddd, "a"'] & & & E_u \arrow[ddd, "\mathsf{id}_{E_u}"] & & \\
 & & Q_{p_1} \arrow[rrr, "v_{p_1}"] \arrow[dl, "b"'] & & & M \arrow[dl, "u_2"] \\
 & A \arrow[dl, "a"'] & & & U \arrow[dl, "e_u"] & \\
Q_{e_u \circ u_1} \arrow[rrr, "v_{e_u \circ u_1}"'] & & & E_u & &
\end{tikzcd}$$

This cube has an $\mathcal{M},\mathcal{N}$-pushout as top and bottom face and all faces beside the frontal one are pullbacks, hence, by $\mathcal{M},\mathcal{N}$-adhesivity it follows that also this last face is a pullback. By Lemma 5.1.4 the rectangle

$$\begin{tikzcd}
U \arrow[r, "a_2"] \arrow[d, "e_u"'] & A \arrow[r, "r"] \arrow[d, "a"] & E_u \arrow[d, "\mathsf{id}_{E_u}"] \\
E_u \arrow[r, "e_2"'] & Q_{e_u \circ u_1} \arrow[r, "v_{e_u \circ u_1}"'] & E_u
\end{tikzcd}$$

is a pullback. Thus e_u is an isomorphism as it is the pullback of id_{E_u}. □

Corollary 5.2.23. *Let* $\mathbf{X}$ *be a category with pullbacks and* $\mathcal{M}$ *a stables system of monos on it. If* $\mathbf{X}$ *is* $\mathcal{M}$*-adhesive, then for every object* X *and every* $[m]$ *and* $[n]$ *in* $\mathcal{M}$-$\mathsf{Sub}(X)$*, their supremum in* $(\mathsf{Sub}(X), \leq)$ *exists and it belongs to* $\mathcal{M}$-$\mathsf{Sub}(X)$.

Combining Theorem 5.2.14 with Theorem 5.2.22 we obtain also the following results.

Corollary 5.2.24. *Let $(\mathcal{M},\mathcal{N})$ be a preadhesive structure on a category $\mathbf{X}$ with pullbacks. If $\mathcal{M}\cap\mathcal{N}$ contains every split mono and every $\mathcal{M},\mathcal{N}$-codiagonal is in $\mathcal{N}$, then the following are equivalent:*

1. *$\mathcal{M}\subseteq\mathcal{N}_{\mathsf{a}}$ and $\mathcal{M}$ is closed under $\mathcal{M},\mathcal{N}$-unions;*
2. *$\mathbf{X}$ is $\mathcal{M},\mathcal{N}$-adhesive.*

Finally Proposition 5.1.21 and Corollaries 5.2.16 and 5.1.26 yield the result below.

Corollary 5.2.25. *Let $\mathcal{M}$ be a stable system of monos on a category $\mathbf{X}$ with pullbacks and suppose that $\mathcal{M}$ contains all split monos. Then the following are equivalent:*

1. *$\mathbf{X}$ is $\mathcal{M}$-adhesive;*
2. *every $\mathcal{M}$-pushout square is Van Kampen and for every object X, every pair $[m],[n]\in\mathcal{M}\text{-}\mathsf{Sub}(X)$ has a supremum in $(\mathsf{Sub}(X),\leq)$ belonging to $\mathcal{M}\text{-}\mathsf{Sub}(X)$;*
3. *$\mathcal{M}$ is stable under pushouts, $\mathcal{M}\subseteq\mathcal{A}(X)_{\mathsf{a}}$ and for every object X, every pair $[m],[n]\in\mathcal{M}\text{-}\mathsf{Sub}(X)$ has a supremum in $(\mathsf{Sub}(X),\leq)$ which is again in $\mathcal{M}\text{-}\mathsf{Sub}(X)$.*

Remark 5.2.26. Notice that, in items 2 and 3 of the previous corollary, the existence of a supremum in $(\mathsf{Sub}(X),\leq)$ for $[m],[n]\in\mathcal{M}\text{-}\mathsf{Sub}(X)$ is guaranteed by the hypothesis that every arrow in $\mathcal{M}$ is adhesive and by Proposition 5.2.8.

5.3 $\mathcal{M},\mathcal{N}$-adhesivity and toposes

In this section we will examine the relationship between $\mathcal{M},\mathcal{N}$-adhesivity and (elementary) toposes. In the first part we will provide a new proof of the fact, first shown in [74], that (elementary) toposes are adhesive. In the second section we will generalize the results of [52] showing that, under suitable hypotheses, an $\mathcal{M},\mathcal{N}$-adhesive category admits a full and faithful embedding into a Grothendieck topos.

5.3.1 Some facts about toposes

Let us recall briefly the definition of a topos and some properties of toposes. The main references about topos theory are [30, 66, 86, 93].

Definition 5.3.1. Let $\mathbf{X}$ be a finitely complete category. A *subobject classifier* is a mono $\mathsf{t}\colon 1\to\Omega$ such that, for every monomorphisms $m\colon M\to X$, there is a unique $\chi_m\colon X\to\Omega$ such that the square below is a pullback

$$\begin{tikzcd} M \arrow[r, "!_M"] \arrow[d, "m"'] & 1 \arrow[d, "\mathsf{t}"] \\ X \arrow[r, dotted, "\chi_m"'] & \Omega \end{tikzcd}$$

A *topos* is a finitely complete, cartesian closed category $\mathbf{X}$ which has a subobject classifier.

Remark 5.3.2. Subobject classifiers are unique up to isomorphism. Indeed, if $\dagger\colon 1 \to \Omega$ and $\hat{\dagger}\colon 1 \to \hat{\Omega}$ are two subobjects classifiers, then we have the two diagrams below, in which every square is a pullback

$$\begin{tikzcd} 1 \arrow[r, "\mathsf{id}_1"] \arrow[d, "\dagger"'] & 1 \arrow[r, "\mathsf{id}_1"] \arrow[d, "\hat{\dagger}"'] & 1 \arrow[d, "\dagger"] \\ \Omega \arrow[r, "\chi_\dagger"'] & \hat{\Omega} \arrow[r, "\chi_{\hat{\dagger}}"'] & \Omega \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "\mathsf{id}_1"] \arrow[d, "\hat{\dagger}"] & 1 \arrow[r, "\mathsf{id}_1"] \arrow[d, "\dagger"'] & 1 \arrow[d, "\hat{\dagger}"] \\ \hat{\Omega} \arrow[r, "\chi_{\hat{\dagger}}"'] & \Omega \arrow[r, "\chi_\dagger"'] & \hat{\Omega} \end{tikzcd}$$

By Lemma 5.1.4 the whole rectangles are pullbacks, showing

$$\mathsf{id}_\Omega = \chi_{\hat{\dagger}} \circ \chi_\dagger \qquad \mathsf{id}_{\hat{\Omega}} = \chi_\dagger \circ \chi_{\hat{\dagger}}$$

Going deep into topos theory will lead us astray, so we rather assume the reader has at least a basic knowledge of the following facts.

Fact 5.3.3. ([65, Sec. A2.2] and [86, Ch. IV, Sec. 5]) If $\mathbf{X}$ is a topos, then it is finitely cocomplete.

Fact 5.3.4. ([48, 65, Sec. A2.3], and [86, Ch. IV, Sec. 7]) If X is an object of a topos $\mathbf{X}$, then the slice category $\mathbf{X}/X$ over X is a topos too.

Fact 5.3.4, the so called "fundamental theorem of topos theory", in particular entails that a topos $\mathbf{X}$ is locally cartesian closed. We can therefore apply Corollary A.3.14 obtaining the corollary below.

Corollary 5.3.5. *Let $f\colon X \to Y$ be a morphism in a topos $\mathbf{X}$, then $\mathsf{pb}_f\colon \mathbf{X}/Y \to \mathbf{X}/X$ has a right adjoint.*

We will also assume familiarity with the notions of coverage, Grothendieck topology, sheaves and Grothendieck topos ([66, Sec. C2.1] and [86, Ch. 3]).

Fact 5.3.6. Every Grothendieck topos is a topos.

Assuming these facts, in the next section, we will nonetheless prove some less known properties of toposes needed to show their adhesivity. The proofs of all these properties are adapted from [65, Ch. A2].

5.3.2 Toposes are adhesive

Our strategy to show that toposes are adhesive is to use Corollary 5.2.25, proving that the class of monos is closed under pushouts and consists of adhesive morphisms.

Proposition 5.3.7. *In a topos $\mathbf{X}$ all pushout squares are stable.*

Proof. Suppose that the following pushout square is given

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "k"] \\ Z \arrow[r, "h"'] & Q \end{tikzcd}$$

$\delta_Q \colon \mathbf{1} \to \mathbf{X}$ trivially preserves pushouts, so Lemma 5.1.35 and Proposition A.3.5 entails that the square

$$\begin{tikzcd} k \circ f \arrow[r, "f"] \arrow[d, "g"'] & k \arrow[d, "k"] \\ h \arrow[r, "h"'] & \mathsf{id}_Q \end{tikzcd}$$

is a pushout in $\mathbf{X}/Q$. By hypothesis $\mathbf{X}$ is a topos. Thus, Corollary 5.3.5, for every $q \colon Q' \to Q$ the functor $\mathsf{pb}_q \colon \mathbf{X}/Q \to \mathbf{X}/Q'$ is a left adjoint .Therefore it preserves colimits and the square

$$\begin{tikzcd} \mathsf{pb}_q(k \circ f) \arrow[r, "\mathsf{pb}_q(f)"] \arrow[d, "\mathsf{pb}_q(g)"'] & \mathsf{pb}_q(k) \arrow[d, "\mathsf{pb}_q(k)"] \\ \mathsf{pb}_q(h) \arrow[r, "\mathsf{pb}_q(h)"'] & \mathsf{pb}_q(\mathsf{id}_Q) \end{tikzcd}$$

is a square in $\mathbf{X}/Q'$. Clearly $\mathsf{id}_{Q'} = \mathsf{pb}_q(\mathsf{id}_Q)$ and we know from Lemma A.3.13 that the functor $\mathsf{dom}_{Q'} \colon \mathbf{X}/Q' \to \mathbf{X}$ is a left adjoint, so that we have another pushout square

$$\begin{tikzcd} \mathsf{pb}_q(X) \arrow[r, "\mathsf{pb}_q(f)"] \arrow[d, "\mathsf{pb}_q(g)"'] & \mathsf{pb}_q(Y) \arrow[d, "\mathsf{pb}_q(k)"] \\ \mathsf{pb}_q(Z) \arrow[r, "\mathsf{pb}_q(h)"'] & Q' \end{tikzcd}$$

We can now construct a cube as the one below, in which all faces are pullbacks

$$\begin{tikzcd}[row sep=small] & \mathsf{pb}_q(X) \arrow[rr, "\mathsf{pb}_q(f)"] \arrow[dl, "\mathsf{pb}_q(g)"'] \arrow[dd, "q_3" near end] & & \mathsf{pb}_q(Y) \arrow[dl, "\mathsf{pb}_q(k)"'] \arrow[dd, "q_1"] \\ \mathsf{pb}_q(Z) \arrow[rr, "\mathsf{pb}_q(h)" near end, crossing over] \arrow[dd, "q_2"'] & & Q' & \\ & X \arrow[rr, "f" near end] \arrow[dl, "g"'] & & Y \arrow[dl, "k"] \\ Z \arrow[rr, "h"'] & & Q \arrow[from=uu, "q" near start, crossing over] & \end{tikzcd}$$

and we already know that the top faces is a pushout, so that Remark 5.1.3 now yields the thesis. □

Let $m \colon X \to Y$ and $f \colon X \to Z$ be two arrows in a topos $\mathbf{X}$, and suppose that m is a mono. Then, since $m = \pi_Y \circ (m, f)$, it follows that (m, f) is a mono $X \to Y \times Z$, and thus, it is classified by $\chi_{(m,f)} \colon Y \times Z \to \Omega$, which, in turn, can be transposed to get $\ulcorner \chi_{(m,f)} \urcorner \colon Y \to \Omega^Z$. In particular, when m and f are both id_X, we will denote by $\{-\}_X$ the arrow $\ulcorner \chi_{\Delta_X} \urcorner \colon X \to \Omega^X$.

Proposition 5.3.8. *Let* $\mathbf{X}$ *be a topos. Then for every* $f \colon X \to Y$*, the following identity holds true*

$$\ulcorner \chi_{(\mathsf{id}_X, f)} \urcorner = \{-\}_Y \circ f$$

Proof. Let us take the solid part in the diagram below

$$\begin{tikzcd}
Z \arrow[dr, dotted, "g"] \arrow[drr, bend left, "g_1"] \arrow[ddr, bend right, "g_2"'] & & \\
& X \arrow[r, "f"] \arrow[d, "(\mathsf{id}_X, f)"] & Y \arrow[d, "\Delta_Y"] \\
& X \times Y \arrow[r, "f \times \mathsf{id}_Y"'] & Y \times Y
\end{tikzcd}$$

Consider the projections $\pi_X \colon X \times Y \to X$, $\pi_Y \colon X \times T \to Y$ and take as g the arrow $\pi_X \circ g_2$. If $\pi_1, \pi_2 \colon Y \rightrightarrows Y$ are the other projections, then we have

$$\begin{aligned}
f \circ g &= f \circ \pi_X \circ g_2 \\
&= \pi_1 \circ (f \times \mathsf{id}_Y) \circ g_2 \\
&= \pi_1 \circ \Delta_Y \circ g_1 \\
&= \mathsf{id}_Y \circ g_1 \\
&= g_1
\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}
\pi_Y \circ g_2 &= \mathsf{id}_Y \circ \pi_Y \circ g_2 \\
&= \pi_2 \circ (f \times \mathsf{id}_Y) \circ g_2 \\
&= \pi_2 \circ \Delta_Y \circ g_1 \\
&= \mathsf{id}_Y \circ g_1 \\
&= g_1
\end{aligned}$$

Therefore we can deduce

$$\begin{aligned}
(\mathsf{id}_X, f) \circ g &= (\mathsf{id}_X \circ g, f \circ g) \\
&= (g, f \circ g) \\
&= (\pi_X \circ g_2, \pi_Y \circ g_2) \\
&= g_2
\end{aligned}$$

Thus g fits in the given diagram. (id_X, f) is mono because $\pi_X \circ (\mathsf{id}_X, f)$ is the identity, thus the previous equalities show that the square below is a pullback.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "(\mathsf{id}_X, f)"'] & Y \arrow[d, "\Delta_Y"] \\
X \times Y \arrow[r, "f \times \mathsf{id}_Y"'] & Y \times Y
\end{tikzcd}$$

We can now use Lemma 5.1.4 to deduce that the whole rectangle

$$\begin{tikzcd}
X \arrow[r, "f"'] \arrow[d, "(\mathsf{id}_X, f)"'] \arrow[rrr, bend left, "!_X"] & Y \arrow[rr, "!_Y"'] \arrow[d, "\Delta_Y"] & & 1 \arrow[d, "\mathsf{t}"] \\
X \times Y \arrow[r, "f \times \mathsf{id}_Y"] & Y \times Y \arrow[r, "\{-\}_Y \times \mathsf{id}_Y"] \arrow[rr, bend right, "\chi_{\Delta_Y}"'] & \Omega^Y \times Y \arrow[r, "\mathsf{ev}_{Y,\Omega}"] & \Omega
\end{tikzcd}$$

is a pullback. Hence we have

$$\begin{aligned}
\chi_{(\mathsf{id}_X, f)} &= \mathsf{ev}_{Y,\Omega} \circ (\{-\}_Y \times \mathsf{id}_Y) \circ (f \times \mathsf{id}_Y) \\
&= \mathsf{ev}_{Y,\Omega} \circ ((\{-\}_Y \circ f) \times \mathsf{id}_Y)
\end{aligned}$$

and the thesis now follows. □

Lemma 5.3.9. *Let $m\colon X \to Y$ and $f\colon X \to Z$ be arrows in a topos $\mathbf{X}$ and suppose that m is a monomorphism, then the following hold true:*

1. *the square*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "m"'] & Z \arrow[d, "\{-\}_Z"] \\
Y \arrow[r, "\ulcorner \chi_{(m,f)} \urcorner"'] & \Omega^Z
\end{tikzcd}$$

   commutes and it is a pullback;

2. *if the square*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "m"'] & Z \arrow[d, "q_1"] \\
Y \arrow[r, "q_2"'] & Q
\end{tikzcd}$$

   is a pushout, then q_1 is a mono and the square is also a pullback.

Proof. 1. Let us start showing that the given square commutes. We can observe that the square

$$\begin{tikzcd}
X \arrow[r, "\mathsf{id}_X"] \arrow[d, "(\mathsf{id}_X, f)"'] & X \arrow[d, "(m, f)"] \\
X \times Z \arrow[r, "m \times \mathsf{id}_Z"'] & Y \times Z
\end{tikzcd}$$

is a pullback. Indeed, let $\pi_1\colon X \times Z \to X$, $\pi_2\colon X \times Z \to Z$, $\pi_1'\colon Y \times Z \to Y'$ and $\pi_2'\colon Y \times Z \to Z$

be projections and suppose that the solid part of the diagram below is given,

$$\begin{tikzcd}
P \arrow[dr, dotted, "g_1"] \arrow[drr, bend left, "g_1"] \arrow[ddr, bend right, "g_2"'] & & \\
& X \arrow[r, "\mathsf{id}_X"] \arrow[d, "(\mathsf{id}_X, f)"] & X \arrow[d, "(m,f)"] \\
& X \times Z \arrow[r, "m \times \mathsf{id}_Z"'] & Y \times Z
\end{tikzcd}$$

Then we get the following two chains of equalities

$$\begin{aligned}
m \circ \pi_1 \circ g_2 &= \pi_1' \circ m \times \mathsf{id}_Z \circ g_2 & \pi_2 \circ g_2 &= \mathsf{id}_Z \circ \pi_2 \circ g \\
&= \pi_1' \circ (m, f) \circ g_1 & &= \pi_2' \circ m \times \mathsf{id}_Z \circ g_2 \\
&= m \circ g_1 & &= \pi_2' \circ (m, f) \circ g_1 \\
& & &= f \circ g_1
\end{aligned}$$

which, since m is a mono, entail that

$$g_2 = (g_1, f \circ g_1)$$

showing that g_1 is the unique arrow which can fill the dotted part of the diagram above. We can combine this observation with Lemma 5.1.4 to conclude that the rectangle

$$\begin{tikzcd}
X \arrow[r, "\mathsf{id}_X"] \arrow[d, "(\mathsf{id}_X, f)"'] & X \arrow[r, "!_X"] \arrow[d, "(m,f)"'] & 1 \arrow[d, "\mathsf{t}"] \\
X \times Z \arrow[r, "m \times \mathsf{id}_Z"'] & Y \times Z \arrow[r, "\chi_{(m,f)}"'] & \Omega
\end{tikzcd}$$

is a pullback, allowing us to conclude that

$$\chi_{(\mathsf{id}_X, f)} = \chi_{(m,f)} \circ (m \times \mathsf{id}_Z)$$

Thanks to this identity, we can build the diagram below

$$\begin{tikzcd}
X \times Z \arrow[r, "m \times \mathsf{id}_Z"'] \arrow[rrr, bend left, "\left(\chi_{\ulcorner (m,f) \urcorner} \circ m\right) \times \mathsf{id}_Z"] \arrow[drr, bend right, "\chi_{(\mathsf{id}_X, f)}"'] & Y \times Z \arrow[rr, "\chi_{\ulcorner (m,f) \urcorner} \times \mathsf{id}_Z"'] \arrow[dr, "\chi_{(m,f)}"'] & & \Omega^Z \times Z \arrow[dl, "\mathsf{ev}_{Z,\Omega}"] \\
& & \Omega &
\end{tikzcd}$$

which shows that

$$\ulcorner \chi_{(\mathsf{id}_X, f)} \urcorner = \chi_{\ulcorner (m,f) \urcorner} \circ m$$

On the other hand, we know by Proposition 5.3.8 that

$$\ulcorner \chi_{(\mathsf{id}_X, f)} \urcorner = \{-\}_Z \circ f$$

and therefore, we obtain the wanted equality

$$\chi_{\ulcorner(m,f)\urcorner} \circ m = \{-\}_Z \circ f$$

To prove the last half of the thesis, suppose that the solid part of the diagram below is given

$$\begin{tikzcd}
Q \arrow[drr, bend left, "g_1"] \arrow[dr, dotted, "g"] \arrow[ddr, bend right, "g_2"'] & & \\
& X \arrow[r, "f"] \arrow[d, "m"'] & Z \arrow[d, "\{-\}_Z"] \\
& Y \arrow[r, "\ulcorner \chi_{(m,f)} \urcorner"'] & \Omega^Z
\end{tikzcd}$$

Since m is a mono it is enough to show that the dotted $g\colon Q \to X$ exists. Computing we get

$$\begin{aligned}
\chi_{(m,f)} \circ (g_2, g_1) &= \mathsf{ev}_{Z,\Omega} \circ \left(\ulcorner \chi_{(m,f)} \urcorner \times \mathsf{id}_Z\right) \circ (g_2, g_1) \\
&= \mathsf{ev}_{Z,\Omega} \circ \left(\ulcorner \chi_{(m,f)} \urcorner \circ g_2, g_1\right) \\
&= \mathsf{ev}_{Z,\Omega} \circ (\{-\}_Z \circ g_1, g_1) \\
&= \mathsf{ev}_{Z,\Omega} \circ (\{-\}_Z \times \mathsf{id}_Z) \circ (g_1, g_1) \\
&= \chi_{\Delta_Z} \circ (g_1, g_1) \\
&= \chi_{\Delta_Z} \circ \Delta_Z \circ g_1 \\
&= \mathsf{t} \circ !_Z \circ g_1 \\
&= \mathsf{t} \circ !_Q
\end{aligned}$$

Thus we get $g\colon Q \to X$ fitting in the diagram below:

$$\begin{tikzcd}
Q \arrow[drr, bend left, "!_Q"] \arrow[dr, "g"] \arrow[ddr, bend right, "(g_2,g_1)"'] & & \\
& X \arrow[r, "!_X"] \arrow[d, "(m,f)"] & 1 \arrow[d, "\mathsf{t}"] \\
& Y \times Z \arrow[r, "\chi_{(m,f)}"'] & \Omega
\end{tikzcd}$$

The commutativity of the left triangle entails

$$g_2 = m \circ g \qquad g_1 = f \circ g$$

as desired.

2. Let us use the previous point to obtain a diagram as the one below

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "m"'] & Z \arrow[d, "q_1"] \arrow[ddr, bend left, "\{-\}_Z"] & \\ Y \arrow[r, "q_2"'] \arrow[drr, bend right, "\ulcorner\chi_{(m,f}\urcorner"'] & Q \arrow[dr, dotted, "g"] & \\ & & \Omega^Z \end{tikzcd}$$

The universal property of pushouts yields the dotted $g\colon Q \to \Omega^Z$. $\{-\}_Z$ is a monomorphism because

$$\Delta_Z = \mathsf{ev}_{Z,\Omega} \circ \{-\}_Z$$

and thus q_1 is a mono too. Too see that the original square is a pullback, take $h_1\colon Q \to Z$ and $h_2\colon Q \to Y$ such that

$$q_1 \circ h_1 = q_2 \circ h_2$$

Composing the two sides othe equation above with g, gives us

$$\begin{aligned} \{-\}_Z \circ h_1 &= g \circ q_1 \circ h_1 \\ &= g \circ q_2 \circ h_2 \\ &= \ulcorner\chi_{(m,f)}\urcorner \circ h_2 \end{aligned}$$

Therefore, applying point 1 again, we get a unique $h\colon Q \to X$ such that

$$h_1 = f \circ h \qquad h_2 = m \circ h$$

which is precisely the thesis. □

A topos $\mathbf{X}$ is finitely cocomplete by Fact 5.3.3. Thus it has all pushouts and from Proposition 5.3.7 and Lemma 5.3.9 we can deduce the following result.

Corollary 5.3.10. *In a topos* $\mathbf{X}$*, every mono is adhesive.*

We can now apply Corollary 5.2.25 and Lemma 5.3.9 together with Remark 5.2.26 to get our result.

Corollary 5.3.11. *Every topos is an adhesive category.*

5.3.3 An embedding theorem

Definition 5.3.12. Let $(\mathcal{M},\mathcal{N})$ be a preadhesive structure for a category $\mathbf{X}$. A $\mathsf{j}_{\mathcal{M},\mathcal{N}}$*-covering family* for an object X is a set $\{p,q\}$ of arrows $p\colon Z \to X$ and $q\colon Y \to X$ such that there exist $m\colon N \to Y$ in $\mathcal{M}$ and $n\colon N \to Z$ in $\mathcal{N}$ making the following square a pushout

$$\begin{tikzcd} N \arrow[r, "n"] \arrow[d, "m"'] & Z \arrow[d, "p"] \\ Y \arrow[r, "q"'] & X \end{tikzcd}$$

We will define $\mathsf{j}_{\mathcal{M},\mathcal{N}}(X)$ as the set of $\mathsf{j}_{\mathcal{M},\mathcal{N}}$-covering families for X.

Proposition 5.3.13. *Let* $\mathbf{X}$ *be a category with pullbacks, for every preadhesive structure* $(\mathcal{M},\mathcal{N})$ *such that* $\mathcal{M} \subseteq \mathcal{N}_{\mathfrak{a}}$*, the family* $\{\mathrm{j}_{\mathcal{M},\mathcal{N}}(X)\}_{X\in\mathbf{X}}$ *defines a coverage* $\mathrm{j}_{\mathcal{M},\mathcal{N}}$ *on* $\mathbf{X}$*.*

Proof. Take p, q in $\mathrm{j}_{\mathcal{M},\mathcal{N}}(Y)$ and $f\colon X \to Y$. By definition of $\mathrm{j}_{\mathcal{M},\mathcal{N}}(Y)$, there exists a pushout square

$$\begin{tikzcd} N \arrow[r, "n"] \arrow[d, "m"'] & Z \arrow[d, "p"] \\ W \arrow[r, "q"'] & Y \end{tikzcd}$$

with $m \in \mathcal{M}$ and $n \in \mathcal{N}$. By Remark 5.1.2, we know that it is stable. We can use Corollary 5.1.5 to build the following cube in which all faces are pullbacks

$$\begin{tikzcd} & X' \arrow[rr, "n'"] \arrow[dl, "m'"'] \arrow[dd, "g"] & & Z' \arrow[dl, "p'"'] \arrow[dd, "z"] \\ W' \arrow[rr, "q'"] \arrow[dd, "w"'] & & X \arrow[dd, "f"] & \\ & N \arrow[rr, "n"] \arrow[dl, "m"'] & & Z \arrow[dl, "p"] \\ W \arrow[rr, "q"'] & & Y & \end{tikzcd}$$

The arrows m and n belong to $\mathcal{M}$ and $\mathcal{N}$, respectively. Thus $m' \in \mathcal{M}$ and $n' \in \mathcal{N}$. The bottom face is stable, therefore the top face witnesses $\{p', q'\} \in \mathrm{j}_{\mathcal{M},\mathcal{N}}(X)$. On the other hand we have squares

$$\begin{tikzcd} W' \arrow[r, "w"] \arrow[d, "q'"'] & W \arrow[d, "q"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \qquad \begin{tikzcd} Z' \arrow[r, "z"] \arrow[d, "p'"'] & Z \arrow[d, "p"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

from which we can deduce the thesis. □

Remark 5.3.14. The coverage $\mathrm{j}_{\mathcal{M},\mathcal{N}}$ is a *cd-structure* in the sense of [120, 121].

Our next step is to characterize sheaves for the site $(\mathbf{X}, \mathrm{j}_{\mathcal{M},\mathcal{N}})$.

Lemma 5.3.15. *Let* $(\mathcal{M},\mathcal{N})$ *be a preadhesive structure for a category* $\mathbf{X}$ *with pullbacks and suppose that every element in* $\mathcal{M}$ *is* $\mathcal{N}$*-adhesive. Then the following are equivalent for a presheaf* $F\colon \mathbf{X}^{op} \to \mathbf{Set}$*:*

1. F *is in* $\mathbf{Sh}(\mathbf{X}, \mathrm{j}_{\mathcal{M},\mathcal{N}})$*;*
2. *given the following two squares, the first of which is an* $\mathcal{M},\mathcal{N}$*-pushout and the other two are pullbacks,*

$$\begin{tikzcd} N \arrow[r, "n"] \arrow[d, "m"'] & Z \arrow[d, "p"] \\ Y \arrow[r, "q"'] & X \end{tikzcd} \qquad \begin{tikzcd} K_q \arrow[r, "y_1"] \arrow[d, "y_2"'] & Y \arrow[d, "q"] \\ Y \arrow[r, "q"'] & X \end{tikzcd}$$

if the solid part of the diagram below is given, then there exists a unique $f\colon S \to F(X)$ fitting in it.

$$\begin{tikzcd}
& S \arrow[dr, dotted, "f"] \arrow[drr, bend left, "f_2"] \arrow[ddr, bend right, "f_1"'] & & \\
& & F(X) \arrow[r, "F(q)"] \arrow[d, "F(p)"'] & F(Z) \arrow[d, "F(m)"] \\
F(K_q) & & F(Y) \arrow[ll, shift right, "F(y_1)"'] \arrow[ll, shift left, "F(y_2)"] \arrow[r, "F(n)"'] & F(N)
\end{tikzcd}$$

Proof. $(1 \Rightarrow 2)$ Let us start showing that, for every $s \in S$, the family $\{f_1(s), f_2(s)\}$ is matching for the $\mathrm{j}_{\mathcal{M},\mathcal{N}}$-cover $\{p, q\}$. Given three commutative squares as the ones below

$$\begin{tikzcd}
A \arrow[r, "a_1"] \arrow[d, "a_2"'] & Y \arrow[d, "q"] \\
Z \arrow[r, "p"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "b_1"] \arrow[d, "b_2"'] & Y \arrow[d, "q"] \\
Y \arrow[r, "q"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "c_1"] \arrow[d, "c_2"'] & Z \arrow[d, "p"] \\
Z \arrow[r, "p"'] & X
\end{tikzcd}$$

p is the pushout of m. Thus it belongs to $\mathcal{M}$ and so is a mono, which implies that

$$c_1 = c_2 \qquad F(c_1) \circ f_2 = F(c_2) \circ f_2$$

Moreover, $m \in \mathcal{N}_{\mathsf{a}}$ and $n \in \mathcal{N}$. Thus in the following diagrams the two inner squares are pullbacks, giving us the dotted arrows $a\colon A \to N$ and $b\colon B \to K_q$.

$$\begin{tikzcd}
A \arrow[dr, dotted, "a"] \arrow[drr, bend left, "a_2"] \arrow[ddr, bend right, "a_1"'] & & \\
& N \arrow[r, "n"] \arrow[d, "m"'] & Z \arrow[d, "p"] \\
& Y \arrow[r, "q"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[dr, dotted, "b"] \arrow[drr, bend left, "b_2"] \arrow[ddr, bend right, "b_1"'] & & \\
& K_q \arrow[r, "y_2"] \arrow[d, "y_1"'] & Y \arrow[d, "q"] \\
& Y \arrow[r, "q"'] & X
\end{tikzcd}$$

Computing we get the following chains of identities

$$\begin{aligned}
F(a_1) \circ f_1 &= F(a) \circ F(m) \circ f_1 \qquad & F(b_1) \circ f_1 &= F(b) \circ F(y_1) \circ f_1 \\
&= F(a) \circ F(n) \circ f_2 & &= F(b) \circ F(y_1) \circ f_1 \\
&= F(a_2) \circ f_2 & &= F(b_2) \circ f_1
\end{aligned}$$

which imply that, for every $s \in S$, $\{f_1(s), f_2(s)\}$ is a matching family for $\{p, q\}$. Since F is a sheaf we can define $f\colon S \to F(X)$ taking as $f(s)$ the unique amalgamation of $\{f_1(s), f_2(s)\}$, by construction

$$f_1 = F(p) \circ f \qquad f_2 = F(q) \circ f$$

For uniqueness it is enough to notice that, if $g\colon S \to F(X)$ is another arrow such that

$$f_1 = F(p) \circ g \qquad f_2 = F(q) \circ g$$

then $g(s)$ is an amalgamation for $\{f_1(s), f_2(s)\}$.
$(2 \Rightarrow 1)$ Let $\{p, q\}$ be a $\mathrm{j}_{\mathcal{M},\mathcal{N}}$-cover of X. By definition there exists an $\mathcal{M},\mathcal{N}$-pushout square

$$\begin{tikzcd} N \arrow[r, "n"] \arrow[d, "m"'] & Z \arrow[d, "p"] \\ Y \arrow[r, "q"'] & X \end{tikzcd}$$

Take a matching family $\{s_1, s_2\}$ for $\{p, q\}$ with $s_1 \in F(Y)$ and $s_2 \in F(Z)$. Applying the matching property to the two squares below

$$\begin{tikzcd} N \arrow[r, "n"] \arrow[d, "m"'] & Z \arrow[d, "p"] \\ Y \arrow[r, "q"'] & X \end{tikzcd} \qquad \begin{tikzcd} K_q \arrow[r, "y_1"] \arrow[d, "y_2"'] & Y \arrow[d, "q"] \\ Y \arrow[r, "q"'] & X \end{tikzcd}$$

we obtain the following identities:

$$F(m)(s_1) = F(n)(s_2) \qquad F(y_1)(s_1) = F(y_2)(s_1)$$

Thus, if $\delta_{s_1} : 1 \to F(Y)$ and $\delta_{s_2} : 1 \to F(Z)$ pick s_1 and s_2, respectively, then we have the solid part of the following commutative diagram and, by hypothesis, also the dotted $\delta : 1 \to F(X)$.

$$\begin{tikzcd} 1 \arrow[dr, dotted, "\delta"] \arrow[drr, bend left, "\delta_{s_2}"] \arrow[ddr, bend right, "\delta_{s_1}"'] & & \\ & F(X) \arrow[r, "F(q)"] \arrow[d, "F(p)"'] & F(Z) \arrow[d, "F(m)"] \\ F(K_q) & F(Y) \arrow[l, shift right, "F(y_1)"'] \arrow[l, shift left, "F(y_2)"] \arrow[r, "F(n)"'] & F(N) \end{tikzcd}$$

Now let s be the element of $F(X)$ picked by δ. Then, by construction s is an amalgamation for $\{s_1, s_2\}$. On the other hand, if s' is another amalgamation, then

$$\delta_{s_1} = F(p) \circ \delta_{s'} \qquad \delta_{s_2} = F(p) \circ \delta_{s'}$$

and so $\delta = \delta_{s'}$, showing that $s = s'$, i.e. that F is a sheaf. □

We can now combine the previous lemma with Lemma 5.2.11 to obtain the following.

Lemma 5.3.16. *Let $(\mathcal{M}, \mathcal{N})$ be a preadhesive structure on a category $\mathbf{X}$ with pullbacks such that $\mathcal{M} \subseteq \mathcal{N}_{\mathrm{a}}$, $\mathcal{M} \cap \mathcal{N}$ contains every split mono and $\mathcal{M}$ is closed under $\mathcal{M},\mathcal{N}$-unions. Then for a presheaf $F : \mathbf{X}^{op} \to \mathbf{Set}$ the following are equivalent:*

1. *F is in $\mathbf{Sh}(\mathbf{X}, \mathrm{j}_{\mathcal{M},\mathcal{N}})$;*
2. *F sends $\mathcal{M},\mathcal{N}$-pushouts to pullbacks.*

Proof. $(1 \Rightarrow 2)$ Given the following two squares, the first of which is an $\mathcal{M},\mathcal{N}$-pushout, while the second is a pullback

$$
\begin{tikzcd}
N \arrow[r, "n"] \arrow[d, "m"'] & Z \arrow[d, "p"] \\
Y \arrow[r, "q"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
K_n \arrow[r, "n_1"] \arrow[d, "n_2"'] & N \arrow[d, "n"] \\
N \arrow[r, "n"'] & Y
\end{tikzcd}
$$

Lemma 5.2.11 gives the following diagrams, in which the common square on the left is an $\mathcal{M},\mathcal{N}$-pushout. In particular, this implies that $\{k, \gamma_q\}$ is a $\mathrm{j}_{\mathcal{M},\mathcal{N}}$-covering family of X

$$
\begin{tikzcd}
N \arrow[r, "\gamma_n"] \arrow[d, "m"'] & K_n \arrow[r, "n_1"] \arrow[d, "k"'] & N \arrow[r, "n"] \arrow[d, "m"'] & Z \arrow[d, "p"] \\
Y \arrow[r, "\gamma_q"'] & K_q \arrow[r, "y_1"'] & Y \arrow[r, "q"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
N \arrow[r, "\gamma_n"] \arrow[d, "m"'] & K_n \arrow[r, "n_2"] \arrow[d, "k"'] & N \arrow[r, "n"] \arrow[d, "m"'] & Z \arrow[d, "p"] \\
Y \arrow[r, "\gamma_q"'] & K_q \arrow[r, "y_2"'] & Y \arrow[r, "q"'] & X
\end{tikzcd}
$$

The arrow k is in $\mathcal{M}$ since it is the pushout of $\mathcal{M}$. Thus, k and γ_q are both mono, so that Lemma 5.3.15 now implies that the square below is a pullback.

$$
\begin{tikzcd}
F(K_q) \arrow[r, "F(k)"] \arrow[d, "F(\gamma_q)"'] & F(K_n) \arrow[d, "F(\gamma_n)"] \\
F(Y) \arrow[r, "F(m)"'] & F(N)
\end{tikzcd}
$$

Suppose that the solid part of the following diagram is given

$$
\begin{tikzcd}
S \arrow[dr, dotted, "f"] \arrow[drr, bend left, "f_2"] \arrow[ddr, bend right, "f_1"'] & & \\
& F(X) \arrow[r, "F(q)"] \arrow[d, "F(p)"'] & F(Z) \arrow[d, "F(n)"] \\
& F(Y) \arrow[r, "F(m)"'] & F(N)
\end{tikzcd}
$$

On the one hand we have at once that

$$
\begin{aligned}
F(\gamma_q) \circ F(y_1) \circ f_1 &= F(y_1 \circ \gamma_q) \circ f_1 \\
&= F(\mathsf{id}_Y) \circ f_1 \\
&= F(y_2 \circ \gamma_q) \circ f_1 \\
&= F(\gamma_q) \circ F(y_2) \circ f_1
\end{aligned}
$$

while, on the other hand, we also have

$$\begin{aligned}
F(k)\circ F(y_1)\circ f_1 &= F(y_1\circ k)\\
&= F(m\circ n_1)\circ f_1\\
&= F(n_1)\circ F(m)\circ f_1\\
&= F(n_1)\circ F(n)\circ f_2\\
&= F(n_2)\circ F(n)\circ f_2\\
&= F(n_2)\circ F(m)\circ f_1\\
&= F(m\circ n_2)\circ f_1\\
&= F(y_2\circ k)\circ f_1\\
&= F(k)\circ F(y_2)\circ f_1
\end{aligned}$$

and we can deduce that $F(y_1)\circ f_1 = F(y_2)\circ f_1$. Lemma 5.3.15 now entails the thesis.
$(2 \Rightarrow 1)$ This follows at once from Lemma 5.3.15. □

We are now ready to deduce our main theorem.

Theorem 5.3.17. *Let $(\mathcal{M},\mathcal{N})$ be a preadhesive structure on a category $\mathbf{X}$ with pullbacks such that $\mathcal{M}\subseteq\mathcal{N}_{\mathfrak{a}}$, $\mathcal{M}\cap\mathcal{N}$ contains every split mono and $\mathcal{M}$ is closed under $\mathcal{M},\mathcal{N}$-unions. Then the following hold true:*

1. *the Yoneda embedding $\text{よ}_{\mathbf{X}}\colon \mathbf{X}\to\mathbf{Set}^{\mathbf{X}^{op}}$ factors through a full and faithful functor $\text{よ}'_{\mathbf{X}}\colon \mathbf{X}\to\mathbf{Sh}(\mathbf{X},\mathrm{j}_{\mathcal{M},\mathcal{N}})$;*
2. *$\text{よ}'_{\mathbf{X}}$ preserves pullbacks and sends $\mathcal{M},\mathcal{N}$-pushouts to pushouts.*

Proof. 1. Since $\mathbf{Sh}(\mathbf{X},\mathrm{j}_{\mathcal{M},\mathcal{N}})$ is a full subcategory of $\mathbf{Set}^{\mathbf{X}^{op}}$, it is enough to show that, for every $X\in\mathbf{X}$, the functor $\mathbf{X}(-,X)$ is a sheaf, but this follows at once from Lemma 5.3.16, since any representable presheaf sends pushouts to pullbacks.

2. The inclusion $\mathbf{Sh}\left(\mathbf{X},\mathrm{j}_{\mathcal{M},\mathcal{N}}\right)\to\mathbf{Set}^{\mathbf{X}^{op}}$ creates limits and $\text{よ}_{\mathbf{X}}$ sends pullbacks to pullbacks. Therefore $\text{よ}'_{\mathbf{X}}$ preserves pullbacks, too. Take now an $\mathcal{M},\mathcal{N}$-pushout

$$\begin{tikzcd}
X \arrow[r, "n"] \arrow[d, "m"'] & Z \arrow[d, "q"] \\
Y \arrow[r, "p"'] & Q
\end{tikzcd}$$

Since $\mathbf{Sh}\left(\mathbf{X},\mathrm{j}_{\mathcal{M},\mathcal{N}}\right)$ is a full subcategory of $\mathbf{Set}^{\mathbf{X}^{op}}$, for every sheaf F, the Yoneda Lemma yields a natural isomorphism $\mathsf{y}\colon \mathbf{Sh}\left(\mathbf{X},\mathrm{j}_{\mathcal{M},\mathcal{N}}\right)\left(\text{よ}'_{\mathbf{X}}(-),F\right)\to F$, so that we obtain a diagram

$$\begin{tikzcd}
\mathbf{Sh}\left(\mathbf{X},\mathrm{j}_{\mathcal{M},\mathcal{N}}\right)\left(\text{よ}'_{\mathbf{X}}(Q),F\right) \arrow[rrr, "(-)\circ\text{よ}'_{\mathbf{X}}(q)"] \arrow[ddd, "(-)\circ\text{よ}'_{\mathbf{X}}(p)"'] \arrow[dr, "\mathsf{y}_Q"'] & & & \mathbf{Sh}\left(\mathbf{X},\mathrm{j}_{\mathcal{M},\mathcal{N}}\right)\left(\text{よ}'_{\mathbf{X}}(Z),F\right) \arrow[ddd, "(-)\circ\text{よ}'_{\mathbf{X}}(n)"] \arrow[dl, "\mathsf{y}_Z"] \\
& F(Q) \arrow[r, "F(q)"] \arrow[d, "F(p)"'] & F(Z) \arrow[d, "F(n)"] & \\
& F(Y) \arrow[r, "F(m)"'] & F(X) & \\
\mathbf{Sh}\left(\mathbf{X},\mathrm{j}_{\mathcal{M},\mathcal{N}}\right)\left(\text{よ}'_{\mathbf{X}}(Y),F\right) \arrow[rrr, "(-)\circ\text{よ}'_{\mathbf{X}}(m)"'] \arrow[ur, "\mathsf{y}_Y"] & & & \mathbf{Sh}\left(\mathbf{X},\mathrm{j}_{\mathcal{M},\mathcal{N}}\right)\left(\text{よ}'_{\mathbf{X}}(X),F\right) \arrow[ul, "\mathsf{y}_X"']
\end{tikzcd}$$

The functor F is a sheaf. Hence, the inner square is a pullback by Lemma 5.3.16, and, thus, the outer one is a pullback, too, proving that $よ'_{\mathbf{X}}$ sends $\mathcal{M}, \mathcal{N}$-pushouts to pushouts. $\square$

Corollary 5.3.18. *Let* $\mathbf{X}$ *be an* $\mathcal{M}, \mathcal{N}$*-adhesive category with pullbacks such that* $\mathcal{M} \cap \mathcal{N}$ *contains all split monomorphisms and* $\mathcal{N}$ *contains all* $\mathcal{M}, \mathcal{N}$*-codiagonals. Then there exists a full and faithful functor from* $\mathbf{X}$ *into a topos. Moroever, such a functor preserves all pullbacks and* $\mathcal{M}, \mathcal{N}$*-pushouts.*

Proof. Apply Theorem 5.2.22 and the previous theorem. $\square$

A zoo of $\mathcal{M},\mathcal{N}$-adhesive categories

CHAPTER 6

Contents

In the previous chapter we have introduced and examined the notion of $\mathcal{M},\mathcal{N}$-adhesivity and provided a criterion, namely Theorem 5.1.31, allowing us to deduce some adhesivity result for a category $\mathbf{X}$ from the existence of a family of functors with suitable properties. This chapter is devoted to exploit this criterion to establish $\mathcal{M},\mathcal{N}$-adhesivity of various categories.

It is well-known that categorical properties are often *prescriptive*, indicating abstractly the presence of some good behaviour of the modelled system. Adhesivity is one such property, as it is highly sought after when it comes to rewriting theories. Thus, our criterion for proving $\mathcal{M},\mathcal{N}$-adhesivity can be seen also as a "litmus test" for the given category. This is the precisely the case of our first important example: *hierarchical graphs*. We roughly can find two alternative proposals for this kind of structures: on the one hand, algebraic formalisms where the edges have some algebraic structures, so that the nesting is a side effect of the term construction; on the other hand, combinatorial approaches where the topology of a standard graph is enriched by some partial order, either on the nodes or on the edges, where the order relation indicates the presence of nesting. By applying our Theorem 5.1.31, we can show that the latter approach yields indeed an $\mathcal{M},\mathcal{N}$-adhesive category, confirming and overcoming the limitations of some previous approaches to hierarchical graphs [99, 101, 102], which we briefly recall next.

The more straightforward proposal is by Palacz [102], using a poset of edges instead of just a set; however, the class of rules has to be restricted in order to apply the approach, which in any case predates the introduction of adhesive categories. Our work allows to rephrase in terms of adhesive properties and generalise Palacz's proposal, dropping the constraint on rules. Another attempt are Mylonakis and Orejas' *graphs with layers* [99], for which $\mathcal{M}$-adhesivity is proved for a class of monomorphisms in the category of symbolic graphs; however, nodes between edges at different layers cannot be shared. Padberg [101] goes for a coalgebraic presentation via a peculiar "superpower set" functor; this gives immediately $\mathcal{M}$-adhesivity provided that this superpower set functor is well-behaved with respect to limits. However, albeit quite general, the approach is rather *ad hoc*, not modular and not very natural for actual modelling.

As a next step, we leverage on the modularity of Theorem 5.1.31 to deal with *hypergraphs* and some variants of them. In this way we are able to introduce *hierarchical hypergraphs*, i.e. hypergraphs in which the edges are organized in some structure, like a tree, a simple graph, or a directed acyclic graph. This, in turn allows us to study two other examples.

The first one is given by a a recently introduced (hyper)graphical formalism for the representation of the internal language of monoidal closed categories. In [11] the authors define a category of labelled hierarchical hypergraphs and use them to represent arrows of a given monoidal closed category. Identities provided by the axioms of a monoidal closed structure are then formalized as rewriting rules. We show that the category of these hypergraphs is $\mathcal{M},\mathcal{M}$-adhesive for a class $\mathcal{M}$ of monos which contains the morphisms appearing in the rules proposed in [11].

Our second hypergraphical examples is given by *term graphs* [38, 108]. These are elements of a particular class of hypergraphs, whose use has been advocated in the past years as a tool for the optimal implementation of terms, with the intuition that the graphical counterpart of trees can allow for the sharing of sub-terms [108]. As a preliminary step we show that two presentations of term graphs appearing in the literature yields isomorphic categories. Next, we provide a new proof of the fact, first proved in [38] with a brute-force approach, that the category of term graphs is quasiadhesive. Our strategy to do so, will be prove that term graphs forms a full subcategory of the category of hypergraphs which is closed under pullbacks and pushouts along regular monos.

This chapter, as the previous one, draws on material previously published in [36]. An extended version of it, including the comparison with the formalism introduced in [11] for monoidal closed categories and the correspondence between the two presentations of term graphs appearing in the literature has been submitted to *Theoretical Computer Science* for publication.

Synopsis In Section 6.1 we apply the results of Chapter 5 to various categories, such as simple graphs, directed graphs, trees and hierarchical graphs. In Section 6.2 we move to hypergraphs, where an edge may join two subsets of nodes, and we investigate the adhesivity of the category of (algebraically) labelled hierarchical graphs. Section 6.3 is devoted to the introduction and study of a category whose objects provide a representation for arrows in monoidal closed categories. Finally, in Section 6.4 we discuss term graphs, which are seen as the standard formalism for the implementation of functional programs.

6.1 $\mathcal{M}, \mathcal{N}$-adhesivity of some categories of graphs

In this section we apply the results provided in Chapter 5, to some important categories of graphs, such as directed (acyclic) graphs and hierarchical graphs. These examples have been chosen for their importance in graph rewriting, and because we can recover their $\mathcal{M}, \mathcal{N}$-adhesivity in a uniform and systematic way. In fact, in the case of hierarchical graphs we give the first proof of $\mathcal{M}, \mathcal{N}$-adhesivity, to our knowledge.

As a preliminary step, let us prove some properties of pushouts in **Set**.

Lemma 6.1.1. *Let the following square be a pushout in* **Set**

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "q_1"] \\ Z \arrow[r, "q_2"'] & Q \end{tikzcd}$$

then the following are true:

1. *the induced arrow $\langle q_1, q_2\rangle \colon Y + Z \to Q$ is surjective;*
2. *If z_1 and z_2 are two distinct elements of Z which do not belong to $g(X)$, then $q_2(z_1) \neq q_2(z_1)$;*
3. *if g is injective then, given $z \in Z$ and $y \in Y$, we have $q_1(z) = q_2(y)$ if and only if there exists a unique $x \in X$ such that $y = f(x)$ and $z = g(x)$.*

Proof. 1. In any category with binary coproducts the following diagram is a coequalizer

$$\begin{tikzcd} X \arrow[r, shift left, "\iota_Y \circ f"] \arrow[r, shift right, "\iota_Z \circ g"'] & Y + Z \arrow[r, "\langle q_1, q_2\rangle"] & Q \end{tikzcd}$$

where $\iota_Y \colon Y \to Y + Z$ and $\iota_Z \colon Z \to Y + Z$ are the coprojections. The thesis now follows since epimorphisms in **Set** are surjective.

2. Consider the functions $h_2 \colon Z \to 2$ which sends $g(X) \cup \{z_1\}$ to 0 and z_2 to 1 and $h_1 \colon Y \to 2$ constant in 0. Then

$$h_1 \circ f = h_2 \circ g$$

and so there exists $h \colon Q \to 2$ such that

$$h_1 = h \circ q_1 \qquad h_2 = h \circ q_2$$

In particular we have that

$$\begin{aligned} h(q_2(z_1)) &= h_2(z_1) & \qquad h(q_2(z_2)) &= h_2(z_2) \\ &= 0 & &= 1 \end{aligned}$$

showing that $q_2(z_1)$ and $q_2(z_2)$ must be different.

3. ($\Rightarrow$) By hypothesis $q_1(z) = q_1(y)$, thus we have the solid part of the diagram below

$$\begin{tikzcd}
1 \arrow[dr, dotted, "\delta"] \arrow[drr, bend left, "\delta_y"] \arrow[ddr, bend right, "\delta_z"'] & & \\
& X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "q_1"] \\
& Z \arrow[r, "q_2"'] & Q
\end{tikzcd}$$

but **Set** is adhesive, thus, by Proposition 5.1.7, the given square is also a pullback and so there is a unique dotted $\delta\colon 1 \to X$. Now it is enough to take as x the element picked by this arrow.

($\Leftarrow$) Obvious. □

6.1.1 Directed (acyclic) graphs

Among visual formalisms, directed simple graphs represent one of the most-used paradigms, since they adhere to the classical view of graphs as relations included in the cartesian product of vertices. It is also well-known that directed graphs are not quasiadhesive [67], not even in their acyclic variant. In this section we are going to exploit Corollary 5.1.34 to show that these categories of (acyclic) graphs have nevertheless adhesivity properties.

Definition 6.1.2. A *directed graph* $\mathcal{G}$ is a 4-tuple $(E_\mathcal{G}, V_\mathcal{G}, s_\mathcal{G}, t_\mathcal{G})$ where $E_\mathcal{G}$ and $V_\mathcal{G}$ are sets, called the set of *edges* and *nodes* respectively, and $s_\mathcal{G}, t_\mathcal{G}\colon E_\mathcal{G} \rightrightarrows V_\mathcal{G}$ are functions, called *source* and *target*. An edge e is *between* v and w if

$$v = s_\mathcal{G}(e) \qquad w = t_\mathcal{G}(e)$$

$\mathcal{G}(v, w)$ will denote the set of edges between v and w.

A morphism $\mathcal{G} \to \mathcal{H}$ is a pair (f, g) of functions $f\colon E_\mathcal{G} \to E_\mathcal{H}$, $g\colon V_\mathcal{G} \to V_\mathcal{H}$ such that the squares below commute. We will denote the category so defined by **Graph**

$$\begin{tikzcd}
E_\mathcal{H} \arrow[r, "s_\mathcal{G}"] \arrow[d, "f"'] & V_\mathcal{F} \arrow[d, "g"] \\
E_\mathcal{H} \arrow[r, "s_\mathcal{H}"'] & V_\mathcal{H}
\end{tikzcd}
\qquad
\begin{tikzcd}
E_\mathcal{H} \arrow[r, "t_\mathcal{G}"] \arrow[d, "f"'] & V_\mathcal{H} \arrow[d, "g"] \\
E_\mathcal{H} \arrow[r, "t_\mathcal{H}"'] & V_\mathcal{H}
\end{tikzcd}$$

A *directed simple graph* is a directed graph in which there is at most one edge between two nodes, **SGraph** will denote the full subcategory of **Graph** made by directed simple graphs.

A *path* $\{e_i\}_{i=1}^n$ in $\mathcal{G}$ is a finite and non empty family of edges such that, for all $1 \leq i \leq n-1$

$$t_\mathcal{G}(e_i) = s_\mathcal{G}(e_{i+1})$$

A path will be called a *cycle* if

$$t_\mathcal{G}(e_n) = s_\mathcal{G}(e_1)$$

A *directed acyclic graph* is a directed simple graph without cycles. Directed acyclic graphs form a full subcategory **DAG** of **SGraph** and **Graph**.

Remark 6.1.3. Let $(f,g)\colon \mathcal{G}\to\mathcal{H}$ be an arrow in **SGraph** with $\mathcal{H}\in$ **DAG**, then $\mathcal{G}$ is in **DAG** too. Given a cycle $\{e_i\}_{i=1}^n$ in $\mathcal{G}$ we have

$$\begin{aligned} t_{\mathcal{H}}(f(e_i)) &= g(t_{\mathcal{G}}(e_i)) \\ &= g(s_{\mathcal{G}}(e_{i+1})) \\ &= t_{\mathcal{H}}(f(e_{i+1})) \end{aligned} \qquad \begin{aligned} t_{\mathcal{H}}(f(e_1)) &= g(t_{\mathcal{G}}(e_1)) \\ &= g(s_{\mathcal{G}}(e_n)) \\ &= t_{\mathcal{H}}(f(e_n)) \end{aligned}$$

so that $\{f(e_i)\}_{i=1}^n$ is a cycle in $\mathcal{H}$.

Proposition 6.1.4. *Let* prod *be the functor* **Set** $\to$ **Set** *defined as*

$$\begin{tikzcd} X \arrow[r, mapsto] \arrow[d, "f"'] & X\times X \arrow[d, "f\times f"] \\ Y \arrow[r, mapsto] & Y\times Y \end{tikzcd}$$

Then **Graph** *is isomorphic to* $\mathsf{id}_{\mathbf{Set}}{\downarrow}\mathsf{prod}$

Proof. Define $F\colon$ **Graph** $\to \mathsf{id}_{\mathbf{Set}}{\downarrow}\mathsf{prod}$ and $G\colon \mathsf{id}_{\mathbf{Set}}{\downarrow}\mathsf{prod}\to$ **Graph** putting

$$\begin{tikzcd} (E_{\mathcal{G}},V_{\mathcal{G}},s_{\mathcal{G}},t_{\mathcal{G}}) \arrow[r, mapsto] \arrow[d, "(f,g)"'] & (E_{\mathcal{G}},V_{\mathcal{G}},(s_{\mathcal{G}},t_{\mathcal{G}})) \arrow[d, "(f,g)"] \\ (E_{\mathcal{H}},V_{\mathcal{H}},s_{\mathcal{H}},t_{\mathcal{H}}) \arrow[r, mapsto] & (E_{\mathcal{H}},V_{\mathcal{H}},(s_{\mathcal{H}},t_{\mathcal{H}})) \end{tikzcd} \qquad \begin{tikzcd} (E_{\mathcal{G}},V_{\mathcal{G}},p_{\mathcal{G}}) \arrow[r, mapsto] \arrow[d, "(f,g)"'] & (E_{\mathcal{G}},V_{\mathcal{G}},\pi_1\circ p_{\mathcal{G}},\pi_2\circ p_{\mathcal{G}}) \arrow[d, "(f,g)"] \\ (E_{\mathcal{H}},V_{\mathcal{H}},p_{\mathcal{H}}) \arrow[r, mapsto] & (E_{\mathcal{H}},V_{\mathcal{H}},\pi_1\circ p_{\mathcal{H}},\pi_2\circ p_{\mathcal{H}}) \end{tikzcd}$$

It is now immediate to see that F and G are mutually inverses. □

Corollary 6.1.5. *The following hold true:*

1. *the functors* $W_{\mathbf{Graph}}, U_{\mathbf{Graph}}\colon$ **Graph** $\rightrightarrows$ **Set** *sending a graph to its set of edges and of nodes, respectively, jointly creates all limits and colimits;*
2. *an arrow* $(f,g)\colon\mathcal{G}\to\mathcal{H}$ *is a mono* **Graph** *if and only if both* f *and* g *are injective;*
3. **Graph** *is an adhesive category.*

Proof. Products commute with limits, thus prod is continuous and the thesis now follows at once from Lemma 5.1.35, Corollaries 5.1.36 and 5.1.37, and Theorem 5.1.38. □

Remark 6.1.6. **Graph** is also equivalent to the category of presheaves on $0\rightrightarrows 1$, the category with just two objects and only two parallel arrows between them (besides the identities).

Remark 6.1.7. As a consequence of point 2 of the previous corollary, if $(f,g)\colon\mathcal{G}\to\mathcal{H}$ is a mono with codomain in **SGraph**, then $\mathcal{G}$ also belongs to **SGraph**.

We can also apply Proposition A.2.3 deducing the following.

Corollary 6.1.8. *The forgetful functor* $U_{\mathbf{Graph}}\colon$ **Graph** $\to$ **Set** *has a left adjoint* $\Delta_{\mathbf{Graph}}\colon$ **Set** $\to$ **Graph**.

$$\begin{tikzcd} X \arrow[r, mapsto] \arrow[d, "f"'] & (\emptyset,X,?_X,?_X) \arrow[d, "(\mathsf{id}_\emptyset, f)"] \\ Y \arrow[r, mapsto] & (\emptyset,Y,?_Y,?_Y) \end{tikzcd}$$

Let us now establish some properties of **SGraph** that will be useful in the following.

Proposition 6.1.9. *If* $(f, g)\colon \mathcal{G} \to \mathcal{H}$ *is an arrow in* **SGraph** *with* g *injective, then* f *is injective too.*

Proof. Let $e_1, e_2 \in E_\mathcal{G}$ be nodes such that $f(e_1) = f(e_2)$, then

$$\begin{aligned} g(s_\mathcal{G}(e_2)) &= s_\mathcal{H}(f(e_2)) & \qquad g(t_\mathcal{G}(e_2)) &= t_\mathcal{H}(f(e_2)) \\ &= s_\mathcal{H}(f(e_1)) & &= t_\mathcal{H}(f(e_1)) \\ &= g(s_\mathcal{G}(e_1)) & &= g(t_\mathcal{G}(e_1)) \end{aligned}$$

so that

$$s_\mathcal{G}(e_1) = s_\mathcal{G}(e_2) \qquad t_\mathcal{G}(e_1) = t_\mathcal{G}(e_2)$$

and the thesis follows since $\mathcal{H}$ is simple. □

Let $\mathcal{G} = (E_\mathcal{G}, V_\mathcal{G}, s_\mathcal{G}, t_\mathcal{G})$ be a directed graph. Define a relation $\sim$ on $E_\mathcal{G}$ putting $e_1 \sim e_2$ if and only if

$$s_\mathcal{G}(e_1) = s_\mathcal{G}(e_2) \qquad t_\mathcal{G}(e_1) = t_\mathcal{G}(e_2)$$

It is immediate to see that $\sim$ is an equivalence relation. If $\pi_\mathcal{G}\colon E_\mathcal{G} \to E_{L(\mathcal{G})}$ denotes the quotient projection, there are two unique functions $s_{L(\mathcal{G})}, t_{L(\mathcal{G})}\colon E_{L(\mathcal{G})} \rightrightarrows V_\mathcal{G}$ such that

$$s_\mathcal{G} = s_{L(\mathcal{G})} \circ \pi_G \qquad t_\mathcal{G} = t_{L(\mathcal{G})} \circ \pi_G$$

We can then consider the graph $L(\mathcal{G})$ given by $(E_{L(\mathcal{G})}, V_\mathcal{G}, s_{L(\mathcal{G})}, t_{L(\mathcal{G})})$ which, by construction is simple.

Proposition 6.1.10. *The inclusion functor* $I\colon$ **SGraph** $\to$ **Graph** *has a left adjoint* $L\colon$ **Graph** $\to$ **SGraph**.

Proof. For every $\mathcal{G}$ in **Graph**, there is an arrow $(\pi_\mathcal{G}, \mathsf{id}_{V_\mathcal{G}})\colon \mathcal{G} \to I(L(\mathcal{G}))$. Let $\mathcal{H}$ be a simple graph and (f, g) an arrow $G \to I(\mathcal{H})$. Since $\mathcal{H}$ is simple, we have that $f(e_1) = f(e_2)$ whenever $e_1 \sim e_2$, thus there exists a unique $\overline{f}\colon E_{L(\mathcal{G})} \to E_\mathcal{H}$ such that $f = \overline{f} \circ \pi_\mathcal{G}$. Moreover

$$\begin{aligned} s_\mathcal{H} \circ \overline{f} \circ \pi_\mathcal{G} &= s_\mathcal{H} \circ f & \qquad t_\mathcal{H} \circ \overline{f} \circ \pi_\mathcal{G} &= t_\mathcal{H} \circ f \\ &= g \circ s_\mathcal{G} & &= g \circ t_\mathcal{G} \\ &= g \circ s_{L(\mathcal{G})} \circ \pi_\mathcal{G} & &= g \circ t_{L(\mathcal{G})} \circ \pi_\mathcal{G} \end{aligned}$$

and, since $\pi_\mathcal{G}$ is surjective, this shows that $(\overline{f}, g)$ is the unique morphism $L(\mathcal{G}) \to \mathcal{H}$ such that

$$\begin{tikzcd} \mathcal{G} \arrow[rr, "(\pi_\mathcal{G}, \mathsf{id}_{V_\mathcal{G}})"] \arrow[dr, "(f,g)"'] & & I(L(\mathcal{G})) \\ & I(\mathcal{H}) \arrow[ur, "(\overline{f},g)"'] & \end{tikzcd}$$

commutes, therefore $(\pi_\mathcal{G}, \mathsf{id}_{V_\mathcal{G}})$ is the unit of $L \dashv I$. □

Remark 6.1.11. $(\pi_\mathcal{G}, \mathsf{id}_{V_\mathcal{G}})$ provides also the component at $\mathcal{G}$ of the counit $L \circ I \to \mathsf{id}_{\mathbf{SGraph}}$, so we can conclude that $L \circ I$ is isomorphic to the identity functor. Notice that this is an instance of the general fact that the counit of an adjunction $F \vdash G$ is an isomorphism if and only if G is full and faithful.

We have proved that I is a full and faithful right adjoint , thus it reflects and preserves monomorphisms, therefore, using Proposition 6.1.9, we can deduce the following result.

Corollary 6.1.12. *Given a morphism $(f,g)\colon \mathcal{G} \to \mathcal{H}$ in* **SGraph***, the following are equivalent*

1. *(f,g) is a mono in* **SGraph***;*
2. *f and g are injective;*
3. *g is injective.*

Corollary 6.1.13. *The functor L preserves monomorphisms.*

Proof. Let $(f,g)\colon \mathcal{G} \to \mathcal{H}$ be a monomorphism in **Graph**, then $L(f,g) = (\overline{f}, g)$ where $\overline{f}$ is the unique arrow $E_{L(\mathcal{G})} \to E_{L(\mathcal{H})}$ fitting in the diagram

$$\begin{tikzcd} E_{\mathcal{G}} \arrow[r, "f"] \arrow[d, "\pi_{\mathcal{G}}"'] & E_{\mathcal{H}} \arrow[d, "\pi_{\mathcal{H}}"] \\ E_{L(\mathcal{G})} \arrow[r, dotted, "\overline{f}"'] & E_{L(\mathcal{H})} \end{tikzcd}$$

By point 2 of Corollary 6.1.5 g is injective and Corollary 6.1.12 yields the thesis. $\square$

Corollary 6.1.14. *Let $D\colon \mathbf{D} \to$* **SGraph** *be a diagram and $(C, \{(f_D, g_D)\}_{D\in\mathbf{D}})$ a colimiting cocone for $I \circ D$, then $\left(L(C), \left\{L(f_D, g_D) \circ \left(\pi_{\mathcal{G}}^{-1}, \mathsf{id}_{V_{\mathcal{G}}}\right)\right\}_{D\in\mathbf{D}}\right)$ is colimiting for $\mathbf{D}$. In particular,* **SGraph** *is cocomplete.*

Proof. L is a left adjoint, thus it preserves colimits and therefore $(L(C), \{L(f_D, g_D)\}_{D\in\mathbf{D}})$ is colimiting for $L \circ I \circ D$ which, by Remark 6.1.11 is naturally isomorphic to D through $\pi * D$. $\square$

Proposition 6.1.15. *The forgetful functor $U_{\mathbf{SGraph}}$ obtained restricting $U_{\mathbf{Graph}}$ has both a left adjoint $\Delta_{\mathbf{SGraph}}$ and a right adjoint $\nabla_{\mathbf{SGraph}}$.*

Proof. For the left adjoint just compose L and $\Delta_{\mathbf{Graph}}$. To see that $U_{\mathbf{SGraph}}$ has a right adjoint, define $\nabla_{\mathbf{SGraph}}(X)$ as $(X \times X, X, \pi_1, \pi_2)$. For every set X we have $\mathsf{id}_X : U_{\mathbf{SGraph}}\left(\nabla_{\mathbf{SGraph}}(X)\right) \to X$. Moreover, if a function $g\colon U_{\mathbf{SGraph}}(\mathcal{G}) \to X$ is given, then we can take $(g \circ s_{\mathcal{G}}, g \circ t_{\mathcal{G}})\colon E_{\mathcal{G}} \to X \times X$. By construction $((g \circ s_{\mathcal{G}}, g \circ t_{\mathcal{G}}), g)$ is the unique arrow $\mathcal{G} \to \nabla_{\mathbf{SGraph}}(X)$ such that

$$g = \mathsf{id}_X \circ U_{\mathbf{SGraph}}\left((g \circ s_{\mathcal{G}}, g \circ t_{\mathcal{G}}), g\right)$$

and we can conclude. $\square$

Corollary 6.1.16. *$\mathcal{M}($***SGraph***$)$ is stable under pushouts.*

Proof. Take a pushout square with (f_1, g_1) in $\mathcal{M}(\mathbf{SGraph})$

$$\begin{tikzcd} \mathcal{H} \arrow[r, "(f_2,g_2)"] \arrow[d, "(f_1,g_1)"'] & \mathcal{K} \arrow[d, "(p_1,q_1)"] \\ \mathcal{G} \arrow[r, "(p_1,q_2)"'] & \mathcal{P} \end{tikzcd}$$

By Proposition 6.1.15 the following square, obtained applying $U_{\mathbf{SGraph}}$ is a pushout in **Set**

$$\begin{tikzcd} V_{\mathcal{H}} \arrow[r, "g_2"] \arrow[d, "g_1"'] & \mathcal{K} \arrow[d, "q_1"] \\ \mathcal{G} \arrow[r, "q_2"'] & \mathcal{P} \end{tikzcd}$$

By Corollary 6.1.12 g_1 is injective, so q_1 is injective too because **Set** is adhesive, thus, using again Corollary 6.1.12 we can conclude that (p_1, q_1) is mono. □

Our next step is to characterize regular monos of **SGraph**.

Definition 6.1.17. An arrow $(f, g) : \mathcal{G} \to \mathcal{H}$ in **Graph** *reflects the edges* if, for every $e \in \mathcal{H}(g(v_1), g(v_2))$ there exists $e' \in \mathcal{G}(v_1, v_2) \to \mathcal{H}(g(v_1), g(v_2))$ such that $e = f(e')$.

Remark 6.1.18. If $(f, g)\colon \mathcal{G} \to \mathcal{H}$ is an arrow of **SGraph**, then it reflects the edges if and only if $\mathcal{G}(v_1, v_2)$ is non empty whenever $\mathcal{H}(g(v_1), g(v_2)) \neq \emptyset$. Indeed, since $\mathcal{H}$ is simple, if e' belongs to $\mathcal{G}(v_1, v_2)$, then necessarily we must have $e = f(e')$.

Proposition 6.1.19. *An arrow $(f, g)\colon \mathcal{G} \to \mathcal{H}$ of* **SGraph** *is a regular monomorphism if and only if it reflects the edges and g is injective.*

Proof. ($\Rightarrow$) Suppose that (f, g) is the equalizer of $(f_1, g_1), (f_2, g_2)\colon \mathcal{H} \rightrightarrows \mathcal{K}$, since I preserves limits, (f, g) is the equalizer of (f_1, g_1) and (f_2, g_2) in **Graph**. Let $\mathcal{G}'$ be the graph where

$$E_{\mathcal{G}'} := \{e \in E_{\mathcal{H}} \mid f_1(e) = f_2(e)\} \qquad V_{\mathcal{G}'} := \{v \in V_{\mathcal{H}} \mid v_1(w) = v_2(w)\}$$

and $s_{\mathcal{G}'}, t_{\mathcal{G}'}$ are the restrictions of $s_{\mathcal{H}}$ and $t_{\mathcal{H}}$. Then, by Corollary 6.1.5 the inclusions $i\colon E_{\mathcal{G}'} \to E_{\mathcal{H}}$, $j\colon V_{\mathcal{G}'} \to V_{\mathcal{H}}$ provide an equalizer $(i, j)\colon \mathcal{G}' \to \mathcal{H}$ of (f_1, g_1) and (f_2, g_2) in **Graph**. By Remark 6.1.7, $\mathcal{G}'$ is an object of **SGraph**. I preserves limits so there exists an isomorphism $(\phi, \psi)\colon \mathcal{G} \to \mathcal{G}'$ such that

$$\begin{tikzcd} \mathcal{G} \arrow[rr, "{(f,g)}"] \arrow[dr, "{(\phi,\psi)}"'] & & \mathcal{H} \\ & \mathcal{G}' \arrow[ur, "{(i,j)}"'] & \end{tikzcd}$$

commutes. If we show that (i, j) is edge-reflecting we are done. For every $e \in \mathcal{H}(i(v_1), i(v_2))$ we have

$$\begin{aligned} s_{\mathcal{K}}(f_1(e)) &= g_1(s_{\mathcal{H}}(e)) & t_{\mathcal{K}}(f_1(e)) &= g_1(t_{\mathcal{H}}(e)) \\ &= g_1(i(v_1)) & &= g_1(i(v_1)) \\ &= g_2(i(v_1)) & &= g_2(i(v_1)) \\ &= g_2(s_{\mathcal{H}}(e)) & &= g_2(t_{\mathcal{H}}(e)) \\ &= s_{\mathcal{K}}(f_2(e)) & &= t_{\mathcal{K}}(f_2(e)) \end{aligned}$$

Thus e is an element of $E_{\mathcal{G}}$ because $\mathcal{K}$ is simple.
($\Leftarrow$) Take the set

$$V := V_{\mathcal{H}} + (V_{\mathcal{H}} \smallsetminus g(V_{\mathcal{G}}))$$

and define $E \subseteq V \times V$ putting $(v, v') \in E$ if and only if one of the following is true

- $v = i_1(w)$, $v' = i_1(w')$ and $\mathcal{H}(w,w') \neq \emptyset$;
- $v = i_2(w)$, $v' = i_2(w')$ and $\mathcal{H}(w,w') \neq \emptyset$;
- $v = i_1(w)$, $v' = i_2(w')$ and $\mathcal{H}(w,w') \neq \emptyset$;
- $v = i_2(w)$, $v' = i_1(w')$ and $\mathcal{H}(w,w') \neq \emptyset$;

where i_1 and i_2 are the inclusion of $V_\mathcal{H}$ and $V_\mathcal{H} \smallsetminus g(V_\mathcal{G})$ into V. Restricting the projections, we get two arrow $s,t\colon E \rightrightarrows V$, let $\mathcal{K}$ be the directed graph (E,V,s,t), which by construction is simple. Now, take

$$f\colon E_\mathcal{G} \to V \qquad e \mapsto (i_1(s_\mathcal{H}(e)), i_1(t_\mathcal{H}(e)))$$

coupled with $i_1\colon V_\mathcal{H} \to V$ it induces a morphism $(f,i_1)\colon \mathcal{H} \to \mathcal{K}$. On the other hand, define

$$i'\colon V_\mathcal{H} \to V \qquad w \mapsto \begin{cases} i_1(w) & w \in g(V_\mathcal{G}) \\ i_2(w) & w \notin g(V_\mathcal{G}) \end{cases}$$

and

$$f'\colon E_\mathcal{H} \to E \qquad e \mapsto \begin{cases} (i_1(s_\mathcal{H}(e)), i_1(t_\mathcal{H}(e))) & s_\mathcal{H}(e), t_\mathcal{H}(e) \in g(V_\mathcal{G}) \\ (i_2(s_\mathcal{H}(e)), i_2(t_\mathcal{H}(e))) & s_\mathcal{H}(e), t_\mathcal{H}(e) \notin g(V_\mathcal{G}) \\ (i_1(s_\mathcal{H}(e)), i_2(t_\mathcal{H}(e))) & s_\mathcal{H}(e) \in g(V_\mathcal{G}) \\ (i_2(s_\mathcal{H}(e)), i_1(t_\mathcal{H}(e))) & t_\mathcal{H}(e) \in g(V_\mathcal{G}) \end{cases}$$

Define the set $A \subseteq E_\mathcal{H}$ as

$$A := \{e \in E_\mathcal{H} \mid s_\mathcal{H}(e), t_\mathcal{H}(e) \in g(V_\mathcal{G})\}$$

with inclusion $i\colon A \to E_\mathcal{H}$. Let also j be the inclusion $g(V_\mathcal{H}) \to V_\mathcal{H}$. By construction there are arrows $s,t\colon A \rightrightarrows g(V_\mathcal{H})$ such that the following diagrams commute:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "s"'] & E_\mathcal{H} \arrow[d, "s_\mathcal{H}"] \\ g(V_\mathcal{G}) \arrow[r, "j"'] & V_\mathcal{H} \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "i"] \arrow[d, "t"'] & E_\mathcal{H} \arrow[d, "t_\mathcal{H}"] \\ g(V_\mathcal{G}) \arrow[r, "j"'] & V_\mathcal{H} \end{tikzcd}$$

Putting $\mathcal{G}' := (A, g(V_\mathcal{G}), s, t)$ we get a (simple) graph, with an inclusion $(i,j)\colon \mathcal{G}' \to \mathcal{G}$ which is the equalizer in **Graph** of (f,i_1) and (f',i').

Now, $g = j \circ \phi$ for some $\phi\colon V_\mathcal{H} \to g(V_\mathcal{G})$ and, since (f,g) is a morphism of **SGraph**, $f = i \circ \psi$ for some $\psi\colon E_\mathcal{H} \to A$. We have the following two chains of identities

$$\begin{aligned} j \circ \phi \circ s_\mathcal{G} &= g \circ s_G \\ &= s_\mathcal{H} \circ f \\ &= s_\mathcal{H} \circ i \circ \psi \\ &= j \circ s \circ \psi \end{aligned} \qquad \begin{aligned} j \circ \phi \circ t_\mathcal{G} &= g \circ t_G \\ &= t_\mathcal{H} \circ f \\ &= t_\mathcal{H} \circ i \circ \psi \\ &= j \circ t \circ \psi \end{aligned}$$

Since j is injective, we obtain a morphism $(\psi,\phi)\colon \mathcal{G} \to \mathcal{G}'$. Moreover, by construction ϕ is surjective and g is injective by hypothesis, thus also ϕ is injective and, by Corollary 6.1.12, we can deduce that ψ is injective too. Let us show that ψ is also surjective. Given $e \in A$, then $e \in \mathcal{H}(g(v_1), g(v_2))$ for some $v_1, v_2 \in V_\mathcal{G}$, thus there exists $e' \in \mathcal{G}(v_1,v_2)$ and, necessarily, $f(e') = e$, but this means that $\psi(e') = e$. $\square$

Example 6.1.20. In [67] it is shown that **SGraph** is not quasiadhesive. Too see this, using Corollary 5.2.25, it is enough to notice that the union of regular monos in **SGraph** is not a regular mono. Take

$$\mathcal{G} := (1, \{0, 1\}, \delta_0, \delta_1) \qquad \mathcal{M} := (\emptyset, 1, ?_1, ?_1)$$

Then we have morphisms $(?_1, \delta_0), (?_1, \delta_1) \colon \mathcal{M} \rightrightarrows \mathcal{G}$ which, by Proposition 6.1.19, are regular monos. Their supremum in $\mathsf{Sub}(\mathcal{G})$ is the inclusion of $(\emptyset, \{0, 1\}, ?_0, ?_1)$ into $\mathcal{G}$ which, again by Proposition 6.1.19, is not a regular monomorphism.

Definition 6.1.21. A monomorphism $(f, g) \colon \mathcal{G} \to \mathcal{H}$ in **Graph** is said to be *downward closed* if, for all $e \in E_{\mathcal{H}}$, $e \in f(E_{\mathcal{G}})$ whenever $t_{\mathcal{H}}(e) \in g(V_{\mathcal{G}})$. We denote by dcl, $\mathsf{dcl_s}$ and $\mathsf{dcl_d}$ the classes of downward closed morphisms in **Graph**, **SGraph** and **DAG** respectively.

Proposition 6.1.22. *Every downward closed morphism in* **SGraph** *is a regular mono.*

Proof. Let $(f, g) \colon \mathcal{G} \to \mathcal{H}$ be an element of $\mathsf{dcl_s}$, we only have to check that it is edge-reflecting. Given $e \in \mathcal{H}(g(v_1), g(v_2))$, since (f, g) is downward closed there exists e' such that $f(e') = e$. But then

$$\begin{aligned} g(s_{\mathcal{G}}(e')) &= s_{\mathcal{H}}(e) \qquad & g(t_{\mathcal{G}}(e')) &= t_{\mathcal{H}}(e) \\ &= g(v_1) & &= g(v_2) \end{aligned}$$

and, since g is injective, it follows that $e' \in \mathcal{G}(v_1, v_2)$. □

Remark 6.1.23. The converse of the previous proposition does not hold. A counterexample is given by the arrow $(?_1, \delta_1) \colon (\emptyset, 1, ?_1, ?_1) \to (1, \{0, 1\}, \delta_0, \delta_1)$.

Proposition 6.1.24. *Take an arrow $(f, g) \colon \mathcal{G} \to \mathcal{H}$ in* **Graph** *and consider the functor $L \colon$* **Graph** $\to$ **SGraph** *left adjoint to the inclusion, then the following hold true:*

1. *if (f, g) is in dcl then $L(f, g)$ is in $\mathsf{dcl_d}$;*
2. *if (f, g) reflects the edges then $L(f, g)$ reflects the edges too.*

Proof. 1. Take an element $(f, g) \colon \mathcal{G} \to \mathcal{H}$ of dcl and let $L(f, g)$ be $(\overline{f}, g)$ as in Corollary 6.1.13. If $t_{L(\mathcal{H})}(\pi_{\mathcal{H}}(e))$ is equal to $g(v)$ for some $v \in V_{\mathcal{G}}$ then we also have

$$t_{\mathcal{H}}(e) = g(v)$$

so that there exists $e' \in E_{\mathcal{G}}$ such that $f(e') = e$. But then

$$\begin{aligned} \overline{f}(\pi_{\mathcal{G}}(e')) &= \pi_{\mathcal{H}}(f(e')) \\ &= \pi_{\mathcal{H}}(e) \end{aligned}$$

which is what we need to conclude.

2. As before, let $L(f, g)$ be $(\overline{f}, g)$ and suppose that $\pi_{\mathcal{G}}(e)$ be an edge between $g(v)$ and $g(v')$ in $L(\mathcal{H})$. Then e is an edge in $\mathcal{H}(g(v), g(v'))$ and thus there exists $e' \in \mathcal{G}(v, v')$ such that $e = f(e')$, but this, by construction, entails $\overline{f}(e') = e$. □

Corollary 6.1.25. $\mathcal{R}$(**SGraph**) *is stable under pushouts.*

Proof. Let $(f_1, g_1)\colon \mathcal{H} \to \mathcal{G}$ be a regular mono in **SGraph**. Given another $(f_2.g_2)\colon \mathcal{H} \to \mathcal{K}$ we can consider the following two diagrams, the first of which is a pushout square in **Graph**, while the second one is a pushout in **SGraph** by Corollary 6.1.14.

$$\begin{tikzcd} \mathcal{H} \arrow[r, "(f_2,g_2)"] \arrow[d, "(f_1,g_1)"'] & \mathcal{K} \arrow[d, "(p_1,q_1)"] \\ \mathcal{G} \arrow[r, "(p_2,q_2)"'] & \mathcal{P} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{H} \arrow[rr, "(f_2,g_2)"] \arrow[dd, "(f_1,g_1)"'] & & \mathcal{K} \arrow[d, "(\pi_{\mathcal{G}}^{-1}, \mathsf{id}_{V_{\mathcal{G}}})"] \\ & & L(I(\mathcal{K})) \arrow[d, "L(p_1,q_1)"] \\ \mathcal{G} \arrow[r, "(\pi_{\mathcal{G}}^{-1}, \mathsf{id}_{V_{\mathcal{G}}})"'] & L(I(\mathcal{G})) \arrow[r, "L(p_2,q_2)"'] & L(\mathcal{P}) \end{tikzcd}$$

Since **Graph** is adhesive, we already know that (p_1, q_1) a monomorphism, thus if we show that it reflects the edges we get the thesis using Corollary 6.1.12 and Propositions 6.1.19 and 6.1.24.

By Corollary 6.1.5 we also know that the squares below are pushouts in **Set** and that $s_{\mathcal{P}}, t_{\mathcal{P}}\colon E_{\mathcal{P}} \rightrightarrows V_{\mathcal{P}}$ are the arrows induced by $q_2 \circ s_{\mathcal{K}}$, $q_1 \circ s_{\mathcal{G}}$ and by $q_2 \circ t_{\mathcal{K}}$, $q_1 \circ t_{\mathcal{K}}$ respectively.

$$\begin{tikzcd} E_{\mathcal{H}} \arrow[r, "f_2"] \arrow[d, "f_1"'] & E_{\mathcal{K}} \arrow[d, "p_1"] \\ E_{\mathcal{G}} \arrow[r, "p_2"'] & E_{\mathcal{P}} \end{tikzcd} \qquad \begin{tikzcd} V_{\mathcal{H}} \arrow[r, "g_2"] \arrow[d, "g_1"'] & V_{\mathcal{K}} \arrow[d, "q_1"] \\ V_{\mathcal{G}} \arrow[r, "q_2"'] & V_{\mathcal{P}} \end{tikzcd}$$

Let us take an edge $e \in \mathcal{P}\,(q_1(v), q_1(v'))$. If $e = p_1(e')$ for some $e' \in E_{\mathcal{K}}$ then

$$\begin{aligned} q_1(s_{\mathcal{K}}(e')) &= s_{\mathcal{P}}(p_1(e')) & \qquad q_1(t_{\mathcal{K}}(e')) &= t_{\mathcal{P}}(p_1(e')) \\ &= s_{\mathcal{P}}(e) & &= t_{\mathcal{P}}(e) \\ &= q_1(v) & &= q_1(v') \end{aligned}$$

showing that e' is an edge between v and v' as wanted. On the other hand, if $e = p_2(e')$ for some $e' \in E_{\mathcal{G}}$, then

$$\begin{aligned} q_2(s_{\mathcal{G}}(e')) &= s_{\mathcal{P}}(p_2(e')) & \qquad q_2(t_{\mathcal{G}}(e')) &= t_{\mathcal{P}}(p_2(e')) \\ &= s_{\mathcal{P}}(e) & &= t_{\mathcal{P}}(e) \\ &= q_1(v) & &= q_1(v') \end{aligned}$$

and by Lemma 6.1.1 this means that there are h_1, h_2 in $V_{\mathcal{H}}$ such that

$$s_{\mathcal{G}}(e') = f_1(h_1) \quad v = f_2(h_1) \quad t_{\mathcal{G}}(e') = f_1(h_2) \quad v' = f_2(h_2)$$

Since (f_1, g_1) reflects the edges we get $\overline{e} \in E_{\mathcal{H}}$ such that $f_1(\overline{e}) = e'$ and so

$$\begin{aligned} p_1(f_2(\overline{e})) &= p_2(f_1(\overline{e})) \\ &= p_2(e') \\ &= e \end{aligned}$$

In particular this means that

$$\begin{aligned} q_1(s_{\mathcal{K}}(f_2(\overline{e}))) &= s_{\mathcal{P}}(p_1(f_2(\overline{e}))) \\ &= s_{\mathcal{P}}(e) \\ &= q_1(v) \end{aligned} \qquad \begin{aligned} q_1(t_{\mathcal{K}}(f_2(\overline{e}))) &= t_{\mathcal{P}}(p_1(f_2(\overline{e}))) \\ &= t_{\mathcal{P}}(e) \\ &= q_1(v') \end{aligned}$$

showing that $f_1(\overline{e})$ belongs to $\mathcal{K}(v, v')$. □

We are now ready to show some closure properties of **DAG** and **SGraph** in **Graph**.

Lemma 6.1.26. *The following are true:*

1. **SGraph** *and* **DAG** *are closed in* **Graph** *under pullbacks;*
2. **SGraph** *is closed in* **Graph** *under* $\mathcal{R}(\mathbf{SGraph}), \mathcal{M}(\mathbf{SGraph})$*-pushouts;*
3. **DAG** *is closed in* **Graph** *under* $\mathsf{dcl_d}, \mathcal{M}(\mathbf{DAG})$*-pushouts.*

Proof. 1. By Corollary 6.1.5, we can construct the pullback $\mathcal{P}$ of $(f_1, g_1)\colon \mathcal{G} \to \mathcal{H}$ along the arrow $(f_2, g_2)\colon \mathcal{K} \to \mathcal{H}$ using the pullbacks

$$\begin{tikzcd} E_{\mathcal{P}} \arrow[r, "p_1"] \arrow[d, "p_2"'] & E_{\mathcal{K}} \arrow[d, "f_2"] \\ E_{\mathcal{G}} \arrow[r, "f_1"'] & E_{\mathcal{H}} \end{tikzcd} \qquad \begin{tikzcd} V_{\mathcal{P}} \arrow[r, "q_1"] \arrow[d, "q_2"'] & V_{\mathcal{K}} \arrow[d, "q_2"] \\ V_{\mathcal{G}} \arrow[r, "g_1"'] & V_{\mathcal{H}} \end{tikzcd}$$

and defining $s_{\mathcal{P}}, t_{\mathcal{P}}\colon E_{\mathcal{P}} \rightrightarrows V_{\mathcal{P}}$ as the arrows induced by $s_{\mathcal{K}} \circ p_1$, $s_{\mathcal{G}} \circ p_2$ and by $t_{\mathcal{K}} \circ p_1$, $t_{\mathcal{G}} \circ p_2$. The colimiting cone is then given by (p_1, q_1) and (p_2, q_2). Now, suppose that $\mathcal{G}$ and $\mathcal{K}$ are simple, then if there are $e, e' \in E_{\mathcal{P}}$ with

$$s_{\mathcal{P}}(e) = s_{\mathcal{P}}(e') \qquad t_{\mathcal{P}}(e) = t_{\mathcal{P}}(e')$$

we also have

$$\begin{aligned} s_{\mathcal{K}}(p_1(e)) &= q_1(s_{\mathcal{P}}(e)) \\ &= q_1(s_{\mathcal{P}}(e')) \\ &= s_{\mathcal{K}}(p_1(e')) \end{aligned} \qquad \begin{aligned} t_{\mathcal{K}}(p_1(e)) &= q_1(t_{\mathcal{P}}(e)) \\ &= q_1(t_{\mathcal{P}}(e')) \\ &= t_{\mathcal{K}}(p_1(e')) \end{aligned}$$

$$\begin{aligned} s_{\mathcal{G}}(p_2(e)) &= q_2(s_{\mathcal{P}}(e)) \\ &= q_2(s_{\mathcal{P}}(e')) \\ &= s_{\mathcal{G}}(p_1(e')) \end{aligned} \qquad \begin{aligned} t_{\mathcal{G}}(p_2(e)) &= q_2(t_{\mathcal{P}}(e)) \\ &= q_2(t_{\mathcal{P}}(e')) \\ &= t_{\mathcal{G}}(p_2(e')) \end{aligned}$$

showing that

$$p_1(e) = p_1(e') \qquad p_2(e) = p_2(e')$$

and so we can conlude that $e = e'$. In particular, **SGraph** is closed in **Graph** under pullbacks. On the other hand, if $\mathcal{G}$ or $\mathcal{H}$ is in **DAG**, then Remark 6.1.3 entails that $\mathcal{P}$ is also in **DAG** and thus also **DAG** is closed in **Graph** under pullbacks.

2. Using again Corollary 6.1.5, we see that, given $(f_1, g_1)\colon \mathcal{H} \to \mathcal{G}$ and $(f_2, g_2)\colon \mathcal{H} \to \mathcal{K}$, their pushout $\mathcal{P}$ is defined taking the two pushout squares

$$\begin{tikzcd} E_{\mathcal{H}} \arrow[r, "f_2"] \arrow[d, "f_1"'] & E_{\mathcal{K}} \arrow[d, "p_1"] \\ E_{\mathcal{G}} \arrow[r, "p_2"'] & E_{\mathcal{P}} \end{tikzcd} \qquad \begin{tikzcd} V_{\mathcal{H}} \arrow[r, "g_2"] \arrow[d, "g_1"'] & V_{\mathcal{K}} \arrow[d, "q_1"] \\ V_{\mathcal{G}} \arrow[r, "q_2"'] & V_{\mathcal{P}} \end{tikzcd}$$

with the arrow induced by $q_2 \circ s_{\mathcal{K}}$ and $q_1 \circ s_{\mathcal{G}}$ as $s_{\mathcal{P}}$, while $t_{\mathcal{P}}$ is the one coming from $q_2 \circ t_{\mathcal{K}}$, $q_1 \circ t_{\mathcal{G}}$. Suppose now that (f_1, g_1) is in $\mathcal{R}(\mathbf{SGraph})$ and (f_2, g_2) in $\mathcal{M}(\mathbf{SGraph})$. By Corollary 6.1.5 and Propositions 6.1.9 and 6.1.19 we know that f_1, f_2, g_1 and g_2 are injective. Since **Set** is adhesive this implies that p_1, p_2, q_1 and q_2 are injective too. Take now two elements e_1 and e_2 of $\mathcal{P}(v, v')$, we can use point 1 of Lemma 6.1.1 to split the cases.

- If $e_1 = p_1(e_1')$ and $e_2 = p_1(e_2')$ for some $e_1', e_2' \in E_{\mathcal{K}}$. Then

$$\begin{aligned} q_1(s_{\mathcal{K}}(e_1')) &= s_{\mathcal{P}}(p_1(e_1')) & \qquad q_1(t_{\mathcal{K}}(e_1')) &= t_{\mathcal{P}}(p_1(e_1')) \\ &= s_{\mathcal{P}}(e_1) & &= t_{\mathcal{P}}(e_1) \\ &= v & &= v' \\ &= s_{\mathcal{P}}(e_2) & &= t_{\mathcal{P}}(e_2) \\ &= s_{\mathcal{P}}(p_1(e_2')) & &= t_{\mathcal{P}}(p_1(e_2')) \\ &= q_1(s_{\mathcal{K}}(e_2')) & &= q_1(t_{\mathcal{K}}(e_2')) \end{aligned}$$

  By the injectivity of q_1 is injective we get

$$s_{\mathcal{K}}(e_1') = s_{\mathcal{K}}(e_2') \qquad t_{\mathcal{K}}(e_1') = t_{\mathcal{K}}(e_2')$$

  therefore, since $\mathcal{K}$ is simple, we know that $e_1' = e_2'$ and thus $e_1 = e_2$.
- Similarly, if $e_1 = p_2(e_1')$ and $e_2 = p_2(e_2')$ for some $e_1', e_2' \in E_{\mathcal{G}}$ we can compute again to get

$$\begin{aligned} q_2(s_{\mathcal{K}}(e_1')) &= s_{\mathcal{P}}(p_2(e_1')) & \qquad q_2(t_{\mathcal{K}}(e_1')) &= t_{\mathcal{P}}(p_2(e_1')) \\ &= s_{\mathcal{P}}(e_1) & &= t_{\mathcal{P}}(e_1) \\ &= v & &= v' \\ &= s_{\mathcal{P}}(e_2) & &= t_{\mathcal{P}}(e_2) \\ &= s_{\mathcal{P}}(p_2(e_2')) & &= t_{\mathcal{P}}(p_2(e_2')) \\ &= q_2(s_{\mathcal{K}}(e_2')) & &= q_2(t_{\mathcal{K}}(e_2')) \end{aligned}$$

  and the thesis now follows using the injectivity of q_2.
- $e_1 = p_1(e_1')$ and $e_2 = p_2(e_2')$ for some $e_1' \in \mathcal{K}$ and $e_2' \in E_{\mathcal{G}}$. Therefore we have

$$\begin{aligned} p_1(s_{\mathcal{K}}(e_1')) &= v & \qquad p_1(t_{\mathcal{K}}(e_1')) &= v' \\ &= p_2(s_{\mathcal{G}}(e_2')) & &= p_2(t_{\mathcal{G}}(e_2')) \end{aligned}$$

  Thus by Lemma 6.1.1 there exist w_1 and $w_2 \in V_{\mathcal{H}}$ such that

$$g_1(w_1) = s_{\mathcal{G}}(e_2'), \quad g_2(w_1) = s_{\mathcal{K}}(e_1') \quad g_1(w_2) = t_{\mathcal{G}}(e_1'), \quad g_2(w_2) = t_{\mathcal{K}}(e_2')$$

Hence $e'_1 \in \mathcal{G}(g_1(w_1), g_1(w_2))$, but (f_1, g_1) is regular, so Proposition 6.1.19 entails the existence of $e \in \mathcal{H}(w_1, w_2)$. Now, $f_1(e) = e'_1$, while

$$\begin{aligned} s_{\mathcal{K}}(f_2(e)) &= g_2(s_{\mathcal{H}}(e)) & t_{\mathcal{K}}(f_2(e)) &= g_2(t_{\mathcal{H}}(e)) \\ &= g_2(w_1) & &= g_2(w_1) \\ &= s_{\mathcal{K}}(e'_1) & &= t_{\mathcal{K}}(e'_1) \end{aligned}$$

and thus $f_2(e) = e'_1$. We conclude that $e_1 = e_2$ in $E_{\mathcal{P}}$

- $e_1 = p_2(e'_1)$ and $e_2 = p_1(e'_2)$ for some $e'_1 \in \mathcal{G}$ and $e'_2 \in E_{\mathcal{K}}$. This is done exactly as in the previous point swapping the roles of e'_1 and e'_2.

3. Now let (f_1, g_1) and (f_2, g_2) be, respectively, a downward closed morphism and a mono in **DAG**, we are going to use agin the explicit construction pushouts in **Graph**. Suppose that a cycle $\{e_i\}_{i=1}^n$ in $\mathcal{P}$ is given. We split again the cases using Lemma 6.1.1.

   - For every $1 \le i \le n$, $e_i = p_1(e'_i)$ for $e'_i \in E_{\mathcal{K}}$. Then

   $$\begin{aligned} q_1(s_{\mathcal{K}}(e'_1)) &= s_{\mathcal{P}}(e_1) & q_1(t_{\mathcal{K}}(e'_i)) &= t_{\mathcal{P}}(e_i) \\ &= t_{\mathcal{P}}(e_n) & &= s_{\mathcal{P}}(e_{i+1}) \\ &= q_1(t_{\mathcal{K}}(e'_n)) & &= q_1(t_{\mathcal{K}}(e'_{i+1})) \end{aligned}$$

   As before, q_1 is injective because is the pushout of an injective function, thus $\{e'_i\}_{i=1}^n$ is a cycle in $\mathcal{K}$, which is absurd.

   - For every $1 \le i \le n$, $e_i = p_2(e'_i)$ for $e'_i \in E_{\mathcal{G}}$. Then

   $$\begin{aligned} q_2(s_{\mathcal{G}}(e'_1)) &= s_{\mathcal{P}}(e_1) & q_2(t_{\mathcal{G}}(e'_i)) &= t_{\mathcal{P}}(e_i) \\ &= t_{\mathcal{P}}(e_n) & &= s_{\mathcal{P}}(e_{i+1}) \\ &= q_2(t_{\mathcal{G}}(e'_n)) & &= q_2(t_{\mathcal{G}}(e'_{i+1})) \end{aligned}$$

   We can conclude again appealing to the injectivity of q_2.

   - To deal with the other cases we can reason in the following way. Take $e = p_1(e')$ for some $e' \in E_{\mathcal{K}}$ and suppose that there exists $a = p_2(a')$ for some $a' \in E_{\mathcal{G}}$ such that $s_{\mathcal{P}}(e) = t_{\mathcal{P}}(a)$. By Lemma 6.1.1 there exists $v \in V_{\mathcal{H}}$ such that

   $$\begin{aligned} q_2(g_1(v)) &= t_{\mathcal{P}}(a) \\ &= q_2(p_2(a')) \end{aligned}$$

   q_2 is injective, thus $g_1(v) = p_2(a')$. Since $(f_1, g_1) \in \mathsf{dcl_d}$ there exists $b \in E_{\mathcal{H}}$ such that $f_1(b) = a'$. Thus $a = p_1(f_2(b))$ belongs to $p_1(E_{\mathcal{K}})$.
   Let us apply this argument to our cycle $\{e_i\}_{i=1}^n$. By Lemma 6.1.1 and the second point above, there must be an index j such that $e_j \in p_1(E_{\mathcal{K}})$. Now, if $j > 1$ the previous argument shows that $e_{j-1} \in p_1(E_{\mathcal{K}})$ too, thus surely $e_1 \in p_1(E_{\mathcal{K}})$. But, since $\{e_i\}_{i=1}^n$ is a cycle, the same argument shows that $e_n \in p_1(E_{\mathcal{K}})$ and this implies that every $e_i \in p_1(E_{\mathcal{K}})$ for every $1 \le i \le n$, but we already know that this is absurd. □

In particular, this implies that the inclusion **DAG** $\to$ **Graph** preserves monomorphisms, since it is a full inclusion we get an analog of Corollary 6.1.12.

Corollary 6.1.27. *Given a morphism $(f,g)\colon \mathcal{G}\to\mathcal{H}$ in* **DAG**, *the following are equivalent*

1. (f,g) *is a mono;*
2. f *and* g *are injective;*
3. g *is injective.*

We can also establish another result, regarding pushouts in **DAG**.

Proposition 6.1.28. *Let J be the inclusion* **DAG** $\to$ **SGraph**, *given a diagram $F\colon \mathbf{D}\to$* **DAG**, *the following are equivalent:*

1. F *has a colimit;*
2. $J\circ F$ *has a colimiting cocone $(\mathcal{C},\{(c_D,d_D)\}_{D\in\mathbf{D}})$ with C acyclic.*

Proof. $(1\Rightarrow 2)$ Let $(\mathcal{A},\{(a_D,b_D)\}_{D\in\mathbf{D}})$ be a colimiting cocone for F in **DAG**. By Corollary 6.1.14 we know that $J\circ F$ also has a colimiting cocone $(\mathcal{C},\{(c_D,d_D)\}_{D\in\mathbf{D}})$. $(J(\mathcal{A}),\{J(a_D,b_D)\}_{D\in\mathbf{D}})$ is a cocone on $J\circ D$ and thus there exists an arrow $(a,b)\colon\mathcal{C}\to\mathcal{A}$ and the thesis follows from Remark 6.1.3.
$(2\Rightarrow 1)$ This follows from the fact that J is full and faithful and thus it creates colimits. □

Corollary 6.1.29. *The inclusion $J\colon$* **DAG** $\to$ **SGraph** *preserves colimits.*

Proof. Let $F\colon\mathbf{D}\to$ **DAG** be a diagram with colimiting cocone $(\mathcal{A},\{(a_D,b_D)\}_{D\in\mathbf{D}})$, by Proposition 6.1.28 in **SGraph** there exists a colimiting cocone $(\mathcal{C},\{(c_D,d_D)_{D\in\mathbf{D}}\})$ for $J\circ F$ with C acyclic. Since J is full and faithful we get that $(\mathcal{C},\{(c_D,d_D)_{D\in\mathbf{D}}\})$ is colimiting for F too and thus there is an isomorphism $(\phi,\psi)\colon\mathcal{C}\to\mathcal{A}$ in **DAG** such that

$$(a_D,b_D)=(\phi,\psi)\circ(c_D,d_D)$$

and this now implies that $(J(\mathcal{A}),\{J(a_D,b_D)\}_{D\in\mathbf{D}})$ is colimiting for $J\circ F$ □

Corollary 6.1.30. $\mathcal{M}(\mathbf{DAG})$ *is stable under pushouts.*

Proof. Let $(f_1,g_1)\colon\mathcal{H}\to\mathcal{G}$ be a mono in **DAG** and take a pushout square

$$\begin{tikzcd}
\mathcal{H} \arrow[r, "{(f_2,g_2)}"] \arrow[d, "{(f_1,g_1)}"'] & \mathcal{K} \arrow[d, "{(p_1,q_1)}"] \\
\mathcal{G} \arrow[r, "{(p_2,q_2)}"'] & \mathcal{P}
\end{tikzcd}$$

By Corollary 6.1.29 the same square is a pushout in **SGraph**, and, by Corollaries 6.1.12 and 6.1.27, (f_1,g_1) is a mono in **SGraph** too, so Corollary 6.1.16 entails that $(p_1,q_1)\in\mathcal{M}(\mathbf{SGraph})$ and we conclude using again Corollaries 6.1.12 and 6.1.27. □

Our next step is to establish some kind of stability also for downward-closed morphisms of **DAG**.

Proposition 6.1.31. *The class* $\mathsf{dcl_d}$ *is stable under pullbacks and pushouts.*

Proof. Let us show the two halves of the thesis separately

- $\mathsf{dcl_d}$ is stable under pullbacks. Take pullback square as the one below with $(f,g) \in \mathsf{dcl_d}$

$$\begin{tikzcd}
\mathcal{P} \arrow[r, "(p_2,q_2)"] \arrow[d, "(p_1,q_1)"'] & \mathcal{K} \arrow[d, "(f_2,g_2)"] \\
\mathcal{G} \arrow[r, "(f_1,g_1)"'] & \mathcal{H}
\end{tikzcd}$$

Let $e_1 \in E_{\mathcal{G}}$ be an edge such that

$$t_{\mathcal{G}}(e_1) = q_1(v)$$

for some $v \in V_{\mathcal{P}}$. We have

$$\begin{aligned}
t_{\mathcal{H}}(f_1(e_1)) &= g_1(t_{\mathcal{G}}(e_1)) \\
&= g_1(q_1(v)) \\
&= g_2(q_2(v))
\end{aligned}$$

By hypothesis, $(f_2, g_2) \in \mathsf{dcl_d}$, and so there exist $e_2 \in E_{\mathcal{K}}$ such that

$$f_2(e_2) = f_1(e_1)$$

But, since $E_{\mathcal{P}}$ is a pullback, this implies the existence of $e \in E_{\mathcal{P}}$ such that

$$e_1 = p_1(e) \qquad e_2 = p_2(e)$$

In particular, we get that (p_1, q_1) is an element of $\mathsf{dcl_d}$.

- $\mathsf{dcl_d}$ is stable under pushouts. Take a pushout square in **SGraph** with (f_1, g_1) in $\mathsf{dcl_d}$.

$$\begin{tikzcd}
\mathcal{H} \arrow[r, "(f_2,g_2)"] \arrow[d, "(f_1,g_1)"'] & \mathcal{K} \arrow[d, "(p'_1,q'_1)"] \\
\mathcal{G} \arrow[r, "(p'_2,q'_2)"'] & \mathcal{P}'
\end{tikzcd}$$

By Corollary 6.1.29 the square above is a pushout in **SGraph** too, which, Corollary 6.1.14, must fit in a diagram

$$\begin{tikzcd}
\mathcal{H} \arrow[r, "(f_2,g_2)"] \arrow[d, "(f_1,g_1)"'] & \mathcal{K} \arrow[d, "(p'_1,q'_1)"'] \arrow[dr, "(\pi_{\mathcal{K}}^{-1},\mathsf{id}_{V_{\mathcal{K}}})"] \arrow[rrrddd, bend left, "(p_1,q_1)"] & & \\
\mathcal{G} \arrow[r, "(p'_2,q'_2)"] \arrow[dr, "(\pi_{\mathcal{G}}^{-1},\mathsf{id}_{V_{\mathcal{G}}})"] \arrow[rrrdd, bend right, "(p_1,q_2)"'] & \mathcal{P}' \arrow[dr, "(\phi,\psi)"] & L(I(\mathcal{K})) \arrow[d, "(L(p'_1,q'_1))"] & \\
& L(I(\mathcal{G})) \arrow[r, "(L(p'_2,q'_2))"'] & L(\mathcal{P}) & \\
& & & \mathcal{P} \arrow[ul, "(\pi_{\mathcal{P}},\mathsf{id}_{V_{\mathcal{P}}})"']
\end{tikzcd}$$

where the outer edges form a pushout in **Graph** and $(\phi,\psi)\colon \mathcal{P}' \to L(\mathcal{P})$ is an isomorphism. If we show that (p_1, q_1) is in dcl, Proposition 6.1.24 yields the thesis.

Suppose that $e \in E_{\mathcal{P}}$ is such that for some $v \in V_{\mathcal{K}}$

$$t_{\mathcal{P}}(e) = q_1(v)$$

If $e \in p_1(E_{\mathcal{K}})$ there is nothing to show. Otherwise, by Lemma 6.1.1 we know that there exists $e' \in E_{\mathcal{G}}$ such that $p_2(e') = e$, but then

$$\begin{aligned} q_1(v) &= t_{\mathcal{P}}(e) \\ &= q_2(t_{\mathcal{G}}(e')) \end{aligned}$$

Thus, again by Lemma 6.1.1 there exists $w \in V_{\mathcal{H}}$ such that

$$g_1(w) = t_{\mathcal{G}}(e') \qquad g_2(w) = v$$

Since, by hypothesis, (f_1, g_1) is in dcl, there exists $e'' \in E_{\mathcal{H}}$ such that $f_1(e'') = e'$, so that

$$\begin{aligned} e &= p_2(e') \\ &= p_2(f_1(e'')) \\ &= p_1(f_2(e'')) \end{aligned}$$

which shows that e is in the image of p_1 as claimed. □

We can now deduce the following results from Theorem 5.1.31 and Lemma 6.1.26.

Corollary 6.1.32. *The following are true*

1. **SGraph** *is* $\mathcal{R}(\mathbf{SGraph}), \mathcal{M}(\mathbf{SGraph})$*-adhesive*
2. **SGraph** *is* $\mathcal{M}(\mathbf{SGraph}), \mathcal{R}(\mathbf{SGraph})$*-adhesive*
3. **DAG** *is* $\mathsf{dcl_d}, \mathcal{M}(\mathbf{DAG})$*-adhesive.*

Proof. We only have to show that the pairs $(\mathcal{R}(\mathbf{SGraph}), \mathcal{M}(\mathbf{SGraph}))$, $(\mathcal{M}(\mathbf{SGraph}), \mathcal{R}(\mathbf{SGraph}))$ are preadhesive structures on **SGraph** and that $(\mathsf{dcl_d}, \mathcal{M}(\mathbf{DAG}))$ is a preadhesive structure on **DAG**. We already know by Corollaries 6.1.16, 6.1.25 and 6.1.30 and Proposition 6.1.31 that all these classes are stable under pullbacks and pushouts and clearly they contains all isomorphisms and are closed under composition. For the decomposition properties: $\mathcal{M}(\mathbf{X})$ is closed under decomposition, and $\mathcal{R}(\mathbf{X})$ is closed under $\mathcal{M}(\mathbf{X})$-decomposition for every category **X**, so $\mathcal{M}(\mathbf{SGraph})$, $\mathcal{R}(\mathbf{SGraph})$ and $\mathcal{R}(\mathbf{DAG})$ are closed under decomposition, $\mathcal{R}(\mathbf{SGraph})$ under $\mathcal{M}(\mathbf{SGraph})$-decomposition, $\mathcal{M}(\mathbf{SGraph})$ under $\mathcal{R}(\mathbf{SGraph})$-decomposition and, finally, the class $\mathcal{M}(\mathbf{DAG})$ under $\mathsf{dcl_d}$-decomposition. □

6.1.2 Tree orders

In this section we present *trees* as partial orders and show that the resulting category is equivalent to a topos of presheaves, and thus, by Corollary 5.3.11, adhesive. This fact will be exploited in Sections 6.1.3 and 6.2.3 to construct two categories of hierarchical graphs, where the hierarchy between edges is modelled by trees.

Definition 6.1.33. A *tree order* is a partial order $(E, \leq)$ such that for every $e \in E$, the set

$$\downarrow e := \{e' \in E \mid e' \leq e\}$$

is a finite set totally ordered by the restriction of $\leq$. Since $\downarrow e$ is a finite chain we can define the *immediate predecessor function*

$$p_E \colon E \to E + 1 \qquad e \mapsto \begin{cases} \max(\downarrow e \smallsetminus \{e\}) & \downarrow e \neq \{e\} \\ \bot & \downarrow e = \{e\} \end{cases}$$

For any $k \in \mathbb{N}_+$ we can define the k^{th} *predecessor function* by induction as follows:

$$p_E^k \colon E \to E + 1 \qquad e \mapsto \begin{cases} p_E\left(p_E^{k-1}(e)\right) & p_E^{k-1}(e) \in E \\ \bot & p_E^{k-1}(e) = \bot \end{cases}$$

We extend this definition to $k \in \mathcal{N}$ taking p_E^0 to be the inclusion $\iota_E \colon E \to E + 1$.

Given a monotone map $f \colon (E, \leq) \to (F, \leq)$ and its extension $f_\bot \colon E + 1 \to F + 1$ sending $\bot$ to $\bot$, we say that f is *strict* if the following diagram commutes

$$\begin{tikzcd} E \arrow[r, "p_E"] \arrow[d, "f"'] & E + 1 \arrow[d, "f_\bot"] \\ F \arrow[r, "p_F"'] & F + 1 \end{tikzcd}$$

Tree will denote the subcategory of the category of posets **Pos** given by tree orders and strict morphisms. $U_{\mathbf{Tree}}$ will denote the functor **Tree** $\to$ **Set** obtained restricting the forgetful functor from **Pos** to **Set**.

Remark 6.1.34. Clearly $p_E^1 = p_E$ and it holds that $p_E^k(e) = \bot$ if and only if $|\downarrow e| \leq k$. In this case an easy induction shows that $|\downarrow p_E^k(e)| = |\downarrow e| - k$.

Example 6.1.35. A strict morphisms is simply a monotone function that preserves immediate predecessors (and thus every predecessor). For instance the function $\{0\} \to \{0, 1\}$ sending 0 to 1 and where we endow the codomain with the order $0 \leq 1$, is not a strict morphism.

Let $(E, \leq)$ be an object of **Tree**, for every $n \in \mathbb{N}$ we can put

$$\widehat{E}(n) := \{e \in E \mid |\downarrow e \smallsetminus \{e\}| = n\}$$

Given another $m \in \mathbb{N}$ such that $n \leq m$, we can define a function

$$p_{n,m}^E \colon \widehat{E}(m) \to \widehat{E}(n) \qquad e \mapsto p_E^{m-n}(e)$$

which is well defined since $|{\downarrow}e| > m-n$ so that

$$\begin{aligned}|{\downarrow}p_E^{m-n}(e)| &= |{\downarrow}e| - m + n\\ &= m+1-m+n\\ &= n+1\end{aligned}$$

Notice, moreover that if $m=n$, $p_E^{m-n}(e)$ is the identity, while for any $k \le n \le m$ we have

$$\begin{aligned}p_{k,n}^E\left(p_{n,m}^E(e)\right) &= p_E^{n-k}\left(p_E^{m-n}(e)\right)\\ &= p_E^{n-k+m-n}(e)\\ &= p_E^{m-k}(e)\\ &= p_E^{m,k}(e)\end{aligned}$$

Thus, taking the category associates to the ordinal $\omega = (\mathbb{N}, \le)$ we get a presheaf $\widehat{E}\colon \omega^{op} \to \mathbf{Set}$.

Proposition 6.1.36. *Let $f\colon (E,\le) \to (F,\le)$ be an arrow in* **Tree**, *for every $n \in \mathbb{N}$, if $e \in \widehat{E}(n)$ then $f(e) \in \widehat{F}(n)$. Moreover, the following equation holds*

$$f_\perp\left(p_E^n(e)\right) = p_F^n\left(f(e)\right)$$

Proof. Let us prove by induction the first half of the proposition.

- If $n=0$ then
$$\begin{aligned}p_F(f(e)) &= f_\perp(p_E(e))\\ &= \perp\end{aligned}$$
so that ${\downarrow}f(e) = \emptyset$ and thus $f(e) \in \widehat{F}(0)$.
- If $n \ge 1$ since $e \in \widehat{E}(n)$, then $p_E(e) \in \widehat{E}(n-1)$ and, by the inductive hypothesis, $f(p_E(e)) \in \widehat{F}(n-1)$, therefore
$$\begin{aligned}f(p_E(e)) &= f_\perp(p_E(e))\\ &= p_F(f(e))\end{aligned}$$
so $p_F(f(e)) \in \widehat{F}(n-1)$ and thus $f(e) \in \widehat{F}(n)$.

For the second half we use again induction.

- Suppose that $n=0$, then
$$\begin{aligned}f_\perp(p_E^0(e)) &= f_\perp(\iota_E(e))\\ &= \iota_F(f(e))\\ &= p_F^0(f(e))\end{aligned}$$
- Let n be greater or equal than 1, then
$$\begin{aligned}f_\perp\left(p_E^n(e)\right) &= f_\perp\left(p_E\left(p_E^{n-1}(e)\right)\right)\\ &= p_F\left(f_\perp\left(p_E^{n-1}(e)\right)\right)\\ &= p_F\left(p_F^{n-1}(f(e))\right)\\ &= p_F^n(f(e))\end{aligned}$$

and we get the thesis. □

We can now prove the main result of this section.

Theorem 6.1.37. *There exists an equivalence of categories* $\widehat{(-)}\colon \mathbf{Tree} \to \mathbf{Set}^{\omega^{op}}$ *sending* $(E, \leq)$ *to* $\widehat{E}$.

Proof. By Proposition 6.1.36, given $f\colon (E, \leq) \to (F, \leq)$ in **Tree** we can define

$$\widehat{f}_n\colon \widehat{F}(n) \to \widehat{G}(n) \qquad e \mapsto f(e)$$

We have to chek naturality. Let $n \leq m$ and $e \in \widehat{E}(m)$, then, using Proposition 6.1.36

$$\begin{aligned}\widehat{f}_n\left(p^E_{n,m}(e)\right) &= f\left(p^{m-n}_E(e)\right)\\ &= f_\perp\left(p^{m-n}_E(e)\right)\\ &= p^{m-n}_F(f(e))\\ &= p^F_{n,m}\left(\widehat{f}_n(e)\right)\end{aligned}$$

Thus we have a functor $\widehat{(-)}\colon \mathbf{Tree} \to \mathbf{Set}^{\omega^{op}}$, we want to show that it is an equivalence. Since every elements e of E belongs $\widehat{E}(n)$ for some $n \in \mathbb{N}$ we can deduce that $\widehat{(-)}$ is faithful. For fullness, take $\alpha\colon \widehat{E} \to \widehat{F}$, and define

$$\overline{\alpha}\colon (E, \leq) \to (F, \leq) \qquad e \mapsto \alpha_{|{\downarrow}e|-1}(e)$$

To see that $\overline{\alpha}$ is strict, notice that, whenever $|{\downarrow}e| = 1$ we have $e \in \widehat{E}(0)$, thus $\alpha_0(e) \in \widehat{F}(0)$, so that

$$\begin{aligned}\overline{\alpha}_\perp(p_E(e)) &= \begin{cases}\overline{\alpha}_\perp(\perp) & |{\downarrow}e| = 1\\ \alpha_{|{\downarrow}p_E(e)|-1}(p_E(e)) & |{\downarrow}e| \geq 2\end{cases}\\ &= \begin{cases}\perp & |{\downarrow}e| = 1\\ \alpha_{|{\downarrow}e|-2}(p_E(e)) & |{\downarrow}e| \geq 2\end{cases}\\ &= \begin{cases}p_F(\alpha_0(e)) & |{\downarrow}e| = 1\\ \alpha_{|{\downarrow}e|-2}\left(p^E_{|{\downarrow}e|-1,|{\downarrow}e|-2}(e)\right) & |{\downarrow}e| \geq 2\end{cases}\\ &= \begin{cases}p_F(\alpha_0(e)) & |{\downarrow}e| = 1\\ p^F_{|{\downarrow}e|-1,|{\downarrow}e|-2}\left(\alpha_{|{\downarrow}e|-1}(e)\right) & |{\downarrow}e| \geq 2\end{cases}\\ &= \begin{cases}p_F(\alpha_0(e)) & |{\downarrow}e| = 1\\ p_F\left(\alpha_{|{\downarrow}e|-1}(e)\right) & |{\downarrow}e| \geq 2\end{cases}\\ &= p_F(\overline{\alpha}(e))\end{aligned}$$

Finally, given $F\colon \omega^{op} \to \mathbf{Set}$ we define $\overline{F}$ as the poset in which

- the underlying set is given by $\sum_{k\in\mathbb{N}} F(k)$;
- if ι_k is the coprojecton $F(k) \to \sum_{k\in\mathbb{N}} F(k)$, we put $\iota_n(x) \leq \iota_m(y)$ whenever

$$n \leq m \qquad x = f_{n,m}(y)$$

where $f_{n,m}\colon F(m) \to F(n)$ is the function corresponding to $n \leq m$.

Given $\iota_m(e) \in \sum_{k\in\mathbb{N}} F(k)$ it holds that

$$\downarrow\iota_m(e) = \left\{ x \in \sum_{k\in\mathbb{N}} F(k) \mid x = \iota_n\left(f_{n,m}(e)\right) \text{ for some } n \leq m \right\}$$

and so $|\downarrow\iota_m(e)| = m+1$. On the other hand if $n \leq k$ and

$$x = \iota_n(f_{n,m}(e)) \qquad y = \iota_k(f_{k,m}(e))$$

then

$$f_{n,m}(e) = f_{n,k}(f_{k,m}(e))$$

showing $x \leq y$. Thus $\downarrow\iota_m(e)$ is totally ordered and $\overline{F}$ is an object of **Tree**. By construction we have

$$|\iota_n(e) \smallsetminus \iota_n(e)| \qquad p_{\overline{F}}(\iota_n(e)) = f_{n-1,n}$$

and this shows that $\overline{F}$ is sent by $\widehat{(-)}$ to F. □

Corollary 6.1.38. **Tree** *is adhesive and the forgetful functor* $U_{\mathbf{Tree}}\colon$ **Tree** $\to$ **Set** *preserves all colimits.*

Proof. Let $\widehat{(-)}$ be the equivalence constructed in the previous theorem, and define $S\colon \mathbf{Set}^{\omega^{op}} \to \mathbf{Set}$ as

$$\begin{array}{ccc} F & \longmapsto & \sum_{n\in\mathbb{N}} F(n) \\ \alpha\big\downarrow & & \big\downarrow \sum_{n\in\omega} \alpha_n \\ G & \longmapsto & \sum_{n\in\mathbb{N}} G(n) \end{array}$$

since colimits are computed component-wise in $\mathbf{Set}^{\omega^{op}}$ and coproducts in **Set** commute with colimits we get that S is cocontinuous. Morover the triangle commutes

$$\begin{tikzcd} \mathbf{Tree} \arrow[rr, "\widehat{(-)}"] \arrow[dr, "U_{\mathbf{Tree}}"'] & & \mathbf{Set}^{\omega^{op}} \arrow[dl, "S"] \\ & \mathbf{Set} & \end{tikzcd}$$

commutes and the thesis follows. □

6.1.3 Hierarchical graphs

We can use trees to produce a category of hierarchical graphs [102], which, in addition, can be equipped with an interface, modelled by a function into the set of nodes.

Definition 6.1.39. A *hierarchical graphs* $\mathcal{G}$ is a 4-uple $((E_{\mathcal{G}},\leq), V_{\mathcal{G}}, s_{\mathcal{G}}, t_{\mathcal{G}})$ made by a tree order $(E_{\mathcal{G}},\leq)$, a set $V_{\mathcal{G}}$ and functions $s_{\mathcal{G}}, t_{\mathcal{G}}\colon E_{\mathcal{G}} \rightrightarrows V_{\mathcal{G}}$. A *morphism* $\mathcal{G}\to\mathcal{H}$ is a pair (f,g) with $f\colon (E,\leq)\to(F,\leq)$ in **Tree** and $g\colon V_{\mathcal{G}} \to V_{\mathcal{H}}$ in **Set** such that the following squares commute

$$\begin{tikzcd} E_{\mathcal{G}} \arrow[r, "s_{\mathcal{G}}"] \arrow[d, "U_{\mathbf{Tree}}(f)"'] & V_{\mathcal{G}} \arrow[d, "g"] \\ E_{\mathcal{G}} \arrow[r, "s_{\mathcal{H}}"'] & V_{\mathcal{H}} \end{tikzcd} \qquad \begin{tikzcd} E_{\mathcal{G}} \arrow[r, "t_{\mathcal{G}}"] \arrow[d, "U_{\mathbf{Tree}}(f)"'] & V_{\mathcal{G}} \arrow[d, "g"] \\ E_{\mathcal{G}} \arrow[r, "t_{\mathcal{H}}"'] & V_{\mathcal{H}} \end{tikzcd}$$

This data, with componentwise composition, form a category **HGraph**.

We can give an analog of Proposition 6.1.4.

Proposition 6.1.40. **HGraph** *is isomorphic to* $U_{\mathbf{Tree}}{\downarrow}\mathsf{prod}$

Proof. Define $F\colon \mathbf{HGraph} \to U_{\mathbf{Tree}}{\downarrow}\mathsf{prod}$ as

$$\begin{array}{ccc} ((E_\mathcal{G}, \leq), V_\mathcal{G}, s_\mathcal{G}, t_\mathcal{G}) & \longmapsto & ((E_\mathcal{G}, \leq), V_\mathcal{G}, (s_\mathcal{G}, t_\mathcal{G})) \\ {\scriptstyle (f,g)}\big\downarrow & & \big\downarrow{\scriptstyle (f,g)} \\ ((E_\mathcal{H}, \leq), V_\mathcal{H}, s_\mathcal{H}, t_\mathcal{H}) & \longmapsto & ((E_\mathcal{H}, \leq), V_\mathcal{H}, (s_\mathcal{H}, t_\mathcal{H})) \end{array}$$

and $G\colon U_{\mathbf{Tree}}{\downarrow}\mathsf{prod} \to \mathbf{HGraph}$

$$\begin{array}{ccc} ((E_\mathcal{G}, \leq), V_\mathcal{G}, p_\mathcal{G}) & \longmapsto & ((E_\mathcal{G}, \leq), V_\mathcal{G}, \pi_1 \circ p_\mathcal{G}, \pi_2 \circ p_\mathcal{G}) \\ {\scriptstyle (f,g)}\big\downarrow & & \big\downarrow{\scriptstyle (f,g)} \\ ((E_\mathcal{H}, \leq), V_\mathcal{H}, p_\mathcal{H}) & \longmapsto & ((E_\mathcal{H}, \leq), V_\mathcal{H}, \pi_1 \circ p_\mathcal{H}, \pi_2 \circ p_\mathcal{H}) \end{array}$$

The thesis follows at once. □

Applying Theorem 5.1.38 and Corollary 6.1.38 we get the following result.

Corollary 6.1.41. **HGraph** *is an adhesive category.*

Given a hierarchical graph $\mathcal{G}$, we can model an *interface* as a function between a set X and the set of nodes $V_\mathcal{G}$. We are then lead to the following definition.

Definition 6.1.42. The category **HIGraph** of *hierarchical graphs with interface* is defined in the following way. Objects are triples $(\mathcal{G}, X, f)$ made by a hierarchical graph $\mathcal{G}$, a set X and a function $f\colon X \to V_\mathcal{G}$. A morphism $(\mathcal{G}, X, f) \to (\mathcal{H}, Y, g)$ is a triple (h, k, l) with $h\colon (E, \leq) \to (F, \leq)$ in **Tree**, $g\colon V_\mathcal{G} \to V_\mathcal{H}$ and $l\colon X \to Y$ in **Set** such that the following squares commute

$$\begin{tikzcd} E_\mathcal{G} \arrow[r, "s_\mathcal{G}"] \arrow[d, "U_{\mathbf{Tree}}(h)"'] & V_\mathcal{G} \arrow[d, "k"] \\ E_\mathcal{G} \arrow[r, "s_\mathcal{H}"'] & V_\mathcal{H} \end{tikzcd} \qquad \begin{tikzcd} E_\mathcal{G} \arrow[r, "t_\mathcal{G}"] \arrow[d, "U_{\mathbf{Tree}}(h)"'] & V_\mathcal{G} \arrow[d, "k"] \\ E_\mathcal{G} \arrow[r, "t_\mathcal{H}"'] & V_\mathcal{H} \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "l"'] & V_\mathcal{G} \arrow[d, "k"] \\ Y \arrow[r, "g"'] & V_\mathcal{H} \end{tikzcd}$$

Now, $U_{\mathbf{Tree}}\colon \mathbf{Tree} \to \mathbf{Set}$ preserves the initial objects by Corollary 6.1.38, thus, Proposition A.2.3 implies that the forgetful functor $\mathbf{HGraph} \to \mathbf{Set}$, which only remembers the set of nodes, has a left adjoint $\Delta_{\mathbf{HGraph}}$ which sends X to $((\emptyset, \leq), X, ?_X, ?_X)$. In particular we get the following.

Proposition 6.1.43. *The category* **HIGraph** *is isomorphic to* $\Delta_{\mathbf{HGraph}}{\downarrow}\mathsf{id}_{\mathbf{HGraph}}$.

Proof. Define $F\colon \mathbf{HIGraph} \to \Delta_{\mathbf{HGraph}}{\downarrow}\mathsf{id}_{\mathbf{HGraph}}$ and $G\colon \Delta_{\mathbf{HGraph}}{\downarrow}\mathsf{id}_{\mathbf{HGraph}} \to \mathbf{HIGraph}$ putting

$$\begin{array}{ccc} (\mathcal{G}, X, f) & \longmapsto & (X, \mathcal{G}, (?_X, f)) \\ {\scriptstyle (h,k,l)}\big\downarrow & & \big\downarrow{\scriptstyle (l,(h,k))} \\ (\mathcal{H}, Y, g) & \longmapsto & (Y, \mathcal{H}, (?_Y, g)) \end{array} \qquad \begin{array}{ccc} (X, \mathcal{G}, (?_X, f)) & \longmapsto & (\mathcal{G}, X, f) \\ {\scriptstyle (l,(h,k))}\big\downarrow & & \big\downarrow{\scriptstyle (h,k,l)} \\ (Y, \mathcal{H}, (?_Y, g)) & \longmapsto & (\mathcal{H}, Y, g) \end{array}$$

which, by inspection, are mutual inverses. □

Corollary 6.1.44. **HIGraph** *is an adhesive category.*

6.2 $\mathcal{M},\mathcal{N}$-adhesivity of some categories of hypergraphs

In this section we will move from the world of graphs to the one of *hypergraphs* allowing an edge to join two arbitrary subsets of nodes. Even in this case, leveraging the modularity provided by Theorem 5.1.31, it is possible to combine sufficiently adhesive categories of preorders or graphs (modelling the hierarchy between the edges) while retaining suitable adhesivity properties. It is worth noticing that, beside hypergraphs or interfaces, this methodology can be extended easily to other settings like Petri nets (see [44]).

6.2.1 An introduction to hypergraphs

We will start this section with the definition of (directed) hypergraph and we will see how label them with an algebraic signature. A pivotal role will be played by the Kleene star $(-)^\star$ the functor $\mathbf{Set} \to \mathbf{Set}$ introduced in Example 2.1.8.

Definition 6.2.1. A *hypergraph* is a 4-uple $\mathcal{G} := (E_\mathcal{G}, V_\mathcal{G}, s_\mathcal{G}, t_\mathcal{G})$ made by two sets $E_\mathcal{G}$ and $V_\mathcal{G}$, whose elements are called respectively *hyperedges* and *nodes*, plus a pair of *source* and *target* functions $s_\mathcal{G}, t_\mathcal{G} \colon E_\mathcal{G} \rightrightarrows V_\mathcal{G}^\star$. A *hypergraph morphism* $(E_\mathcal{G}, V_\mathcal{G}, s_\mathcal{G}, t_\mathcal{G}) \to (E_\mathcal{H}, V_\mathcal{H}, s_\mathcal{H}, t_\mathcal{H})$ is a pair (h,k) of functions $h\colon E_\mathcal{G} \to E_\mathcal{H}$, $k\colon V_\mathcal{G} \to V_\mathcal{H}$ such that the following diagrams commute.

$$
\begin{tikzcd}
E_\mathcal{G} \arrow[r, "s_\mathcal{G}"] \arrow[d, "h"'] & V_\mathcal{G}^\star \arrow[d, "k^\star"] \\
E_\mathcal{G} \arrow[r, "s_\mathcal{H}"'] & V_\mathcal{H}^\star
\end{tikzcd}
\qquad
\begin{tikzcd}
E_\mathcal{G} \arrow[r, "t_\mathcal{G}"] \arrow[d, "h"'] & V_\mathcal{G}^\star \arrow[d, "k^\star"] \\
E_\mathcal{G} \arrow[r, "t_\mathcal{H}"'] & V_\mathcal{H}^\star
\end{tikzcd}
$$

We define $\mathbf{Hyp}$ to be the resulting category.

Let $\mathsf{prod}^\star$ be the composition $\mathsf{prod} \circ (-)^\star$, then we can prove the following result analogous to Propositions 6.1.4 and 6.1.40.

Proposition 6.2.2. $\mathbf{Hyp}$ *is isomorphic to* $\mathsf{id}_{\mathbf{Set}}{\downarrow}\mathsf{prod}^\star$

Proof. This is done exactly as in Propositions 6.1.4 and 6.1.40. Define two functors $F\colon \mathbf{Hyp} \to \mathsf{id}_{\mathbf{Set}}{\downarrow}\mathsf{prod}^\star$ and $G\colon \mathsf{id}_{\mathbf{Set}}{\downarrow}\mathsf{prod}^\star \to \mathbf{Hyp}$ as follows

$$
\begin{array}{ccc}
(E_\mathcal{G}, V_\mathcal{G}, s_\mathcal{G}, t_\mathcal{G}) & \longmapsto & (E_\mathcal{G}, V_\mathcal{G}, (s_\mathcal{G}, t_\mathcal{G})) \\
(f,g)\big\downarrow & & \big\downarrow(f,g) \\
(E_\mathcal{H}, V_\mathcal{H}, s_\mathcal{H}, t_\mathcal{H}) & \longmapsto & (E_\mathcal{H}, V_\mathcal{H}, (s_\mathcal{H}, t_\mathcal{H}))
\end{array}
\qquad
\begin{array}{ccc}
(E_\mathcal{G}, V_\mathcal{G}, p_\mathcal{G}) & \longmapsto & (E_\mathcal{G}, V_\mathcal{G}, \pi_1 \circ p_\mathcal{G}, \pi_2 \circ p_\mathcal{G}) \\
(f,g)\big\downarrow & & \big\downarrow(f,g) \\
(E_\mathcal{H}, V_\mathcal{H}, p_\mathcal{H}) & \longmapsto & (E_\mathcal{H}, V_\mathcal{H}, \pi_1 \circ p_\mathcal{H}, \pi_2 \circ p_\mathcal{H})
\end{array}
$$

Now it is enough to notice that they are one the inverse of the other. □

We can show that the Kleene star preserves pullbacks (see also [35, Sec. 3] and [77, Ch.4] for details and a deeper and more conceptual approach).

Proposition 6.2.3. *The functor* $(-)^\star$ *preserves pullbacks.*

Proof. Suppose that a pullbacks square as the one below is given.

$$
\begin{tikzcd}
P \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, "f"] \\
Y \arrow[r, "g"'] & Z
\end{tikzcd}
$$

and consider the solid part of the following diagram.

$$\begin{tikzcd}
Z \arrow[dr, dotted, "t"] \arrow[drr, bend left, "t_1"] \arrow[ddr, bend right, "t_2"'] & & \\
& P^\star \arrow[r, "p_1^\star"] \arrow[d, "p_2^\star"'] & X^\star \arrow[d, "f^\star"] \\
& Y^\star \arrow[r, "g^\star"'] & Z^\star
\end{tikzcd}$$

For every $z \in Z$ we have arrows $t_1(z)\colon n \to X$ and $t_2(z)\colon m \to Y$ such that

$$f^\star(t_1(x)) = g^\star(t_2(z))$$

In particular this entails that $n = m$ and that there is $t(z)\colon n \to P$ as in the diagram below

$$\begin{tikzcd}
n \arrow[dr, "t(z)"] \arrow[drr, bend left, "t_1(z)"] \arrow[ddr, bend right, "t_2(z)"'] & & \\
& P \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

But this is equivalent to say that the dotted $t\colon Z \to P$ exists, while its uniqueness follows at once from the universal property of the pullback with which we started. ☐

Remark 6.2.4. Preservation of pullbacks implies that $(-)^\star$ sends monos to monos.

Corollary 6.2.5. **Hyp** *is an adhesive category.*

Proof. $(-)^\star$ preserves pullbacks by Proposition 6.2.3, while prod is continuous by definition, thus the thesis follows from This follows from Theorem 5.1.38 and Proposition 6.2.2. ☐

Propositions 6.2.2 and A.2.3 allows us to deduce immediately the following.

Proposition 6.2.6. *The forgetful functor* $U_{\mathbf{Hyp}}\colon \mathbf{Hyp} \to \mathbf{Set}$ *which sends an hypergraph* $\mathcal{G}$ *to its set of nodes has a left adjoint* $\Delta_{\mathbf{Hyp}}$.

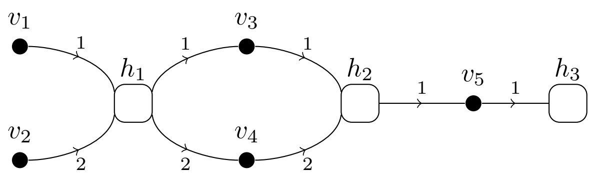
Remark 6.2.7. Since the initial object of **Set** is the empty set, $\Delta_{\mathbf{Hyp}}(X)$ is the hypergraph which has X as set of nodes and $\emptyset$ as set of hyperedges and $?_X$ as both source and target function.

Hypergraphs, can be represented graphically. We will use dots to denote nodes and squares to denote hyperedges, the name of a node or of an hyperedge will be put near the corresponding dot or square. Sources and targets are represented by lines between dots and squares: the lines from the sources of an hyperedge will have an arrowhead in the middle pointing towards the hyperedge, while the lines to the targets will have arrowheads pointing to the target nodes. We will decorate the arrow corresponding to the i^{th} letter (i.e. its value at $i-1$) of a target or a source with a label i.

Example 6.2.8. Take $V_{\mathcal{G}}$ to be be $\{v_1, v_2, v_3, v_4, v_5\}$ and $E_{\mathcal{G}}$ to be $\{h_1, h_2, h_3\}$. Sources and targets are given by:

$$\begin{array}{llllll} s_{\mathcal{G}}(h_1)\colon 2 \to V_{\mathcal{G}} & \begin{array}{l} 0 \mapsto v_1 \\ 1 \mapsto v_2 \end{array} & s_{\mathcal{G}}(h_2)\colon 2 \to V_{\mathcal{G}} & \begin{array}{l} 0 \mapsto v_3 \\ 1 \mapsto v_4 \end{array} & s_{\mathcal{G}}(h_3)\colon 1 \to V_{\mathcal{G}} & 0 \mapsto v_5 \\ t_{\mathcal{G}}(h_1)\colon 2 \to V_{\mathcal{G}} & \begin{array}{l} 0 \mapsto v_3 \\ 1 \mapsto v_4 \end{array} & t_{\mathcal{G}}(h_2)\colon 2 \to V_{\mathcal{G}} & 0 \mapsto v_5 & t_{\mathcal{G}}(h_3)\colon 0 \to V_{\mathcal{G}} & t_{\mathcal{G}}(h_3) = ?_{V_{\mathcal{G}}} \end{array}$$

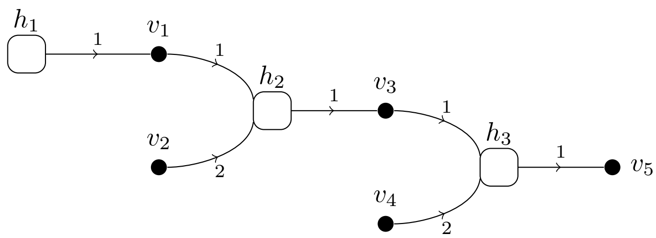
We can draw the resulting $\mathcal{G}$ as follows:



Example 6.2.9. Let $V_{\mathcal{G}}$ be as in the previous example and $E_{\mathcal{G}} = \{h_1, h_2, h_3\}$. Then we define

$$\begin{array}{llllll} s_{\mathcal{G}}(h_1)\colon 0 \to V_{\mathcal{G}} & s_{\mathcal{G}}(h_1) = ?_{V_{\mathcal{G}}} & s_{\mathcal{G}}(h_2)\colon 2 \to V_{\mathcal{G}} & \begin{array}{l} 0 \mapsto v_1 \\ 1 \mapsto v_2 \end{array} & s_{\mathcal{G}}(h_3)\colon 2 \to V_{\mathcal{G}} & \begin{array}{l} 0 \mapsto v_1 \\ 1 \mapsto v_4 \end{array} \\ t_{\mathcal{G}}(h_1)\colon 1 \to V_{\mathcal{G}} & 0 \mapsto v_1 & t_{\mathcal{G}}(h_2)\colon 1 \to V_{\mathcal{G}} & 0 \mapsto v_3 & t_{\mathcal{G}}(h_3)\colon 1 \to V_{\mathcal{G}} & 1 \mapsto v_5 \end{array}$$

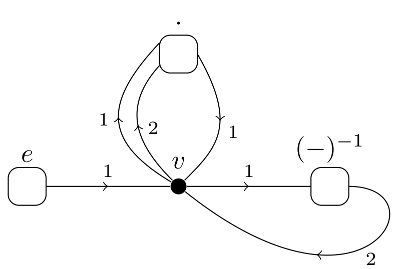
Now we can depict $\mathcal{G}$ as



Example 6.2.10. Let $\Sigma = (O_\Sigma, \mathsf{ar}_\Sigma)$ be an algebraic signature, we can construct the hypergraph $\mathcal{G}^\Sigma$ taking $V_{\mathcal{G}^\Sigma}$ and $E_{\mathcal{G}^\Sigma}$ to be respectively the singleton $\{\heartsuit\}$ and the set O_Σ. We put

$$s_{\mathcal{G}^\Sigma} : O_\Sigma \to \{\heartsuit\}^\star \quad o \mapsto \delta_\heartsuit^{\mathsf{ar}_\Sigma(o)} \qquad t_{\mathcal{G}^\Sigma} : O_\Sigma \to \{\heartsuit\}^\star \quad o \mapsto \delta_\heartsuit$$

For instance let Σ_G be the signature of groups of Example 2.2.41, then $\mathcal{G}^{\Sigma_G}$ is depicted as:



Hyp as a topos of presheaves By Corollary 5.1.36 we already know that **Hyp** has all pullbacks and by Corollary 6.2.5 we know that it is adhesive. Actually more can be proved about it: we can realize **Hyp** as a topos of presheaves [26].

Definition 6.2.11. Let $\mathbf{H}$ be the category in which:

- the set of objects is given by $(\mathbb{N}\times\mathbb{N})\cup\{\bullet\}$
- arrows are given by the identities $\mathsf{id}_{k,l}$ and $\mathsf{id}_\bullet$ and exactly $k+l$ arrows $f_i\colon (k,l)\to\bullet$, where i ranges from 0 to $k+l-1$;
- composition is defined simply putting, for every $f_i\colon (k,l)\to\bullet$:

$$f_i = f_i\circ \mathsf{id}_{k,l} \qquad f_i=\mathsf{id}_\bullet\circ f_i$$

Now, given $F\colon \mathbf{H}\to\mathbf{Set}$ we can define

$$E_F := \sum_{k,l\in\mathbb{N}} F(k,l)$$

For every element x of $F(k,l)$ we can put

$$s^F_{k,l}(x)\colon k\to F(\bullet)\quad i\mapsto F(f_i)(x) \qquad t^F_{k,l}(x)\colon l\to F(\bullet)\quad i\mapsto F(f_{i+k})(x)$$

obtaining $s_F, t_F\colon E_F\rightrightarrows \mathcal{F}(\bullet)^\star$. Let $\mathcal{G}_F$ be the resulting hypergraph. Now, every $\eta\colon F\to H$ in $\mathbf{Set}^{\mathbf{H}}$ has components $\eta_{k,l}\colon F(k,l)\to H(k,l)$, $\eta_\bullet\colon F(\bullet)\to H(\bullet)$, thus it induces a function $\hat\eta\colon E_F\to E_H$ such that the following squares commute

$$\begin{tikzcd} E_F \arrow[r, "s_F"] \arrow[d, "\hat\eta"'] & F(\bullet)^\star \arrow[d, "\eta^\star_\bullet"] \\ E_H \arrow[r, "s_H"'] & H(\bullet)^\star \end{tikzcd} \qquad \begin{tikzcd} E_F \arrow[r, "t_F"] \arrow[d, "\hat\eta"'] & F(\bullet)^\star \arrow[d, "\eta^\star_\bullet"] \\ E_H \arrow[r, "t_H"'] & H(\bullet)^\star \end{tikzcd}$$

This is equivalent to say that η induces a morphism $(\hat\eta,\eta_\bullet)\colon \mathcal{G}_F\to\mathcal{G}_H$. It is now clear that sending F to $\mathcal{G}_F$ and η to $(\hat\eta,\eta_\bullet)$ defines a faithful functor $\mathcal{G}_{-}\colon \mathbf{Set}^{\mathbf{H}}\to\mathbf{Hyp}$.

Proposition 6.2.12. **Hyp** *is equivalent to the category* $\mathbf{Set}^{\mathbf{H}}$.

Proof. Let X be a set, for every $n\in\mathbb{N}$ define

$$X_n := \{w\in X^\star \mid \mathsf{dom}(w)=n\}$$

In particular, if $F\colon\mathbf{H}\to\mathbf{Set}$ then the image of the coprojection $\iota^F_{k,l}\colon F(k,l)\to E_F$ is the intersection

$$s_F^{-1}\left(F(\bullet)_k\right)\cap t_F^{-1}\left(F(\bullet)_l\right)$$

We are now ready to that $\mathcal{G}_{-}$ is full and essentially surjective.

- For fullness, let $(f, g)\colon \mathcal{G}_F \to \mathcal{G}_H$ be a morphism of hypergraphs and define $f_{k,l}$ to be $f \circ \iota^F_{k,l}$, the composition of h with Now, if $x \in F(k,l)$ then

$$\begin{aligned} s_H(f_{k,l}(x)) &= s_H\left(f\left(\iota^F_{k,l}(x)\right)\right) & \qquad t_H(f_{k,l}(x)) &= s_t\left(f\left(\iota^F_{k,l}(x)\right)\right) \\ &= g^\star\left(s_F\left(\iota^F_{k,l}(x)\right)\right) & &= g^\star\left(t_F\left(\iota^F_{k,l}(x)\right)\right) \end{aligned}$$

Therefore there exists $\eta_{k,l}\colon F(k,l) \to H(k,l)$ fitting in the diagram below

$$\begin{tikzcd} F(k,l) \arrow[r, "\iota^F_{k,l}"] \arrow[d, dotted, "\eta_{k,l}"'] & E_F \arrow[d, "f_{k,l}"] \\ H(k,l) \arrow[r, "\iota^H_{k,l}"'] & E_H \end{tikzcd}$$

Define $\eta_\bullet\colon F(\bullet) \to H(\bullet)$ simply as $g^\star$, then the collection of all the $\eta_{k,l}$ and of $\eta_\bullet$ defines a natural transformation $\eta\colon F \to H$. Indeed, if $f_i\colon (k,l) \to \bullet$ we have:

$$\begin{tikzcd} F(k,l) \arrow[r, "\iota^F_{k,l}"'] \arrow[rr, bend left, "s^F_{k,l}"] \arrow[d, "\eta_{k,l}"'] & E_F \arrow[r, "s_F"'] \arrow[d, "f_{k,l}"'] & F(\bullet)^\star \arrow[d, "g^\star"] \\ H(k,l) \arrow[r, "\iota^H_{k,l}"] \arrow[rr, bend right, "s^H_{k,l}"'] & E_H \arrow[r, "s_H"] & H(\bullet)^\star \end{tikzcd} \qquad \begin{tikzcd} F(k,l) \arrow[r, "\iota^F_{k,l}"'] \arrow[rr, bend left, "t^F_{k,l}"] \arrow[d, "\eta_{k,l}"'] & E_F \arrow[r, "t_F"'] \arrow[d, "f_{k,l}"'] & F(\bullet)^\star \arrow[d, "g^\star"] \\ H(k,l) \arrow[r, "\iota^H_{k,l}"] \arrow[rr, bend right, "t^H_{k,l}"'] & E_H \arrow[r, "t_H"] & H(\bullet)^\star \end{tikzcd}$$

Thus if $i < k$ then

$$\begin{aligned} \eta_\bullet(F(f_i)(x)) &= g(F(f_i)(x)) \\ &= g\left(s^F_{k,l}(x)(i)\right) \\ &= g^\star\left(s^F_{k,l}(x)\right)(i) \\ &= s^H_{k,l}(\eta_{k,l}(x))(i) \\ &= F(f_i)(\eta_{k,l}(x)) \end{aligned}$$

while, if $k \le i < k + l - 1$

$$\begin{aligned} \eta_\bullet(F(f_i)(x)) &= g(F(f_i)(x)) \\ &= g\left(t^F_{k,l}(x)(i)\right) \\ &= g^\star\left(t^F_{k,l}(x)\right)(i) \\ &= t^H_{k,l}(\eta_{k,l}(x))(i) \\ &= F(f_i)(\eta_{k,l}(x)) \end{aligned}$$

Finally, by contruction it is clear that $(\hat{\eta}, \eta_\bullet) = (f, g)$.

- Given an hypergraph $\mathcal{G} = (E_\mathcal{G}, V_\mathcal{G}, s_\mathcal{G}, t_\mathcal{G})$ we can define
$$F_\mathcal{G}(k,l) := s_\mathcal{G}^{-1}(V_k) \cap t_\mathcal{G}^{-1}(V_l) \qquad F_\mathcal{G}(\bullet) := V_\mathcal{G}$$
Given $f_i\colon (k,l) \to \bullet$ we put
$$F_\mathcal{G}(f_i)\colon F_\mathcal{G}(k,l) \to F_\mathcal{G}(\bullet) \qquad x \mapsto \begin{cases} s_\mathcal{G}(x)(i) & i < k \\ t_\mathcal{G}(x)(i-k) & i \leq k < k+l-1 \end{cases}$$
$F_\mathcal{G}$ so defined is a functor $\mathbf{H} \to \mathbf{Set}$ and for every $h \in E_\mathcal{G}$ there exists a unique pair (k,l) such that $h \in F_\mathcal{G}(k,l)$, namely the pair $(\mathsf{dom}(s_\mathcal{G})(h), \mathsf{dom}(t_\mathcal{G})(h))$thus
$$\sum_{k,l \in \mathbb{N}} F_\mathcal{G}(k,l) \simeq E$$
Moreover, by construction $s_{F_\mathcal{G}} = s$ and $t_{F_\mathcal{G}} = t$, from which the thesis follows. □

As a corollary we get immediately the following.

Corollary 6.2.13. **Hyp** *is a complete category.*

6.2.2 Labelled hypergraphs

We will end this section examining two different kinds of labelings for hypergraphs. We need the first one in Section 6.3, while the second one will be used in Section 6.4 for term graphs.

Labeling edges and nodes

Let us start with labeling both edges and nodes. In order to do so we will fix two sets L_E and L_V). Their elements will be the *labels* for the edges and for the nodes respectively. Notice that $\mathbf{Set}/L_E$ and $\mathbf{Set}/L_V$ are adhesive thanks to Corollary 5.1.39. We have two forgetful functors
$$U_E\colon \mathbf{Set}/L_E \to \mathbf{Set} \qquad U_V\colon \mathbf{Set}/L_V \to \mathbf{Set}$$
which, by Lemma 5.1.35 and since **Set** is complete, preserve pullbacks.

Definition 6.2.14. A *labelled hypergraph* $\mathcal{G}$ is a 6-uple $(X_\mathcal{G}, Y_\mathcal{G}, l_{E,\mathcal{G}}, l_{V,\mathcal{G}}, s_{E,\mathcal{G}}, t_{E,\mathcal{G}})$ made by: two sets $X_\mathcal{G}$ and $Y_\mathcal{G}$, *labelling functions* $l_{E,\mathcal{G}}\colon X_\mathcal{G} \to L_E$ and $l_{V,\mathcal{G}}\colon Y_\mathcal{G} \to L_V$, and, finally *source and target functions* $s_\mathcal{G}, t_\mathcal{G}\colon X_\mathcal{G}^\star \rightrightarrows Y_\mathcal{G}^\star$. A morphism $(h,k)\colon \mathcal{G} \to \mathcal{H}$ is given by $f\colon h_\mathcal{G} \to X_\mathcal{H}$ and $k\colon Y_\mathcal{G} \to Y_\mathcal{H}$ such that the following diagrams commute.

$$\begin{tikzcd} X_\mathcal{G} \arrow[r, "s_\mathcal{G}"] \arrow[d, "h"'] & Y_\mathcal{G}^\star \arrow[d, "k^\star"] \\ X_\mathcal{H} \arrow[r, "s_\mathcal{H}"'] & Y_\mathcal{H}^\star \end{tikzcd} \quad \begin{tikzcd} X_\mathcal{G} \arrow[r, "t_\mathcal{G}"] \arrow[d, "h"'] & Y_\mathcal{G}^\star \arrow[d, "k^\star"] \\ X_\mathcal{H} \arrow[r, "t_\mathcal{H}"'] & Y_\mathcal{H}^\star \end{tikzcd} \quad \begin{tikzcd} X_\mathcal{G} \arrow[rr, "h"] \arrow[dr, "l_{E,\mathcal{G}}"'] & & X_\mathcal{H} \arrow[dl, "l_{E,\mathcal{G}}"] \\ & L_E & \end{tikzcd} \quad \begin{tikzcd} Y_\mathcal{H} \arrow[rr, "k"] \arrow[dr, "l_{V,\mathcal{G}}"'] & & Y_\mathcal{H} \arrow[dl, "l_{V,\mathcal{H}}"] \\ & L_V & \end{tikzcd}$$

Remark 6.2.15. Notice that there is a forgetful functor $U_{\mathbf{LHyp}}\colon \mathbf{LHyp} \to \mathbf{Hyp}$:

$$\begin{tikzcd} (X_\mathcal{G}, Y_\mathcal{G}, l_{E,\mathcal{G}}, l_{V,\mathcal{G}}, s_{E,\mathcal{G}}, t_{E,\mathcal{G}}) \arrow[r, mapsto] \arrow[d, "(h,k)"'] & (X_\mathcal{G}, Y_\mathcal{G}, s_\mathcal{G}, t_\mathcal{G}) \arrow[d, "(h,k)"] \\ (X_\mathcal{G}, Y_\mathcal{G}, l_{E,\mathcal{G}}, l_{V,\mathcal{G}}, s_{E,\mathcal{G}}, t_{E,\mathcal{G}}) \arrow[r, mapsto] & (X_\mathcal{H}, Y_\mathcal{H}, s_\mathcal{H}, t_\mathcal{H}) \end{tikzcd}$$

Now, consider the functor $\mathsf{lprod}\colon \mathbf{Set}/L_V \to \mathbf{Set}$ given by the composition $\mathsf{prod}^\star \circ U_V$, on the one hand we can define a functor $\mathbf{LHyp} \to U_E{\downarrow}\mathsf{lprod}$:

$$\begin{array}{ccc} (X_\mathcal{G}, Y_\mathcal{G}, l_{E,\mathcal{G}}, l_{V,\mathcal{G}}, s_{E,\mathcal{G}}, t_{E,\mathcal{G}}) & \longmapsto & (l_{E,\mathcal{G}}, l_{V,\mathcal{G}}, (s_\mathcal{G}, t_\mathcal{G})) \\ (h,k)\big\downarrow & & \big\downarrow(h,k) \\ (X_\mathcal{G}, Y_\mathcal{G}, l_{E,\mathcal{G}}, l_{V,\mathcal{G}}, s_{E,\mathcal{G}}, t_{E,\mathcal{G}}) & \longmapsto & (l_{E,\mathcal{H}}, l_{V,\mathcal{H}}, (s_\mathcal{H}, t_\mathcal{H})) \end{array}$$

while, on the other hand, we can define

$$\begin{array}{ccc} (l_{E,\mathcal{G}}, l_{V,\mathcal{G}}, p_\mathcal{G}) & \longmapsto & (X_\mathcal{G}, Y_\mathcal{G}, l_{E,\mathcal{G}}, l_{V,\mathcal{G}}, \pi_1 \circ p_\mathcal{G}, \pi_2 \circ p_\mathcal{G}) \\ (h,k)\big\downarrow & & \big\downarrow(h,k) \\ (l_{E,\mathcal{H}}, l_{V,\mathcal{H}}, p_\mathcal{H}) & \longmapsto & (X_\mathcal{G}, Y_\mathcal{G}, l_{E,\mathcal{G}}, l_{V,\mathcal{G}}, \pi_1 \circ p_\mathcal{H}, \pi_2 \circ p_\mathcal{H}) \end{array}$$

By inspection these two functors are one the inverse of the other, thus we have just proved the following.

Proposition 6.2.16. **LHyp** *and* $U_E{\downarrow}\mathsf{lprod}$ *are isomorphic.*

Noticing that U_E preserves pushouts we get at once an adhesivity result.

Corollary 6.2.17. **LHyp** *is adhesive.*

Labelling hypergraph with an algebraic signature

Let $\Sigma = (O_\Sigma, \mathsf{ar}_\Sigma)$ be an algebraic signature, we are going to use the hypergraph $\mathcal{G}^\Sigma$ of Example 6.2.10 in order to label hyperedges with operations.

Definition 6.2.18. Let $\Sigma = (O, \mathsf{ar})$ be an algebraic signature, the category $\mathbf{Hyp}_\Sigma$ of *algebraically labelled hypergraphs* is the slice category $\mathbf{Hyp}/\mathcal{G}^\Sigma$.

Corollary 5.1.37 and Corollary 5.1.39 give us immediately an adhesivity result for $\mathbf{Hyp}_\Sigma$ and a characterization of monomorphisms in it.

Proposition 6.2.19. *For every algebraic signature* Σ*,* $\mathbf{Hyp}_\Sigma$ *is an adhesive category. Moreover a morphism* (h,k) *between two object of* $\mathbf{Hyp}_\Sigma$ *is a mono if and only if* h *and* k *are injective functions.*

Remark 6.2.20. Let $\mathcal{H} = (E, V, s, t)$ be an hypergraph, since $U_{\mathbf{Hyp}}(\mathcal{G}^\Sigma)$ is the singleton an arrow $\mathcal{H} \to \mathcal{G}^\Sigma$, is determined by a function $h\colon E_\mathcal{H} \to O_\Sigma$ such that, for every $e \in E_\mathcal{H}$

$$\mathsf{ar}_\Sigma(h(e)) = s_\mathcal{H}(e)$$

On the other hand, if $\mathcal{H}$ has an hyperedge e such that $t_\mathcal{H}(e)$ has a length different from 1, then there is no morphism $\mathcal{H} \to \mathcal{G}^\Sigma$. Indeed, if such a morphism $(h, !_{V_\mathcal{H}})\colon \mathcal{H} \to \mathcal{G}^\Sigma$ exists, then, for every $e \in E_\mathcal{H}$ we have

$$\begin{aligned} f^\star(t_\mathcal{H}(h)) &= t_{\mathcal{G}^\Sigma}(f(h)) \\ &= \delta_\heartsuit \end{aligned}$$

and so $\mathsf{dom}(t_\mathcal{H}(h)) = 1$.

$\mathbf{Hyp}_\Sigma$, as any slice category, has a forgetful functor $U_\Sigma\colon \mathbf{Hyp}_\Sigma \to \mathbf{Set}$ which sends $(h,k)\colon \mathcal{H} \to \mathcal{G}^\Sigma$ to $U_{\mathbf{Hyp}}(\mathcal{H})$. Now, $U_{\mathbf{Hyp}}(\mathcal{G}^\Sigma) = \{v\}$ thus, for every set X, there is only one arrow $X \to U_{\mathbf{Hyp}}(\mathcal{G}^\Sigma)$. Define $\Delta_\Sigma(X)\colon \Delta_{\mathbf{Hyp}}(X) \to \mathcal{G}^\Sigma$ to be the transpose of this arrow.

Proposition 6.2.21. *U_Σ has a left adjoint Δ_Σ.*

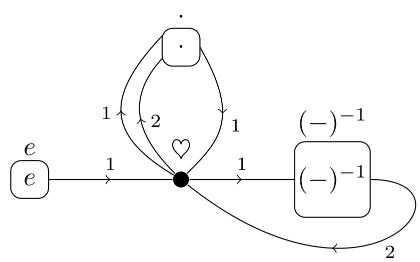
Proof. Let $(h, !_{V_\mathcal{H}})\colon \mathcal{H} \to \mathcal{G}^\Sigma$ be an object of $\mathbf{Hyp}_\Sigma$, and suppose that there exists $f\colon X \to U_\Sigma(\mathcal{H})$. Since $U_\Sigma(\mathcal{H}) = U_{\mathbf{Hyp}}(\mathcal{H})$ and $\mathsf{id}_{\mathbf{Set}}$ is the unit of $\Delta_{\mathbf{Hyp}} \dashv U_{\mathbf{Hyp}}$, there exists a unique morphism $(k,f)\colon \Delta_{\mathbf{Hyp}}(X) \to \mathcal{H}$ of $\mathbf{Hyp}$. Since the set of hyperedges of $\Delta_{\mathbf{Hyp}}(X)$ is empty, k must be $?_{E_\mathcal{H}}$ and the commutativity of each of the two triangles below is equivalent to that of the other

$$\begin{tikzcd}
\Delta_{\mathbf{Hyp}}(X) \arrow[rr, "(?_{E_\mathcal{H}}, f)"] \arrow[dr, "\Delta_\Sigma(X)"'] & & \mathcal{H} \arrow[dl, "(h, !_{V_\mathcal{H}})"] \\
& \mathcal{G}^\Sigma &
\end{tikzcd}
\qquad
\begin{tikzcd}
U_{\mathbf{Hyp}}(\Delta_{\mathbf{Hyp}}(X)) \arrow[rr, "f"] \arrow[dr, "U_{\mathbf{Hyp}}(\Delta_\Sigma(X))"'] & & U_{\mathbf{Hyp}}(X) \arrow[dl, "!_{V_\mathcal{H}}"] \\
& U_{\mathbf{Hyp}}(\mathcal{G}^\Sigma) &
\end{tikzcd}$$

But the triangle on the right commutes because $U_{\mathbf{Hyp}}(\mathcal{G}^\Sigma)$ is terminal. □

We will extend our graphical notation of hypergraphs to labeled ones putting the label of an hyperedge h inside its corresponding square.

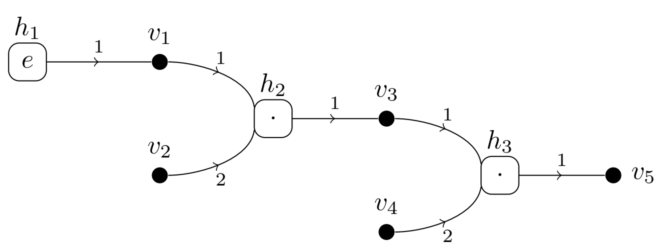
Example 6.2.22. The simplest example is given by the identity $\mathsf{id}_{\mathcal{G}^\Sigma}\colon \mathcal{G}^\Sigma \to \mathcal{G}^\Sigma$. If Σ is the signature of groups $\Sigma_G$we get



Example 6.2.23. Take again Σ_G the signature of groups, then the hypergraph $\mathcal{G}$ of Example 6.2.9 can be labeled defining

$$e = f(h_1) \quad \cdot = f(h_2) \quad \cdot = f(h_3)$$

In this case we get the following picture



Remark 6.2.24. There is a *colored* (or *typed*) version of these last constructions. Start with a *colored* algebraic signature: this is a triple (C, O, ar) where C is the set of *colors*, O is the set of *operations* and $\mathsf{ar}\colon O \to C^\star \times C^\star$ assigns to every operations f an arity and a coarity given by strings of colors. We can still construct an hypergraph $\mathcal{G}^\Sigma$ with C as set of nodes using the operations as hyperedges. In this context an object in the slice $\mathbf{Hyp}/\mathcal{G}^\Sigma$ is an hypergraph in which both the hyperedges and the nodes are labeled, the formers with an elemento of O and the latters with an element of C [26].

6.2.3 Hierarchical hypergraphs

We can leverage on the modularity of Theorem 5.1.31 and Theorem 5.1.38 to give hypergraphical variants for Corollaries 6.1.41 and 6.1.44. This is done replacing the set $E_\mathcal{G}$ of hyperedges with a tree order $(E_\mathcal{G}, \leq)$ and $\mathsf{id}_{\mathbf{Set}}$ with the forgetful functor $U_{\mathbf{Tree}}\colon \mathbf{Tree} \to \mathbf{Set}$.

Definition 6.2.25. A *hierarchical hypergraph* $\mathcal{G}$ is a triple $((E_\mathcal{G}, \leq), V_\mathcal{G}, s_\mathcal{G}, t_\mathcal{G})$ where $(E_\mathcal{G}, \leq)$ is a tree order, $V_\mathcal{G}$ a set and $s_\mathcal{G}, t_\mathcal{G}\colon E_\mathcal{G} \rightrightarrows V_\mathcal{G}^\star$ two functions. A *morphism* $\mathcal{G} \to \mathcal{H}$ is a pair (h, k) made by $h\colon (E_\mathcal{G}, \leq) \to (E_\mathcal{H}, \leq)$ in $\mathbf{Tree}$ and by $k\colon V \to W$ in $\mathbf{Set}$ such that the following squares commute

$$
\begin{tikzcd}
E_\mathcal{G} \arrow[r, "s_\mathcal{G}"] \arrow[d, "U_{\mathbf{Tree}}(h)"'] & V_\mathcal{G}^\star \arrow[d, "k^\star"] \\
E_\mathcal{H} \arrow[r, "t_\mathcal{G}"'] & V_\mathcal{H}^\star
\end{tikzcd}
\qquad
\begin{tikzcd}
E_\mathcal{G} \arrow[r, "t_\mathcal{G}"] \arrow[d, "U_{\mathbf{Tree}}(h)"'] & V_\mathcal{G}^\star \arrow[d, "k^\star"] \\
E_\mathcal{H} \arrow[r, "t_\mathcal{H}"'] & V_\mathcal{H}^\star
\end{tikzcd}
$$

Taking componentwise composition we get a category $\mathbf{HHGraph}$.

Proposition 6.2.26. $\mathbf{HHGraph}$ *is isomorphic to* $U_{\mathbf{Tree}} \downarrow \mathsf{prod}^\star$

Proof. Define $F\colon \mathbf{HHGraph} \to U_{\mathbf{Tree}} \downarrow \mathsf{prod}^\star$

$$
\begin{array}{ccc}
((E_\mathcal{G}, \leq), V_\mathcal{G}, s_\mathcal{G}, t_\mathcal{G}) & \longmapsto & ((E_\mathcal{G}, \leq), V_\mathcal{G}, (s_\mathcal{G}, t_\mathcal{G})) \\
(h,k) \big\downarrow & & \big\downarrow (h,k) \\
((E_\mathcal{H}, \leq), V_\mathcal{H}, s_\mathcal{H}, t_\mathcal{H}) & \longmapsto & ((E_\mathcal{H}, \leq), V_\mathcal{H}, (s_\mathcal{H}, t_\mathcal{H}))
\end{array}
$$

and $G\colon U_{\mathbf{Tree}} \downarrow \mathsf{prod}^\star \to \mathbf{HHGraph}$ as

$$
\begin{array}{ccc}
((E_\mathcal{G}, \leq), V_\mathcal{G}, p_\mathcal{G}) & \longmapsto & ((E_\mathcal{G}, \leq), V_\mathcal{G}, \pi_1 \circ p_\mathcal{G}, \pi_2 \circ p_\mathcal{G}) \\
(f,g) \big\downarrow & & \big\downarrow (f,g) \\
((E_\mathcal{H}, \leq), V_\mathcal{H}, p_\mathcal{H}) & \longmapsto & ((E_\mathcal{H}, \leq), V_\mathcal{H}, \pi_1 \circ p_\mathcal{H}, \pi_2 \circ p_\mathcal{H})
\end{array}
$$

The thesis follows immediately. $\square$

Corollary 6.2.27. $\mathbf{HHGRaph}$ *is adhesive. Moreover, the functor* $\mathbf{HHGraph} \to \mathbf{Set}$, *which sends a hierarchical hypergraph to its set of nodes, has a left adjoint* $\Delta_{\mathbf{HHGraph}}$.

Proof. The first half of the thesis follows from Theorem 5.1.38 and Proposition 6.2.26, while the second one is entailed by Proposition A.2.3. $\square$

Remark 6.2.28. $\Delta_{\mathbf{HHGraph}}$ sends a set X to the hierarchical hypergraph $((\emptyset, \leq), X, ?_{X^\star}, ?_{X^\star})$.

To add interface we proceed exactly as in Section 6.1.3, using the previous corollary.

Definition 6.2.29. The category **HHIGraph** of *hierarchical hypergraphs with interface* is the category in which objects are triples $(\mathcal{G}, X, f)$ made by a hierarchical hypergraph $\mathcal{G} = ((E_{\mathcal{G}}, \leq), V_{\mathcal{G}}, s_{\mathcal{G}}, t_{\mathcal{G}})$, a set X and a function $f\colon X \to V_{\mathcal{G}}$. A morphism $(\mathcal{G}, X, f) \to (\mathcal{H}, Y, g)$ is a triple (h, k, l) with $h\colon (E_{\mathcal{G}}, \leq) \to (E_{\mathcal{H}}, \leq)$ in **Tree**, $k\colon V_{\mathcal{G}} \to V_{\mathcal{H}}$ and $l\colon X \to Y$ in **Set** such that the following squares commute.

$$
\begin{tikzcd}
E_{\mathcal{G}} \arrow[r, "s_{\mathcal{G}}"] \arrow[d, "U_{\mathbf{Tree}}(h)"'] & V_{\mathcal{G}}^{\star} \arrow[d, "k^{\star}"] \\
E_{\mathcal{G}} \arrow[r, "s_{\mathcal{H}}"'] & V_{\mathcal{H}}^{\star}
\end{tikzcd}
\qquad
\begin{tikzcd}
E_{\mathcal{G}} \arrow[r, "t_{\mathcal{G}}"] \arrow[d, "U_{\mathbf{Tree}}(h)"'] & V_{\mathcal{G}}^{\star} \arrow[d, "k^{\star}"] \\
E_{\mathcal{G}} \arrow[r, "t_{\mathcal{H}}"'] & V_{\mathcal{H}}^{\star}
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "l"'] & V_{\mathcal{G}} \arrow[d, "k"] \\
Y \arrow[r, "g"'] & V_{\mathcal{H}}
\end{tikzcd}
$$

Remark 6.2.30. This category of hypergraphs whose edges form a tree order, corresponds to Milner's (pure) bigraphs [96], with possibly infinite edges[1].

Proposition 6.2.31. *The category* **HHIGraph** *is isomorphic to* $\Delta_{\mathbf{HHGraph}}\!\downarrow\!\mathsf{id}_{\mathbf{Hyp}}$

Proof. Define $F\colon \mathbf{HHIGraph} \to \Delta_{\mathbf{HHGraph}}\!\downarrow\!\mathsf{id}_{\mathbf{Hyp}}$ and $G\colon \Delta_{\mathbf{HHGraph}}\!\downarrow\!\mathsf{id}_{\mathbf{Hyp}} \to \mathbf{HHIGraph}$ putting

$$
\begin{array}{ccc}
(\mathcal{G}, X, f) & \longmapsto & (X, \mathcal{G}, (?_X, f)) \\
{\scriptstyle (h,k,l)}\big\downarrow & & \big\downarrow{\scriptstyle (l,(h,k))} \\
(\mathcal{H}, Y, g) & \longmapsto & (Y, \mathcal{H}, (?_Y, g))
\end{array}
\qquad
\begin{array}{ccc}
(X, \mathcal{G}, (?_X, f)) & \longmapsto & (\mathcal{G}, X, f) \\
{\scriptstyle (l,(h,k))}\big\downarrow & & \big\downarrow{\scriptstyle (h,k,l)} \\
(Y, \mathcal{H}, (?_Y, g)) & \longmapsto & (\mathcal{H}, Y, g)
\end{array}
$$

The thesis now follows at once. □

Corollary 6.2.32. **HHIGraph** *is adhesive.*

6.2.4 SGraph and DAG-hypergraphs

We can consider more general relations between edges, besides tree orders. An interesting case is when edges form a directed acyclic graph, yielding the category of **DAG**-*hypergraphs*; this corresponds to (possibly infinite) *bigraphs with sharing*, where an edge can have more than one parent, as in [117] (see also Fig. 6.1, left). Even more generally, we can consider any relation between edges, i.e., the edges form a generic directed graph possibly with cycles, yielding the category of **SGraph**-*hypergraphs*. These can be seen as "recursive bigraphs", i.e., bigraphs which allow for cyclic dependencies between controls, like in recursive processes; an example is in Fig. 6.1 (right).

Definition 6.2.33. A **SGraph**-*hypergraph* (respectively **DAG**-*hypergraphs*) is a triple $(\mathcal{G}, V, s, t)$ where $\mathcal{G}$ is in **SGraph** (in **DAG**), V is a set and s, t functions $V_{\mathcal{G}} \rightrightarrows V^{\star}$. A *morphism* of **SGraph**-*hypergraph* (**DAG**-*hypergraphs*) is a pair $((h_1, h_2), k)\colon (\mathcal{G}, V, s, t) \to (\mathcal{H}, W, s', t')$ with $(h_1, h_2)\colon \mathcal{G} \to \mathcal{H}$ in **SGraph** (in **DAG**) and $k\colon V \to W$ in **Set** such that the following squares commute

$$
\begin{tikzcd}
V_{\mathcal{G}} \arrow[r, "s"] \arrow[d, "h_2"'] & V^{\star} \arrow[d, "k^{\star}"] \\
V_{\mathcal{H}} \arrow[r, "s'"'] & W^{\star}
\end{tikzcd}
\qquad
\begin{tikzcd}
V_{\mathcal{G}} \arrow[r, "t"] \arrow[d, "h_2"'] & V^{\star} \arrow[d, "k^{\star}"] \\
V_{\mathcal{H}} \arrow[r, "t'"'] & W^{\star}
\end{tikzcd}
$$

These data give rise to the categories **SHGraph** and **DAGHGraph** respectively.

[1] In bigraph terminology, "controls" and "edges" correspond to our edges and nodes.

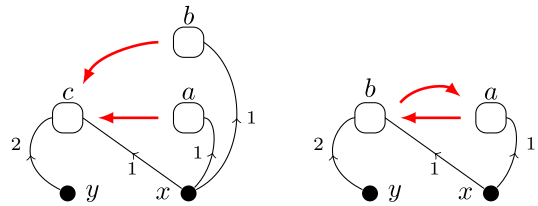


Figure 6.1: A **DAG**-hypergraph (left) and a **SGraph**-hypergraph corresponding to the CCS process $P = a(x).b(xy).P$ (right). The red arrows denotes the graph structure of the edges.

SHGraph and **DAGHGraph** isomorphic to, respectively $U_{\mathbf{SGraph}}{\downarrow}\mathsf{prod}^\star$ and $U_{\mathbf{DAG}}{\downarrow}\mathsf{prod}^\star$. This is easily proved considering the four functors:

$$F_1\colon \mathbf{SHGraph} \to U_{\mathbf{SGraph}}{\downarrow}\mathsf{prod}^\star \qquad G_1\colon U_{\mathbf{SGraph}}{\downarrow}\mathsf{prod}^\star \to \mathbf{SHGraph}$$

$$\begin{tikzcd} (\mathcal{G},V,s,t) \arrow[r, mapsto] \arrow[d, "{((h_1,h_2),k)}"'] & (\mathcal{G},V,(s,t)) \arrow[d, "{((h_1,h_2),k)}"] \\ (\mathcal{H},W,s',t') \arrow[r, mapsto] & (\mathcal{H},W,(s',t')) \end{tikzcd} \qquad \begin{tikzcd} (\mathcal{G},V,p) \arrow[r, mapsto] \arrow[d, "{((h_1,h_2),k)}"'] & (\mathcal{G},V,\pi_1\circ p,\pi_2\circ p) \arrow[d, "{((h_1,h_2),k)}"] \\ (\mathcal{H},W,p') \arrow[r, mapsto] & (\mathcal{H},W,\pi_1\circ p',\pi_2\circ p') \end{tikzcd}$$

$$F_2\colon \mathbf{DAGHGraph} \to U_{\mathbf{DAG}}{\downarrow}\mathsf{prod}^\star \qquad G_2\colon U_{\mathbf{DAG}}{\downarrow}\mathsf{prod}^\star \to \mathbf{DAGHGraph}$$

$$\begin{tikzcd} (\mathcal{G},V,s,t) \arrow[r, mapsto] \arrow[d, "{((h_1,h_2),k)}"'] & (\mathcal{G},V,(s,t)) \arrow[d, "{((h_1,h_2),k)}"] \\ (\mathcal{H},W,s',t') \arrow[r, mapsto] & (\mathcal{H},W,(s',t')) \end{tikzcd} \qquad \begin{tikzcd} (\mathcal{G},V,p) \arrow[r, mapsto] \arrow[d, "{((h_1,h_2),k)}"'] & (\mathcal{G},V,\pi_1\circ p,\pi_2\circ p) \arrow[d, "{((h_1,h_2),k)}"] \\ (\mathcal{H},W,p') \arrow[r, mapsto] & (\mathcal{H},W,\pi_1\circ p',\pi_2\circ p') \end{tikzcd}$$

Theorem 6.2.34. **SHGraph** *is $\mathcal{M},\mathcal{N}$-adhesive with respect to the classes*

$$\mathcal{M} := \{((h_1,h_2),k) \in \mathcal{A}(\mathbf{SHGraph}) \mid (h_1,h_2) \in \mathcal{R}(\mathbf{SGraph}), k \in \mathcal{M}(\mathbf{Set})\}$$
$$\mathcal{N} := \{((h_1,h_2),k) \in \mathcal{A}(\mathbf{SHGraph}) \mid (h_1,h_2) \in \mathcal{M}(\mathbf{SGraph})\}$$

while **DAGHGraph** *is adhesive with respect to the classes*

$$\{((h_1,h_2),k) \in \mathcal{A}(\mathbf{DAGHGraph}) \mid (h_1,h_2) \in \mathsf{dcl_d}, k \in \mathcal{M}(\mathbf{Set})\}$$
$$\{((h_1,h_2),k) \in \mathcal{A}(\mathbf{DAGHGraph}) \mid (h_1,h_2) \in \mathcal{M}(\mathbf{DAG})\}$$

Moreover, the functors **DHGraph** $\to$ **Set** *and* **DAGHGraph** $\to$ **Set***, which assign to an hypergraph its set of nodes, have left adjoints* $\Delta_{\mathbf{DHGraph}}$ *and* $\Delta_{\mathbf{DAGHGraph}}$*.*

Remark 6.2.35. Let $\mathcal{I}$ be the initial object of **Graph**, i.e. $(\emptyset,\emptyset,\mathsf{id}_\emptyset,\mathsf{id}_\emptyset)$. $\mathcal{I}$ is both in **SGraph** and in **DAG**, thus it is initial in these categories too. Thus $\Delta_{\mathbf{DHGraph}}$ and $\Delta_{\mathbf{DAGHGraph}}$ assign to a set X the **DAG** and **SGraph**-hypergraph $(\mathcal{I}, X, ?_{X^\star}, ?_{X^\star})$.

As in Sections 6.1.3 and 6.2.3, we can exploit these two last corollaries to add interfaces.

Definition 6.2.36. The category **SHIGraph** (**DAGIHGraph**) of **SGraph**-hypergraphs (resp. of **DAG**-hypergraphs) *with interfaces* has as objects triples $((\mathcal{G},V,s,t),X,f)$ made by a **SGraph**-hypergraph (a **DAG**-hypergraph) $(\mathcal{G},V,s,t)$ and a function $f\colon X \to V$. An arrow between $((\mathcal{G},V,s,t),X,f)$ and

$((\mathcal{H}, w, s', t'), Y, g)$ is a triple $((h_1, h_2), k, l)$ made by a morphism $((h_1, h_2), k)\colon \mathcal{G} \to \mathcal{H}$ in **SHGraph** (in **DAGHGraph**), and a function $l\colon X \to Y$ in **Set** such that the following squares commute

$$\begin{tikzcd} V_{\mathcal{G}} \arrow[r, "s"] \arrow[d, "h_2"'] & V^\star \arrow[d, "k^\star"] \\ V_{\mathcal{H}} \arrow[r, "s'"'] & W^\star \end{tikzcd} \qquad \begin{tikzcd} V_{\mathcal{G}} \arrow[r, "t"] \arrow[d, "h_2"'] & V^\star \arrow[d, "k^\star"] \\ V_{\mathcal{H}} \arrow[r, "t'"'] & W^\star \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "l"'] & V \arrow[d, "k"] \\ Y \arrow[r, "g"'] & W \end{tikzcd}$$

As before we can consider functors

$$\begin{array}{rcl} & F_1\colon \textbf{SHIGraph} \to \Delta_{\textbf{SHGraph}}{\downarrow}\mathsf{id}_{\textbf{SHGraph}} & \\ ((\mathcal{G}, V, s, t), X, f) & \longmapsto & (X, (\mathcal{G}, V, s, t), (?_{\mathcal{G}}, f)) \\ ((h_1, h_2), k, l)\big\downarrow & & \big\downarrow(l, ((h_1, h_2), k)) \\ ((\mathcal{H}, W, s', t'), Y, g) & \longmapsto & (Y, (\mathcal{H}, W, s', t'), (?_{\mathcal{H}}, g)) \\ & G_1\colon \Delta_{\textbf{SHGraph}}{\downarrow}\mathsf{id}_{\textbf{SHGraph}} \to \textbf{SHIGraph} & \\ (X, (\mathcal{G}, V, s, t), (?_{\mathcal{G}}, f)) & \longmapsto & ((\mathcal{G}, V, s, t), X, f) \\ (l, ((h_1, h_2), k))\big\downarrow & & \big\downarrow((h_1, h_2), k, l) \\ (Y, (\mathcal{H}, W, s', t'), (?_{\mathcal{H}}, g)) & \longmapsto & ((\mathcal{H}, W, s', t'), Y, g) \end{array}$$

showing that **SHIGRaph** and $\Delta_{\textbf{SHGraph}}{\downarrow}\mathsf{id}_{\textbf{SHGraph}}$ are isomorphic.

We have another pair of functors (defined in the same way):

$$\begin{array}{rcl} & F_2\colon \textbf{DAGHIGraph} \to \Delta_{\textbf{DAGHGraph}}{\downarrow}\mathsf{id}_{\textbf{DAGHGraph}} & \\ ((\mathcal{G}, V, s, t), X, f) & \longmapsto & (X, (\mathcal{G}, V, s, t), (?_{\mathcal{G}}, f)) \\ ((h_1, h_2), k, l)\big\downarrow & & \big\downarrow(l, ((h_1, h_2), k)) \\ ((\mathcal{H}, W, s', t'), Y, g) & \longmapsto & (Y, (\mathcal{H}, W, s', t'), (?_{\mathcal{H}}, g)) \\ & G_2\colon \Delta_{\textbf{DAGHGraph}}{\downarrow}\mathsf{id}_{\textbf{DAGHGraph}} \to \textbf{DAGHIGraph} & \\ (X, (\mathcal{G}, V, s, t), (?_{\mathcal{G}}, f)) & \longmapsto & ((\mathcal{G}, V, s, t), X, f) \\ (l, ((h_1, h_2), k))\big\downarrow & & \big\downarrow((h_1, h_2), k, l) \\ (Y, (\mathcal{H}, W, s', t'), (?_{\mathcal{H}}, g)) & \longmapsto & ((\mathcal{H}, W, s', t'), Y, g) \end{array}$$

which shows that and **DAGHIGraph** is isomorphic to $\Delta_{\textbf{DAGHGraph}}{\downarrow}\mathsf{id}_{\textbf{DAGHGraph}}$.

Summing up we can get a last adhesivity result.

Theorem 6.2.37. **SHIGraph** *is $\mathcal{M},\mathcal{N}$-adhesive with respect to the classes*

$$\mathcal{M} := \{((h_1, h_2), k, l) \in \mathcal{A}(\textbf{SHIGraph}) \mid (h_1, h_2) \in \mathcal{R}(\textbf{SGraph}), k, l \in \mathcal{M}(\textbf{Set})\}$$
$$\mathcal{N} := \{((h_1, h_2), k, l) \in \mathcal{A}(\textbf{SHIGraph}) \mid (h_1, h_2) \in \mathcal{M}(\textbf{SGraph})\}$$

while **DAGIHGraph** *is $\mathcal{M},\mathcal{N}$-adhesive with respect to the classes*

$$\mathcal{M} := \{((h_1, h_2), k, l) \in \mathcal{A}(\textbf{DAGHIGraph}) \mid (h_1, h_2) \in \mathsf{dcl_d}, k, l \in \mathcal{M}(\textbf{Set})\}$$
$$\mathcal{N} := \{((h_1, h_2), k, l) \in \mathcal{A}(\textbf{DAGHIGraph}) \mid (h_1, h_2) \in \mathcal{M}(\textbf{DAG})\}$$

6.3 A graphical formalism for monoidal closed categories

In [11], the authors use a kind of hierarchical graphs to implement rewriting of arrows in a monoidal closed categories in terms of the double pushout approach. In this section we will prove some adhesivity property of this category of hierarchical graphs

6.3.1 The category HHG and labelled DAG-hypergraphs

In this section we will start introducing the objects used in [11]. We will also show that the category so obtained can be realized fully and faithfully embedded into a category of *labelled* **DAG**-*hypergraphs* of which we know some adhesivity properties.

Definition 6.3.1 ([11, Def. 16]). We define the category **HHG** in the following way.

- Objects are 8-uples $\mathsf{G} := (E_\mathsf{G}, V_\mathsf{G}, s_\mathsf{G}, t_\mathsf{G}, l_{E,\mathsf{G}}, l_{V,\mathsf{G}}, p_{E,\mathsf{G}}, p_{V,\mathsf{G}})$ where $(E_\mathsf{G}, V_\mathsf{G}, s_\mathsf{G}, t_\mathsf{G}, l_{E,\mathsf{G}}, l_{V,\mathsf{G}})$ is an object of **LHyp** such that E_G and V_G are finite, $p_{E,\mathsf{G}}$ is a function $E_\mathsf{G} \to E_\mathsf{G} + 1$ and $p_{V,\mathsf{G}}$ one $V \to E_\mathsf{G} + 1$. Moreover we ask that:

  1. if $\iota_1 \colon E_\mathsf{G} \to E_\mathsf{G} + 1$, $\iota_2 \colon 1 \to E_\mathsf{G} + 1$ are the coprojections and $p^*_{E,\mathsf{G}} \colon E_\mathsf{G} + 1 \to E_\mathsf{G} + 1$ is the unique arrow fitting in the diagram

$$\begin{tikzcd} E_\mathsf{G} \arrow[dr, "\iota_1"'] \arrow[drr, bend left, "p_{E,\mathsf{G}}"] & & \\ & E_\mathsf{G} + 1 \arrow[r, dotted, "p^*_{E,\mathsf{G}}"] & E_\mathsf{G} + 1 \\ 1 \arrow[ur, "\iota_2"] \arrow[urr, bend right, "\iota_2"'] & & \end{tikzcd}$$

  then for every $e \in E_\mathsf{G}$ there exists a natural number $k \geq 1$ such that

$$\perp = \left(p^*_{E,\mathsf{G}}\right)^k (\iota_1(e))$$

  where $\perp$ is the element picked by $\iota_2 \colon 1 \to E_\mathsf{G} + 1$;

  2. for every $v \in V_\mathsf{G}$, if v is in the image of $s_\mathsf{G}(e)$ or in that of $t_\mathsf{G}(e)$ for some $e \in E_\mathsf{G}$ then

$$p_{V,\mathsf{G}}(v) = p_{E,\mathsf{G}}(e)$$

Given an object G of **HHG**, we will define the sets

$$S_{E,\mathsf{G}} := \{e \in E_\mathsf{G} \mid \text{ there exists } \overline{p}_{E,\mathsf{G}}(e) \in E_\mathsf{G} \text{ such that } p_{E,\mathsf{G}}(e) = \iota_1(\overline{p}_{E,\mathsf{G}}(e))\}$$
$$S_{V,\mathsf{G}} := \{v \in V_\mathsf{G} \mid \text{ there exists } \overline{p}_{V,\mathsf{G}}(v) \in E_\mathsf{G} \text{ such that } p_{V,\mathsf{G}}(v) = \iota_1(\overline{p}_{V,\mathsf{G}}(v))\}$$

By construction, there are $\overline{p}_{E,\mathsf{G}} \colon S_{E,\mathsf{G}} \to E_\mathsf{G}$ and $\overline{p}_{V,\mathsf{G}} \colon S_{V,\mathsf{G}} \to E_\mathsf{G}$ fitting in the diagrams below

$$\begin{tikzcd} S_{E,\mathsf{G}} \arrow[r, "\overline{p}_{E,\mathsf{G}}"] \arrow[d, "i_{E,\mathsf{G}}"'] & E_\mathsf{G} \arrow[d, "\iota_1"] \\ E_\mathsf{G} \arrow[r, "p_{E,\mathsf{G}}"'] & E_\mathsf{G} + 1 \end{tikzcd} \qquad \begin{tikzcd} S_{V,\mathsf{G}} \arrow[r, "\overline{p}_{V,\mathsf{G}}"] \arrow[d, "i_{V,\mathsf{G}}"'] & E_\mathsf{G} \arrow[d, "\iota_1"] \\ E_\mathsf{G} \arrow[r, "p_{E,\mathsf{G}}"'] & E_\mathsf{G} + 1 \end{tikzcd}$$

where $i_{E,\mathsf{G}}\colon S_{E,\mathsf{G}} \to E_{\mathsf{G}}$, $i_{V,\mathsf{G}}\colon S_{V,\mathsf{G}} \to E_{\mathsf{G}}$ are inclusions. We are now ready to define arrows of **HHG**.

- An arrow $(h,k)\colon (E_{\mathsf{G}}, V_{\mathsf{G}}, s_{\mathsf{G}}, t_{\mathsf{G}}, l_{E,\mathsf{G}}, l_{V,\mathsf{G}}) \to (E_{\mathsf{H}}, V_{\mathsf{H}}, s_{\mathsf{H}}, t_{\mathsf{H}}, l_{E,\mathsf{H}}, l_{V,\mathsf{H}})$ of **LHyp** is a morphism $\mathsf{G} \to \mathsf{H}$ if there are $\overline{h}\colon S_{E,\mathsf{G}} \to S_{E,\mathsf{H}}$ and $\overline{k}\colon S_{V,\mathsf{G}} \to S_{V,\mathsf{H}}$ which fit in the diagrams below.

$$\begin{tikzcd}
E_{\mathsf{G}} \arrow[d, "h"'] & S_{E,\mathsf{G}} \arrow[l, "i_{E,\mathsf{G}}"'] \arrow[r, "\overline{p}_{E,\mathsf{G}}"] \arrow[d, dotted, "\overline{h}"'] & E_{\mathsf{G}} \arrow[d, "h"] \\
E_{\mathsf{H}} & S_{E,\mathsf{H}} \arrow[l, "i_{E,\mathsf{H}}"] \arrow[r, "\overline{p}_{E,\mathsf{H}}"'] & E_{\mathsf{H}}
\end{tikzcd}
\qquad
\begin{tikzcd}
V_{\mathsf{G}} \arrow[d, "k"'] & S_{V,\mathsf{G}} \arrow[l, "i_{V,\mathsf{G}}"'] \arrow[r, "\overline{p}_{V\mathsf{G}}"] \arrow[d, dotted, "\overline{k}"'] & E_{\mathsf{G}} \arrow[d, "k"] \\
V_{\mathsf{H}} & S_{V,\mathsf{H}} \arrow[l, "i_{V,\mathsf{H}}"] \arrow[r, "\overline{p}_{V,\mathsf{H}}"'] & E_{\mathsf{H}}
\end{tikzcd}$$

As in Section 6.2.2, we will use L_E and L_V for the set of labels for edges and the one for nodes.

Notation. The exponential $\left(p^*_{E,\mathsf{G}}\right)^k$ appearing in the first point of the definition of the objects of **HHG** means the composition of $p^*_{E,\mathsf{G}}$ with itself k times.

The request on $p_{E,\mathsf{G}}$ suggest some kind of relationship between **HHG** and a category of hierarchical graphs in which the hierarchy is given by a directed acyclic graph. First of all we have to adapt the results of Section 6.2.2 in order to equip **DAGHGraph** with labels.

First of all we can notice the following.

Proposition 6.3.2. $U_{\mathbf{DAG}}\colon \mathbf{DAG} \to \mathbf{Set}$ *preserves limits and* $\mathsf{dcl_d}, \mathcal{M}(\mathbf{DAG})$*-pushouts.*

Proof. This follows at once since $\Delta_{\mathbf{DAG}} \dashv U_{\mathbf{DAG}}$ and from Corollary 6.1.5 and Lemma 6.1.26. □

We are now ready to define a category of labelled **DAG**-hypergraphs.

Definition 6.3.3. The category of *labelled* **DAG**-*hypergraphs* **LDAGHGraph** is the category in which object are 6-uples $(\mathcal{G}, X, s, t, l_X, l_{\mathcal{G}})$ made a $\mathcal{G} \in \mathbf{DAG}$, a set X, labelling functions $l_X\colon X \to L_V$, $l_{\mathcal{G}}\colon V_{\mathcal{G}} \to L_E + 1$ and source and target functions $s,t\colon V_{\mathcal{G}} \rightrightarrows X^\star$ and . A morphism $(\mathcal{G}, X, s, t, l_X, l_{\mathcal{G}}) \to (\mathcal{H}, Y, s', t', l_Y, l_{\mathcal{H}})$ is a pair $((h_1,h_2),k)$ where (h_1,h_2) is a morphism $\mathcal{G} \to \mathcal{H}$ of **DAG** and k a function $X \to Y$ such that the following diagrams commute

$$\begin{tikzcd}
V_{\mathcal{G}} \arrow[r, "s"] \arrow[d, "h_2"'] & X^\star \arrow[d, "k^\star"] \\
V_{\mathcal{H}} \arrow[r, "s'"'] & Y^\star
\end{tikzcd}
\quad
\begin{tikzcd}
V_{\mathcal{G}} \arrow[r, "t"] \arrow[d, "h_2"'] & X^\star \arrow[d, "k^\star"] \\
V_{\mathcal{H}} \arrow[r, "t'"'] & Y^\star
\end{tikzcd}
\quad
\begin{tikzcd}
V_{\mathcal{G}} \arrow[rr, "h_2"] \arrow[dr, "l_{\mathcal{G}}"'] & & V_{\mathcal{H}} \arrow[dl, "l_{\mathcal{H}}"] \\
& L_E + 1 &
\end{tikzcd}
\quad
\begin{tikzcd}
X \arrow[rr, "k"] \arrow[dr, "l_X"'] & & Y \arrow[dl, "l_Y"] \\
& L_V &
\end{tikzcd}$$

Notation. We will denote by k_{L_E} and by $k_{\spadesuit}$ the coprojections $L_E \to L_E+1$ and $1 \to L_E+1$. Moreover, we will use $\spadesuit$ for the element of $L_E + 1$ picked by $k_{\spadesuit}$.

We want now to show that the category **LDAGHGraph** has some adhesivity property. We can define a continuous functor $\mathsf{ps}\colon \mathbf{Set} \to \mathbf{Set}$ putting

$$\begin{array}{rcl}
X & \longmapsto & (L_E+1) \times X^\star \times X^\star \\
f \big\downarrow & & \big\downarrow \mathsf{id}_{L_E+1} \times f^\star \times f^\star \\
Y & \longmapsto & (L_E+1) \times Y^\star \times Y^\star
\end{array}$$

Proposition 6.3.4. **LDAGHGraph** *is isomorphic to* $U_{\mathbf{DAG}}{\downarrow}\mathsf{ps} \circ U_V$*, where* U_V *is the forgetful functor* $U_V\colon \mathbf{Set}/L_V \to \mathbf{Set}$.

Proof. In one direction, define $G_1\colon \mathbf{LDAGHGraph} \to U_{\mathbf{DAG}}{\downarrow}\mathsf{ps} \circ U_V$ putting

$$\begin{array}{ccc} (\mathcal{G}, X, s, t, l_X, l_{\mathcal{G}}) & \longmapsto & (\mathcal{G}, l_X, (l_{\mathcal{G}}, s, t)) \\ {\scriptstyle ((h_1,h_2),k)}\big\downarrow & & \big\downarrow{\scriptstyle ((h_1,h_2),k)} \\ (\mathcal{H}, Y, s', t', l_Y, l_{\mathcal{H}}) & \longmapsto & (\mathcal{H}, l_Y, (l_{\mathcal{H}}, s', t')) \end{array}$$

In the other direction we can take $G_2\colon U_{\mathbf{DAG}}{\downarrow}\mathsf{ps} \circ U_V \to \mathbf{LDAGHGraph}$ as

$$\begin{array}{ccc} (\mathcal{G}, l, p) & \longmapsto & (\mathcal{G}, \mathsf{dom}(l), \pi_2 \circ p, \pi_3 \circ p, l, \pi_1 \circ p) \\ {\scriptstyle ((h_1,h_2),k)}\big\downarrow & & \big\downarrow{\scriptstyle ((h_1,h_2),k)} \\ (\mathcal{H}, l', p') & \longmapsto & (\mathcal{H}, \mathsf{dom}(l'), \pi_2 \circ p', \pi_3 \circ p', l', \pi_1 \circ p') \end{array}$$

It is now immediate to see that these functors give the thesis. □

From Proposition 6.3.2 now we can obtain the following result.

Corollary 6.3.5. **LDAGHGraph** *is* $\mathcal{M},\mathcal{N}$*-adhesive with respect to the classes*

$$\mathcal{M} := \{((h_1,h_2),k) \in \mathcal{A}(\mathbf{LDAGHGraph}) \mid (h_1,h_2) \in \mathsf{dcl_d}, k \in \mathcal{M}(\mathbf{Set})\}$$
$$\mathcal{N} := \{((h_1,h_2),k) \in \mathcal{A}(\mathbf{LDAGHGraph}) \mid (h_1,h_2) \in \mathcal{M}(\mathbf{DAG}), k \in \mathcal{M}(\mathbf{Set})\}$$

Take now an object G of **HHG**, we can use $p_{E,\mathsf{G}}$ and $p_{V,\mathsf{G}}$ to define a labelled **DAG**-hypergraph $F(\mathsf{G})$. First of all we need to define a directed acyclic graph $\mathcal{G}$ of edges. Define two sets

$$E^1_{\mathcal{G}} := \{(e,e') \in E_{\mathsf{G}} \times E_{\mathsf{G}} \mid \iota_1(e) = p_{E,\mathsf{G}}(e')\} \qquad E^2_{\mathcal{G}} := \{(e,v) \in E_{\mathsf{G}} \times V_{\mathsf{G}} \mid \iota_1(e) = p_{V,\mathsf{G}}(v)\}$$

and notice that they come with the restrictions of the projections

$$s^1_{\mathcal{G}}\colon E^1_{\mathcal{G}} \to E_{\mathsf{G}} \quad (e,e') \mapsto e \qquad s^2_{\mathcal{G}}\colon E^2_{\mathcal{G}} \to E_{\mathsf{G}} \quad (e,v) \mapsto e$$

$$t^1_{\mathcal{G}}\colon E^1_{\mathcal{G}} \to E_{\mathsf{G}} \quad (e,e') \mapsto e' \qquad t^2_{\mathcal{G}}\colon E^2_{\mathcal{G}} \to V_{\mathsf{G}} \quad (e,v) \mapsto v$$

Take $E_{\mathcal{G}}$ and $V_{\mathcal{G}}$ to be, respectively, $E^1_{\mathcal{G}} + E^2_{\mathcal{G}}$ and $E_{\mathsf{G}} + V_{\mathsf{G}}$, then we can define $s_{\mathcal{G}}, t_{\mathcal{G}}\colon E_{\mathcal{G}} \rightrightarrows V_{\mathcal{G}}$ as

$$s_{\mathcal{G}} := s^1_{\mathcal{G}} + s^2_{\mathcal{G}} \qquad t_{\mathcal{G}} := t^1_{\mathcal{G}} + t^2_{\mathcal{G}}$$

Notation. We will denote be $j^1_{\mathcal{G}}$ and $j^2_{\mathcal{G}}$ the coprojections $E^1_{\mathcal{G}} \to E_{\mathcal{G}}$, $E^2_{\mathcal{G}} \to E_{\mathcal{G}}$, while $j_{E,\mathcal{G}}$ and $j_{V,\mathcal{G}}$ will denote those $E_{\mathsf{G}} \to V_{\mathcal{G}}$, $V_{\mathsf{G}} \to V_{\mathcal{G}}$.

Remark 6.3.6. Let us notice two facts:

1. if (e,e') belongs to $E^1_{\mathcal{G}}$ then $e' \in S_{E,\mathcal{G}}$, similarly, if (e,v) is an element of $E^2_{\mathcal{G}}$ then v is in $S_{E,\mathcal{G}}$;
2. the image of $s_{\mathcal{G}}$ is contained into the image of $j_{E,\mathcal{G}}$.

Proposition 6.3.7. *The graph* $\mathcal{G} = (E_{\mathcal{G}}, V_{\mathcal{G}}, s_{\mathcal{G}}, t_{\mathcal{G}})$ *is an object of* **DAG**.

Proof. First of all let us show that $\mathcal{G}$ is simple. Take $v \in V_{\mathcal{G}}$ and $x_1, x_2 \in \mathcal{G}(w_1, w_2)$, we have four cases.

- $x_1 = j^1_{\mathcal{G}}(e_1, e'_1)$ and $x_2 = j^1_{\mathcal{G}}(e_2, e'_2)$ for some (e_1, e'_1) and $(e_2, e'_2) \in E^1_{\mathcal{G}}$. Thus

$$\begin{aligned} j_{E,\mathcal{G}}(e_1) &= j_{E,\mathcal{G}}\left(s^1_{\mathcal{G}}(e_1, e'_1)\right) & j_{E,\mathcal{G}}(e'_1) &= t^1_{\mathcal{G}}(e_1, e'_1) \\ &= s_{\mathcal{G}}\left(j^1_{\mathcal{G}}(e_1, e'_1)\right) & &= t_{\mathcal{G}}\left(j^1_{\mathcal{G}}(e_1, e'_1)\right) \\ &= s_{\mathcal{G}}(x_1) & &= t_{\mathcal{G}}(x_1) \\ &= w_1 & &= w_2 \\ &= s_{\mathcal{G}}(x_2) & &= t_{\mathcal{G}}(x_2) \\ &= s_{\mathcal{G}}\left(j^1_{\mathcal{G}}(e_2, e'_2)\right) & &= t_{\mathcal{G}}\left(j^1_{\mathcal{G}}(e_2, e'_2)\right) \\ &= j_{E,\mathcal{G}}\left(s^1_{\mathcal{G}}(e_2, e'_2)\right) & &= j_{E,\mathcal{G}}\left(t^1_{\mathcal{G}}(e_2, e'_2)\right) \\ &= j_{E,\mathcal{G}}(e_2) & &= j_{E,\mathcal{G}}(e'_2) \end{aligned}$$

  and thus $x_1 = x_2$.

- $x_1 = j^2_{\mathcal{G}}(e_1, v_1)$ and $x_2 = j^2_{\mathcal{G}}(e_2, v_2)$ for some (e_1, v_1) and $(e_2, v_2) \in E^2_{\mathcal{G}}$.

$$\begin{aligned} j_{E,\mathcal{G}}(e_1) &= j_{E,\mathcal{G}}\left(s^1_{\mathcal{G}}(e_1, v_1)\right) & j_{E,\mathcal{G}}(v_1) &= t^1_{\mathcal{G}}(e_1, v_1) \\ &= s_{\mathcal{G}}\left(j^1_{\mathcal{G}}(e_1, v_1)\right) & &= t_{\mathcal{G}}\left(j^1_{\mathcal{G}}(e_1, v_1)\right) \\ &= s_{\mathcal{G}}(x_1) & &= t_{\mathcal{G}}(x_1) \\ &= w_1 & &= w_2 \\ &= s_{\mathcal{G}}(x_2) & &= t_{\mathcal{G}}(x_2) \\ &= s_{\mathcal{G}}\left(j^1_{\mathcal{G}}(e_2, v_2)\right) & &= t_{\mathcal{G}}\left(j^1_{\mathcal{G}}(e_2, v_2)\right) \\ &= j_{E,\mathcal{G}}\left(s^1_{\mathcal{G}}(e_2, v_2)\right) & &= j_{E,\mathcal{G}}\left(t^1_{\mathcal{G}}(e_2, v_2)\right) \\ &= j_{E,\mathcal{G}}(e_2) & &= j_{E,\mathcal{G}}(v_2) \end{aligned}$$

  Hence, even in this case we can conclude that $x_1 = x_2$

- $x_1 = j^1_{\mathcal{G}}(e_1, e'_1)$ and $x_2 = j^2_{\mathcal{G}}(e_2, v_2)$ for some $(e_1, e'_1) \in E^2_{\mathcal{G}}$ and $(e_2, v_2) \in E^2_{\mathcal{G}}$. This case is impossible: indeed we must have

$$\begin{aligned} w_2 &= t_{\mathcal{G}}(x_1)) & w_2 &= t_{\mathcal{G}}(x_2)) \\ &= t_{\mathcal{G}}\left(j^1_{\mathcal{G}}(e_1, e'_1)\right) & &= t_{\mathcal{G}}\left(j^2_{\mathcal{G}}(e_2, v_2)\right) \\ &= j_{E,\mathcal{G}}\left(t^1_{\mathcal{G}}(e_1, e'_1)\right) & &= j_{V,\mathcal{G}}\left(t^1_{\mathcal{G}}(e_2, v_2)\right) \\ &= j_{E,\mathcal{G}}(e'_1) & &= j_{V,\mathcal{G}}(v_2) \end{aligned}$$

  but the images of $j_{E,\mathcal{G}}$ and $j_{V,\mathcal{G}}$ are disjoint.

- $x_1 = j^2_{\mathcal{G}}(e_1, v_1)$ and $x_2 = j^2_{\mathcal{G}}(e_2, e'_2)$ for some $(e_1, v_1) \in E^1_{\mathcal{G}}$ and $(e_2, e'_2) \in E^2_{\mathcal{G}}$. Swapping x_1 and x_2 we fall back in the previous case.

Next, suppose that $\{x_i\}_{i=1}^n$ is a cycle in $\mathcal{G}$, we have two cases.

- For every $1 \leq i \leq n$ there exists $(e_i, e'_i) \in E^1_{\mathcal{G}}$ such that

$$x_i = j^1_{\mathcal{G}}(e_i, e'_i)$$

  The cycle condition implies that, for very $1 \leq i < n$

$$e'_n = e_1 \qquad e'_{i+1} = e_i$$

and we know by definition that

$$\iota_1(e_i) = p_{E,\mathsf{G}}(e'_i)$$

In particular, this implies that, for every $k \geq 1$ we have $\left(p^*_{E,\mathsf{G}}\right)^k (\iota_1(e'_1))$ in the image of ι_1, which contradicts Definition 6.3.1.

- There exists an index j such that $x_j = j^2_{\mathcal{G}}(e_j, v_j)$ for some $(e_j, v_j) \in E^2_{\mathcal{G}}$. This is impossible: indeed, if this were the case, the cycle condition would imply that

$$j_{V,\mathcal{G}}(v_j) = \begin{cases} s_{\mathcal{G}}(x_{j+1}) & j \neq n \\ s_{\mathcal{G}}(x_1) & j = n \end{cases}$$

and this is absurd by Remark 6.3.6 and the fact that the images of $j_{V,\mathcal{G}}$ and $j_{E,\mathcal{G}}$ are disjoint. $\square$.

Remark 6.3.8. Notice that the images of $t_{\mathcal{G}} \circ j^1_{\mathcal{G}}$ and $t_{\mathcal{G}} \circ j^2_{\mathcal{G}}$ are contained in, respectively, $j_{E,\mathcal{G}}(S_{E,\mathsf{G}})$ and $j_{V,\mathcal{G}}(S_{V,\mathsf{G}})$. Since $j_{E,\mathcal{G}}$ is injective and $\mathcal{G}$ is simple, in particulat implies that, for every $e \in E_{\mathsf{G}}$, if $j_{E,\mathcal{G}}(e) = t_{\mathcal{G}}(x)$ for some $x \in E_{\mathcal{G}}$ then $e \in S_{E,\mathsf{G}}$ and

$$x = j^1_{\mathcal{G}}(\overline{p}_{E,\mathsf{G}}(e), e)$$

Similarly, for every $v \in V_{\mathsf{G}}$, if $j_{V,\mathcal{G}}(v) = t_{\mathcal{G}}(y)$ for some $y \in E_{\mathcal{G}}$, then $v \in S_{V,\mathsf{G}}$ and

$$y = j^2_{\mathcal{G}}(\overline{p}_{V,\mathsf{G}}(v), v)$$

Next, we have to define source and targets $s_{F(\mathsf{G})}, t_{F(\mathsf{G})} \colon V_{\mathcal{G}} \rightrightarrows V^\star_{\mathsf{G}}$. To do so it is enough to take the arrows induced by, respectively, s_{G} and t_{G}, paired with the unit $\eta_{V_{\mathsf{G}}} \colon V_{\mathsf{G}} \to V^\star_{\mathsf{G}}$ coming from Example 2.1.8.

$$\begin{tikzcd}
E_{\mathsf{G}} \arrow[dr, "j_{E,\mathcal{G}}"'] \arrow[drr, bend left, "s_{\mathsf{G}}"] & & \\
& V_{\mathcal{G}} \arrow[r, dotted, "s_{F(\mathsf{G})}"] & V^\star_{\mathsf{G}} \\
V_{\mathsf{G}} \arrow[ur, "j_{V,\mathcal{G}}"] \arrow[urr, bend right, "\eta_{V_{\mathsf{G}}}"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
E_{\mathsf{G}} \arrow[dr, "j_{E,\mathcal{G}}"'] \arrow[drr, bend left, "t_{\mathsf{G}}"] & & \\
& V_{\mathcal{G}} \arrow[r, dotted, "t_{F(\mathsf{G})}"] & V^\star_{\mathsf{G}} \\
V_{\mathsf{G}} \arrow[ur, "j_{V,\mathcal{G}}"] \arrow[urr, bend right, "\eta_{V_{\mathsf{G}}}"'] & &
\end{tikzcd}$$

Finally to label nodes and edges, we can take as $l_{V_{\mathsf{G}}} \colon V_{\mathsf{G}} \to L_V$ simply the function $l_{V,\mathsf{G}}$, while as $l_{\mathcal{G}} \colon V_{\mathcal{G}} \to L_E + 1$ we take the function induced by $l_{E,\mathsf{G}}$ and the constant function in $\spadesuit$.

$$\begin{tikzcd}
E_{\mathsf{G}} \arrow[r, "l_{E,\mathsf{G}}"] \arrow[dr, "j_{E,\mathcal{G}}"'] & L_E \arrow[dr, bend left, "k_{L_E}"] & \\
& V_{\mathcal{G}} \arrow[r, dotted, "l_{\mathcal{G}}"] & L_E + 1 \\
V_{\mathsf{G}} \arrow[r, "!_{V_{\mathsf{G}}}"'] \arrow[ur, "j_{V,\mathcal{G}}"] & 1 \arrow[ur, bend right, "k_{\spadesuit}"'] &
\end{tikzcd}$$

Let us define $F(\mathsf{G})$ as $(\mathcal{G}, V_{\mathsf{G}}, s_{F(\mathsf{G})}, t_{F(\mathsf{G})}, l_{V_{\mathsf{G}}}, l_{\mathcal{G}})$. We have no to extend this construction to morphisms. Take an arrow $(h,k)\colon \mathsf{G} \to \mathsf{H}$ in **HHG**, by definition k is a function $V_{\mathsf{G}} \to V_{\mathsf{H}}$ such that

$$\begin{aligned} l_{V_{\mathsf{H}}} \circ k &= l_{V,\mathsf{H}} \circ k \\ &= l_{V,\mathsf{G}} \\ &= l_{V_{\mathsf{G}}} \end{aligned}$$

Moreover, we can define $h_2\colon V_{\mathcal{G}} \to V_{\mathcal{H}}$ as the coproduct of h and k, so that we have a diagram

$$\begin{tikzcd} E_{\mathsf{G}} \arrow[r, "j_{E,\mathcal{G}}"] \arrow[d, "h"'] & V_{\mathcal{G}} \arrow[d, "h_2"'] & V_{\mathsf{G}} \arrow[l, "j_{V,\mathcal{G}}"'] \arrow[d, "k"] \\ E_{\mathsf{H}} \arrow[r, "j_{E,\mathcal{H}}"'] & V_{\mathcal{H}} & V_{\mathsf{G}} \arrow[l, "j_{V,\mathcal{H}}"] \end{tikzcd}$$

To get a morphism $(h_1,h_2)\colon \mathcal{G} \to \mathcal{H}$ of **DAG** we have to define another function $h_1\colon E_{\mathcal{G}} \to E_{\mathcal{H}}$. Now, given $(e,e') \in E^1_{\mathcal{G}}$ and $(e,v) \in E^2_{\mathcal{G}}$ we have

$$\begin{aligned} h(e) &= h(p_{E,\mathsf{G}}(e')) \\ &= p_{E,\mathsf{H}}(h(e')) \end{aligned} \qquad \begin{aligned} h(e) &= h(p_{V,\mathsf{G}}(v)) \\ &= p_{V,\mathsf{H}}(k(v)) \end{aligned}$$

so that we can put

$$h^1_1\colon E^1_{\mathcal{G}} \to E^1_{\mathcal{H}} \quad (e,e') \mapsto (h(e), h(e')) \qquad h^2_1\colon E^2_{\mathcal{G}} \to E^2_{\mathcal{H}} \quad (e,v) \mapsto (h(e), k(v))$$

and define h_1 as the coproduct of these two functions. Moreover, we can check that

$$\begin{aligned} s_{\mathcal{H}}\left(h_1\left(j^1_{\mathcal{G}}(e,e')\right)\right) &= s_{\mathcal{H}}\left(j_{E,\mathcal{H}}\left(h^1_1(e,e')\right)\right) \\ &= s^1_{\mathcal{H}}\left(h^1_1(e,e')\right) \\ &= s^1_{\mathcal{H}}(h(e),h(e')) \\ &= h(e) \\ &= h(s^1_{\mathcal{G}}(e,e')) \\ &= h_2\left(j_{E,\mathcal{G}}\left(s^1_{\mathcal{G}}(e,e')\right)\right) \\ &= h_2\left(s_{\mathcal{G}}\left(j^1_{\mathcal{G}}(e,e')\right)\right) \end{aligned} \qquad \begin{aligned} s_{\mathcal{H}}\left(h_1\left(j^2_{\mathcal{G}}(e,v)\right)\right) &= s_{\mathcal{H}}\left(j_{V,\mathcal{H}}\left(h^2_1(e,v)\right)\right) \\ &= s^2_{\mathcal{H}}\left(h^2_1(e,v)\right) \\ &= s^2_{\mathcal{H}}(h(e),k(v)) \\ &= h(e) \\ &= h(s^2_{\mathcal{G}}(e,v)) \\ &= h_2\left(j_{E,\mathcal{G}}\left(s^2_{\mathcal{G}}(e,v)\right)\right) \\ &= h_2\left(s_{\mathcal{G}}\left(j^2_{\mathcal{G}}(e,v)\right)\right) \end{aligned}$$

$$\begin{aligned} t_{\mathcal{H}}\left(h_1\left(j^1_{\mathcal{G}}(e,e')\right)\right) &= t_{\mathcal{H}}\left(j_{E,\mathcal{H}}\left(h^1_1(e,e')\right)\right) \\ &= t^1_{\mathcal{H}}\left(h^1_1(e,e')\right) \\ &= t^1_{\mathcal{H}}(h(e),h(e')) \\ &= h(e') \\ &= h(t^1_{\mathcal{G}}(e,e')) \\ &= h_2\left(j_{E,\mathcal{G}}\left(t^1_{\mathcal{G}}(e,e')\right)\right) \\ &= h_2\left(t_{\mathcal{G}}\left(j^1_{\mathcal{G}}(e,e')\right)\right) \end{aligned} \qquad \begin{aligned} t_{\mathcal{H}}\left(h_1\left(j^2_{\mathcal{G}}(e,v)\right)\right) &= t_{\mathcal{H}}\left(j_{V,\mathcal{H}}\left(h^2_1(e,v)\right)\right) \\ &= t^2_{\mathcal{H}}\left(h^2_1(e,v)\right) \\ &= t^2_{\mathcal{H}}(h(e),k(v)) \\ &= k(v) \\ &= k(t^2_{\mathcal{G}}(e,v)) \\ &= h_2\left(j_{V,\mathcal{G}}\left(t^2_{\mathcal{G}}(e,v)\right)\right) \\ &= h_2\left(t_{\mathcal{G}}\left(j^2_{\mathcal{G}}(e,v)\right)\right) \end{aligned}$$

and we can therefore conclude that (h_1, h_2) is really a morphism $\mathcal{G} \to \mathcal{H}$ of **DAG**. We claim now that $((h_1, h_2), k)$ is a morphism of **LDAGHGraph**, so that sending (h, k) to it we get a functor $F\colon$ **HHG** $\to$ **LDAGHGraph**. We already know that $l_{V_\mathsf{G}} = l_{V_\mathsf{H}} \circ k$, while the other three equalities follow at once from the definition of $V_\mathcal{G}$ and from the computations below.

$$\begin{aligned} k^\star \circ s_{F(\mathsf{G})} \circ j_{E,\mathcal{G}} &= k^\star \circ s_\mathsf{G} \\ &= s_\mathsf{H} \circ h \\ &= s_{F(\mathsf{H})} \circ j_{E,\mathcal{H}} \circ h \\ &= s_{F(\mathsf{H})} \circ h_2 \circ j_{E,\mathcal{G}} \end{aligned} \qquad \begin{aligned} k^\star \circ s_{F(\mathsf{G})} \circ j_{V,\mathcal{G}} &= k^\star \circ \eta_{V_\mathsf{G}} \\ &= \eta_{V_\mathsf{H}} \circ k \\ &= s_{F(\mathsf{H})} \circ j_{V,\mathcal{H}} \circ k \\ &= s_{F(\mathsf{H})} \circ h_2 \circ j_{V,\mathcal{G}} \end{aligned}$$

$$\begin{aligned} k^\star \circ t_{F(\mathsf{G})} \circ j_{E,\mathcal{G}} &= k^\star \circ t_\mathsf{G} \\ &= t_\mathsf{H} \circ h \\ &= t_{F(\mathsf{H})} \circ j_{E,\mathcal{H}} \circ h \\ &= t_{F(\mathsf{H})} \circ h_2 \circ j_{E,\mathcal{G}} \end{aligned} \qquad \begin{aligned} k^\star \circ t_{F(\mathsf{G})} \circ j_{V,\mathcal{G}} &= k^\star \circ \eta_{V_\mathsf{G}} \\ &= \eta_{V_\mathsf{H}} \circ k \\ &= t_{F(\mathsf{H})} \circ j_{V,\mathcal{H}} \circ k \\ &= t_{F(\mathsf{H})} \circ h_2 \circ j_{V,\mathcal{G}} \end{aligned}$$

$$\begin{aligned} l_\mathcal{H} \circ h_2 \circ j_{E,\mathcal{G}} &= l_\mathcal{H} \circ j_{E,\mathcal{H}} \circ h \\ &= k_{L_E} \circ l_{E,\mathsf{H}} \circ k \\ &= k_{L_E} \circ l_{E,\mathsf{G}} \\ &= l_\mathcal{G} \circ j_{E,\mathcal{G}} \end{aligned} \qquad \begin{aligned} l_\mathcal{H} \circ h_2 \circ j_{V,\mathcal{G}} &= l_\mathcal{H} \circ j_{V,\mathcal{H}} \circ k \\ &= k_\spadesuit \circ !_{V_\mathsf{H}} \circ k \\ &= k_\spadesuit \circ !_{V_\mathsf{G}} \\ &= l_\mathcal{G} \circ j_{V,\mathcal{G}} a \end{aligned}$$

We are now ready to prove the first properties of F in which we are interested.

Proposition 6.3.9. *The functor $F\colon$* **HHG** $\to$ **LDAGHGraph** *defined above is full and faithful.*

Proof. For faithfulness: if $(h, k), (h', k')\colon \mathsf{G} \rightrightarrows \mathsf{H}$ are arrows of **HHG** and suppose that

$$F(h, k) = ((h_1, h_2), k) \qquad F(h', k') = ((h'_1, h'_2), k')$$

are equal. By definition of F we have this entails at once that $k = k'$. On the other hand, by hypothesis $h_2 = h'_2$, thus

$$\begin{aligned} j_{E,\mathcal{H}} \circ h &= h_2 \circ j_{E,\mathcal{G}} \\ &= h'_2 \circ \circ j_{E,\mathcal{G}} \\ &= j_{E,\mathcal{H}} \circ h' \end{aligned}$$

and, since $j_{E,\mathcal{H}}$ is mono, we can conclude that $h = h'$.

Let us prove fullness. Let $((h_1, h_2), k)$ be an arrow $F(\mathsf{G}) \to F(\mathsf{H})$. By construction

$$\begin{aligned} l_\mathcal{H} \circ h_2 \circ j_{E,\mathcal{G}} &= l_\mathcal{G} \circ j_{E,\mathcal{G}} \\ &= k_{L_E} \circ l_{E,\mathsf{G}} \end{aligned}$$

Since the images of k_{L_E} and $k_\spadesuit$ are disjoint, this shows that there exists unique $h\colon E_\mathsf{G} \to E_\mathsf{H}$ and $f\colon V_\mathsf{G} \to V_\mathsf{H}$ as in the diagram below.

$$\begin{tikzcd} E_\mathsf{G} \arrow[r, "j_{E,\mathcal{G}}"] \arrow[d, dotted, "h"'] & V_\mathcal{G} \arrow[d, "h_2"] & V_\mathsf{G} \arrow[l, "j_{V,\mathcal{G}}"'] \arrow[d, dotted, "f"] \\ E_\mathsf{H} \arrow[r, "j_{E,\mathcal{H}}"'] & V_\mathcal{H} & V_\mathsf{H} \arrow[l, "j_{V,\mathcal{H}}"] \end{tikzcd}$$

Moreover, computing we get

$$\begin{aligned}
\eta_{V_{\mathsf{H}}} \circ f &= s_{F(\mathsf{H})} \circ j_{V,\mathcal{H}} \circ f \\
&= s_{F(\mathsf{H})} \circ h_2 \circ j_{V,\mathcal{G}} \\
&= k^\star \circ s_{F(\mathsf{G})} \circ j_{V,\mathcal{G}} \\
&= k^\star \circ \eta_{V_G} \\
&= \eta_{V_{\mathsf{H}}} \circ k
\end{aligned}$$

which entalis that $f = k$ and that h_2 is the coproduct of h and k. This in turn implies that

$$\begin{aligned}
s_{\mathcal{H}} \circ h_1 &= h_2 \circ s_{\mathcal{G}} & t_{\mathcal{H}} \circ h_1 &= h_2 \circ t_{\mathcal{G}} \\
&= (h+k) \circ (s^1_{\mathcal{G}} + s^2_{\mathcal{G}}) & &= (h+k) \circ (t^1_{\mathcal{G}} + t^2_{\mathcal{G}}) \\
&= (h \circ s^1_{\mathcal{G}} + k \circ s^2_{\mathcal{G}}) & &= (h \circ t^1_{\mathcal{G}} + k \circ t^2_{\mathcal{G}})
\end{aligned}$$

But then for every $(e, e') \in E^1_{\mathcal{G}}$ and $(\overline{e}, v) \in E^2_{\mathcal{G}}$ we have

$$\begin{aligned}
s_{\mathcal{H}}\left(h_1\left(j^1_{\mathcal{G}}(e,e')\right)\right) &= j_{E,\mathcal{H}}\left(h\left(s^1_{\mathcal{G}}(e,e')\right)\right) & t_{\mathcal{H}}\left(h_1\left(j^1_{\mathcal{G}}(e,e')\right)\right) &= j_{E,\mathcal{H}}\left(h\left(t^1_{\mathcal{G}}(e,e')\right)\right) \\
&= j_{E,\mathcal{H}}(h(e)) & &= j_{E,\mathcal{H}}(h(e'))
\end{aligned}$$

$$\begin{aligned}
s_{\mathcal{H}}\left(h_1\left(j^2_{\mathcal{G}}(\overline{e},v)\right)\right) &= j_{E,\mathcal{H}}\left(h\left(s^2_{\mathcal{G}}(\overline{e},v)\right)\right) & t_{\mathcal{H}}\left(h_1\left(j^2_{\mathcal{G}}(\overline{e},v)\right)\right) &= j_{V,\mathcal{V}}\left(k\left(t^2_{\mathcal{G}}(\overline{e},v)\right)\right) \\
&= j_{E,\mathcal{H}}(h(\overline{e})) & &= j_{V,\mathcal{H}}(k(v))
\end{aligned}$$

The previous identities, together with Remark 6.3.8 and the injectivity of $j^1_{\mathcal{H}}$ and $j^2_{\mathcal{H}}$ entail that $h(e')$ is an element of $S_{E,\mathsf{H}}$, $k(v)$ belongs to $S_{V,\mathsf{H}}$ and

$$\begin{aligned}
\overline{p}_{E,\mathsf{H}}(h(e')) &= h(e) & \overline{p}_{V,\mathsf{H}}(k(v)) &= h(\overline{e}) \\
&= h\left(\overline{p}_{E,\mathsf{G}}(e')\right) & &= h\left(\overline{p}_{V,\mathsf{G}}(v)\right)
\end{aligned}$$

Moreover, the same identities show that h_1 is the coproduct of

$$h^1_1 \colon E^1_{\mathcal{G}} \to E^1_{\mathcal{H}} \quad (e,e') \mapsto (h(e), h(e')) \qquad h^2_1 \colon E^2_{\mathcal{G}} \to E^2_{\mathcal{H}} \quad (e,v) \mapsto (h(e), k(v))$$

Given the previous remarks, if we show that (h, k) is an arrow $\mathsf{G} \to \mathsf{H}$ we are done. The only thing left to show is that (h, k) is an arrow of labelled hypergraphs between $(E_{\mathsf{G}}, V_{\mathsf{G}}, s_{\mathsf{G}}, t_{\mathsf{G}}, l_{E,\mathsf{G}}, l_{V,\mathsf{G}})$ and

$(E_{\mathsf{H}}, V_{\mathsf{H}}, s_{\mathsf{H}}, t_{\mathsf{H}}, l_{E,\mathsf{H}}, l_{V,\mathsf{H}})$. By construction we have diagrams

$$
\begin{tikzcd}
E_{\mathsf{G}} \arrow[r, "j_{E,\mathcal{G}}"'] \arrow[d, "h"'] \arrow[rr, bend left, "s_{\mathsf{G}}"] & V_{\mathcal{G}} \arrow[r, "s_{F(\mathsf{G})}"'] \arrow[d, "h_2"'] & V^{\star}_{\mathsf{G}} \arrow[d, "k^{\star}"] \\
E_{\mathsf{H}} \arrow[r, "j_{E,\mathcal{H}}"] \arrow[rr, bend right, "s_{\mathsf{H}}"'] & V_{\mathcal{H}} \arrow[r, "s_{F(\mathsf{H})}"] & V^{\star}_{\mathsf{H}}
\end{tikzcd}
\qquad
\begin{tikzcd}
E_{\mathsf{G}} \arrow[r, "j_{E,\mathcal{G}}"'] \arrow[d, "h"'] \arrow[rr, bend left, "t_{\mathsf{G}}"] & V_{\mathcal{G}} \arrow[r, "t_{F(\mathsf{G})}"'] \arrow[d, "h_2"'] & V^{\star}_{\mathsf{G}} \arrow[d, "k^{\star}"] \\
E_{\mathsf{H}} \arrow[r, "j_{E,\mathcal{H}}"] \arrow[rr, bend right, "t_{\mathsf{H}}"'] & V_{\mathcal{H}} \arrow[r, "t_{F(\mathsf{H})}"] & V^{\star}_{\mathsf{H}}
\end{tikzcd}
$$

$$
\begin{tikzcd}
E_{\mathsf{G}} \arrow[r, "j_{E,\mathcal{G}}"'] \arrow[dd, "h"'] \arrow[drrr, bend left, "l_{E,\mathsf{G}}"] & V_{\mathcal{G}} \arrow[dr, "l_{\mathcal{G}}"] \arrow[dd, "h_2"'] & & \\
& & L_E + 1 & L_E \arrow[l, "k_{L_E}"'] \\
E_{\mathsf{H}} \arrow[r, "j_{E,\mathcal{H}}"] \arrow[urrr, bend right, "l_{E,\mathsf{H}}"'] & V_{\mathcal{H}} \arrow[ur, "l_{\mathcal{H}}"'] & &
\end{tikzcd}
$$

and the thesis follows since k_{L_E} is a monomorphism. $\square$

6.3.2 Adhesivity properties of HHG

We ended the last section proving that we have a full and faithful functor $F\colon \mathbf{HHG} \to \mathbf{LDAGHGraph}$. We are now going to characterize the essential image of F and show that it is closed in **LDAGHGraph** under pullbacks and some kinds of pushouts, allowing us to deduce an adhesivity result regarding **HHG**.

Proposition 6.3.10. *Let* G *be an object of* **HHG**, *then* $F(\mathsf{G})$ *has the following properties:*

(a) $V_{\mathcal{G}}$ *and* $E_{\mathcal{G}}$ *are finite;*

(b) $t_{\mathcal{G}}$ *is injective;*

(c) for every v *in* V_{G} *there is a unique* $v_{\spadesuit} \in V_{\mathcal{G}}$ *such that*

$$l_{\mathcal{G}}(v_{\spadesuit}) = \spadesuit \quad \delta_v = s_{F(\mathsf{G})}(v_{\spadesuit}) \quad \delta_v = t_{F(\mathsf{G})}(v_{\spadesuit})$$

moreover, for every $x \in V_{\mathcal{G}}$, *if* $l_{\mathcal{G}}(x) = \spadesuit$ *then* $x = v_{\spadesuit}$ *for some* $v \in V_{\mathsf{G}}$;

(d) for every $v \in V_{\mathsf{G}}$, $v_{\spadesuit}$ *does not belong to the image of* $s_{\mathcal{G}}$;

(e) for every $v \in V_{\mathsf{G}}$, *and* $x \in V_{\mathcal{G}}$ *such that* v *is in the image of* $s_{F(\mathsf{G})}(x)$ *or* $t_{F(\mathsf{G})}(x)$ *the following are true:*

(e_1) if there is $y \in E_{\mathcal{G}}$ *with* $t_{\mathcal{G}}(y) = v_{\spadesuit}$ *then there exists* $y' \in E_{\mathcal{G}}$ *such that*

$$s_{\mathcal{G}}(y') = s_{\mathcal{G}}(y) \qquad x = t_{\mathcal{G}}(y')$$

(e_2) if there is $y \in E_{\mathcal{G}}$ *such that* $x = t_{\mathcal{G}}(y)$ *then there exists* $y' \in E_{\mathcal{G}}$ *such that*

$$s_{\mathcal{G}}(y') = s_{\mathcal{G}}(y) \qquad v_{\spadesuit} = t_{\mathcal{G}}(y')$$

Proof. (a) By definition $V_{\mathcal{G}}$ is $E_{\mathsf{G}} + V_{\mathsf{G}}$ and so it is finite. On the other hand $E_{\mathcal{G}}$ is the coproduct of $E^1_{\mathcal{G}}$ and $E^2_{\mathcal{G}}$, but they are subsets of, respectively, $E_{\mathsf{G}} \times E_{\mathsf{G}}$ and $E_{\mathsf{G}} \times V_{\mathsf{G}}$.

(b) Let $(e_1, e'_1), (e_2, e'_2)$ in $E^1_{\mathcal{G}}$ and $(\overline{e}_1, v_1), (\overline{e}_2, v_2)$ in $E^2_{\mathcal{G}}$ such that

$$t^1_{\mathcal{G}}(e_1, e'_1) = t^1_{\mathcal{G}}(e_2, e'_2) \qquad t^2_{\mathcal{G}}(\overline{e}_1, v_1) = t^2_{\mathcal{G}}(\overline{e}_2, v_2)$$

then, by definition we have

$$e_1 = p_{E,\mathsf{G}}(e'_1) \quad e_2 = p_{E,\mathsf{G}}(e'_2) \quad e'_1 = e'_2 \quad v_1 = v_2 \quad \overline{e}_1 = p_{V,\mathsf{G}}(v_1) \quad \overline{e}_2 = p_{V,\mathsf{G}}(v_2)$$

and so $t^1_{\mathcal{G}}$ and $t^2_{\mathcal{G}}$ are injectives. The thesis now follows since $t_{\mathcal{G}} = t^1_{\mathcal{G}} + t^2_{\mathcal{G}}$.

(c) For existence, take $j_{V,\mathcal{G}}(v)$, then $l_{\mathcal{G}}(j_{V,\mathcal{G}}(v)) = \spadesuit$ and:

$$\begin{aligned} \delta_v &= \eta_{V_G}(v) \\ &= s_{F(\mathsf{G})}(j_{V,\mathcal{G}}(v)) \end{aligned} \qquad \begin{aligned} \delta_v &= \eta_{V_G}(v) \\ &= t_{F(\mathsf{G})}(j_{V,\mathcal{G}}(v)) \end{aligned}$$

On the other hand if $x \in V_{\mathcal{G}}$ is such that $l_{\mathcal{G}}(x) = \spadesuit$ then there must exists $v \in V_{\mathcal{G}}$ such that

$$x = j_{V,\mathcal{G}}(v)$$

and this proves uniqueness of $v_\spadesuit$ and the last half of the thesis.

(d) This follows from the previous point and Remark 6.3.6.

(e) Let us prove (e_1) and (e_2).

(e_1) Let y be an edge in $\mathcal{G}$ with target $v_\spadesuit$, by point (c) above we know that

$$t_{\mathcal{G}}(y) = j_{V,\mathcal{G}}(v)$$

thus, by Remark 6.3.8 we can further deduce that $v \in S_{V,\mathsf{G}}$ and that

$$y = j^2_{\mathcal{G}}(\overline{p}_{V,\mathsf{G}}(v), v)$$

We have now two cases.

- If $x = j_{V,\mathcal{G}}(w)$ for some $w \in V_{\mathsf{G}}$ then

$$\delta_w = s_{F(\mathsf{G})}(x) \qquad \delta_w = s_{F(\mathsf{G})}(x)$$

  so that $w = v$ and we can take as y' the y with which we have started.
- If, instead, $x = j_{E,\mathcal{G}}(e)$ for some $e \in E_{\mathsf{G}}$, then, by hypothesis and by the definition of $s_{F(\mathsf{G})}(x)$ and, $t_{F(\mathsf{G})}(x)$ we know that v must be in the image of $s_{\mathsf{G}}(e)$ or in that of $t_{\mathsf{G}}(e)$. Therefore, by point 2 of Definition 6.3.1 we also know that

$$p_{E,\mathsf{G}}(e) = p_{V,\mathsf{G}}(v)$$

  In particular this implies that $e \in S_{E,\mathsf{G}}$ and that $(\overline{p}_{V,\mathsf{G}}(v), e)$ is an element of $E^1_{\mathcal{G}}$ and the thesis follows taking as y' its image through $j^1_{\mathcal{G}}$.

(e_2) Let us split the cases as in the proof of (e_1).

- As before, if $x = j_{V,\mathcal{G}}(w)$ for some $w \in V_{\mathsf{G}}$ then w must coincide con x. Moreover, by Remark 6.3.8 this implies that $v \in S_{V,\mathsf{G}}$ and that
$$y = j^2_{\mathcal{G}}(\overline{p}_{V,\mathsf{G}}(v), v)$$
In particular we can take as y' the same y.
- Suppose that $x = j_{E,\mathcal{G}}(e)$ for some $e \in E_{\mathsf{G}}$, this implies that v is a letter of $s_{\mathsf{G}}(e)$ or of $t_{\mathsf{G}}(e)$, then the second point of Definition 6.3.1 entails that
$$p_{E,\mathsf{G}}(e) = p_{V,\mathsf{G}}(v)$$
By hypothesis there is $y \in E_{\mathcal{G}}$ such that
$$t_{\mathcal{G}}(y) = j_{E,\mathcal{G}}(e)$$
and so, again by Remark 6.3.8, we can conclude that $e \in S_{E,\mathsf{G}}$ and that $v \in S_{V,\mathsf{G}}$, therefore as y' we can take $j^1_{\mathcal{G}}(\overline{p}_{V,\mathsf{G}}(v), v)$. □

Lemma 6.3.11. *An object $(\mathcal{G}, X, s, t, l_X, l_{\mathcal{G}})$ of* **LDAGHGraph** *is in the essential image of F if and only if*

(a) the sets of nodes and edges of $\mathcal{G}$ are both finite;

(b) $t_{\mathcal{G}}$ is injective;

(c) for every x in X there is a unique $x_\spadesuit \in V_{\mathcal{G}}$ such that
$$l_{\mathcal{G}}(x_\spadesuit) = \spadesuit \quad \delta_x = s(x_\spadesuit) \quad \delta_x = t(x_\spadesuit)$$
moreover, for every $v \in V_{\mathcal{G}}$, if $l_{\mathcal{G}}(v) = \spadesuit$ then $v = x_\spadesuit$ for some $x \in X$;

(d) for every $e \in E_{\mathcal{G}}$ and $x \in X$, $s_{\mathcal{G}}(e) \neq x_\spadesuit$;

(e) for every $x \in X$, and $v \in V_{\mathcal{G}}$ such that x is in the image of $s(v)$ or $t(v)$ the following are true:

(e$_1$) if there is $e \in E_{\mathcal{G}}$ with $t_{\mathcal{G}}(e) = x_\spadesuit$ then there exists $e' \in E_{\mathcal{G}}$ such that
$$s_{\mathcal{G}}(e') = s_{\mathcal{G}}(e) \qquad v = t_{\mathcal{G}}(e')$$

(e$_2$) if there is $e \in E_{\mathcal{G}}$ such that $v = t_{\mathcal{G}}(e)$ then there exists $e' \in E_{\mathcal{G}}$ such that
$$s_{\mathcal{G}}(e') = s_{\mathcal{G}}(e) \qquad x_\spadesuit = t_{\mathcal{G}}(e')$$

Remark 6.3.12. Point (c), in particular, entails that for every $v \in V_{\mathcal{G}}$, if $s(v) \neq t(v)$ then $l_{\mathcal{G}}(v) \neq \spadesuit$.

Proof. ($\Rightarrow$). It is immediate to notice that all the properties (a)–(e) are invariant under isomorphisms, so this implication follows from Proposition 6.3.10.

($\Leftarrow$). Start defining
$$V_{\mathsf{G}} := X \qquad E_{\mathsf{G}} := \{v \in V_{\mathcal{G}} \mid l_{\mathcal{G}}(v) \neq \spadesuit\}$$

As source and target functions $s_{\mathsf{G}}, t_{\mathsf{G}} \colon E_{\mathsf{G}} \rightrightarrows V_{\mathsf{G}}^{\star}$ we can take the restrictions of s and t. For labelings, use l_X as $l_{V,\mathsf{G}}$ and take $l_{E,\mathsf{G}}$ as the unique arrow such that the square below commutes

$$\begin{tikzcd} E_{\mathsf{G}} \arrow[r, dotted, "l_{E,\mathsf{G}}"] \arrow[d, "i"'] & L_E \arrow[d, "k_{L_E}"] \\ V_{\mathcal{G}} \arrow[r, "l_{\mathcal{G}}"'] & L_E + 1 \end{tikzcd}$$

where $i \colon E_{\mathsf{G}} \to V_{\mathcal{G}}$ is the inclusion function.

Now, property (b) entails that for every $v \in V_{\mathcal{G}}$ there exists at most one e such that

$$v = t_{\mathcal{G}}(e)$$

while points (c) and (d) imply that the source of such an e must be in E_{G}, so that we can put:

$$p_{E,\mathsf{G}} \colon E_{\mathsf{G}} \to E_{\mathsf{G}} + 1 \qquad v \mapsto \begin{cases} \iota_1(s_{\mathcal{G}}(e)) & \text{there exists } e \in E_{\mathcal{G}} \text{ such that } t_{\mathcal{G}}(e) = v \\ \perp & \text{otherwise} \end{cases}$$

$$p_{V,\mathsf{G}} \colon V_{\mathsf{G}} \to E_{\mathsf{G}} + 1 \qquad x \mapsto \begin{cases} \iota_1(s_{\mathcal{G}}(e)) & \text{there exists } e \in E_{\mathcal{G}} \text{ such that } t_{\mathcal{G}}(e) = x \\ \perp & \text{otherwise} \end{cases} \spadesuit$$

We have to prove that these data satisfies the two points of Definition 6.3.1.

1. Suppose that there exists $v_0 \in E_{\mathsf{G}}$ such that, for every natural k greater or equal than 1

   $$\perp \neq \left(p^*_{E,\mathsf{G}}\right)^k(\iota_1(v_0))$$

   thus for every such κ there must be $v_k \in E_{\mathsf{G}}$ such that

   $$\iota_1(v_k) = \left(p^*_{E,\mathsf{G}}\right)^k(\iota_1(v_0))$$

   In this way we get a succession $\{v_i\}_{i\in\mathbb{N}}$ of elements of E_{G} which, by point (a) is finite so that there must be $h, k \in \mathbb{N}$ with $h < k$ such that $v_h = v_k$. Notice that every v_i is in $S_{E,\mathsf{G}}$ and

   $$v_{i+1} = \overline{p}_{E,\mathsf{G}}(v_i)$$

   and that, by definition of $p_{E,\mathsf{G}}$, for every index $i \geq 1$ there is $e_i \in \mathcal{G}(v_i, v_{i-1})$, thus $\{e_{k-i}\}_{i=1}^{k-h}$ is a cycle in $\mathcal{G}$, which is absurd.
2. Let $x \in V_{\mathsf{G}}$ and $v \in E_{\mathsf{G}}$ be such that v is in the image of $s_{\mathsf{G}}(v)$ or in that of $t_{\mathsf{G}}(v)$. Notice that, by definition

   $$s_{\mathsf{G}}(v) = s(v) \qquad t_{\mathsf{G}}(v) = t(v)$$

   thus we can use property (e) to see the following two facts

   - If $x \in S_{E,\mathsf{G}}$ then $v \in S_{V,\mathsf{G}}$ and

     $$\overline{p}_{E,\mathsf{G}}(v) = \overline{p}_{V,\mathsf{G}}(x)$$

The definition of $p_{V,\mathsf{G}}$ implies that there is $e \in E_{\mathcal{G}}$ such that $t_{\mathcal{G}}(e) = x_{\spadesuit}$ so that we can use property (e_1) to obtain another edge $e' \in E_{\mathcal{G}}$ satisfying

$$s_{\mathcal{G}}(e') = s_{\mathcal{G}}(e) \qquad v = t_{\mathcal{G}}(e')$$

Then an easy computation shows

$$\begin{aligned} p_{E,\mathsf{G}}(v) &= \iota_1(s_{\mathcal{G}}(e')) \\ &= \iota_1(s_{\mathcal{G}}(e')) \\ &= p_{V,\mathsf{G}}(x) \end{aligned}$$

which is precisely what we need to conclude.

- If $v \in S_{E,\mathsf{G}}$ then $x \in S_{V,\mathsf{G}}$ and

$$\overline{p}_{E,\mathsf{G}}(v) = \overline{p}_{V,\mathsf{G}}(x)$$

By hypothesis there exists $e \in E_{\mathcal{G}}$ such that $t_{\mathcal{G}}(e) = v$, then, by (e_2) there is $e' \in E_{\mathcal{G}}$ such that

$$s_{\mathcal{G}}(e') = s_{\mathcal{G}}(e) \qquad x_{\spadesuit} = t_{\mathcal{G}}(e')$$

As before these two equalities now entail

$$\begin{aligned} p_{V,\mathsf{G}}(x) &= \iota_1(s_{\mathcal{G}}(e')) \\ &= \iota_1(s_{\mathcal{G}}(e)) \\ &= p_{E,\mathsf{G}}(v) \end{aligned}$$

and we are done.

Now it is immediate to see that $p_{V,\mathsf{G}}(x) = p_{E,\mathsf{G}}(v)$ as wanted.

Thus we have constructed an object G of **HHG**, let us show that its image

$$F(\mathsf{G}) := (\mathcal{G}', X, s_{F(\mathsf{G})}, s_{F(\mathsf{G})}, l_X, l_{\mathcal{G}'})$$

through F is isomorphic to the original object $(\mathcal{G}, X, s, t, l_{\mathcal{G}}, l_X)$ of **LDAGHGraph**.

On the one hand, consider the inclusion function $i\colon E_{\mathsf{G}} \to V_{\mathcal{G}}$ and

$$(-)_{\spadesuit}\colon V_{\mathsf{G}} \to \{x_{\spadesuit}\}_{x\in X} \qquad x \mapsto x_{\spadesuit}$$

Notice that property (c) implies that $V_{\mathcal{G}} = E_{\mathsf{G}} \cup \{x_{\spadesuit}\}_{x\in X}$ and, because of Remark 6.3.12, the images of i and $(-)_{\spadesuit}$ are disjoint, so that the induced function$\phi\colon E_{\mathsf{G}} + V_{\mathsf{G}} \to V_{\mathcal{G}}$ is a bijection $V_{\mathcal{G}'} \to V_{\mathcal{G}}$.

On the other hand we have

$$\begin{aligned} E^1_{\mathcal{G}'} &= \{(v, v') \in V_{\mathcal{G}} \times V_{\mathcal{G}} \mid l_{\mathcal{G}}(v) \neq \spadesuit, l_{\mathcal{G}}(v') \neq \spadesuit, v = p_{E,\mathsf{G}}(v')\} \\ E^2_{\mathcal{G}'} &= \{(v, x) \in V_{\mathcal{G}} \times X \mid l_{\mathcal{G}}(v) \neq \spadesuit, v = p_{V,\mathsf{G}}(x)\} \end{aligned}$$

Now, by the definitions of $p_{E,\mathsf{G}}$ and $p_{V,\mathsf{G}}$ and by hypothesis (b), for every $(v, v') \in E^1_{\mathcal{G}'}$ and $(w, x) \in E^2_{\mathcal{G}'}$ there exist unique $\psi_1(v, v')$ and $\psi_2(w, x)$ in $E_{\mathcal{G}}$ such that

$$v = s_{\mathcal{G}}(\psi_1(v, v')) \quad v' = t_{\mathcal{G}}(\psi_1(v, v')) \quad w = s_{\mathcal{G}}(\psi_2(w, x)) \quad x_{\spadesuit} = t_{\mathcal{G}}(\psi_2(w, x))$$

This allows us to define functions $\psi_1\colon E^1_{\mathsf{G}} \to E_{\mathcal{G}}$, $\psi_2\colon E^1_{\mathsf{G}} \to E_{\mathcal{G}}$ which, in turn, induce an arrow $\psi\colon E_{\mathcal{G}'} \to E_{\mathcal{G}}$, which, by construction, is a morphism $\mathcal{G}' \to \mathcal{G}$ of **DAG**, which, by Corollary 6.1.27 is a mono and thus ψ is injective. On the other hand if e is in $E_{\mathcal{G}}$ we have two cases:

- if $t_\mathcal{G}(e)$ is in E_G then $(s_\mathcal{G}(e), t_\mathcal{G}(e))$ is an element of $E^1_{\mathcal{G}'}$ sent by ψ_1 to e;
- if there exists $x \in X$ such that $t_\mathcal{G}(e) = x_\spadesuit$ then $(s_\mathcal{G}(e), x)$ is in $E^2_{\mathcal{G}'}$ and $e = \psi_2(s_\mathcal{G}(e), x)$

This shows that ψ is a bijection and thus that $\mathcal{G}$ and $\mathcal{G}'$ are isomorphic. Notice, moreover, that

$$\begin{aligned} l_\mathcal{G} \circ \phi \circ j_{E,\mathcal{G}'} &= l_\mathcal{G} \circ i \\ &= k_{L_E} \circ l_{E,\mathsf{G}} \\ &= l_{\mathcal{G}'} \circ j_{E,\mathcal{G}'} \end{aligned}$$

while, by point (c),

$$l_\mathcal{G} \circ \phi \circ j_{V,\mathcal{G}'} = l_\mathcal{G} \circ (-)_\spadesuit$$

is the constant function in $\spadesuit$, so we can conclude that $l_{\mathcal{G}'} = l_\mathcal{G} \circ \phi$

To conclude it is now enough to check that $((\psi, \phi), \mathsf{id}_X)$ is really a morphism of **LDAGHGraph**. In particular, the only equalities left to us to prove are

$$s_{F(\mathsf{G})} = s \circ \phi \qquad t_{F(\mathsf{G})} = t \circ \phi$$

To see this, notice that property (c) entails, in particular that

$$\eta_X = s \circ (-)_\spadesuit \qquad \eta_X = t \circ (-)_\spadesuit$$

so that, remembering that $X = V_\mathsf{G}$, we can compute to get

$$\begin{aligned} s \circ \phi \circ j_{E,\mathcal{G}'} &= s \circ i & \qquad t \circ \phi \circ j_{E,\mathcal{G}'} &= t \circ i \\ &= s_\mathsf{G} & &= t_\mathsf{G} \\ &= s_{F(\mathsf{G})} \circ j_{E,\mathcal{G}'} & &= t_{F(\mathsf{G})} \circ j_{E,\mathcal{G}'} \end{aligned}$$

$$\begin{aligned} s \circ \phi \circ j_{V,\mathcal{G}'} &= s \circ (-)_\spadesuit & \qquad t \circ \phi \circ j_{V,\mathcal{G}'} &= t \circ (-)_\spadesuit \\ &= \eta_X & &= \eta_X \\ &= s_{F(\mathsf{G})} \circ j_{V,\mathcal{G}'} & &= t_{F(\mathsf{G})} \circ j_{V,\mathcal{G}'} \end{aligned}$$

and we are done. □

So equipped we can establish that the essential image of F is closed under pullbacks.

Proposition 6.3.13. *Given a pullback square in* **LDAGHGraph**

$$\begin{tikzcd} (\mathcal{P}, P, s, t, l_P, l_\mathcal{P}) \arrow[r, "((a_1,b_1),p_1)"] \arrow[d, "((a_2,b_2),p_2)"'] & F(\mathsf{H}) \arrow[d, "F(h_2,k_2)"] \\ F(\mathsf{K}) \arrow[r, "F(h_1,k_1)"'] & F(\mathsf{G}) \end{tikzcd}$$

$(\mathcal{P}, P, s, t, l_P, l_\mathcal{P})$ is in the essential image of F.

Proof. Let $F(\mathsf{G}) = (\mathcal{G}, G, s_G, t_G, l_G, l_\mathcal{G})$, $F(\mathsf{H}) = (\mathcal{H}, H, s_H, t_H, l_H, l_\mathcal{H})$, $F(\mathsf{K}) = (\mathcal{K}, K, s_K, t_K, l_K, l_\mathcal{K})$, by Proposition 6.3.4 we know that in **Set** we have three pullback squares

$$
\begin{tikzcd}
E_\mathcal{P} \arrow[r, "a_1"] \arrow[d, "a_2"'] & E_\mathcal{H} \arrow[d, "f_2"] \\
E_\mathcal{K} \arrow[r, "f_1"'] & E_\mathcal{G}
\end{tikzcd}
\qquad
\begin{tikzcd}
V_\mathcal{P} \arrow[rr, "b_1"] \arrow[dd, "b_2"'] \arrow[dr, "l_\mathcal{P}"] & & V_\mathcal{H} \arrow[dd, "g_2"] \arrow[dl, "l_\mathcal{H}"'] \\
& L_E + 1 & \\
V_\mathcal{K} \arrow[rr, "g_1"'] \arrow[ur, "l_\mathcal{K}"] & & V_\mathcal{G} \arrow[ul, "l_\mathcal{G}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
P \arrow[rr, "p_1"] \arrow[dd, "p_2"'] \arrow[dr, "l_P"] & & H \arrow[dd, "k_2"] \arrow[dl, "l_H"'] \\
& L_V & \\
K \arrow[rr, "k_1"'] \arrow[ur, "l_K"] & & G \arrow[ul, "l_G"']
\end{tikzcd}
$$

plus four other diagrams defining the remaining of the structure of $(\mathcal{P}, P, s, t, l_P, l_\mathcal{P})$:

$$
\begin{tikzcd}
V_\mathcal{P} \arrow[rrr, "b_1"] \arrow[ddd, "b_2"'] & & & V_\mathcal{H} \arrow[ddd, "g_2"] \\
& E_\mathcal{P} \arrow[ul, "s_\mathcal{P}"'] \arrow[r, "a_1"] \arrow[d, "a_2"'] & E_\mathcal{H} \arrow[ur, "s_\mathcal{H}"] \arrow[d, "f_2"] & \\
& E_\mathcal{K} \arrow[dl, "s_\mathcal{K}"] \arrow[r, "f_1"'] & E_\mathcal{G} \arrow[dr, "s_\mathcal{G}"'] & \\
V_\mathcal{K} \arrow[rrr, "g_1"'] & & & V_\mathcal{G}
\end{tikzcd}
\qquad
\begin{tikzcd}
V_\mathcal{P} \arrow[rrr, "b_1"] \arrow[ddd, "b_2"'] & & & V_\mathcal{H} \arrow[ddd, "g_2"] \\
& E_\mathcal{P} \arrow[ul, "t_\mathcal{P}"'] \arrow[r, "a_1"] \arrow[d, "a_2"'] & E_\mathcal{H} \arrow[ur, "t_\mathcal{H}"] \arrow[d, "f_2"] & \\
& E_\mathcal{K} \arrow[dl, "t_\mathcal{K}"] \arrow[r, "f_1"'] & E_\mathcal{G} \arrow[dr, "t_\mathcal{G}"'] & \\
V_\mathcal{K} \arrow[rrr, "g_1"'] & & & V_\mathcal{G}
\end{tikzcd}
$$

$$
\begin{tikzcd}
P^\star \arrow[rrr, "p_1^\star"] \arrow[ddd, "p_2^\star"'] & & & H^\star \arrow[ddd, "k_2^\star"] \\
& V_\mathcal{P} \arrow[ul, "s"'] \arrow[r, "b_1"] \arrow[d, "b_2"'] & V_\mathcal{H} \arrow[ur, "s_H"] \arrow[d, "g_2"] & \\
& V_\mathcal{K} \arrow[dl, "s_K"] \arrow[r, "g_1"'] & V_\mathcal{G} \arrow[dr, "s_G"'] & \\
K^\star \arrow[rrr, "k_1^\star"'] & & & G^\star
\end{tikzcd}
\qquad
\begin{tikzcd}
P^\star \arrow[rrr, "p_1^\star"] \arrow[ddd, "p_2^\star"'] & & & H^\star \arrow[ddd, "k_2^\star"] \\
& V_\mathcal{P} \arrow[ul, "t"'] \arrow[r, "b_1"] \arrow[d, "b_2"'] & V_\mathcal{H} \arrow[ur, "t_H"] \arrow[d, "g_2"] & \\
& V_\mathcal{K} \arrow[dl, "t_K"] \arrow[r, "g_1"'] & V_\mathcal{G} \arrow[dr, "t_G"'] & \\
K^\star \arrow[rrr, "k_1^\star"'] & & & G^\star
\end{tikzcd}
$$

We are now going to show that $(\mathcal{P}, P, s, t, l_P, l_\mathcal{P})$ satisfies conditions (a)–(e) of Lemma 6.3.11.

(a) $V_\mathcal{H}$, $V_\mathcal{K}$, $E_\mathcal{H}$ and $E_\mathcal{K}$ are finite, so $E_\mathcal{P}$ and $V_\mathcal{P}$ are finite.

(b) Let $e, e' \in E_\mathcal{P}$ such that $t_\mathcal{P}(e) = t_\mathcal{P}(e')$, then

$$
\begin{aligned}
t_\mathcal{H}(a_1(e)) &= b_1(t_\mathcal{P}(e)) \\
&= b_1(t_\mathcal{P}(e')) \\
&= t_\mathcal{H}(a_1(e'))
\end{aligned}
\qquad\qquad
\begin{aligned}
t_\mathcal{K}(a_2(e)) &= b_2(t_\mathcal{P}(e)) \\
&= b_2(t_\mathcal{P}(e')) \\
&= t_\mathcal{K}(a_2(e'))
\end{aligned}
$$

hence $a_1(e) = a_1(e')$ and $a_2(e) = a_2(e')$ and thus $e = e'$.

(c) Given $p \in P$, we can consider $p_1(p)_\spadesuit \in V_{\mathcal{H}}$ and $p_2(p)_\spadesuit \in V_{\mathcal{K}}$. Then

$$\begin{aligned} f_2(p_1(p)_\spadesuit) &= k_2(p_1(p))_\spadesuit \\ &= k_1(p_2(p))_\spadesuit \\ &= f_1(p_2(p)_\spadesuit) \end{aligned}$$

thus there exists $p_\spadesuit \in V_{\mathcal{P}}$ such that

$$p_1(p)_\spadesuit = b_1(p_\spadesuit) \qquad p_2(p)_\spadesuit = b_2(p_\spadesuit)$$

Now, we have identities

$$s_H(b_1(p_\spadesuit)) = p_1(p) \quad s_K(b_2(p_\spadesuit)) = p_2(p) \quad t_H(b_1(p_\spadesuit)) = p_1(p) \quad t_K(b_2(p_\spadesuit)) = p_2(p)$$

so that both $s(p_\spadesuit)$ and $t(p_\spadesuit)$ are equal to δ_p. Moreover,

$$\begin{aligned} l_{\mathcal{P}}(p_\spadesuit) &= l_{\mathcal{K}}(b_1(p_\spadesuit)) \\ &= \spadesuit \end{aligned}$$

For uniqueness, suppose that $x \in V_{\mathcal{P}}$ is such that

$$\delta_p = s(x) \quad \delta_p = t(x) \quad l_{\mathcal{P}}(x) = \spadesuit$$

then, we must have

$$b_1(x) = p_1(p)_\spadesuit \qquad b_2(x) = p_2(p)_\spadesuit$$

and thus $y = p_\spadesuit$. Finally, if $x \in V_{\mathcal{P}}$ is such that $l_{\mathcal{P}}(x) = \spadesuit$, then

$$\begin{aligned} l_{\mathcal{H}}(b_1(x)) &= l_{\mathcal{P}}(x) \qquad & l_{\mathcal{K}}(b_2(x)) &= l_{\mathcal{P}}(x) \\ &= \spadesuit & &= \spadesuit \end{aligned}$$

so that, since $F(\mathsf{H})$ and $F(\mathsf{K})$ saisfy property (c) of Proposition 6.3.10 we must have

$$h_\spadesuit = b_1(x) \qquad k_\spadesuit = b_2(x)$$

for some $h \in H$ and $k \in K$. In particular, this means that

$$\delta_h = t_H(b_1(x)) \qquad \delta_k = t_K(b_2(x))$$

so that

$$\begin{aligned} \delta_{k_2(h)} &= k_2 \circ \delta_h \\ &= k_2^\star(\delta_h) \\ &= k_2^\star(t_H(b_1(x))) \\ &= t_G(g_2(b_1(x))) \\ &= t_G(g_1(b_2(x))) \\ &= k_1^\star(t_K(b_2(x))) \\ &= k_1^\star(\delta_k) \\ &= k_1 \circ \delta_k \\ &= \delta_{k_1(k)} \end{aligned}$$

and thus we can deduce that $k_2(h) = k_1(k)$, therefore there exists $p \in P$ such that

$$h = p_1(p) \qquad k = p_2(p)$$

On the other hand we have

$$\begin{aligned} p_1 \circ s(x) &= p_1^\star(s(x)) & p_2 \circ s(x) &= p_2^\star(s(x)) \\ &= s_H(b_1(x)) & &= s_K(b_2(x)) \\ &= \delta_h & &= \delta_k \end{aligned}$$

$$\begin{aligned} p_1 \circ t(x) &= p_1^\star(t(x)) & p_2 \circ t(x) &= p_2^\star(t(x)) \\ &= t_H(b_1(x)) & &= t_K(b_2(x)) \\ &= \delta_h & &= \delta_k \end{aligned}$$

showing that $\mathsf{dom}(t(x)) = 1$ and that $t = \delta_p$ which now implies $x = p_\spadesuit$.

(d) Let $e \in E_\mathcal{P}$ such that exists $s_\mathcal{P}(e) = p_\spadesuit$ for some $p \in P$, then,

$$\begin{aligned} l_\mathcal{H}(s_\mathcal{H}(a_1(e))) &= l_\mathcal{H}(b_1(s_\mathcal{P}(e))) \\ &= l_\mathcal{H}(b_1(p_\spadesuit)) \\ &= l_\mathcal{P}(p_\spadesuit) \\ &= \spadesuit \end{aligned}$$

which, by point (c) and (d) of Proposition 6.3.10 applied to $F(\mathsf{H})$ is absurd.

(e) Fix an element p of P and a vertex $v \in V_\mathcal{P}$ such that p is in the image of $s(v)$ or $t(v)$. Notice that $p_1(p)$ must then be in the image of $s_H(b_1(v))$ or in that of $t_H(b_1(v))$ and, similarly $p_2(p)$ or is a letter of $s_K(b_2(v))$ or one of $t_K(b_2(v))$.

(e_1) Suppose that there is $e \in E_\mathcal{P}$ be such that $t_\mathcal{P}(e) = p_\spadesuit$, then

$$\begin{aligned} l_\mathcal{H}(t_\mathcal{H}(a_1(e))) &= l_\mathcal{H}(b_1(t_\mathcal{P}(e))) & l_\mathcal{K}(t_\mathcal{K}(a_2(e))) &= l_\mathcal{K}(b_2(t_\mathcal{P}(e))) \\ &= l_\mathcal{H}(b_1(p_\spadesuit)) & &= l_\mathcal{K}(b_2(p_\spadesuit)) \\ &= l_\mathcal{P}(p_\spadesuit) & &= l_\mathcal{P}(p_\spadesuit) \\ &= \spadesuit & &= \spadesuit \end{aligned}$$

$$\begin{aligned} s_H(t_\mathcal{H}(a_1(e))) &= s_H(b_1(t_\mathcal{P}(e))) & s_K(t_\mathcal{K}(a_2(e))) &= s_K(b_2(t_\mathcal{P}(e))) \\ &= s_H(b_1(p_\spadesuit)) & &= s_K(b_2(p_\spadesuit)) \\ &= p_1^\star(s(p_\spadesuit)) & &= p_2^\star(s(p_\spadesuit)) \\ &= p_1^\star(\delta_p) & &= p_2^\star(\delta_p) \\ &= p_1 \circ \delta_p & &= p_2 \circ \delta_p \\ &= \delta_{p_1(p)} & &= \delta_{p_2(p)} \\ t_H(t_\mathcal{H}(a_1(e))) &= t_H(b_1(t_\mathcal{P}(e))) & t_K(t_\mathcal{K}(a_2(e))) &= t_K(b_2(t_\mathcal{P}(e))) \\ &= t_H(b_1(p_\spadesuit)) & &= t_K(b_2(p_\spadesuit)) \\ &= p_1^\star(t(p_\spadesuit)) & &= p_2^\star(t(p_\spadesuit)) \\ &= p_1^\star(\delta_p) & &= p_2^\star(\delta_p) \\ &= p_1 \circ \delta_p & &= p_2 \circ \delta_p \\ &= \delta_{p_1(p)} & &= \delta_{p_2(p)} \end{aligned}$$

and thus we must have

$$p_1(p)_\spadesuit = t_{\mathcal{H}}(a_1(e)) \qquad p_2(p)_\spadesuit = t_{\mathcal{K}}(a_2(e))$$

We know from Proposition 6.3.10 that $F(\mathsf{H})$ and $F(\mathsf{K})$ satisfy property (e_1), so there exist $e_H \in E_{\mathcal{H}}$ and $e_K \in E_{\mathcal{K}}$ with the property that

$$s_{\mathcal{H}}(e_H) = s_{\mathcal{H}}(a_1(e)) \quad s_{\mathcal{K}}(e_K) = s_{\mathcal{K}}(a_2(e)) \quad t_{\mathcal{H}}(e_H) = b_1(v) \quad t_{\mathcal{K}}(e_K) = b_2(v)$$

Now, if we compute we have

$$\begin{aligned}
t_{\mathcal{G}}(f_2(e_H)) &= g_2(t_{\mathcal{H}}(e_H)) \\
&= g_2(p_1(p)_\spadesuit) \\
&= k_2(p_1(p))_\spadesuit \\
&= k_1(p_2(p))_\spadesuit \\
&= g_1(p_2(p)_\spadesuit) \\
&= g_2(t_{\mathcal{K}}(e_K)) \\
&= t_{\mathcal{G}}(f_1(e_K))
\end{aligned}$$

and we know that $t_{\mathcal{G}}$ is injective, so that

$$f_2(e_H) = f_1(e_K)$$

This equality in turn implies the existence of $e' \in E_{\mathcal{P}}$ such that

$$e_H = a_1(e') \qquad e_K = a_2(e')$$

To see that $s_{\mathcal{P}}(e') = s_{\mathcal{P}}(e)$ and $t_{\mathcal{P}}(e') = v$ it is enough to compute:

$$\begin{aligned}
b_1(s_{\mathcal{P}}(e')) &= s_{\mathcal{H}}(a_1(e')) & \qquad b_1(t_{\mathcal{P}}(e')) &= t_{\mathcal{H}}(a_1(e')) \\
&= s_{\mathcal{H}}(e_H) & &= t_{\mathcal{H}}(e_H) \\
&= s_{\mathcal{H}}(a_1(e)) & &= b_1(v) \\
&= b_1(s_{\mathcal{P}}(e))
\end{aligned}$$

$$\begin{aligned}
b_2(s_{\mathcal{P}}(e')) &= s_{\mathcal{K}}(a_2(e')) & \qquad b_2(t_{\mathcal{P}}(e')) &= t_{\mathcal{K}}(a_2(e')) \\
&= s_{\mathcal{K}}(e_K) & &= t_{\mathcal{K}}(e_K) \\
&= s_{\mathcal{K}}(a_2(e)) & &= b_2(v) \\
&= b_2(s_{\mathcal{P}}(e))
\end{aligned}$$

(e_2) Take $e \in E_{\mathcal{P}}$ such that $t_{\mathcal{P}}(e) = v$, then $a_1(e)$ and $a_2(e)$ are such that

$$t_{\mathcal{H}}(a_1(e')) = b_1(v) \qquad t_{\mathcal{K}}(a_2(e')) = b_2(v)$$

hence there are $e_H \in E_{\mathcal{H}}$ and $e_K \in E_{\mathcal{K}}$ such that

$$s_{\mathcal{H}}(e_H) = s_{\mathcal{H}}(a_1(e')) \quad s_{\mathcal{K}}(e_K) = s_{\mathcal{K}}(a_2(e')) \quad t_{\mathcal{H}}(e_H) = p_1(p)_\spadesuit \quad t_{\mathcal{K}}(e_H) = p_2(p)_\spadesuit$$

We can proceed as in the proof of (e_1): a computation yields

$$\begin{aligned} t_{\mathcal{G}}(f_2(e_H)) &= g_2(t_{\mathcal{H}}(e_H)) \\ &= g_2(p_1(p)_\spadesuit) \\ &= k_2(p_1(p))_\spadesuit \\ &= k_1(p_2(p))_\spadesuit \\ &= g_1(p_2(p)_\spadesuit) \\ &= g_2(t_{\mathcal{K}}(e_K)) \\ &= t_{\mathcal{G}}(f_1(e_K)) \end{aligned}$$

and by the injectivity of $t_{\mathcal{G}}$ this gives us the existence of $e' \in E_{\mathcal{P}}$ such that

$$e_H = a_1(e') \qquad e_K = a_2(e')$$

so that

$$\begin{aligned} b_1(s_{\mathcal{P}}(e')) &= s_{\mathcal{H}}(a_1(e')) & b_2(s_{\mathcal{P}}(e')) &= s_{\mathcal{K}}(a_2(e')) \\ &= s_{\mathcal{H}}(e_H) & &= s_{\mathcal{K}}(e_K) \\ &= s_{\mathcal{H}}(a_1(e)) & &= s_{\mathcal{K}}(a_2(e)) \\ &= b_1(s_{\mathcal{P}}(e)) & &= b_2(s_{\mathcal{P}}(e)) \end{aligned}$$

$$\begin{aligned} b_1(t_{\mathcal{P}}(e')) &= t_{\mathcal{H}}(a_1(e')) & b_2(t_{\mathcal{P}}(e')) &= t_{\mathcal{K}}(a_2(e')) \\ &= t_{\mathcal{H}}(e_H) & &= t_{\mathcal{K}}(e_K) \\ &= p_1(p)_\spadesuit & &= p_2(p)_\spadesuit \end{aligned}$$

By the proof of point (c) we know that

$$p_1(p)_\spadesuit = b_1(p_\spadesuit) \qquad p_2(p)_\spadesuit = b_2(p_\spadesuit)$$

therefore identities implies that $s_{\mathcal{P}}(e) = s_{\mathcal{P}}(e')$ and$t_{\mathcal{P}}(e') = p_\spadesuit$. □

Proposition 6.3.14. *Suppose that a pushout square in* **LDAGHGraph**

$$\begin{tikzcd} F(\mathsf{G}) \arrow[r, "F(h_2,k_2)"] \arrow[d, "F(h_1,k_1)"'] & F(\mathsf{H}) \arrow[d, "((c_1,d_1),q_1)"] \\ F(\mathsf{K}) \arrow[r, "((c_2,d_2),q_2)"'] & (\mathcal{Q}, Q, s, t, l_Q, l_{\mathcal{P}}) \end{tikzcd}$$

is given. Suppose also that

$$F(h_1, k_1) = ((f_1, g_1), k_1) \qquad F(h_2, k_2) = ((f_2, g_2), k_2)$$

with k_1 and k_2 injective and $(f_1, g_1), (f_2, g_2) \in \mathsf{dcl_d}$. Then $(\mathcal{Q}, Q, s, t, l_Q, l_{\mathcal{Q}})$ is in the essential image of F.

Proof. As in the proof of Proposition 6.3.13, let $F(\mathsf{G}) = (\mathcal{G}, G, s_G, t_G, l_G, l_{\mathcal{G}})$, $F(\mathsf{H}) = (\mathcal{H}, H, s_H, t_H, l_H, l_{\mathcal{H}})$, $F(\mathsf{K}) = (\mathcal{K}, K, s_K, t_K, l_K, l_{\mathcal{K}})$, so that we have three pushout squares in **Set**

$$
\begin{tikzcd}
E_{\mathcal{G}} \arrow[r, "f_2"] \arrow[d, "f_1"'] & E_{\mathcal{H}} \arrow[d, "c_1"] \\
E_{\mathcal{K}} \arrow[r, "c_2"'] & E_{\mathcal{Q}}
\end{tikzcd}
\qquad
\begin{tikzcd}
V_{\mathcal{G}} \arrow[rr, "g_2"] \arrow[dd, "g_1"'] \arrow[dr, "l_{\mathcal{G}}"] & & V_{\mathcal{H}} \arrow[dd, "d_1"] \arrow[dl, "l_{\mathcal{H}}"'] \\
& L_E + 1 & \\
V_{\mathcal{K}} \arrow[rr, "d_2"'] \arrow[ur, "l_{\mathcal{K}}"] & & V_{\mathcal{Q}} \arrow[ul, "l_{\mathcal{Q}}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[rr, "k_2"] \arrow[dd, "k_1"'] \arrow[dr, "l_G"] & & H \arrow[dd, "q_1"] \arrow[dl, "l_H"'] \\
& L_V & \\
K \arrow[rr, "q_2"'] \arrow[ur, "l_K"] & & Q \arrow[ul, "l_Q"']
\end{tikzcd}
$$

We also have four other diagrams

$$
\begin{tikzcd}
V_{\mathcal{G}} \arrow[rrr, "g_2"] \arrow[ddd, "g_1"'] & & & V_{\mathcal{H}} \arrow[ddd, "d_1"] \\
& E_{\mathcal{G}} \arrow[ul, "s_{\mathcal{G}}"'] \arrow[r, "f_2"] \arrow[d, "f_1"'] & E_{\mathcal{H}} \arrow[ur, "s_{\mathcal{H}}"] \arrow[d, "c_1"] & \\
& E_{\mathcal{K}} \arrow[dl, "s_{\mathcal{K}}"] \arrow[r, "c_2"'] & E_{\mathcal{Q}} \arrow[dr, "s_{\mathcal{Q}}"'] & \\
V_{\mathcal{K}} \arrow[rrr, "d_2"'] & & & V_{\mathcal{Q}}
\end{tikzcd}
\qquad
\begin{tikzcd}
V_{\mathcal{G}} \arrow[rrr, "g_2"] \arrow[ddd, "g_1"'] & & & V_{\mathcal{H}} \arrow[ddd, "d_1"] \\
& E_{\mathcal{G}} \arrow[ul, "t_{\mathcal{G}}"'] \arrow[r, "f_2"] \arrow[d, "f_1"'] & E_{\mathcal{H}} \arrow[ur, "t_{\mathcal{H}}"] \arrow[d, "c_1"] & \\
& E_{\mathcal{K}} \arrow[dl, "t_{\mathcal{K}}"] \arrow[r, "c_2"'] & E_{\mathcal{Q}} \arrow[dr, "t_{\mathcal{Q}}"'] & \\
V_{\mathcal{K}} \arrow[rrr, "d_2"'] & & & V_{\mathcal{G}}
\end{tikzcd}
$$

$$
\begin{tikzcd}
G^\star \arrow[rrr, "k_2^\star"] \arrow[ddd, "k_1^\star"'] & & & H^\star \arrow[ddd, "q_1^\star"] \\
& V_{\mathcal{G}} \arrow[ul, "s_G"'] \arrow[r, "g_2"] \arrow[d, "g_1"'] & V_{\mathcal{H}} \arrow[ur, "s_H"] \arrow[d, "d_1"] & \\
& V_{\mathcal{K}} \arrow[dl, "s_K"] \arrow[r, "d_2"'] & V_{\mathcal{Q}} \arrow[dr, "s"'] & \\
K^\star \arrow[rrr, "q_2^\star"'] & & & Q^\star
\end{tikzcd}
\qquad
\begin{tikzcd}
G^\star \arrow[rrr, "k_2^\star"] \arrow[ddd, "k_1^\star"'] & & & H^\star \arrow[ddd, "q_1^\star"] \\
& V_{\mathcal{G}} \arrow[ul, "t_G"'] \arrow[r, "g_2"] \arrow[d, "g_1"'] & V_{\mathcal{H}} \arrow[ur, "t_H"] \arrow[d, "d_1"] & \\
& V_{\mathcal{K}} \arrow[dl, "t_K"] \arrow[r, "d_2"'] & V_{\mathcal{Q}} \arrow[dr, "t"'] & \\
K^\star \arrow[rrr, "q_2^\star"'] & & & Q^\star
\end{tikzcd}
$$

It is now enough to show that $(\mathcal{Q}, Q, s, t, l_Q, l_{\mathcal{Q}})$ satisfies the conditions of Lemma 6.3.11.

(a) $V_{\mathcal{H}}$, $V_{\mathcal{K}}$, $E_{\mathcal{H}}$ and $E_{\mathcal{K}}$ are finite, so $E_{\mathcal{P}}$ and $V_{\mathcal{P}}$ are finite.

(b) Let $e, e' \in E_{\mathcal{Q}}$ such that $t_{\mathcal{Q}}(e) = t_{\mathcal{Q}}(e')$, by Lemma 6.1.1 we have four cases.

- $e = c_1(h)$ and $e' = c_1(h')$ for some $h, h' \in E_{\mathcal{H}}$. Then

$$
\begin{aligned}
d_1(t_{\mathcal{H}}(h)) &= t_{\mathcal{Q}}(c_1(h)) \\
&= t_{\mathcal{Q}}(e) \\
&= t_{\mathcal{Q}}(e') \\
&= t_{\mathcal{Q}}(c_1(h')) \\
&= d_1(t_{\mathcal{H}}(h'))
\end{aligned}
$$

d_1 is the pushout of g_1 and so it is injective, therefore we get $h = h'$.

- $e = c_2(k)$ and $e' = c_1(k')$ for some $k, k' \in E_{\mathcal{K}}$. This is done as in the previous point:

$$\begin{aligned} d_2(t_{\mathcal{K}}(k)) &= t_{\mathcal{Q}}(c_2(k)) \\ &= t_{\mathcal{Q}}(e) \\ &= t_{\mathcal{Q}}(e') \\ &= t_{\mathcal{Q}}(c_2(k')) \\ &= d_2(t_{\mathcal{K}}(k')) \end{aligned}$$

  and d_2 is injective, so that $k = k'$.
- $e = c_1(h)$ and $e' = c_2(k)$ for some $h \in V_{\mathcal{H}}$ and $k \in V_{\mathcal{K}}$, thus

$$\begin{aligned} d_2(t_{\mathcal{K}}(k)) &= t_{\mathcal{Q}}(c_2(k)) \\ &= t_{\mathcal{Q}}(c_1(h)) \\ &= d_1(t_{\mathcal{H}}(h)) \end{aligned}$$

  By Lemma 6.1.1 there exists $w \in V_{\mathcal{G}}$ such that $g_1(w) = t_{\mathcal{K}}(k)$ and $g_2(w) = t_{\mathcal{H}}(h)$. Since (f_1, g_1) is downward closed there exists $g \in E_{\mathcal{G}}$ such that $f_1(g) = k$, and so

$$\begin{aligned} g_1(t_{\mathcal{G}}(g)) &= t_{\mathcal{K}}(f_1(g)) \\ &= t_{\mathcal{K}}(k) \\ &= g_1(w) \end{aligned}$$

  g_1 is injective by hypothesis therefore we have $t_{\mathcal{G}}(g) = w$. Therefore

$$\begin{aligned} t_{\mathcal{H}}(f_2(g)) &= g_2(t_{\mathcal{G}}(g)) \\ &= g_2(w) \\ &= t_{\mathcal{H}}(h) \end{aligned}$$

  from which it follows that $f_2(g) = h$ and that $e = e'$.
- If $e = c_2(k)$ and $e' = c_1(h)$ for some $k \in V_{\mathcal{K}}$ and $h \in V_{\mathcal{H}}$ it is enough to swap e with e' and apply the previous point.

(c) Let q be an element of Q, by Lemma 6.1.1 we have two cases.

- $q = q_1(h)$ for some $h \in H$. If we start from $h_\spadesuit$, on the one hand we obtain

$$\begin{aligned} l_{\mathcal{Q}}(d_1(h_\spadesuit)) &= l_{\mathcal{H}}(h_\spadesuit) \\ &= \spadesuit \end{aligned}$$

  while on the other we have

$$\begin{aligned} s(d_1(h_\spadesuit)) &= q_1^*(s_H(h_\spadesuit)) & \qquad t(d_1(h_\spadesuit)) &= q_1^*(t_H(h_\spadesuit)) \\ &= q_1^*(\delta_h) & &= q_1^*(\delta_h) \\ &= q_1 \circ \delta_h & &= q_1 \circ \delta_h \\ &= \delta_{q_1(h)} & &= \delta_{q_1(h)} \\ &= \delta_q & &= \delta_q \end{aligned}$$

and so we can take $d_1(h_\spadesuit)$ as $q_\spadesuit$. For uniqueness, suppose that $y \in E_{\mathcal{Q}}$ is such that

$$\spadesuit = l_{\mathcal{Q}}(y) \quad q = s(y) \quad q = s(y)$$

We have again two cases.

- If $y = d_1(h')$ for some other $h' \in V_{\mathcal{H}}$ then

$$\begin{aligned} l_{\mathcal{H}}(h') &= l_{\mathcal{Q}}(d_1(h')) \\ &= l_{\mathcal{Q}}(y) \\ &= \spadesuit \end{aligned}$$

  so that $h' = x_\spadesuit$ for some $x \in H$. On the other hand we have

$$\begin{aligned} \delta_{q_1(x)} &= q_1 \circ \delta_x \\ &= q_1^\star(\delta_x q_1^\star(t_H(h')) \\ &= t(d_1(h')) \\ &= t(y) \\ &= \delta_q \\ &= \delta_{q_1(h)} \end{aligned}$$

  Thus $q_1(x) = q_1(h)$, but q_1 is injective as it is pushout of k_1, so $x = h$ and $h' = h_\spadesuit$.
- $y = d_2(k)$ for some other $k \in V_{\mathcal{K}}$. Then

$$\begin{aligned} l_{\mathcal{K}}(k) &= l_{\mathcal{Q}}(d_2(k)) \\ &= l_{\mathcal{Q}}(y) \\ &= \spadesuit \end{aligned}$$

  and therefore $k = x_\spadesuit$ for some $x \in K$. Notice, moreover, that

$$\begin{aligned} \delta_{q_2(x)} &= q_2 \circ \delta_x \\ &= q_2^\star(\delta_x) \\ &= q_2^\star(t_K(k)) \\ &= t(d_2(k)) \\ &= t(y) \\ &= \delta_q \end{aligned}$$

  Thus $q_2(x) = q_1(h)$ and then, by Lemma 6.1.1, there exists $g \in G$ such that

$$x = k_1(g) \qquad h = k_2(g)$$

  Now, notice that,

$$\begin{aligned} l_{\mathcal{K}}(g_1(g_\spadesuit) &= l_{\mathcal{G}}(g_\spadesuit) \\ &= \spadesuit \end{aligned} \qquad \begin{aligned} l_{\mathcal{H}}(g_2(g_\spadesuit) &= l_{\mathcal{G}}(g_\spadesuit) \\ &= \spadesuit \end{aligned}$$

$$\begin{aligned} s_K(g_1(g_\spadesuit)) &= k_1^\star(s_G(g_\spadesuit)) \\ &= k_1^\star(\delta_g) \\ &= k_1^\star \circ \delta_g \\ &= \delta_{k_1(g)} \\ &= \delta_x \end{aligned} \qquad \begin{aligned} s_H(g_2(g_\spadesuit)) &= k_2^\star(s_G(g_\spadesuit)) \\ &= k_2^\star(\delta_g) \\ &= k_2^\star \circ \delta_g \\ &= \delta_{k_2(g)} \\ &= \delta_h \end{aligned}$$

$$\begin{aligned} t_K(g_1(g_\spadesuit)) &= k_1^\star(t_G(g_\spadesuit)) \\ &= k_1^\star(\delta_g) \\ &= k_1^\star \circ \delta_g \\ &= \delta_{k_1(g)} \\ &= \delta_x \end{aligned} \qquad \begin{aligned} t_K(g_2(g_\spadesuit)) &= k_2^\star(t_G(g_\spadesuit)) \\ &= k_2^\star(\delta_g) \\ &= k_2^\star \circ \delta_g \\ &= \delta_{k_2(g)} \\ &= \delta_h \end{aligned}$$

so that

$$x_\spadesuit = g_1(g_\spadesuit) \qquad h_\spadesuit = g_2(g_\spadesuit)$$

and this in turn implies that $d_2(k) = d_1(h_\spadesuit)$.

- If $q = q_2(k)$ for some $k \in K$ we can repeat almost verbatim the same argument to obtain that $d_2(k_\spadesuit)$ is the unique $q_\spadesuit$ we wanted. Clearly

$$\begin{aligned} l_{\mathcal{Q}}(d_2(k_\spadesuit)) &= l_{\mathcal{K}}(k_\spadesuit) \\ &= \spadesuit \end{aligned}$$

and

$$\begin{aligned} s(d_2(k_\spadesuit)) &= q_2^*(s_K(k_\spadesuit)) \\ &= q_2^*(\delta_k) \\ &= q_2 \circ \delta_k \\ &= \delta_{q_2(k)} \\ &= \delta_q \end{aligned} \qquad \begin{aligned} t(d_2(k_\spadesuit)) &= q_2^*(t_K(k_\spadesuit)) \\ &= q_2^*(\delta_k) \\ &= q_2 \circ \delta_k \\ &= \delta_{q_2(k)} \\ &= \delta_q \end{aligned}$$

To prove uniqueness, take again $y \in E_{\mathcal{Q}}$ such that

$$\spadesuit = l_{\mathcal{Q}}(y) \quad q = s(y) \quad q = s(y)$$

and split the cases.

  - If $y = d_2(k')$ for some other $k' \in V_{\mathcal{K}}$ then

$$\begin{aligned} l_{\mathcal{K}}(k') &= l_{\mathcal{Q}}(d_2(k')) \\ &= l_{\mathcal{Q}}(y') \\ &= \spadesuit \end{aligned}$$

thus $k' = x_\spadesuit$ for some $x \in K$, but then:

$$\begin{aligned}\delta_{q_2(x)} &= q_2 \circ \delta_x \\ &= q_2^\star(\delta_x) \\ &= q_2^\star(t_K(k')) \\ &= t(d_2(k')) \\ &= t(y) \\ &= \delta_q \\ &= \delta_{q_2(k)}\end{aligned}$$

As above, since q_2 is injective this impliws $x = k$ and $k' = k_\spadesuit$.

- $y = d_1(h)$ for some $h \in V_\mathcal{H}$. Then

$$\begin{aligned}l_\mathcal{H}(h) &= l_\mathcal{Q}(d_1(h)) \\ &= l_\mathcal{Q}(y) \\ &= \spadesuit\end{aligned}$$

therefore there is some $x \in H$ such that $h = x_\spadesuit$. Computing we get

$$\begin{aligned}\delta_{q_1(x)} &= q_1 \circ \delta_x \\ &= q_1^\star(\delta_x) \\ &= q_1^\star(t_H(h)) \\ &= t(d_1(h)) \\ &= t(y) \\ &= \delta_q\end{aligned}$$

entailing $q_2(k) = q_1(x)$ and the existence of $g \in G$ such that

$$k = k_1(g) \qquad x = k_2(g)$$

We can observe again that,

$$\begin{aligned}l_K(g_1(g_\spadesuit) &= l_\mathcal{G}(g_\spadesuit) & l_\mathcal{H}(g_2(g_\spadesuit) &= l_\mathcal{G}(g_\spadesuit) \\ &= \spadesuit & &= \spadesuit \\ s_K(g_1(g_\spadesuit)) &= k_1^\star(s_G(g_\spadesuit)) & s_H(g_2(g_\spadesuit)) &= k_2^\star(s_G(g_\spadesuit)) \\ &= k_1^\star(\delta_g) & &= k_2^\star(\delta_g) \\ &= k_1^\star \circ \delta_g & &= k_2^\star \circ \delta_g \\ &= \delta_{k_1(g)} & &= \delta_{k_2(g)} \\ &= \delta_k & &= \delta_x \\ t_K(g_1(g_\spadesuit)) &= k_1^\star(t_G(g_\spadesuit)) & t_K(g_2(g_\spadesuit)) &= k_2^\star(t_G(g_\spadesuit)) \\ &= k_1^\star(\delta_g) & &= k_2^\star(\delta_g) \\ &= k_1^\star \circ \delta_g & &= k_2^\star \circ \delta_g \\ &= \delta_{k_1(g)} & &= \delta_{k_2(g)} \\ &= \delta_k & &= \delta_x\end{aligned}$$

Therefore we have identities

$$k_\spadesuit = g_1(g_\spadesuit) \qquad x_\spadesuit = g_2(g_\spadesuit)$$

from which it follows that $d_1(h) = d_2(k_\spadesuit)$.

We are left with the last half of the thesis. Take $v \in V_\mathcal{Q}$ and suppose that $l_\mathcal{Q}(v) = \spadesuit$. We have:

$$\begin{aligned} \spadesuit &= l_\mathcal{Q}(v) \\ &= \begin{cases} l_\mathcal{Q}(d_1(h)) & v = d_1(h) \text{ for some } h \in V_\mathcal{H} \\ l_\mathcal{Q}(d_2(k)) & v = d_2(k) \text{ for some } k \in V_\mathcal{K} \end{cases} \\ &= \begin{cases} l_\mathcal{H}(h) & v = d_1(h) \text{ for some } h \in V_\mathcal{H} \\ l_\mathcal{K}(k) & v = d_2(k) \text{ for some } k \in V_\mathcal{K} \end{cases} \end{aligned}$$

So v is equal to $d_1(x_\spadesuit)$ or to $d_2(y_\spadesuit)$ for some $x \in H$ or $y \in K$ and the thesis now follows.

(d) It is worth to notice explicitly that the proof of the previous point entails that, for every $h \in H$ and $k \in K$:

$$d_1(h_\spadesuit) = q_1(h)_\spadesuit \qquad d_2(k_\spadesuit) = q_2(k)_\spadesuit$$

Take now $e \in E_\mathcal{Q}$ such that $s_\mathcal{Q}(e) = q_\spadesuit$ for some $q \in Q$, using Lemma 6.1.1 we have four cases.

- $e = c_1(e_H)$ and $q = q_1(h)$ for some $e_H \in E_\mathcal{H}$ and $h \in H$. Then

$$\begin{aligned} d_1(h_\spadesuit) &= q_1(h)_\spadesuit \\ &= q_\spadesuit \\ &= s_\mathcal{Q}(e) \\ &= s_\mathcal{Q}(c_1(e_H)) \\ &= d_1(s_\mathcal{H}(e_H)) \end{aligned}$$

  and, since d_1 is injective, this entail $s_\mathcal{H}(e_H) = h_\spadesuit$, which is absurd.
- $e = c_2(e_K)$ and $q = q_2(k)$ for some $e_K \in E_\mathcal{K}$ and $k \in K$. We proceed as above:

$$\begin{aligned} d_2(k_\spadesuit) &= q_2(k)_\spadesuit \\ &= q_\spadesuit \\ &= s_\mathcal{Q}(e) \\ &= s_\mathcal{Q}(c_2(e_K)) \\ &= d_2(s_\mathcal{K}(e_K)) \end{aligned}$$

  The injectivity of d_2 implies $s_\mathcal{H}(e_K) = k_\spadesuit$.
- $e = c_1(e_H)$ and $q = q_2(k)$ for some $e_H \in E_\mathcal{H}$ and $k \in K$. Let w be $s_H(s_\mathcal{H}(e_H))$, then

$$\begin{aligned} \delta_q &= s(q_\spadesuit) \\ &= s(s_\mathcal{Q}(e)) \\ &= s(s_\mathcal{Q}(c_1(e_H))) \\ &= s(d_1(s_\mathcal{H}(e_H))) \\ &= q_1^\star(s_H(s_\mathcal{H}(e_H))) \\ &= q_1^\star(w) \end{aligned}$$

Thus w is a function $1 \to H$ such that $q_1 \circ w = \delta_q$. This implies that there exists $h \in H$ such that $q_1(h) = q$ and we fall back in the first point.

- $e = c_2(e_K)$ and $q = q_1(h)$ for some $e_K \in E_{\mathcal{K}}$ and $h \in H$. This is proved as in the previous point. If w is $s_L(s_{\mathcal{K}}(e_K))$, then

$$\begin{aligned}\delta_q &= s(q_\spadesuit)\\ &= s(s_{\mathcal{Q}}(e))\\ &= s(s_{\mathcal{Q}}(c_2(e_K)))\\ &= s(d_2(s_{\mathcal{K}}(e_K)))\\ &= q_2^\star(s_K(s_{\mathcal{K}}(e_K)))\\ &= q_2^\star(w)\end{aligned}$$

Hence there exists $k \in K$ such that $q_2(k) = q$, bringing us back to the second point.

(e) Let $q \in Q$ and $v \in V_{\mathcal{Q}}$ such that q is a letter of $s(v)$ or $t(v)$. We can make some preliminary observations.

- If $v = d_1(v_H)$ and $q = q_1(h)$ for some $v_H \in V_{\mathcal{H}}$ and $h \in H$, then:

$$\begin{aligned}s(v) &= s(d_1(v_H)) & t(v) &= t(d_1(v_H))\\ &= q_1^\star(s_H(v_H)) & &= q_1^\star(t_H(v_H))\end{aligned}$$

therefore, by the injectivity of q_1, h must be a in the image of $s_H(v_H)$ or of $t_H(v_H)$.

- Similarly, if there are $v_K \in V_{\mathcal{K}}$ and $k \in K$ such that

$$v = d_2(v_K) \qquad q = q_2(k)$$

then we have

$$\begin{aligned}s(v) &= s(d_2(v_K)) & t(v) &= t(d_2(v_K))\\ &= q_2^\star(s_K(v_K)) & &= q_2^\star(t_K(v_K))\end{aligned}$$

and the injectivity of q_2 entails that k has to be a letter of $s_K(v_K)$ or of $t_K(v_K)$.

- Suppose that $v = d_1(v_H)$ and $q = q_2(k)$ for some $v_H \in V_{\mathcal{H}}$ and $k \in K$, then, as before:

$$\begin{aligned}s(v) &= s(d_1(v_H)) & t(v) &= t(d_1(v_H))\\ &= q_1^\star(s_H(v_H)) & &= q_1^\star(t_H(v_H))\end{aligned}$$

So, since q is a letter of $s(v)$ or of $t(v)$, there must be a, unique, letter h of $s_H(v_H)$ or of $t_H(v_H)$ such that $q_1(h) = q$. By Lemma 6.1.1, this implies that there exists $g \in G$ such that

$$k = k_1(g) \qquad h = k_2(g)$$

- Simmetrically, if $v = d_1(v_K)$ and $q = q_1(h)$ for some $v_K \in V_{\mathcal{K}}$ and $h \in H$ from

$$\begin{aligned}s(v) &= s(d_2(v_K)) & t(v) &= t(d_2(v_K))\\ &= q_2^\star(s_K(v_K)) & &= q_2^\star(t_K(v_K))\end{aligned}$$

we can deduce that there is a letter k of $s_K(v_K)$ or of $t_K(v_K)$ such that $q_2(k) = q$ and, therefore there also is a $g \in G$ such that

$$k = k_1(g) \qquad h = k_2(g)$$

We are now ready to prove properties (e_1) and (e_2).

(e_1) Let e be an element of $E_{\mathcal{Q}}$ such that $t_{\mathcal{Q}}(e) = q_{\spadesuit}$, we have eight cases.

- $e = c_1(e_H)$, $q = q_1(h)$ and $v = d_1(v_H)$ for some $e_H \in E_{\mathcal{H}}$, $h \in H$ and $v_H \in V_{\mathcal{H}}$. We have already noticed that
$$d_1(h_{\spadesuit}) = q_1(h)_{\spadesuit}$$
hence we have a chain of equalities:
$$\begin{aligned} d_1(h_{\spadesuit}) &= q_1(h)_{\spadesuit} \\ &= q_{\spadesuit} \\ &= t_{\mathcal{Q}}(e) \\ &= t_{\mathcal{Q}}(c_1(e_H)) \\ &= d_1(t_{\mathcal{H}}(e_H)) \end{aligned}$$
and d_1 is injective, showing that
$$h_{\spadesuit} = t_{\mathcal{H}}(e_H)$$
We also know that h is a letter of $s_H(v_H)$ or of $t_H(v_H)$, so that there is $e'_H \in E_{\mathcal{H}}$ such that
$$s_{\mathcal{H}}(e'_H) = s_{\mathcal{H}}(e_H) \qquad t_{\mathcal{H}}(e'_H) = v_H$$
Now it is enough to take $c_1(e'_H)$ and compute:
$$\begin{aligned} s_{\mathcal{Q}}(c_1(e'_H)) &= d_1(s_{\mathcal{H}}(e'_H)) \\ &= d_1(s_{\mathcal{H}}(e_H)) \\ &= s_{\mathcal{Q}}(c_1(e_H)) \\ &= s_{\mathcal{Q}}(e) \end{aligned} \qquad \begin{aligned} t_{\mathcal{Q}}(c_1(e'_H)) &= d_1(t_{\mathcal{H}}(e'_H)) \\ &= d_1(v_H) \\ &= v \end{aligned}$$

- $e = c_2(e_K)$, $q = q_2(k)$ and $v = d_2(v_K)$ for some $e_K \in E_{\mathcal{K}}$, $k \in K$ and $v_K \in V_{\mathcal{K}}$. This is done as in the previous point. Start with
$$\begin{aligned} d_2(k_{\spadesuit}) &= q_2(k)_{\spadesuit} \\ &= q_{\spadesuit} \\ &= t_{\mathcal{Q}}(e) \\ &= t_{\mathcal{Q}}(c_2(e_K)) \\ &= d_2(t_{\mathcal{K}}(e_K)) \end{aligned}$$
so that we can conclude that
$$k_{\spadesuit} = t_{\mathcal{K}}(e_K)$$
Since k is a letter of $s_K(v_K)$ or of $t_K(v_K)$, there is $e'_K \in E_{\mathcal{K}}$ such that
$$s_{\mathcal{K}}(e'_K) = s_{\mathcal{K}}(e_K) \qquad t_{\mathcal{K}}(e'_K) = v_K$$
and the thesis now follows considering $c_2(e'_K)$.

- $e = c_1(e_H)$, $q = q_1(h)$ and $v = d_2(v_K)$ for some $e_H \in E_{\mathcal{H}}$, $h \in H$ and $v_K \in V_{\mathcal{K}}$. Notice that

$$\begin{aligned} d_1(h_\spadesuit) &= q_1(h)_\spadesuit \\ &= q_\spadesuit \\ &= t_{\mathcal{Q}}(e) \\ &= t_{\mathcal{Q}}(c_1(e_H)) \\ &= d_1(t_{\mathcal{H}}(e_H)) \end{aligned}$$

thus $t_{\mathcal{H}}(e_H) = h_\spadesuit$. On the other hand, we already know that $s_K(v_K)$ or $t_K(v_K)$ must have a letter $k \in K$ such that

$$k = k_1(g) \qquad h = k_2(g)$$

for some $g \in G$, so that $q = q_2(k)$. Moreover

$$\begin{aligned} l_{\mathcal{H}}(g_2(g_\spadesuit)) &= l_{\mathcal{G}}(g_\spadesuit) \\ &= \spadesuit \end{aligned}$$

and

$$\begin{aligned} s_H(g_2(g_\spadesuit)) &= k_2^\star(s_G(g_\spadesuit)) \\ &= k \star_2 (\delta_g) \\ &= k_2 \circ \delta_g \\ &= \delta_{k_2(g)} \\ &= \delta_h \end{aligned} \qquad \begin{aligned} t_H(g_2(g_\spadesuit)) &= k_2^\star(t_G(g_\spadesuit)) \\ &= k \star_2 (\delta_g) \\ &= k_2 \circ \delta_g \\ &= \delta_{k_2(g)} \\ &= \delta_h \end{aligned}$$

showing thar $g_2(g_\spadesuit) = h_\spadesuit$. Since (f_2, g_2) is downward closed, we can deduce the existence of $e_G \in E_{\mathcal{G}}$ such that $f_2(e_G) = e_H$. This in turn implies that

$$e = c_2(f_1(e_G))$$

so that we fall back to the previous point.
- $e = c_2(e_K)$, $q = q_2(k)$ and $v = d_1(v_H)$ for some $e_K \in E_{\mathcal{K}}$, $k \in K$ and $v_H \in V_{\mathcal{H}}$. As in the point above, we know that $d_2(k_\spadesuit) = d_2(t_{\mathcal{K}}(e_K))$, so that

$$t_{\mathcal{K}}(e_K) = k_\spadesuit$$

We also know that there are $g \in G$ and $h \in H$ such that h is in the image of $s_H(v_H)$ or of $t_H(v_H)$ and

$$k = k_1(g) \qquad h = k_2(g)$$

In turn this implies that $g_1(g_\spadesuit) = k_\spadesuit$ and (f_1, g_1) is in $\mathsf{dcl_d}$, thus there is $e_G \in E_{\mathcal{G}}$ such that $f_1(e_G) = e_K$ and the thesis now follows from the first point.
- $e = c_1(e_H)$, $q = q_2(k)$ and $v = d_1(v_H)$ for some $e_H \in E_{\mathcal{H}}$, $k \in K$ and $v_H \in V_{\mathcal{H}}$. We have remarked at the beginning of this proof that our hypotheses entails the existence of $g \in G$ such that

$$k = k_1(g) \qquad q = q_1(k_2(g))$$

Hence we can conclude using the first point.

- $e = c_2(e_K)$, $q = q_1(h)$ and $v = d_2(v_K)$ for some $e_K \in E_{\mathcal{K}}$, $h \in H$ and $v_K \in V_{\mathcal{K}}$. This is done as before noticing that there must be $g \in G$ such that

$$q = q_2(k_1(g)) \qquad h = k_2(g)$$

allowing us to appeal to the second point of this list.

- $e = c_1(e_H)$, $q = q_2(k)$ and $v = d_2(v_K)$ for some $e_H \in E_{\mathcal{H}}$, $k \in K$ and $v_K \in V_{\mathcal{K}}$. We have

$$\begin{aligned} q_1^\star(t_H(t_{\mathcal{H}}(e_H))) &= t(d_1(t_{\mathcal{H}}(e_H))) \\ &= t(t_{\mathcal{Q}}(c_1(e_H))) \\ &= t(t_{\mathcal{Q}}(e)) \\ &= t(q_\spadesuit) \\ &= \delta_q \end{aligned}$$

This implies that q is in the image of q_1 and the thesis follows from the third item of this list.

- $e = c_2(e_K)$, $q = q_1(h)$ and $v = d_1(v_H)$ for some $e_K \in E_{\mathcal{K}}$, $h \in H$ and $v_H \in V_{\mathcal{H}}$. Computing:

$$\begin{aligned} q_2^\star(t_K(t_{\mathcal{K}}(e_K))) &= t(d_2(t_{\mathcal{K}}(e_K))) \\ &= t(t_{\mathcal{Q}}(c_2(e_K))) \\ &= t(t_{\mathcal{Q}}(e)) \\ &= t(q_\spadesuit) \\ &= \delta_q \end{aligned}$$

Thus q is in the image of q_2 and the thesis now follows from the fourth point.

(e_2) Suppose now that there exists $e \in E_{\mathcal{Q}}$ with $t_{\mathcal{Q}}(e) = v$. We have eight other cases to examine.

- $e = c_1(e_H)$, $q = q_1(h)$ and $v = d_1(v_H)$ for some $e_H \in E_{\mathcal{H}}$, $h \in H$ and $v_H \in V_{\mathcal{H}}$. Then

$$\begin{aligned} d_1(v_H) &= v \\ &= t_{\mathcal{Q}}(e) \\ &= t_{\mathcal{Q}}(c_1(e_H)) \\ &= d_1(t_{\mathcal{H}}(e_H)) \end{aligned}$$

and d_1 is injective, so $t_{\mathcal{H}}(e'_H) = v_H$. Since we know h is in the image $s_H(v_H)$ or of $t_H(v_H)$, we conclude that there exists $e'_H \in E_{\mathcal{H}}$ such that

$$s_{\mathcal{H}}(e_H) = s_{\mathcal{H}}(e'_H) \qquad h_\spadesuit = t_{\mathcal{H}}(e'_H)$$

therefore $c_1(e'_H)$ satisfies

$$\begin{aligned} s_{\mathcal{Q}}(c_1(e'_H)) &= d_1(s_{\mathcal{H}}(e'_H)) \\ &= d_1(s_{\mathcal{H}}(e_H)) \\ &= s_{\mathcal{Q}}(c_1(e_H)) \\ &= s_{\mathcal{Q}}(e) \end{aligned} \qquad \begin{aligned} t_{\mathcal{Q}}(c_1(e'_H)) &= d_1(t_{\mathcal{H}}(e'_H)) \\ &= d_1(h_\spadesuit) \\ &= q_1(h)_\spadesuit \\ &= q_\spadesuit \end{aligned}$$

and we can conclude.

- $e = c_2(e_K)$, $q = q_2(k)$ and $v = d_2(v_K)$ for some $e'_K \in E_{\mathcal{K}}$, $k \in K$ and $v_K \in V_{\mathcal{K}}$. As above we have

$$\begin{aligned} d_2(v_K) &= v \\ &= t_{\mathcal{Q}}(e) \\ &= t_{\mathcal{Q}}(c_2(e_K)) \\ &= d_2(t_{\mathcal{K}}(e_K)) \end{aligned}$$

implying $t_{\mathcal{K}}(e_K) = v_K$ and the existence of $e'_K \in E_{\mathcal{K}}$ such that

$$s_{\mathcal{H}}(e_K) = s_{\mathcal{K}}(e'_K) \qquad k_{\spadesuit} = t_{\mathcal{K}}(e'_K)$$

We can conclude considering $c_2(e'_K)$.

- $e = c_1(e_H)$, $q = q_1(h)$ and $v = d_2(v_K)$ for some $e_H \in E_{\mathcal{H}}$, $h \in H$ and $v_K \in V_{\mathcal{K}}$. Since

$$\begin{aligned} v &= t_{\mathcal{Q}}(e) \\ &= t_{\mathcal{Q}}(c_1(e_H)) \\ &= d_1(t_{\mathcal{H}}(e_H)) \end{aligned}$$

we know that v is in the image of d_1 and we can appeal to the first point.

- $e = c_2(e_K)$, $q = q_2(k)$ and $v = d_1(v_H)$ for some $e_K \in E_{\mathcal{K}}$, $k \in K$ and $v_H \in V_{\mathcal{H}}$. As above

$$\begin{aligned} v &= t_{\mathcal{Q}}(e) \\ &= t_{\mathcal{Q}}(c_2(e_K)) \\ &= d_2(t_{\mathcal{K}}(e_K)) \end{aligned}$$

shows that v is in the image of d_2 so that we fall back to the second point.

- $e = c_1(e_H)$, $q = q_2(k)$ and $v = d_1(v_H)$ for some $e_H \in E_{\mathcal{H}}$, $k \in K$ and $v_H \in V_{\mathcal{H}}$. We have

$$\begin{aligned} s(v) &= s(d_1(v_H)) \\ &= q_1^{\star}(s_H(v_H)) \\ &= q_1 \circ s_H(v_H) \end{aligned}$$

By hypothesis q is in the image of $s(v)$, thus it is also in the image of $q_1 \circ s_H(v_H)$. In particular this implies that $q = q_1(h)$ for some $h \in H$, so the first point applies.

- $e = c_2(e_K)$, $q = q_1(h)$ and $v = d_2(v_K)$ for some $e_K \in E_{\mathcal{K}}$, $h \in H$ and $v_K \in V_{\mathcal{K}}$. Since

$$\begin{aligned} s(v) &= s(d_2(v_K)) \\ &= q_2^{\star}(s_K(v_K)) \\ &= q_2 \circ s_K(v_K) \end{aligned}$$

q is in the image of q_2 and we can conclude.

- $e = c_1(e_H)$, $q = q_2(k)$ and $v = d_2(v_K)$ for some $e_H \in E_{\mathcal{H}}$, $k \in K$ and $v_K \in V_{\mathcal{K}}$. This point is proved appealing to the fifth item of this list and noticing that

$$\begin{aligned} v &= t_{\mathcal{Q}}(c_1(e_H)) \\ &= d_1(t_{\mathcal{H}}(e_H)) \end{aligned}$$

- $e = c_2(e_K)$, $q = q_1(h)$ and $v = d_1(v_H)$ for some $e_K \in E_{\mathcal{K}}$, $h \in H$ and $v_H \in V_{\mathcal{H}}$. Then

$$\begin{aligned} v &= t_{\mathcal{Q}}(c_2(e_K)) \\ &= d_2(t_{\mathcal{K}}(e_K)) \end{aligned}$$

and the sixth item of this list applies. □

We know by Corollary 6.3.5 that **LDAGHGraph** is $\mathcal{M}, \mathcal{N}$-adhesive with respect to the classes

$$\mathcal{M} := \{((h_1, h_2), k) \in \mathcal{A}(\mathbf{LDAGHGraph}) \mid (h_1, h_2) \in \mathsf{dcl_d}, k \in \mathcal{M}(\mathbf{Set})\}$$
$$\mathcal{N} := \{((h_1, h_2), k) \in \mathcal{A}(\mathbf{LDAGHGraph}) \mid (h_1, h_2) \in \mathcal{M}(\mathbf{DAG}), k \in \mathcal{M}(\mathbf{Set})\}$$

In particular, since $\mathcal{M}$ is a subclass of $\mathcal{N}$, we know that **LDAGHGraph** is also an $\mathcal{M}, \mathcal{M}$-adhesive category. Applying point 3 of Theorem 5.1.31 together with Propositions 6.3.13 and 6.3.14 we get the following.

Corollary 6.3.15. **HHG** *is* $\mathcal{M}', \mathcal{M}'$*-adhesive, where*

$$\mathcal{M}' := \{(h, k) \in \mathbf{HHG} \mid F(h, k) \in \mathcal{M}\}$$

6.4 Term graphs

A brute force proof of quasiadhesivity of the category of term graphs was given in [38]. In this section we will present the category of term graphs as a subcategory of labelled hypergraphs. First of all we will prove that this presentation is equivalent to the traditional one. Next, we will recoverthe result of [38] by means of our Theorem 5.1.31.

6.4.1 Two categories of term graphs

Let us start using labelled hypergraphs to define term graphs.

Definition 6.4.1. Let Σ be an algebraic signature, a labelled hypergraph $(l, !_{V_\mathcal{G}}) \colon \mathcal{G} \to \mathcal{G}^\Sigma$ is a *term graph* if $t_\mathcal{G}$ is injective. We define $\mathbf{TG}_\Sigma$ to be the full subcategory of $\mathbf{Hyp}_\Sigma$ and denote by I_Σ the inclusion. Restricting $U_\Sigma \colon \mathbf{Hyp}_\Sigma \to \mathbf{Set}$ we get a forgetful functor $U_{\mathbf{TG}_\Sigma} \colon \mathbf{TG}_\Sigma \to \mathbf{Set}$.

Remark 6.4.2. Notice that, by Remark 6.2.20, if $\mathcal{G}$ is a term graph then $t_\mathcal{G}(h)$ is a word of length 1.

Example 6.4.3. Of the examples of Section 6.2.2, only Example 6.2.23 is a term graph.

In the literature there are two definitions of term graphs: Definition 6.4.1 is different from the classical one that was adopted in [38], and it is in turn more in tune with the current interests in string diagrams. The aim of this section is to prove that the categories arising from the two definitions are in fact equivalent.

Definition 6.4.4. Let $\Sigma = (O_\Sigma, \mathsf{ar}_\Sigma)$ be an algebraic signature. The category $\mathbf{TeGr}_\Sigma$ is defined as follows:

- an object is a triple (V, l, s) where V is a set of *nodes*, $l \colon V \rightharpoonup O_\Sigma$, $s \colon V \rightharpoonup V^\star$ are partial functions such that $\mathsf{dom}(l) = \mathsf{dom}(s)$ and, for each $v \in \mathsf{dom}(l)$

$$\mathsf{ar}_\Sigma(l(v)) = \mathsf{dom}(s(v))$$

- A morphism $(V, l, s) \to (W, p, r)$ is a function $f \colon V \to W$ such that, for every $v \in \mathsf{dom}(l)$, $f(v)$ belongs to $\mathsf{dom}(p)$ and

$$p(f(v)) = l(v) \qquad r(f(v)) = f^\star(s(v))$$

A node v not in $\mathsf{dom}(l)$ is called *empty*

We can define a functor from $G\colon \mathbf{TeGr}_\Sigma \to \mathbf{TG}_\Sigma$. Given (V, l, s) in $\mathbf{TeGr}_\Sigma$, the set of nodes of $\mathcal{G}(V, l, s)$ is V, while the set of edges is $\mathsf{dom}(s)$. If $i\colon \mathsf{dom}(s) \to V$ is the inclusion, we obtain $s_{\mathcal{G}(V,l,s)}, t_{\mathcal{G}(V,l,s)}\colon \mathsf{dom}(s) \rightrightarrows V^\star$ putting

$$s_{\mathcal{G}(V,l,s)} := s \circ i \qquad t_{\mathcal{G}(V,l,s)} := \eta_V \circ i$$

where η is the natural transformation $\mathsf{id}_{\mathbf{Set}} \to (-)^\star$ defined in Example 2.1.8. Now $\mathsf{dom}(s) = \mathsf{dom}(l)$ and computing we have

$$\begin{aligned} s_{\mathcal{G}^\Sigma}(l(v)) &= \delta_\heartsuit^{\mathsf{ar}_\Sigma(l(v))} \\ &= \delta_\heartsuit^{\mathsf{dom}(s(v))} \\ &= !_V^\star(s(v)) \\ &= !_V^\star(s_{\mathcal{G}(V,l,s)}(v)) \end{aligned}$$

thus $(l, !_V)$ defines an algebraic labelled hypergraph $G(V, l, s)\colon \mathcal{G}(V, l, s) \to \mathcal{G}^\Sigma$. .

Proposition 6.4.5. *For every (V, l, s) in $\mathbf{TeGr}_\Sigma$, $G(V, l, s)\colon \mathcal{G}(V, l, s) \to \mathcal{G}^\Sigma$ is a term graph.*

Proof. This follows at once since η_V and i are injective. □

We have now to define the action of G on arrows of $\mathbf{TeGr}_\Sigma$. Given $f\colon (V, l, s) \to (W, p, r)$ we know by definition that $f(v) \in \mathsf{dom}(r)$ for every $v \in \mathsf{dom}(l)$, therefore we can restrict $f\colon V \to W$ to a function $g\colon \mathsf{dom}(s) \to \mathsf{dom}(r)$. If we compute we get:

$$\begin{aligned} f^\star\left(t_{\mathcal{G}(V,l,s)}(v)\right) &= f \circ \delta_v \\ &= \delta_{f(v)} \\ &= \delta_{g(v)} \\ &= t_{\mathcal{G}(W,p,r)}(g(v)) \end{aligned} \qquad \begin{aligned} f^\star\left(s_{\mathcal{G}(V,l,s)}(v)\right) &= f^\star(s(v)) \\ &= r(f(v)) \\ &= s_{\mathcal{G}(W,p,r)}(g(v)) \end{aligned}$$

thus (g, f) defines a morphism of hypergraphs $\mathcal{G}(V, l, s) \to \mathcal{G}(W, p, r)$. Moreover, for every $v \in \mathsf{dom}(l)$

$$\begin{aligned} p(g(v)) &= p(f(v)) \\ &= l(v) \end{aligned}$$

and so(g, f) is a morphism in the category in $\mathbf{Hyp}_\Sigma$.

Theorem 6.4.6. *The functor $G\colon \mathbf{TeGr}_\Sigma \to \mathbf{TG}_\Sigma$ defined above is an equivalence.*

Proof. Faithfulness of G follows immediately from the definition. For fullness, let (g, f) be a morphism between $G(V, l, s) \to G(W, p, r)$, then, for every $v \in \mathsf{dom}(s)$, we must have

$$\begin{aligned} t_{\mathcal{G}(W,p,r)}(g(v)) &= f^\star(t_{\mathcal{G}(V,l,s)}(v)) \\ &= f^\star(\delta_v) \\ &= f \circ \delta_v \\ &= \delta_f(v) \\ &= t_{\mathcal{G}(W,p,r)}(f(v)) \end{aligned}$$

Since $t_{\mathcal{G}(W,p,r)}$ is injective, this shows that $g\colon \mathsf{dom}(s) \to \mathsf{dom}(r)$ must coincide with the restriction of f, so it's now enough to show that $f\colon V \to W$ is a morphism of $\mathbf{TeGr}_\Sigma$. Take $v \in \mathsf{dom}(s)$, since, by definition

$$G(V,l,s) = (l, !_V) \qquad G(W,p,r) := (p, !_W)$$

the fact that (g,f) is a morphism of $\mathbf{Hyp}_\Sigma$ entails at once the identity

$$p(f(v)) = l(v)$$

for every $v \in \mathsf{dom}(s)$. On the other hand

$$\begin{aligned} r(f(v)) &= s_{\mathcal{G}(W,p,r)}(g(v)) \\ &= f^\star(s_{\mathcal{G}(V,l,s)}(v)) \\ &= f^\star(s(v)) \end{aligned}$$

Thus we are left with essential surjectivity of G. Let $(h, !_{V_\mathcal{G}})\colon \mathcal{G} \to \mathcal{G}^\Sigma$ be a term graph, we can define an object of $\mathbf{TeGr}_\Sigma$ as follows.

- The set of nodes is $V_\mathcal{G}$.
- Given v be in $V_\mathcal{G}$, by definition there is at most one $e \in E_\mathcal{G}$ such that $t_\mathcal{G}(e) = \delta_v$ so we can define

$$l\colon V \rightharpoonup O_\Sigma \qquad v \mapsto \begin{cases} h(e) & \text{there exists } e \text{ such that } t_\mathcal{G}(e) = \delta_v \\ \text{undefined} & \text{otherwise} \end{cases}$$
$$s\colon V \rightharpoonup V^\star \qquad v \mapsto \begin{cases} s_\mathcal{G}(e) & \text{there exists } e \text{ such that } t_\mathcal{G}(e) = \delta_v \\ \text{undefined} & \text{otherwise} \end{cases}$$

By construction $\mathsf{dom}(l) = \mathsf{dom}(s)$ and $\mathsf{ar}_\Sigma(l(v)) = \mathsf{dom}(s(v))$, so that $(V_\mathcal{G}, l, s)$ is an object of $\mathbf{TeGr}_\Sigma$.

We have to show that $G(V_\mathcal{G}, l, s)$ is isomorphic to $(h, !_{V_\mathcal{G}})$. For every $e \in E_\mathcal{G}$ there is exactly one $\phi(e) \in \mathsf{dom}(s)$ such that $t_\mathcal{G}(e) = \delta_{\phi(e)}$, thus we get a bijection $\phi\colon E_\mathcal{G} \to \mathsf{dom}(s)$. Now, we have

$$\begin{aligned} t_{\mathcal{G}(V,l,s)}(\phi(e)) &= \delta_{\phi(e)} \\ &= t_\mathcal{G}(e) \end{aligned} \qquad \begin{aligned} s_{\mathcal{G}(V,l,s)}(\phi(e)) &= s_{\mathcal{G}(V,l,s)}(t_\mathcal{G}(e)) \\ &= s(t_\mathcal{G}(e)) \\ &= s_\mathcal{G}(e) \end{aligned}$$

so that $(\phi, \mathsf{id}_{V_\mathcal{G}})$ is an isomorphism from $\mathcal{G}$ to $\mathcal{G}(V_\mathcal{G}, l, s)$. Moreover l sends $\phi(e)$ to $h(e)$ by construction, thus $(\phi, \mathsf{id}_{V_\mathcal{G}})$ lies in $\mathbf{Hyp}_\Sigma$ and we are done. □

6.4.2 $\mathbf{TG}_\Sigma$ is quasiadhesive

We are now going back to examine the properties of $\mathbf{TG}_\Sigma$, with the purpose of proving its quasidhesivity.

Proposition 6.4.7. *The forgetful functor* $U_{\mathbf{TG}_\Sigma}\colon \mathbf{TG}_\Sigma \to \mathbf{Set}$ *has a left adjoint* $\Delta_{\mathbf{TG}_\Sigma}$.

Proof. This follows at once noticing that, for every set X, $\Delta_\Sigma(X)$ is a term graph. □

Take now a mono $(i,j)\colon \mathcal{H}\to\mathcal{G}$ between $(l,!_{V_\mathcal{G}})\colon \mathcal{G}\to\mathcal{G}^\Sigma$ and $(l',!_{V'_\mathcal{G}})\colon \mathcal{H}\to\mathcal{G}^\Sigma$ in $\mathbf{Hyp}_\Sigma$. In particular we have a commutative square

$$\begin{tikzcd} E_\mathcal{G} \arrow[r, "t_\mathcal{G}"] \arrow[d, "i"'] & V^\star_\mathcal{G} \arrow[d, "j"] \\ E_{\mathcal{G}'} \arrow[r, "t_{\mathcal{G}'}"'] & V^\star_{\mathcal{G}'} \end{tikzcd}$$

By Proposition 6.2.19 i and j are injective, thus if $t_{\mathcal{G}'}$ is injective then $t_\mathcal{G}$ is injective too. This show that if $(l',!_{V'_\mathcal{G}})$ is a term graph then $(l,!_{V_\mathcal{G}})$ belongs to $\mathbf{TG}_\Sigma$ too. We can apply this argument when (i,j) is the equalizer in $\mathbf{Hyp}_\Sigma$ of two parallel arrows between term graphs to get the following.

Proposition 6.4.8. $\mathbf{TG}_\Sigma$ *has equalizers and* I_Σ *creates them.*

We have a similar result also for binary products.

Proposition 6.4.9. $\mathbf{TG}_\Sigma$ *has binary products and* I_Σ *creates them.*

Proof. Let $(l,!_{V_\mathcal{G}})\colon \mathcal{G}\to\mathcal{G}^\Sigma$ and $(l',!_{V_\mathcal{H}})\colon \mathcal{H}\to\mathcal{G}^\Sigma$ be two term graphs, their product in $\mathbf{Hyp}_\Sigma$ is given by $(p,!_{V_\mathcal{P}})\colon \mathcal{P}\to\mathcal{G}^\Sigma$, where the square below is a pullback in $\mathbf{Hyp}$ and $(p,!_{V_\mathcal{P}})$ is the diagonal filling it.

$$\begin{tikzcd} \mathcal{P} \arrow[r, "(p_E,p_V)"] \arrow[d, "(q_E,e_V)"'] & \mathcal{G} \arrow[d, "(l,!_{V_\mathcal{G}})"] \\ \mathcal{H} \arrow[r, "(l',!_{V_\mathcal{H}})"'] & \mathcal{G}^\Sigma \end{tikzcd}$$

By Proposition 6.2.2 we have two pullback square in $\mathbf{Set}$:

$$\begin{tikzcd} E_\mathcal{P} \arrow[r, "p_E"] \arrow[d, "q_E"'] & E_\mathcal{G} \arrow[d, "l"] \\ E_\mathcal{H} \arrow[r, "l'"'] & O_\Sigma \end{tikzcd} \qquad \begin{tikzcd} V_\mathcal{P} \arrow[r, "p_V"] \arrow[d, "q_V"'] & V_\mathcal{G} \arrow[d, "!_{V_\mathcal{G}}"] \\ V_\mathcal{H} \arrow[r, "!_{V_\mathcal{H}}"'] & \{\heartsuit\} \end{tikzcd}$$

Moreover $t_\mathcal{P}$ fits in the following diagram.

$$\begin{tikzcd} V^\star_\mathcal{P} \arrow[rrr, "p^\star_V"] \arrow[ddd, "q^\star_V"'] & & & V^\star_\mathcal{G} \arrow[ddd, "!^\star_{V_\mathcal{G}}"] \\ & E_\mathcal{P} \arrow[ul, "t_\mathcal{P}"] \arrow[r, "p_E"] \arrow[d, "q_E"'] & E_\mathcal{G} \arrow[ur, "t_\mathcal{G}"'] \arrow[d, "l"] & \\ & E_\mathcal{H} \arrow[dl, "t_\mathcal{H}"'] \arrow[r, "l'"'] & O_\Sigma \arrow[dr, "t_{\mathcal{G}^\Sigma}"] & \\ V^\star_\mathcal{H} \arrow[rrr, "!^\star_{V_\mathcal{H}}"'] & & & \{\heartsuit\}^\star \end{tikzcd}$$

If we show that $\mathcal{P}$ is a term graph we are done. Take $h_1, h_2 \in E_\mathcal{P}$ with the same image through $t_\mathcal{P}$, then we get the following chains of equalities

$$\begin{aligned} t_\mathcal{G}(p_E(h_1)) &= p_V^\star(t_\mathcal{P}(h_1)) \\ &= p_V^\star(t_\mathcal{P}(h_2)) \\ &= t_\mathcal{G}(p_E(h_2)) \end{aligned} \qquad \begin{aligned} t_\mathcal{H}(q_E(h_1)) &= q_V^\star(t_\mathcal{P}(h_1)) \\ &= q_V^\star(t_\mathcal{P}(h_2)) \\ &= t_\mathcal{H}(q_E(h_2)) \end{aligned}$$

since $t_\mathcal{G}$ and $t_\mathcal{H}$ are injective we get

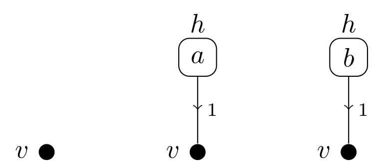
$$p_E(h_1) = p_E(h_2) \qquad q_E(h_1) = q_E(h_2)$$

which, in turn, imply $h_1 = h_2$. □

Since pullbacks can be computed from products and equalizers we also get the following.

Corollary 6.4.10. $\mathbf{TG}_\Sigma$ *has pullbacks and they are created by* I_Σ.

Remark 6.4.11. $\mathbf{TG}_\Sigma$ in general does not have terminal objects. Since $U_{\mathbf{TG}_\Sigma}$ preserves limits, if a terminal object exists it must have the singleton as set of nodes, therefore the set of hyperedges must be empty or a singleton $\{h\}$. Now take as signature the one given by two operations a and b, both of arity 0; we have three term graphs with only one node v: $\Delta_{\mathbf{TG}_\Sigma\Sigma}(\{v\})$, $(l_a, !_{V_\mathcal{G}}) \colon \mathcal{G}_a \to \mathcal{G}^\Sigma$ and $(l_b, !_{V_\mathcal{G}}) \colon \mathcal{G}_b \to \mathcal{G}^\Sigma$.



There are no morphisms in $\mathbf{TG}_\Sigma$ between the last two and from the last two to the first one, therefore none of them can be terminal.

Remark 6.4.12. $\mathbf{TG}_\Sigma$ is not an adhesive category. In particular it does not have pushouts along all monomorphisms. Take the three term graphs of the previous remark, we have two arrows $(?_{\{h\}}, \mathsf{id}_{\{v\}}) \colon \Delta_{\mathbf{TG}_\Sigma}(\{v\}) \to (l_a, !_{V_{\mathcal{G}_a}})$ and $(?_{\{h\}}, \mathsf{id}_{\{v\}}) \colon \Delta_{\mathbf{TG}_\Sigma}(\{v\}) \to (l_b, !_{V_{\mathcal{G}_a}})$ which cannot be completed to a square. Indeed if $(q, !_{V_\mathcal{H}}) \colon \mathcal{H} \to \mathcal{G}^\Sigma$ is another term graph with $(g_E, g_V) \colon (l_a, !_{V_\mathcal{G}}) \to (q, !_{V_\mathcal{H}})$ and $(k_E, k_V) \colon (l_a, !_{V_\mathcal{G}}) \to (q, !_{V_\mathcal{H}})$ such that

$$(g_E, g_V) \circ (?_{\{h\}}, \mathsf{id}_{\{v\}}) = (k_E, k_V) \circ (?_{\{h\}}, \mathsf{id}_{\{v\}})$$

then $g_V = k_V$ and

$$\begin{aligned} t_\mathcal{H}(g_E(h)) &= g_V^\star(t_\mathcal{G}(h)) \\ &= g_V^\star(\delta_v) \\ &= k_V^\star(\delta_V) \\ &= k_V^\star(t_\mathcal{G}(h)) \\ &= t_\mathcal{H}(k_E(h)) \end{aligned}$$

so that we also have $g_E = k_E$, but then

$$\begin{aligned} a &= l_a(h) \\ &= q(g_E(h)) \\ &= q(k_E(h)) \\ &= l_b(h) \\ &= b \end{aligned}$$

Definition 6.4.13. Given a labelled hypergraph $(l, !_{V_\mathcal{G}})\colon \mathcal{G} \to \mathcal{G}^\Sigma$, we will say that $v \in V_\mathcal{G}$ is an *input node* if δ_v does not belong to the image of $t_\mathcal{G}$.

Proposition 6.4.14. *Let $(l, !_{V_\mathcal{H}})\colon \mathcal{H} \to \mathcal{G}^\Sigma$ be a term graph and $(f, g)\colon \mathcal{G} \to \mathcal{H}$ an arrow of* **Hyp** *sending input nodes to input nodes. For every $h \in E_\mathcal{H}$, if $t_\mathcal{H}(h) = \delta_{g(v)}$ for some $v \in V_\mathcal{G}$ then $h \in f(E_\mathcal{G})$.*

Proof. Take $v \in V_\mathcal{G}$ such that $\delta_{g(v)} = t_\mathcal{H}(h)$, since (f, g) sends input nodes to input nodes, δ_v must be in the image of $t_\mathcal{G}$, thus there exists a $k \in E_\mathcal{G}$ such that $t_\mathcal{G}(k) = \delta_v$. Now,

$$\begin{aligned} t_\mathcal{H}(f(k)) &= g^\star(t_\mathcal{G}(k)) \\ &= g^\star(\delta_v) \\ &= g \circ \delta_v \\ &= \delta_{g(v)} \\ &= t_\mathcal{H}(h) \end{aligned}$$

Buy $\mathcal{H}$ is a term graph, therefore we can conclude that $f(k) = h$. □

We are now ready to show that regular monos are exactly monos sending input nodes to input nodes.

Lemma 6.4.15. *A mono (i, j) between two term graphs $(l, !_{V_\mathcal{G}})\colon \mathcal{G} \to \mathcal{G}^\Sigma$ and $(l', !_{V_\mathcal{H}})\colon \mathcal{H} \to \mathcal{G}^\Sigma$ is regular if and only if it sends input nodes to input nodes.*

Proof. ($\Rightarrow$). If (i, j) is a regular mono in $\mathbf{TG}_\Sigma$ then, by Proposition 6.4.8 it is so also in $\mathbf{Hyp}_\Sigma$. By Corollaries 6.1.5 and 5.1.36 if (f_1, g_1) and (f_2, g_2) are arrows from $(l', !_{V_\mathcal{H}})$ to $(k, !_{V_\mathcal{K}})\colon \mathcal{K} \to \mathcal{G}^\Sigma$ in $\mathbf{Hyp}_\Sigma$, then their equalizer $(e, !_{V_\mathcal{E}})\colon \mathcal{E} \to \mathcal{G}^\Sigma$ is such that the two diagrams below are equalizer in **Set**.

$$E_\mathcal{E} \xrightarrow{\iota_E} E_\mathcal{H} \overset{f_1}{\underset{f_2}{\rightrightarrows}} E_\mathcal{K} \qquad V_\mathcal{E} \xrightarrow{\iota_V} V_\mathcal{H} \overset{g_1}{\underset{g_2}{\rightrightarrows}} V_\mathcal{K}$$

Moreover, the target function of $\mathcal{E}$ fits into the diagram

$$\begin{tikzcd} E_\mathcal{E} \arrow[r, "\iota_E"] \arrow[d, dotted, "t_\mathcal{E}"'] & E_\mathcal{H} \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] \arrow[d, "t_\mathcal{H}"'] & E_\mathcal{K} \\ V_\mathcal{E}^\star \arrow[r, "\iota_V^\star"'] & V_\mathcal{H}^\star \arrow[r, shift left, "g_1^\star"] \arrow[r, shift right, "g_2^\star"'] & V_\mathcal{K}^\star \end{tikzcd}$$

and the arrow $(\iota_E, \iota_V) \colon (e, !_{V_\mathcal{E}}) \to (l', !_{V_\mathcal{H}})$ is given by the inclusions. In particular, if $v \in V_\mathcal{E}$ is such that $\delta_{\iota_V(v)}$ is the target of $h \in \mathcal{H}$, then

$$\begin{aligned} t_\mathcal{K}(f_1(h)) &= g_1^\star(t_\mathcal{H}) \\ &= g_1^\star(\delta_{\iota_V(v)}) \\ &= g_1 \circ \delta_{\iota_V(v)} \\ &= \delta_{g_1(\iota_V(v))} \\ &= \delta_{g_2(\iota_V(v))} \\ &= g_2 \circ \delta_{\iota_V(v)} \\ &= g_2^\star(\delta_{\iota_V(v)}) \\ &= t_\mathcal{K}(f_2(h)) \end{aligned}$$

thus $f_1(h) = f_2(h)$ and $h = \iota_E(h')$ for some $h' \in E_\mathcal{E}$. By construction

$$\begin{aligned} \iota_V^\star(t_\mathcal{E}(h')) &= t_\mathcal{H}(\iota_E(h')) \\ &= t_\mathcal{H}(h) \\ &= \delta_{\iota_V(v)} \\ &= \iota_V^\star(\delta_V) \end{aligned}$$

thus, by Remark 6.2.4, $t_\mathcal{E}(h') = \delta_v$, showing that (ι_E, ι_V) sends input nodes to input nodes.
$(\Leftarrow)$. Take V and E to be, respectively, $V_\mathcal{H} + (V_\mathcal{H} \smallsetminus j(V_\mathcal{G}))$ and $E_\mathcal{H} + (E_\mathcal{H} \smallsetminus i(E_\mathcal{G}))$, with inclusions

$$j_1 \colon V_\mathcal{H} \to V \qquad j_2 \colon V_\mathcal{H} \smallsetminus j(V_\mathcal{G}) \to V \qquad i_1 \colon E_\mathcal{H} \to E \qquad i_2 \colon E_\mathcal{H} \smallsetminus i(E_\mathcal{G}) \to E$$

Now, we are going to use another auxiliary function

$$r \colon V_\mathcal{H} \to V \qquad v \mapsto \begin{cases} j_1(v) & v \in j(V_\mathcal{G}) \\ j_2(v) & v \notin j(V_\mathcal{G}) \end{cases}$$

which is clearly injective. We can now define $s, t \colon E \rightrightarrows V^\star$ as the functions induced by

$$\begin{array}{llll} s_1 \colon E_\mathcal{H} \to V^\star & h \mapsto j_1^\star(s_\mathcal{H}(h)) & t_1 \colon E_\mathcal{H} \to V^\star & h \mapsto j_1^\star(t_\mathcal{H}(h)) \\ s_2 \colon E_\mathcal{H} \smallsetminus i(E_\mathcal{G}) \to V^\star & h \mapsto r^\star(s_\mathcal{H}(h)) & t_2 \colon E_\mathcal{H} \smallsetminus i(E_\mathcal{G}) \to V^\star & h \mapsto r^\star(t_\mathcal{H}(h)) \end{array}$$

We have just constructed an hypergraph $\mathcal{K} := (E, V, s, t)$, which we can label taking $(q, !_V) \colon \mathcal{K} \to \mathcal{G}^\Sigma$, where $q \colon E \to O_\Sigma$ is the morphism induced by $l' \colon E_\mathcal{H} \to O_\Sigma$ and its restriction to $E_\mathcal{H} \smallsetminus i(E_\mathcal{G})$. We have now to check that $(q, !_V) \colon \mathcal{K} \to \mathcal{G}^\Sigma$ is actually a term graph, i.e. that t is injective. Suppose that $t(h_1) = t(h_2)$, we have four cases.

- $h_1 = i_1(h)$ and $h_2 = i_1(k)$ for some h, k in $E_\mathcal{H}$. Then

$$\begin{aligned} j_1^\star(t_\mathcal{H}(h)) &= t(i_1(h)) \\ &= t(h_1) \\ &= t(h_2) \\ &= t(i_1(k)) \\ &= j_1^\star(t_\mathcal{H}(k)) \end{aligned}$$

But $j_1^\star$ is injective so $t_\mathcal{H}(h) = t_\mathcal{H}(k)$ and the thesis follows since $(l', !_{V_\mathcal{H}}) \colon \mathcal{H} \to \mathcal{G}^\Sigma$ is a term graph.

- $h_1 = i_2(h)$ and $h_2 = i_2(k)$ for some h, k in $E_{\mathcal{H}} \smallsetminus i(E_{\mathcal{G}})$. As before we can compute to get

$$\begin{aligned} r^\star(t_{\mathcal{H}}(h)) &= t(i_2(h)) \\ &= t(h_1) \\ &= t(h_2) \\ &= t(i_2(k)) \\ &= r^\star(t_{\mathcal{H}}(k)) \end{aligned}$$

  and thus, exploiting Remark 6.2.4, $h_1 = h_2$.

- $h_1 = i_1(h)$ and $h_2 = i_2(k)$ for some $h \in E_{\mathcal{H}}$, k in $E_{\mathcal{H}} \smallsetminus i(E_{\mathcal{G}})$. By the definition of t, this can happen only if $t_{\mathcal{H}}(k) \in j(V_{\mathcal{G}})$, therefore, using Proposition 6.4.14, k must be an element of $i(E_{\mathcal{G}})$, which is absurd.

- $h_1 = i_2(h)$ and $h_2 = i_1(k)$ for some $h \in E_{\mathcal{H}}$, k in $E_{\mathcal{H}} \smallsetminus i(E_{\mathcal{G}})$. This is done as in the previous point, switching the roles of h_1 and h_2.

Now, by construction (i_1, j_1) defines an arrow $\mathcal{H} \to \mathcal{K}$, which is also a morphism $(l', !_{V_{\mathcal{H}}}) \to (q, !_V)$ of $\mathbf{TG}_\Sigma$. On the other hand we can construct another arrow (f, r) parallel to it defining

$$f \colon E_{\mathcal{H}} \to E \quad h \mapsto \begin{cases} i_1(h) & h \in i(E_{\mathcal{G}}) \\ i_2(h) & h \notin i(E_{\mathcal{G}}) \end{cases}$$

and noticing that

$$\begin{aligned} s(f(h)) &= \begin{cases} s_1(h) & h \in i(E_{\mathcal{G}}) \\ s_2(h) & h \notin i(E_{\mathcal{G}}) \end{cases} \\ &= \begin{cases} j_1^\star(s_{\mathcal{H}}(h)) & h \in i(E_{\mathcal{G}}) \\ r^\star(s_{\mathcal{H}}(h)) & h \notin i(E_{\mathcal{G}}) \end{cases} \\ &= r^\star(s_{\mathcal{H}}(h)) \end{aligned} \qquad \begin{aligned} t(f(h)) &= \begin{cases} t_1(h) & h \in i(E_{\mathcal{G}}) \\ t_2(h) & h \notin i(E_{\mathcal{G}}) \end{cases} \\ &= \begin{cases} j_1^\star(t_{\mathcal{H}}(h)) & h \in i(E_{\mathcal{G}}) \\ r^\star(t_{\mathcal{H}}(h)) & h \notin i(E_{\mathcal{G}}) \end{cases} \\ &= r^\star(t_{\mathcal{H}}(h)) \end{aligned}$$

Where the last equalities follows since $h \in i(E_{\mathcal{G}})$ implies that

$$\begin{aligned} s_{\mathcal{H}}(h) &= s_{\mathcal{H}}(i(k)) \\ &= j^\star(s_{\mathcal{G}}(k)) \end{aligned} \qquad \begin{aligned} t_{\mathcal{H}}(h) &= t_{\mathcal{H}}(i(k)) \\ &= j^\star(t_{\mathcal{G}}(k)) \end{aligned}$$

By construction we have

$$q(f(h)) = l'(h)$$

thus (f, r) is a morphism in $\mathbf{TG}_\Sigma$. Now, $(i, j)\mathcal{G} \to \mathcal{H}$ is the equalizer of $(f, r), (i, j) \colon \mathcal{H} \rightrightarrows \mathcal{K}$ in $\mathbf{Hyp}$, thus it is also the equalizer of $(f, r), (i, j) \colon (l', !_{V_{\mathcal{H}}}) \rightrightarrows (q, !_{V_{\mathcal{K}}})$ in $\mathbf{Hyp}_\Sigma$.The thesis follows from Proposition 6.4.8. □

Lemma 6.4.16. *Let $(l_0, !_{V_{\mathcal{G}}}) \colon \mathcal{G} \to \mathcal{G}^\Sigma$, $(l_1, !_{V_{\mathcal{H}}}) \colon \mathcal{H} \to \mathcal{G}^\Sigma$ and $(l_2, !_{V_{\mathcal{K}}}) \colon \mathcal{K} \to \mathcal{G}^\Sigma$ be term graphs. Given $(f_1, g_1) \colon (l_0, !_{V_{\mathcal{G}}}) \to (l_1, !_{V_{\mathcal{H}}})$, $(f_2, g_2) \colon (l_0, !_{V_{\mathcal{G}}}) \to (l_2, !_{V_{\mathcal{K}}})$, if (f_1, g_1) is a regular mono in $\mathbf{TG}_\Sigma$, then their pushout $(p, !_{V_{\mathcal{P}}}) \colon \mathcal{P} \to \mathcal{G}^\Sigma$ in $\mathbf{Hyp}_\Sigma$ is a term graph too.*

Proof. $\mathbf{Hyp}_\Sigma$ is isomorphic to $\mathsf{id}_{\mathbf{Hyp}}\downarrow\delta_{\mathcal{G}^\Sigma}$, and $\delta_{\mathcal{G}^\Sigma}$ preserves pushouts. In particular this implies that $\mathcal{P}$ is a pushout of (f_1, g_1) along (f_2, g_2) in **Hyp**, equipped with the labeling induced by l_1 and l_2. More precisely, we have pushout squares in **Set**u

$$\begin{tikzcd} E_{\mathcal{G}} \arrow[r, "f_2"] \arrow[d, "f_1"'] & E_{\mathcal{K}} \arrow[d, "k_E"] \\ E_{\mathcal{H}} \arrow[r, "h_E"'] & E_{\mathcal{P}} \end{tikzcd} \qquad \begin{tikzcd} V_{\mathcal{G}} \arrow[r, "g_2"] \arrow[d, "g_1"'] & V_{\mathcal{K}} \arrow[d, "k_V"] \\ V_{\mathcal{H}} \arrow[r, "h_V"'] & V_{\mathcal{K}} \end{tikzcd}$$

And diagrams

$$\begin{tikzcd} V^\star_{\mathcal{G}} \arrow[rrr, "g^\star_2"] \arrow[ddd, "g^\star_1"'] & & & V^\star_{\mathcal{K}} \arrow[ddd, "k^\star_V"] \\ & E_{\mathcal{G}} \arrow[ul, "s_{\mathcal{G}}"'] \arrow[r, "f_2"] \arrow[d, "f_1"'] & E_{\mathcal{K}} \arrow[ur, "s_{\mathcal{K}}"'] \arrow[d, "k_E"] & \\ & E_{\mathcal{H}} \arrow[dl, "s_{\mathcal{H}}"] \arrow[r, "h_E"'] & E_{\mathcal{P}} \arrow[dr, "s_{\mathcal{P}}"] & \\ V^\star_{\mathcal{H}} \arrow[rrr, "h^\star_V"'] & & & V^\star_{\mathcal{P}} \end{tikzcd} \qquad \begin{tikzcd} V^\star_{\mathcal{G}} \arrow[rrr, "g^\star_2"] \arrow[ddd, "g^\star_1"'] & & & V^\star_{\mathcal{K}} \arrow[ddd, "k^\star_V"] \\ & E_{\mathcal{G}} \arrow[ul, "t_{\mathcal{G}}"'] \arrow[r, "f_2"] \arrow[d, "f_1"'] & E_{\mathcal{K}} \arrow[ur, "t_{\mathcal{K}}"'] \arrow[d, "k_E"] & \\ & E_{\mathcal{H}} \arrow[dl, "t_{\mathcal{H}}"] \arrow[r, "h_E"'] & E_{\mathcal{P}} \arrow[dr, "t_{\mathcal{P}}"] & \\ V^\star_{\mathcal{H}} \arrow[rrr, "h^\star_V"'] & & & V^\star_{\mathcal{P}} \end{tikzcd} \qquad \begin{tikzcd} E_{\mathcal{G}} \arrow[r, "f_2"] \arrow[d, "f_1"'] & E_{\mathcal{K}} \arrow[d, "k_E"] \arrow[ddr, bend left, "l_2"] & \\ E_{\mathcal{H}} \arrow[r, "h_E"'] \arrow[drr, bend right, "l_1"'] & E_{\mathcal{P}} \arrow[dr, "p"] & \\ & & O_\Sigma \end{tikzcd}$$

Notice that k_V and k_E are injective as they are the pushout of injective arrows, hence by Remark 6.2.4 we know that $k^\star_V$ is injective too. Suppose now that there exists $h_1, h_2 \in E_{\mathcal{P}}$ such that $t_{\mathcal{P}}(h_1) = t_{\mathcal{P}}(h_2)$, by Remark 6.2.20 we know that there must be $v \in V_{\mathcal{P}}$ such that

$$\delta_v = t_{\mathcal{P}}(h_1) \qquad \delta_v = t_{\mathcal{P}}(h_2)$$

Using Lemma 6.1.1 we can split the cases.

- $h_1 = k_E(k_1)$ and $h_2 = k_E(k_2)$ for some k_1 and $k_2 \in E_{\mathcal{K}}$. Then we have

$$\begin{aligned} k^\star_V(t_{\mathcal{K}}(k_1)) &= t_{\mathcal{P}}(k_E(k_1)) \\ &= t_{\mathcal{P}}(h_1) \\ &= t_{\mathcal{P}}(h_2) \\ &= t_{\mathcal{P}}(k_E(k_2)) \\ &= k^\star_V(t_{\mathcal{K}}(k_2)) \end{aligned}$$

  $k^\star_V$ and $t_{\mathcal{K}}$ are injective, thus $k_1 = k_2$ and so $h_1 = h_2$.

- $h_1 = k_E(k)$ and $h_2 = h_E(h')$ for some $k \in E_{\mathcal{K}}$ and $h' \in E_{\mathcal{H}}$. Let $w_1 \in V_{\mathcal{K}}$ and $w_2 \in V_{\mathcal{H}}$ be the nodes such that

$$\delta_{w_1} = t_{\mathcal{K}}(k) \qquad \delta_{w_2} = t_{\mathcal{H}}(h')$$

then we have

$$
\begin{aligned}
\delta_{k_V(w_1)} &= k_V \circ \delta_{w_1} \\
&= k_V^\star(\delta_{w_1}) \\
&= k_V^\star(t_{\mathcal{K}}(k)) \\
&= t_{\mathcal{P}}(k_E(k)) \\
&= t_{\mathcal{P}}(h_1) \\
&= t_{\mathcal{P}}(h_2) \\
&= t_{\mathcal{P}}(h_E(h')) \\
&= h_V^\star(t_{\mathcal{H}}(h')) \\
&= h_V^\star(\delta_{w_2}) \\
&= h_V \circ \delta_{w_2} \\
&= \delta_{h_V(w_2)}
\end{aligned}
$$

and thus we can deduce that

$$k_V(w_1) = h_V(w_2)$$

By the third point of Lemma 6.1.1 there must be a $w_3 \in V_{\mathcal{G}}$ such that

$$w_1 = g_2(w_3) \qquad w_2 = g_1(w_3)$$

Proposition 6.4.14 and Lemma 6.4.15 now entail that there exists $e \in \mathcal{E}_{\mathcal{G}}$ such that $h' = f_1(e)$. Notice that

$$
\begin{aligned}
g_1^\star(t_{\mathcal{G}}(e)) &= t_{\mathcal{H}}(f_1(e)) \\
&= t_{\mathcal{H}}(h') \\
&= \delta_{w_2} \\
&= \delta_{g_1(w_3)} \\
&= g_1^\star(\delta_{w_3})
\end{aligned}
$$

and so

$$\delta_{w_3} = t_{\mathcal{G}}(e)$$

But then we also have

$$
\begin{aligned}
t_{\mathcal{K}}(f_2(e)) &= g_2^\star(t_{\mathcal{G}}(e)) \\
&= g_2^\star(\delta_{w_3}) \\
&= g_2 \circ \delta_{w_3} \\
&= \delta_{g_2(w_3)} \\
&= \delta_{w_1} \\
&= t_{\mathcal{K}}(k)
\end{aligned}
$$

Since $(l_2, !_{V_{\mathcal{K}}})\colon \mathcal{K} \to \mathcal{G}^{\Sigma}$ is a term graph this entails that

$$f_2(k) = k$$

and we can conclude that $h_1 = h_2$.

- $h_1 = h_E(h')$ and $h_2 = k_E(k)$ for some $k \in E_{\mathcal{K}}$ and $h' \in E_{\mathcal{H}}$. This is done as in the previous case swapping the role of h_1 and h_2.
- $h_1 = h_E(h'_1)$ and $h_2 = h_E(h'_2)$ for some h'_1 and $h'_2 \in E_{\mathcal{H}}$. Let x_1 and x_2 be the unique elements of $V_{\mathcal{H}}$ such that

$$\delta_{x_1} = t_{\mathcal{H}}(h'_1) \qquad \delta_{x_2} = t_{\mathcal{H}}(h'_2)$$

Then it must be that

$$\begin{aligned}
\delta_{h_V(x_1)} &= h_V \circ \delta_{x_1} \\
&= h_V^\star(\delta_{x_1}) \\
&= h_V^\star(t_{\mathcal{H}}(h'_1)) \\
&= t_{\mathcal{P}}(h_1) \\
&= t_{\mathcal{P}}(h_2) \\
&= h_V^\star(t_{\mathcal{H}}(h'_2)) \\
&= h_V^\star(\delta_{x_2}) \\
&= h_V \circ \delta_{x_2} \\
&= \delta_{h_V(x_2)}
\end{aligned}$$

showing that $h_V(x_1) = h_V(x_2)$. By the second point of Lemma 6.1.1 we know that at least one between x_1 or x_2 must belong to $g_1(V_{\mathcal{G}})$. Without loss of generality we can suppose that it is x_1 (otherwise just swap it with x_2). Using Proposition 6.4.14 and Lemma 6.4.15 we know that h'_1 is in the image of f_1, i.e. that there exists $e \in E_{\mathcal{G}}$ such that $h'_1 = f_1(e)$, but then

$$\begin{aligned}
k_E(f_2(e)) &= h_E(f_1(e)) \\
&= h_E(h'_1) \\
&= h_1
\end{aligned}$$

so we fall back to the third case and we can conclude. □

Corollary 5.1.34 and Lemmas 6.4.15 and 6.4.16 allow us to recover the following result, previously proved by direct computation in [38, Thm. 4.2].

Corollary 6.4.17. *The category* $\mathbf{TG}_\Sigma$ *is quasiadhesive.*

Conclusions for Part II

CHAPTER 7

The second part of the thesis is devoted to the study of $\mathcal{M},\mathcal{N}$*-adhesivity*, a crucial property in the algebraic treatment of rewriting theories.

In Chapter 5, we have first provided a brief definition and some fundamental properties of $\mathcal{M},\mathcal{N}$-adhesive categories. Then, we presented a novel criterion for verifying $\mathcal{M},\mathcal{N}$-adhesivity, which involves analyzing certain properties of functors that connect the category of interest to a family of categories possessing suitable adhesive properties. This criterion can be seen as a distilled abstraction of several ad hoc proofs of adhesivity found in the literature. By using this criterion, we were able to systematically and uniformly establish some results concerning the adhesivity of categories formed by products, exponents, and comma constructions.

Next, we have proceeded to generalize three well-known results from the theory of *(quasi)adhesive categories* to the $\mathcal{M},\mathcal{N}$-adhesive setting, adapting the techniques developed in [52].

The first result pertains to binary suprema in the poset of subobjects of an $\mathcal{M},\mathcal{N}$-adhesive category. We have demonstrated that given a mono in $\mathcal{M}$ and one in $\mathcal{M}\cap\mathcal{N}$, then their supremum, called a $\mathcal{M},\mathcal{N}$-*union*, exists and it is computed as the pushout of the pullbacks of the two given monos.

We have then proved a kind of converse of the previous result: in the presence of $\mathcal{M},\mathcal{N}$-unions, we can guarantee $\mathcal{M},\mathcal{N}$-adhesivity if we know that $\mathcal{M}$ is contained in the class of $\mathcal{N}$-adhesive morphisms. This enables us to reduce the proof of the Van Kampen condition to demonstrating the stability of some squares and that some pullbacks are pushouts. As an example, adhesivity of toposes can be easily proven using this method.

Finally, we showed that under some mild hypotheses about $\mathcal{M}$ and $\mathcal{N}$, an $\mathcal{M},\mathcal{N}$-adhesive category can be embedded in a Grothendieck topos via a functor that preserves all relevant structure (i.e. pullbacks and $\mathcal{M},\mathcal{N}$-pushouts). Therefore, the slogan "an adhesive category is one whose pushouts of monomorphisms exist and behave more or less as they do in a topos" holds true even for $\mathcal{M},\mathcal{N}$-adhesive categories.

In Chapter 6, we have applied the criterion established in Chapter 5 to various significant examples, such as term graphs and directed (acyclic) graphs. Furthermore, due to the modularity of our approach, we could easily establish appropriate adhesivity properties for categories formed by combining simpler ones. In particular, we tackled the adhesivity issue for several categories of hierarchical (hyper)graphs, including Milner's bigraphs, bigraphs with sharing, and a new version of bigraphs with recursion. Additionally, we proved an adhesivity property for a category of hierarchical hypergraphs employed in [11] to provide a graphical semantics for monoidal closed categories.

As future work, we plan to analyse other categories of graph-like objects using our criterion; an interesting case is that of *directed bigraphs* [14, 34, 57, 58]. Moreover, it is worth to verify whether the $\mathcal{M},\mathcal{N}$-adhesivity that we obtain from the results of this thesis is suited for modelling specific rewriting systems, e.g. based on the double pushout approach. As an example, $\mathbf{TG}_\Sigma$ is quasiadhesive yet the left-hand side of rules typically adopted in applications is often a non regular mono, thus questioning the relative usefulness of the adhesivity property [38].

Our discussion on a criterion for adhesivity begs the question of its meaning for a rewriting system at hand. Namely, which is the right notion of $\mathcal{M},\mathcal{N}$-adhesivity, given a set of rewriting rules? More specifically, given some set of rewriting rules, the question of devising the right kind of adhesivity properties that should be proven is still open and an ongoing subject of work. In particular, we are planning to investigate if the presence of *conditions* [21, 43, 59] in a rewriting system can suggest some canonical choice of $\mathcal{M}$ and $\mathcal{N}$ for which $\mathcal{M},\mathcal{N}$-adhesivity can be proved.

One may also notice that, if a category is $\mathcal{M},\mathcal{N}$-adhesive, then $\mathcal{M}$ must be contained in the class of $\mathcal{N}$-adhesive morphisms. In particular, in the $\mathcal{M}$-adhesive case, $\mathcal{M}$ must be a subclass of the class of adhesive morphisms. Hence, the preadhesive structures for which $\mathbf{X}$ is $\mathcal{M},\mathcal{N}$-adhesive form a bounded family in the poset of all preadhesive structures. This suggests to study such poset, in order to characterize the largest preadhesive structure, suited for the specific problem, for which $\mathbf{X}$ is $\mathcal{M},\mathcal{N}$-adhesive.

Categorical preliminaries

APPENDIX A

Contents

The aim of this appendix is to prove some basic results of category theory which we used throughout all this thesis. Doing so we also fix some notation. The material contained in Appendices A.1 to A.4 is standard and can be found in any textbook on category theory [5, 12, 85, 93]. While the standard references for Appendix A.5 are [81, 85].

A.1 Remarks on limits and colimits

Let us start pointing out some results about limits and colimits.

Definition A.1.1. [5] Let $G\colon \mathbf{D} \to \mathbf{X}$ be a diagram, given a functor $F\colon \mathbf{X} \to \mathbf{Y}$ we we say that F:

1. *preserves (co)limits* of G if given a (co)limiting (co)cone $(L, \{l_D\}_{D\in\mathbf{D}})$ for G, $(F(L), \{F(l_D)\}_{D\in\mathbf{D}})$ is (co)limiting for $F \circ G$;
2. *reflects (co)limits* of G if a (co)cone $(L, \{l_D\}_{D\in\mathbf{D}})$ is (co)limiting for the functor G whenever the (co)cone $(F(L), \{F(l_D)\}_{D\in\mathbf{D}})$ is (co)limiting for $F \circ G$;
3. *creates (co)limits* of G if G has a (co)limit in $\mathbf{X}$ whenever $F \circ G$ has one, and F preserves and reflects (co)limits along G.

Remark A.1.2. Notice that our notion of creation is laxer than, e.g., [85, Def. V.1].

Proposition A.1.3. *Let $G \colon \mathbf{D} \to \mathbf{X}$ be a diagram and $F \colon \mathbf{X} \to \mathbf{Y}$ a functor. The following are equivalent:*

1. *F creates (co)limits along G;*
2. *given a (co)limiting (co)cone $(L, \{l_D\}_{D \in \mathbf{D}})$ for $F \circ G$, there exists a (co)cone $(X, \{x_D\}_{D \in \mathbf{D}})$ which is (co)limiting for G and such that $(F(X), \{F(x_D)\}_{D \in \mathbf{D}})$ is (co)limiting for $F \circ G$. Moreover, for every other (co)cone $(Y, \{y_D\}_{D \in \mathbf{D}})$ such that $(F(Y), \{F(y_D)\}_{D \in \mathbf{D}})$ is (co)limiting, there is a (unique) isomorphism $f \colon X \to Y$ such that , for every $D \in \mathbf{D}$:*

$$x_D = y_D \circ f$$

Proof. We prove the thesis for limits, the case of colimits follows by duality.
$(1 \Rightarrow 2)$ By hypothesis $F \circ G$ has a limit, thus there exists a limiting cone $(X, \{x_D\}_{D \in \mathbf{D}})$ for G. Since F preserves limits of G we know that $(F(X), \{F(x_D)\}_{D \in \mathbf{D}})$ is a limit cone. If $(Y, \{y_D\}_{D \in \mathbf{D}})$ is another cone such that $(F(Y), \{F(y_D)\}_{D \in \mathbf{D}})$ is a limit, then, by reflection, it is limiting and thus the thesis follows.
$(2 \Rightarrow 1)$ Let $(L, \{l_D\}_{D \in \mathbf{D}})$ be a limiting cone for $F \circ G$, by hypothesis, we can pick a cone $(X, \{x_D\}_{D \in \mathbf{D}})$ in $\mathbf{X}$ which is limiting for G. Since $(F(X), \{F(x_D)\}_{D \in \mathbf{D}})$ is a limit, we get also an isomorphism $h \colon F(X) \to L$ such that, for every $D \in \mathbf{D}$

$$F(x_d) = l_D \circ h$$

Take now a limiting cone $(Y, \{y_D\}_{D \in \mathbf{D}})$ on G, then there exists an isomorphism $g \colon Y \to X$ such that

$$y_D = x_D \circ g$$

Thus $h \circ F(g)$ is an isomorphism $F(Y) \to L$ such that

$$\begin{aligned} F(y_D) &= F(x_D \circ g) \\ &= F(x_D) \circ F(g) \\ &= l_D \circ h \circ F(g) \end{aligned}$$

showing that $(F(Y), \{F(y_D)\}_{D \in \mathbf{D}})$ is limiting, so that F preserves limits along G.

For reflection: suppose that $(Y, \{y_D\}_{D \in \mathbf{D}})$ is a cone on G such that $(F(Y), \{F(y_D)\}_{D \in \mathbf{D}})$ is limiting. By hypothesis we have an isomorphism $f \colon Y \to Y$ such that

$$x_D = y_D \circ f$$

and the thesis now follows because we already know that $(X, \{x_D\}_{D \in \mathbf{D}})$ is a limit. $\square$

Proposition A.1.4. *If $F \colon \mathbf{X} \to \mathbf{Y}$ is a full and faithful functor then it reflects all limits and colimits.*

Proof. Fix a diagram $G \colon \mathbf{D} \to \mathbf{X}$ and suppose that a cone $(L, \{l_D\}_{D \in \mathbf{D}})$ for G is given with the property that $(F(L), \{F_j(l_D)\}_{D \in \mathbf{D}})$ is limiting for $F \circ G$. Let $(X, \{x_D\}_{D \in \mathbf{D}})$ be another cone in $\mathbf{X}$, by hypothesis we have a unique arrow $f \colon F(X) \to F(L)$ such that, for every $D \in \mathbf{D}$

$$F(x_D) = F(l_D) \circ f$$

Since F is full and faithful, f is equal to $F(x)$ for a unique $x \colon X \to L$. Faithfulness also implies that

$$x_D = l_D \circ x$$

Moreover, if $x' \colon X \to L$ is another arrow such that $l_D \circ x'$ is equal to x_D, then $F(x')$ must be f, proving that $x' = x$ and thus that $(L, \{l_D\}_{D \in \mathbf{D}})$ is limiting for G.

The thesis for colimits follows by duality. $\square$

We end this section recalling a classical construction of basic category theory. Let $\mathbf{X}$ be a category with arbitrary coproducts, then for every object X and set S we can construct the coproduct $S \bullet X$ of the family $\{X_s\}_{s\in S}$, where $X_s = X$ for every $s \in S$. Now, if $\iota_s \colon X \to S \bullet X$ is the coprojection corresponding to $s \in S$, then we have a function

$$\eta_S \colon S \to \mathbf{X}(X, S \bullet X) \qquad s \mapsto \iota_S$$

On the other hand, for every function $f \colon S \to \mathbf{X}(X, Y)$, there exists a unique $\hat{f} \colon S \bullet X \to Y$ such that

$$f(s) = \hat{f} \circ \iota_s$$

In particular, this means that we have a commutative triangle

$$\begin{tikzcd} S \arrow[r, "\eta_S"] \arrow[dr, "f"'] & \mathbf{X}(X, S \bullet X) \arrow[d, "\hat{f}\circ(-)"] \\ & \mathbf{X}(X, Y) \end{tikzcd}$$

Thus we have showed the following.

Proposition A.1.5. *If $\mathbf{X}$ is a category with coproducts, then, for every $X \in \mathbf{X}$, the representable functor $\mathbf{X}(X, -) \colon \mathbf{X} \to \mathbf{Set}$ has a left adjoint $(-) \bullet X$.*

A.1.1 Colimits in Set

We will now recall a general recipe to compute colimits in the category of sets and functions.

Lemma A.1.6. *Let $F \colon \mathbf{D} \to \mathbf{Set}$ be a functor with a small domain, for every $D \in \mathbf{D}$ consider the coprojection $i_D \colon F(D) \to \sum_{D\in\mathbf{D}} F(D)$. Let also $\sim$ be the relation on $\sum_{D\in\mathbf{D}} F(D)$ defined by $i_{D_1}(x) \sim i_{D_2}(y)$ if and only if there exists $n \in \mathbb{N}$ and families $\{E_i\}_{i=0}^{n}$,$\{G_i\}_{i=0}^{n+1}$ $\{f_i\}_{i=0}^{2n+1}$, $\{e_i\}_{i=0}^{n}$, $\{g_i\}_{i=0}^{n+1}$ such that:*

- *every E_i and every G_i is an object of $\mathbf{D}$, moreover $e_i \in F(E_i)$ and $g_i \in F(G_i)$;*
- *$G_0 = D_1$, $G_{n+1} = D_2$, $g_0 = x$ and $g_{2n+1} = y$;*
- *$\{f_i\}_{i=0}^{2n+1}$ is a family of arrows of $\mathbf{D}$ such that $f_{2i+1} \colon E_k \to G_{k+1}$ and $f_{2i} \colon E_i \to F_i$, moreover the following equations hold*
$$F(f_{2i})(e_i) = g_i \qquad F(f_{2i+1})(e_i) = g_{i+1}$$

Then the following hold true:

1. *$\sim$ is an equivalence relation;*
2. *if C is the quotient $\sum_{D\in\mathbf{D}} F(D)/ \sim$ and $\pi \colon \sum_{D\in\mathbf{D}} F(D) \to C$ is the quotient function, then a colimiting cocone for F is given by $(C, \{j_D\}_{D\in\mathbf{D}})$ where $j_D := \pi \circ i_D$.*

Proof. 1. Let x be an element of $F(D)$ and put take $n = 0$, $E_0 = D$, $f_0 = f_1 = \mathsf{id}_D$, then $i_D(x) \sim i_D x$, proving reflexivity. For simmetry, let $\{E_i\}_{i=0}^{n}$,$\{G_i\}_{i=0}^{n+1}$ $\{f_i\}_{i=0}^{2n+1}$, $\{e_i\}_{i=0}^{n}$, $\{g_i\}_{i=0}^{n+1}$ be families witnessing $i_{D_1}(x) \sim i_{D_2}(y)$, then we can define

$$E'_i := E_{n-i} \quad e'_i := e_{n-1} \quad G'_i := G_{n+1-i} \quad g'_i := g_{n+1-i} \quad f'_i := f_{2n+1-i}$$

and these families witness $i_{D_2}(y) \sim i_{D_1}(x)$. We are left with transitivity: take $\{E_i\}_{i=0}^{n}$,$\{G_i\}_{i=0}^{n+1}$ $\{f_i\}_{i=0}^{2n+1}$, $\{e_i\}_{i=0}^{n}$, $\{g_i\}_{i=0}^{n+1}$, $\{E'_i\}_{i=0}^{m}$,$\{G'_i\}_{i=0}^{m+1}$ $\{f'_i\}_{i=0}^{2m+1}$, $\{e'_i\}_{i=0}^{m}$, $\{g'_i\}_{i=0}^{m+1}$ which witness , respectively, $i_{D_1}(x) \sim i_{D_2}(y)$ and $i_{D_2}(y) \sim i_{D_3}(z)$, then we get $i_{D_1}(x) \sim i_{D_3}(z)$ putting:

$$E''_i := \begin{cases} E_i & i \leq n \\ E'_{i-n-1} & n+1 \leq i \leq n+m \end{cases} \qquad e''_i := \begin{cases} e_i & i \leq n \\ e'_{i-n-1} & n+1 \leq i \leq n+m \end{cases}$$

$$G''_i := \begin{cases} G_i & i \leq n+1 \\ G'_{i-n-1} & n+2 \leq i \leq n+m+1 \end{cases} \qquad g''_i := \begin{cases} g_i & i \leq n+1 \\ g'_{i-n-1} & n+2 \leq i \leq n+m+1 \end{cases}$$

$$f''_i := \begin{cases} f_i & i \leq 2n+1 \\ f'_i & 2n+2 \leq i \leq 2(n+m) \end{cases}$$

2. First of all we have to prove that $(C, \{j_D\}_{D \in \mathbf{D}})$ is a cocone. Given $f \colon D_1 \to D_2$ in $\mathbf{D}$, we can put

$$E_0 := D_1 \quad G_0 := D_1 \quad G_1 := D_2 \quad f_0 := F(\mathsf{id}_{D_1}) \quad f_1 := F(f)$$

which witness that, for every $x \in F(D_1)$, $i_{D_1}(x) \sim i_{D_2}(f(x))$, and so $j_{D_2} \circ F(f) = j_{D_1}$.

Let $(A, \{c_D\}_{D \in \mathbf{D}})$ be another cocone, there is a unique arrow $c \colon \sum_{D \in \mathbf{D}} F(D) \to A$ making the following diagram commutative.

$$\begin{tikzcd}
 & F(D) \arrow[dl, bend right, "i_D"'] \arrow[dr, bend left, "c_D"] & \\
\sum\limits_{D \in \mathbf{D}} F(D) \arrow[rr, "c"'] & & A
\end{tikzcd}$$

Take now $x \in F(D_1)$ and $y \in F(D_2)$ such that $i_{D_1}(x) \sim i_{D_2}(y)$ and let $\{E_i\}_{i=0}^{n}$,$\{G_i\}_{i=0}^{n+1}$ $\{f_i\}_{i=0}^{2n+1}$, $\{e_i\}_{i=0}^{n}$, $\{g_i\}_{i=0}^{n+1}$ be families witnessing it, then

$$\begin{aligned} c(i_{D_1}(x)) &= c_{D_1}(x) \\ &= c_{D_1}(f_0(e_0)) \\ &= c_{E_0}(e_0) \\ &= c_{D_2}(f_1(e_0)) \end{aligned}$$

By induction this argument entails

$$c(i_{D_1}(x)) = c(i_{D_2}(y))$$

therefore we can conclude that there exists a unique $q \colon C \to A$ such that $q \circ \pi = c$, but then

$$\begin{aligned} q \circ j_D &= q \circ \pi \circ i_D \\ &= c \circ i_D \\ &= c_D \end{aligned}$$

On the other hand, if $k\colon C \to A$ is another arrow such that $k \circ j_D = c_D$ then

$$k \circ \pi \circ i_D = c_D$$

for every $D \in \mathbf{D}$, thus $\kappa \circ \pi = c$ and we can conclude that $k = q$. □

Corollary A.1.7. *Let D_0 be an object of a small category $\mathbf{D}$, then $(1, \{!_{\mathbf{D}(D_0,D)}\}_{D\in\mathbf{D}})$ is a colimiting cocone for $\mathbf{D}(D_0, -)\colon \mathbf{D} \to \mathbf{Set}$, where $!_{\mathbf{D}(D_0,D)}$ is the unique arrow $\mathbf{D}(D_0, D) \to 1$.*

Proof. For every $f \in \mathbf{D}(D_0, D)$ we can take

$$E_0 := D_0 \quad e_0 := \mathsf{id}_{D_0} \quad G_i := \begin{cases} D & i = 0 \\ D_0 & i = 1 \end{cases} \quad g_i := \begin{cases} f & i = 0 \\ \mathsf{id}_{D_0} & i = 1 \end{cases} \quad f_i := \begin{cases} f & i = 0 \\ \mathsf{id}_{D_0} & i = 1 \end{cases}$$

and we have

$$\mathbf{D}(D_0, f)(\mathsf{id}_{D_0}) = f \qquad \mathbf{D}(D_0, \mathsf{id}_{D_0})(\mathsf{id}_{D_0}) = \mathsf{id}_{D_0}$$

showing that $i_D(f) \sim i_{D_0}(\mathsf{id}_{D_0})$, from which the thesis follows. □

A.2 Comma categories

In this section we will briefly recall the definition of the comma category associated to two functors and some of its properties.

Definition A.2.1. Let $L\colon \mathbf{A} \to \mathbf{X}$ and $R\colon \mathbf{B} \to \mathbf{X}$ be two functors with the same codomain, the *comma category* $L{\downarrow}R$ is the category in which

- objects are triples (A, B, f) with $A \in \mathbf{A}$, $B \in \mathbf{B}$, and $f\colon L(A) \to R(B)$;
- a morphism $(A, B, f) \to (A', B', g)$ is a pair (h, k) with $h\colon A \to A'$ in $\mathbf{A}$ and $k\colon B \to B'$ in $\mathbf{B}$ such that the following diagram commutes

$$\begin{tikzcd} L(A) \arrow[r, "L(h)"] \arrow[d, "f"'] & L(A') \arrow[d, "g"] \\ R(B) \arrow[r, "R(k)"'] & R(B') \end{tikzcd}$$

We have two forgetful functors $U_L\colon L{\downarrow}R \to \mathbf{A}$ and $U_R\colon L{\downarrow}R \to \mathbf{B}$ given, respectively by

$$\begin{array}{ccc} (A, B, f) & \longmapsto & A \\ (h,k)\big\downarrow & & \big\downarrow h \\ (A', B', g) & \longmapsto & A' \end{array} \qquad \begin{array}{ccc} (A, B, f) & \longmapsto & B \\ (h,k)\big\downarrow & & \big\downarrow k \\ (A', B', g) & \longmapsto & B' \end{array}$$

Given $L\colon \mathbf{A} \to \mathbf{X}$ and $R\colon \mathbf{B} \to \mathbf{X}$, we can also consider their duals $L^{op}\colon \mathbf{A}^{op} \to \mathbf{X}^{op}$ and $R^{op}\colon \mathbf{B}^{op} \to \mathbf{X}^{op}$. An arrow $f\colon L(A) \to R(B)$ in $\mathbf{X}$ is the same ting as an arrow $f\colon R^{op}(B) \to L^{op}(A)$ in $\mathbf{X}^{op}$, thus

$(L{\downarrow}R)$ and $R^{op}{\downarrow}L^{op}$ have the same objects. Moreover, the commutativity in $\mathbf{X}$ of the square

$$\begin{array}{ccc} L(A) & \xrightarrow{L(h)} & L(A') \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ R(B) & \xrightarrow[R(k)]{} & R(B') \end{array}$$

is tantamount to the commutativity in $\mathbf{X}^{op}$ of the square

$$\begin{array}{ccc} R(B') & \xrightarrow{R(k)} & R(B) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ L(A') & \xrightarrow[L(h)]{} & L(A) \end{array}$$

Summing up we have just proved the following fact.

Proposition A.2.2. *$(L{\downarrow}R)^{op}$ is equal to $R^{op}{\downarrow}L^{op}$, moreover $U_L^{op} = U_{L^{op}}$ and $U_R^{op} = U_{R^{op}}$.*

We can notice another useful fact, showing that in some cases we can guarantee the existence of a left adjoint to U_R.

Proposition A.2.3. *If $\mathbf{A}$ has initial objects and L preserves them then the forgetful functor $U_R\colon L{\downarrow}R \to \mathbf{B}$ has a left adjoint Δ.*

Proof. For an object $B \in \mathbf{B}$ we can define $\Delta(B)$ as $(0, B, ?_B)$, where 0 is an initial object in $\mathbf{A}$ and $?_{R(B)}$ is the unique arrow $L(0) \to R(B)$. Consider $\mathsf{id}_B\colon B \to U_R(\Delta(B))$ be the identity, and suppose that a $k\colon B \to U_R(A, B', f)$ in $\mathbf{B}$ is given. By initiality of 0, there is only one arrow $?_A\colon 0 \to A$ in $\mathbf{A}$ and, since L preserves initial objects, the following square commutes.

$$\begin{array}{ccc} L(0) & \xrightarrow{L(?_A)} & L(A) \\ {\scriptstyle ?_{R(B)}}\downarrow & & \downarrow{\scriptstyle f} \\ R(B) & \xrightarrow[R(k)]{} & R(B') \end{array}$$

Thus (h, k) is the unique morphism $\Delta(B) \to (A, B', f)$ such that $U_R(h, k) = k$. $\square$

Dualizing we get immediately the following.

Corollary A.2.4. *If $\mathbf{B}$ has terminal objects preserved by R then $U_L\colon L{\downarrow}R \to \mathbf{A}$ has a right adjoint.*

A.3 Slice categories

This section is devoted to recall some basic facts about the so called *slice categories*.

Definition A.3.1. Let X be an object of a category $\mathbf{X}$, we will define the following two categories.

- The *slice category over* X is the category $\mathbf{X}/X$ which has as objects arrows $f\colon Y \to X$ and in which an arrow $h\colon f \to g$ is $h\colon Y \to Y'$ in $\mathbf{X}$ such that the following triangle commutes.

$$\begin{tikzcd} Y \arrow[rr, "h"] \arrow[dr, "f"'] & & Y' \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

- Dually, the *slice category under* X is the category $X/\mathbf{X}$ in which objects are arrows $f\colon X \to Y$ with domain X and a morphism $h\colon f \to g$ is an arrow of $\mathbf{X}$ fitting in a triangle as the one below.

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\ Y \arrow[rr, "h"'] & & Y' \end{tikzcd}$$

Remark A.3.2. For every $X \in X$ we have forgetful functors

$$\begin{array}{ccc} \mathsf{dom}_X\colon \mathbf{X}/X \to \mathbf{X} & \qquad & \mathsf{cod}_X\colon X/\mathbf{X} \to \mathbf{X} \\ \begin{array}{ccc} f & \longmapsto & \mathsf{dom}(f) \\ h\big\downarrow & & \big\downarrow h \\ g & \longmapsto & \mathsf{dom}(g) \end{array} & & \begin{array}{ccc} f & \longmapsto & \mathsf{cod}(f) \\ h\big\downarrow & & \big\downarrow h \\ g & \longmapsto & \mathsf{cod}(g) \end{array} \end{array}$$

Lemma A.3.3. *For every $f\colon Y \to X$ the categories $(\mathbf{X}/X)/f$ and $\mathbf{X}/Y$ are the same category.*

Proof. Given $g\colon Z \to X$, an object of $(\mathbf{X}/X)/f$ is an arrow $h\colon Z \to Y$ in $\mathbf{X}$ thus, in particular, it is an object of $\mathbf{X}/Y$. On the other hand, any object $k\colon Z \to Y$ defines an arrow $f \circ k \to f$ in $(\mathbf{X}/X)/f$, showing that the two categories have the same objects. Take an arrow $k\colon h \to h'$ in $(\mathbf{X}/X)/f$ with $h\colon Z \to Y$ and $h'\colon Z' \to Y$, by definition it is an arrow of $Z \to Z'$ in $\mathbf{X}$ such that $h = h' \circ k$, that is

$$((\mathbf{X}/X)/f)\,(h,h') = (\mathbf{X}/Y)\,(h,h')$$

and the thesis follows. $\square$

Remark A.3.4. In this situation, the functor $\mathsf{dom}_f\colon (\mathbf{X}/X)/f \to \mathbf{X}/X$ becomes $f \circ (-)\colon \mathbf{X}/Y \to \mathbf{X}/X$

$$\begin{array}{ccc} h & \longmapsto & f \circ h \\ k\big\downarrow & & \big\downarrow k \\ h' & \longmapsto & f \circ h' \end{array}$$

We can realize the slice over and under an object $X \in \mathbf{X}$ as comma categories.

Proposition A.3.5. *For every object X in a category $\mathbf{X}$, if $\delta_X\colon \mathbf{1} \to \mathbf{X}$ is the constant functor of value X from the category with only one object $*$, then $\mathbf{X}/X$ and $X/\mathbf{X}$ are isomorphic to, respectively, $\mathsf{id}_X \downarrow \delta_X$ and $\delta_X \downarrow \mathsf{id}_X$*

.

Proof. Define functors $F_1 \colon \mathsf{id}_X {\downarrow} \delta_X \to \mathbf{X}/X$ and $G_1 \colon \mathbf{X}/X \to \mathsf{id}_X {\downarrow} \delta_X$ as follows

$$\begin{tikzcd}
(Y,*,f) \arrow[r, mapsto] \arrow[d, "{(h,\mathsf{id}_*)}"'] & f \arrow[d, "h"] \\
(Y',*,g) \arrow[r, mapsto] & g
\end{tikzcd}
\qquad
\begin{tikzcd}
f \arrow[r, mapsto] \arrow[d, "h"'] & (\mathsf{dom}(f),*,f) \arrow[d, "{(h,\mathsf{id}_*)}"] \\
g \arrow[r, mapsto] & (\mathsf{dom}(g),*,g)
\end{tikzcd}$$

Similarly, we have $F_2 \colon \delta_X {\downarrow} \mathsf{id}_X \to X/\mathbf{X}$ and $G_2 \colon X/\mathbf{X} \to \delta_X {\downarrow} \mathsf{id}_X$

$$\begin{tikzcd}
(*,Y,f) \arrow[r, mapsto] \arrow[d, "{(\mathsf{id}_*,h)}"'] & f \arrow[d, "h"] \\
(*,Y',g) \arrow[r, mapsto] & g
\end{tikzcd}
\qquad
\begin{tikzcd}
f \arrow[r, mapsto] \arrow[d, "h"'] & (*,\mathsf{cod}(f),f) \arrow[d, "{(\mathsf{id}_*,h)}"] \\
g \arrow[r, mapsto] & (*,\mathsf{cod}(g),g)
\end{tikzcd}$$

It is now obvious to see that F_1, G_1 and F_2, G_2 are pairs of inverses. □

A straightforward application of Corollary 5.1.36 now yields the following.

Corollary A.3.6. *If $\mathbf{X}$ has pullbacks, then for every object X, the slice $\mathbf{X}/X$ has pullbacks too.*

Let us turn to products.

Proposition A.3.7. *Let $f \colon Y \to X$ and $g \colon Z \to X$ be two arrow in a category $\mathbf{X}$ with a common codomain, then f has a pullback along g if and only if f and g have a product in $\mathbf{X}/X$.*

Proof. $(\Rightarrow)$ Take a pullback square as the one below and define $p \colon P \to X$ as its diagonal.

$$\begin{tikzcd}
P \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, "p"] & Y \arrow[d, "f"] \\
Z \arrow[r, "g"'] & X
\end{tikzcd}$$

Then p_1 and p_2 are arrows $p \to f$ and $p \to g$. Moreover, for every other $q \colon W \to X$ with arrows $w_1 \colon q \to f$ and $w_2 \colon q \to g$, it must be that

$$\begin{aligned} f \circ w_1 &= q \\ &= f \circ w_2 \end{aligned}$$

thus there exists a unique $w \colon Z \to P$ such that

$$w_1 = p_1 \circ w \qquad w_2 = p_2 \circ w$$

so that

$$\begin{aligned} p \circ w &= f \circ p_1 \circ w \\ &= f \circ w_1 \\ &= q \end{aligned}$$

and so w is a morphism of $\mathbf{X}/X$ and (p, p_1, p_2) is a product of f and g.

($\Leftarrow$) Let $p\colon P \to X$ with projections $p_1\colon P \to Y$ and $p_2\colon P \to Z$ be the product of f and g, then we must have a square

$$\begin{tikzcd} P \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, "p"] & Y \arrow[d, "f"] \\ Z \arrow[r, "g"'] & X \end{tikzcd}$$

To see that this is a pullback square, let $w_1\colon W \to Y$ and $w_2\colon W \to Z$ such that

$$g \circ w_2 = f \circ w_1$$

then w_1 and w_2 are, respectively, arrows $f \circ w_1 \to f$ and $g \circ w_2 \to g$ in $\mathbf{X}/X$. By hypothesis the domains of these arrows are the same, therefore there exists a unique $w\colon f \circ w_1 \to p$ such that

$$w_1 = p_1 \circ w \qquad w_2 = p_2 \circ w$$

Such a w is, in particular, an arrow $W \to P$, thus we only have to check is uniqueness in $\mathbf{X}$. Now, if $w'\colon W \to P$ is such that

$$w_1 = p_1 \circ w' \qquad w_2 = p_2 \circ w'$$

then

$$\begin{aligned} p \circ w' &= f \circ p_1 \circ w' \\ &= f \circ w_1 \end{aligned}$$

thus w' defines a morphism $f \circ w_1 \to p$ and it must therefore coincide with w. $\square$

Notation. Given two arrows $f\colon X \to Y$ and $g\colon X \to Y$, we will denote by $\mathsf{pb}_f(g) : \mathsf{pb}_f(G) \to X$ any choosen representative of the pullback of g along f. Dually, given $f\colon Y \to X$ and $g\colon Y \to Z$, we will use $\mathsf{po}_f(g)$ to denote any representative of the pushout of g along g.

Proposition A.3.8. *Let* $\mathbf{X}$ *be a category with pullbacks. Given an arrow* $f\colon X \to Y$ *there exists a functor* $\mathsf{pb}_f\colon \mathbf{X}/Y \to \mathbf{X}/X$ *sending* g *to* $\mathsf{pb}_f(g)$.

Proof. Let $k\colon G \to H$ be an arrow between $g\colon G \to Y$ and $h\colon H \to Y$, then in $\mathbf{X}$ we have a diagram

$$\begin{tikzcd} \mathsf{pb}_f(G) \arrow[rr, dotted, "\mathsf{pb}_f(k)"] \arrow[dr, "\mathsf{pb}_f(g)"'] \arrow[ddd, "p"'] & & \mathsf{pb}_f(H) \arrow[dl, "\mathsf{pb}_f(h)"] \arrow[ddd, "q"] \\ & X \arrow[d, "f"] & \\ & Y & \\ G \arrow[ur, "g"] \arrow[rr, "k"'] & & H \arrow[ul, "h"'] \end{tikzcd}$$

in which the two diagonal inner trapezoids are pullbacks. Now,

$$\begin{aligned} h \circ k \circ p &= g \circ p \\ &= f \circ \mathsf{pb}_f(g) \end{aligned}$$

so that we can guarantee the existence of the dotted arrow $\mathsf{pb}_f(k)$. Clearly

$$\mathsf{pb}_f(\mathsf{id}_g) = \mathsf{id}_{\mathsf{pb}_f(g)}$$

while, on the other hand, given another $s\colon h \to t$ in $\mathbf{X}/Y$, the diagram

$$\begin{tikzcd}
\mathsf{pb}_f(G) \arrow[r, "\mathsf{pb}_f(k)"] \arrow[drr, bend right, "\mathsf{pb}_f(g)"'] \arrow[ddd, "p"'] & \mathsf{pb}_f(H) \arrow[rr, "\mathsf{pb}_f(s)"] \arrow[dr, "\mathsf{pb}_f(h)"] \arrow[ddd, "q"] & & \mathsf{pb}_f(T) \arrow[dl, "\mathsf{pb}_f(t)"'] \arrow[ddd, "u"] \\
& & X \arrow[d, "f"] & \\
& & Y & \\
G \arrow[r, "k"'] \arrow[urr, bend left, "g"] & H \arrow[rr, "s"'] \arrow[ur, "h"] & & T \arrow[ul, "t"']
\end{tikzcd}$$

witness $\mathsf{pb}_f(s \circ k) = \mathsf{pb}_f(t) \circ \mathsf{pb}_f(k)$ and the thesis now follows. □

We can dualize this to get the following.

Proposition A.3.9. *Let $\mathbf{X}$ be a category with pushouts. For every arrow $f\colon X \to Y$ there exists a functor $\mathsf{po}_f\colon \mathbf{X}/X \to \mathbf{X}/Y$ sending g to $\mathsf{po}_f(g)$.*

Let X be an object in a category $\mathbf{X}$ binary products, for any other object Y in $\mathbf{X}$ we can consider the second projection $L_X(Y)\colon Y \times X \to X$ as an object of $\mathbf{X}/X$. The following lemma guarantees that in this way we get a right adjoint $L_X\colon \mathbf{X} \to \mathbf{X}/X$ to dom_X.

Lemma A.3.10. *Let $\mathbf{X}$ be a category with binary product. For every object X there exists a functor $L_X\colon \mathbf{X} \to \mathbf{X}/X$, sending an object Y to the second projection $Y \times X \to X$, such that $\mathsf{dom}_X \dashv L_X$.*

Proof. By definition given, for every object $Y \in \mathbf{X}$

$$\mathsf{dom}_X(L_X(Y)) = Y \times X$$

and we could define $\epsilon_Y\colon \mathsf{dom}_X(L_X(Y)) \to Y$ simply as the first projection. Given $f \in \mathbf{X}/X$ and $g\colon \mathsf{dom}_X(f) \to Y$ we have a diagram in $\mathbf{X}$ as below

$$\begin{tikzcd}
Y \times X \arrow[r, "\epsilon_Y"] \arrow[d, "L_X(Y)"'] & Y \\
X & \mathsf{dom}_X(f) \arrow[l, "f"] \arrow[u, "g"'] \arrow[ul, "(g,f)"']
\end{tikzcd}$$

Clearly (g, f) defines an arrow $f \to L_X(Y)$ such that

$$g = \epsilon_Y \circ \mathsf{dom}_X(g, f)$$

Viceversa, if $z\colon f \to L_X(Y)$ is such that

$$g = \epsilon_Y \circ \mathsf{dom}_X(z)$$

then it must coincide with (g, f), showing that ϵ_Y is the component of the counit of $\mathsf{dom}_X \dashv L_X$. $\square$

Remark A.3.11. More explicitly, if $f\colon Z \to Y$ is an arrow in $\mathbf{X}$, then $L_X(f)$ is the transpose of $f \circ \epsilon_Z\colon Z \times X \to Y$, that is $L_X(f) := f \times \mathsf{id}_X$.

Take now an arrow $f\colon X \to Y$ in a category $\mathbf{X}$ with pullbacks. Then, by Proposition A.3.7 the slice $\mathbf{X}/Y$ has all products so that Lemma A.3.10 gives us a functor $L_f\colon \mathbf{X}/Y \to (\mathbf{X}/Y)/f$. Now, the codomain of L_f is $\mathbf{X}/X$ by Lemma A.3.3. Using again Proposition A.3.7 it is immediate to see that L_f must coincide with pb_f, therefore we have just established the following result.

Corollary A.3.12. *If $\mathbf{X}$ is a category with pullbacks, then for every $f\colon X \to Y$*

$$f \circ (-) \dashv \mathsf{pb}_f$$

If we now take $\mathbf{X}$ to be cartesian closed we can prove the existence of another adjunction $L_X \dashv R_X$.

Notation. Let us fix some notation. Given $f\colon Y \times X \to Z$ in a cartesian closed category $\mathbf{X}$, we will denote by $\ulcorner f \urcorner$ the transpose $Y \to Z^X$. If ev_X is the counit of $(-) \times X \dashv (-)^X$, $\ulcorner f \urcorner$ is the unique morphism who fits in the diagram below

$$\begin{tikzcd}
Y \times X \arrow[d, "\ulcorner f \urcorner \times \mathsf{id}_X"'] \arrow[dr, "f"] & \\
Z^X \times X \arrow[r, "\mathsf{ev}_{X,Z}"'] & Z
\end{tikzcd}$$

In particular, and with a slight abuse of notation, every $f\colon X \to X$ induces $\ulcorner f \urcorner\colon 1 \to X^X$ which is the unique one fitting in the diagram

$$\begin{tikzcd}
1 \times X \arrow[r, "\pi_X"] \arrow[d, "\ulcorner f \urcorner \times \mathsf{id}_X"'] & X \arrow[d, "f"] \\
X^X \times X \arrow[r, "\mathsf{ev}_{X,X}"'] & X
\end{tikzcd}$$

Lemma A.3.13. *Given a cartesian closed category $\mathbf{X}$ with pullbacks, for every $X \in \mathbf{X}$ there exists a functor $R_X\colon \mathbf{X}/X \to \mathbf{X}$ which is right adjoint to L_X.*

Proof. Given $f\colon Y \to X$, we can consider the following pullback square

$$\begin{tikzcd}
R_X(f) \arrow[r, "p"] \arrow[d, "!_{R_X(f)}"'] & Y^X \arrow[d, "f^X"] \\
1 \arrow[r, "\ulcorner \mathsf{id}_X \urcorner"'] & X^X
\end{tikzcd}$$

If we apply $(-) \times X$ and paste with the naturality square of ev_X, we get

$$\begin{tikzcd}
R_X(f) \times X \arrow[r, "p \times \mathsf{id}_X"] \arrow[d, "!_{R_X(f)} \times \mathsf{id}_X"'] & Y^X \times X \arrow[r, "\mathsf{ev}_{X,Y}"] \arrow[d, "f^X \times \mathsf{id}_X"'] & Y \arrow[d, "f"] \\
1 \times X \arrow[r, "\ulcorner \mathsf{id}_X \urcorner \times \mathsf{id}_X"] \arrow[rr, bend right, "\pi_X"'] & X^X \times X \arrow[r, "\mathsf{ev}_{X,X}"] & X
\end{tikzcd}$$

We can now notice that

$$L_X(R_X(f)) = \pi_X \circ \left(!_{R_X(f)} \times \mathsf{id}_X\right)$$

so that $\mathsf{ev}_{X,Y} \circ (p \times \mathsf{id}_X)$ defines an arrow $L_X(R_X(f)) \to f$ in $\mathbf{X}/X$. To show that in this way we get a counit for $L_X \dashv R_X$, take $Z \in \mathbf{X}$ and $h\colon L_X(Z) \to f$. In particular, h is an arrow $Z \times X \to Y$, so that it has a transpose $\ulcorner h \urcorner\colon Z \to Y^X$. First of all, let us notice that the diagram below commutes.

$$\begin{tikzcd}
Z \times X \arrow[r, "L_X(Z)"] \arrow[d, "!_{L_X(Z)} \times \mathsf{id}_X"'] & X \\
1 \times X \arrow[r, "\ulcorner \mathsf{id}_X \urcorner \times \mathsf{id}_X"'] & X^X \times X \arrow[u, "\mathsf{ev}_{X,X}"']
\end{tikzcd}$$

On the other hand, we know that $L_X(Z) = f \circ h$, thus we can build:

$$\begin{tikzcd}
 & Y^X \times X \arrow[r, "f^X \times \mathsf{id}_X"] \arrow[d, "\mathsf{ev}_{X,Y}"] & X^X \times X \arrow[d, "\mathsf{ev}_{X,X}"] \\
Z \times X \arrow[ur, bend left, "\ulcorner h \urcorner \times \mathsf{id}_X"] \arrow[r, "h"'] \arrow[dr, bend right, "!_{L_X(Z)} \times \mathsf{id}_X"'] & Y \arrow[r, "f"'] & X \\
 & 1 \times X \arrow[r, "\ulcorner \mathsf{id}_X \urcorner \times \mathsf{id}_X"'] & X^X \times X \arrow[u, "\mathsf{ev}_{X,X}"]
\end{tikzcd}$$

showing that

$$f^X \circ \ulcorner h \urcorner = \ulcorner \mathsf{id}_X \urcorner \circ !_{L_X(Z)}$$

so that we get a unique $k\colon Z \to R_X(f)$ such that

$$\ulcorner h \urcorner = p \circ k$$

and thus

$$\begin{aligned}
\mathsf{ev}_{X,Y} \circ (p \times \mathsf{id}_X) \circ L_X(k) &= \mathsf{ev}_{X,Y} \circ (p \times \mathsf{id}_X) \circ (k \times \mathsf{id}_X) \\
&= \mathsf{ev}_{X,Y} \circ ((p \circ k) \times \mathsf{id}_X) \\
&= \mathsf{ev}_{X,Y} \circ (\ulcorner h \urcorner \times \mathsf{id}_X) \\
&= h
\end{aligned}$$

On the other hand, if $k'\colon Z \to R_X(f)$ is such that

$$h = \mathsf{ev}_{X,Y} \circ (p \times \mathsf{id}_X) \circ L_X(k')$$

then $p \circ k'$ must coincide with $\ulcorner h \urcorner$, implying $k = k'$. $\square$

Take now $\mathbf{X}$ to be *locally cartesian closed*: that is a category such that $\mathbf{X}/X$ is cartesian closed for every object X. Notice that by Proposition A.3.7 this implies that $\mathbf{X}$ has all pullbacks, thus Corollary A.3.6 entails that every slice $\mathbf{X}/X$ also has pullbacks. Take now an arrow $f\colon X \to Y$, by Lemmas A.3.10 and A.3.13 we have functors: $\mathsf{dom}_f, R_f\colon (\mathbf{X}/Y)/f \rightrightarrows \mathbf{X}/Y$, $L_f\colon \mathbf{X}/Y \to (\mathbf{X}/Y)/f$ such that

$$\mathsf{dom}_f \dashv L_f \dashv R_f$$

We have already noticed that L_f coincides with pb_f, thus we can deduce at once the following

Corollary A.3.14. *If $f\colon X \to Y$ is a morphism in a locally cartesian closed category $\mathbf{X}$, then the pullback functor pb_f is both a left and a right adjoint.*

A.4 Subobjects and quotients

We are now going to recall the notion of *quotients* and of *subobjects*, in order to fix a uniform notation.

Definition A.4.1. Let $\mathbf{X}$ be a category and $\mathcal{M} \subseteq \mathcal{M}(\mathbf{X})$ a class of monomorphisms. Ff $m\colon M \to X$ and $m'\colon M' \to X$ are two elements of $\mathcal{M}$ with the same codomain, then we say that $m \leq m'$ if and only if there exists a, necessarily unique $h\colon M \to M'$ such that the following diagram commute

$$\begin{tikzcd} M \arrow[rr, dotted, "h"] \arrow[dr, "m"'] & & N \arrow[dl, "m'"] \\ & X & \end{tikzcd}$$

We define $m \equiv m'$ if and only if $m \leq m'$ and $m' \leq m$. This is an equivalence relation on the class

$$\mathcal{M}/X = \{m \in \mathcal{M} \mid \mathsf{cod}(m) = X\}$$

A $\mathcal{M}$*-subobject* of X is an equivalence class $[m]$ with respect to the relation $\equiv$, we will denote by $\mathcal{M}$-$\mathsf{Sub}(X)$ the class of $\mathcal{M}$-subobjects. $\mathbf{X}$ is $\mathcal{M}$*-wellpowered* if, for every object X, $\mathcal{M}$-$\mathsf{Sub}(X)$ is a set.

Dually, if $\mathcal{E}$ a class of epis in $\mathbf{X}$, and $e\colon X \to Y$, $e'\colon X \to Y'$ are two elements of it, we say that $e \leq e'$ if and only if there exists a, necessarily unique, $h\colon Y \to Y'$ such that the following diagram commute

$$\begin{tikzcd} & X \arrow[dl, "e"'] \arrow[dr, "e'"] & \\ Y \arrow[rr, dotted, "h"'] & & Y' \end{tikzcd}$$

We put $e \equiv e'$ if and only if $e \leq e'$ and $e' \leq e$, getting an equivalence relation on the class

$$X/\mathcal{E} = \{e \in \mathcal{E} \mid \mathsf{dom}(e) = X\}$$

A $\mathcal{E}$*-quotient* of X is an equivalence class $[e]$ with respect to the relation $\equiv$ and we will denote by $\mathcal{E}$-$\mathsf{Quot}(X)$ the class of $\mathcal{E}$-quotients. $\mathbf{X}$ is $\mathcal{E}$*-cowellpowered* if, for every object X, $\mathcal{E}$-$\mathsf{Quot}(X)$ is a set.

Notation. We will drop the prefixes "$\mathcal{M}$-" and "$\mathcal{E}$-" when considering the classes of all monomorphisms or of all epimorphisms.

Remark A.4.2. $\mathcal{M}$-$\mathsf{Sub}(X)$ and $\mathcal{E}$-$\mathsf{Quot}(X)$ can be naturally equipped with orders putting, respectively $[m] \leq [m']$ if and only if $m \leq m'$ and $[e] \leq [e']$ if and only if $e \leq e$. the class of id_X is a maximum in $\mathcal{M}$-$\mathsf{Sub}(X)$, while it is a minimum in $\mathcal{E}$-$\mathsf{Quot}(X)$. Notice, moreover, that $m \equiv m'$ if and only if there is a isomorphism h such that $m' \circ h = m$ and, similarly, $e \equiv e'$ if and only if there exists an isomorphism h such that $h \circ e = e'$.

Remark A.4.3. If X is an object of a $\mathcal{M}$-wellpowered category $\mathbf{X}$, then, assuming the axiom of choice for classes, there exists a set $R(X) \subseteq \mathcal{M}/X$ of representatives for $\equiv$. Similarly, if $\mathbf{X}$ is $\mathcal{E}$-cowellpowered, we can find a set of representatives in $X/\mathcal{E}$ for $\equiv$.

Notation. Let $m\colon M \to X$ and $f: Y \to X$ be arrows, we will denote by $\mathsf{pb}_f(m) : \mathsf{pb}_f(M) \to Y$ any representative of the pullback of m along f. Dually, given $e\colon X \to E$ and $g\colon X \to Y$, we will use $\mathsf{po}_g(e)$ to denote any representative of the pushout of e along g.

Proposition A.4.4. *Let $\mathbf{X}$ be a category and $\mathcal{M}$ be a class of monos closed under pullbacks: i.e. for every $m\colon M \to X$ in it and $f: Y \to X$, $\mathsf{pb}_f(m)$ belongs to $\mathcal{M}$. Then the following hold true:*

1. *if $m\colon M \to X$ and $n\colon N \to X$ are elements of $\mathcal{M}/X$ such that $m \leq n$, then*
$$\mathsf{pb}_f(m) \leq \mathsf{pb}_f(n)$$
for every arrow $f: Y \to X$;
2. *if $\mathbf{X}$ is wellpowered then there exists a functor $\mathcal{M}$-$\mathsf{Sub}\colon \mathbf{X}^{op} \to \mathbf{Pos}$.*

Proof. 1. By definition, there exists $h\colon M \to N$ such that $n \circ h = m$, thus we have the solid part of the following diagram

$$\begin{tikzcd}
\mathsf{pb}_f(M) \arrow[rr, "p"] \arrow[dr, dotted, "k"] \arrow[ddr, "\mathsf{pb}_f(m)"'] & & M \arrow[dr, "h"] \arrow[ddr, "m"] & \\
& \mathsf{pb}_f(N) \arrow[rr, "q"] \arrow[d, "\mathsf{pb}_f(n)"] & & N \arrow[d, "n"] \\
& Y \arrow[rr, "f"'] & & X
\end{tikzcd}$$

This implies the existence of the dotted k and the thesis follows.

2. Given $f\colon Y \to X$ we can define a function
$$\mathbf{Pb}_f\colon \mathcal{M}\text{-}\mathsf{Sub}(X) \to \mathcal{M}\text{-}\mathsf{Sub}(Y) \qquad [m] \mapsto [\mathsf{pb}_f(m)]$$
By the previous point this is a well-defined and monotone function and, for every other $g\colon Z \to Y$
$$\mathsf{pb}_{\mathsf{id}_X}(m) \equiv m \qquad \mathsf{pb}_{f\circ g}(m) \equiv \mathsf{pb}_g\big(\mathsf{pb}_f(m)\big)$$
from which the thesis follows. □

Dualizing we get the following corollary.

Corollary A.4.5. *Let $\mathbf{X}$ be a category and $\mathcal{E}$ be a class of epis closed under pushouts: i.e. for every $e\colon X \to E$ in it and $g\colon X \to Y$, $\mathsf{po}_g(e)$ belongs to $\mathcal{E}$, then the following hold true:*

1. *if* $e\colon X \to E$ *and* $f\colon X \to F$ *are elements of* $X/\mathcal{E}$ *such that* $e \leq f$*, then*

$$\mathsf{po}_g(e) \leq \mathsf{po}_g(f)$$

for every arrow $g\colon X \to Y$*;*

2. *if* $\mathbf{X}$ *is cowellpowered then there exists a functor* $\mathcal{E}\text{-}\mathsf{Quot}\colon \mathbf{X} \to \mathbf{Pos}$*.*

In the presence of limits, we can easily compute infima in the poset of subobjects.

Proposition A.4.6. *Let* $\{[m_i]\}_{i\in I}$ *be a subset of* $\mathsf{Sub}(X)$*, and suppose that the diagram defined by the arrows* $\{m_i\}_{i\in I}$ *admits a wide pullback. Then* $\{[m_i]\}_{i\in I}$ *has an infimum.*

Proof. By definition of limit, for every $i \in I$ we have a triangle

$$\begin{tikzcd} M \arrow[r, "l_i"] \arrow[dr, "m"'] & M_i \arrow[d, "m_i"] \\ & X \end{tikzcd}$$

where $(M, \{l_i\}_{i\in I} \cup \{m\})$ is a limiting cone. Notice that m is monic, indeed if $f, g\colon A \rightrightarrows M$ are such that

$$m \circ f = m \circ g$$

then, for every $i \in I$ we have an equality

$$m_i \circ l_i \circ f = m_i \circ l_i \circ g$$

which, since every m_i is a mono, allows us to deduce that

$$l_i \circ f = l_i \circ g$$

and therefore $f = g$. Clearly $[m] \leq [m_i]$ for every i. Let $[n]$ be another lower bound, with $n\colon N \to X$, then there must be $k_i\colon N \to M_i$ such that, for every $i \in I$, $m_i \circ k_i = n$ and thus there exists $\phi\colon N \to M$ such that $l_i \circ \phi = k_i$. Composing with any m_i we get $m \circ \phi = n$, i.e. $[n] \leq [m]$. □

A.5 A crash course on coends and Kan extensions

We are now going to briefly introduce the concept of *coends* and the notion of *left Kan extension*.

Definition A.5.1. Let $F\colon \mathbf{A}^{op} \times \mathbf{A} \to \mathbf{B}$ be a functor, a *cowedge* ω for F is a (large) family $\{\omega_A\}_{A\in\mathbf{A}}$ formed by arrows $\omega_A\colon F(A,A) \to B$ with a common codomain B and such that , for every $f\colon A' \to A$ the following square commutes

$$\begin{tikzcd} & F(A,A) \arrow[dr, "\omega_A"] & \\ F(A,A') \arrow[ur, "F(\mathsf{id}_A, f)"] \arrow[dr, "F(f, \mathsf{id}_{A'})"'] & & B \\ & F(A',A') \arrow[ur, "\omega_{A'}"'] & \end{tikzcd}$$

A cowedge ω with codomain $\int^{A\in\mathbf{A}} F(A,A)$ is *initial*, or a *coend* for F, if for every other cowedge ω', with codomain B, there exists a unique $f\colon \int^{A\in\mathbf{A}} F(A,A) \to B$ fitting in the diagram below.

$$\begin{tikzcd}
& F(A,A) \arrow[d, "\omega_A"] \arrow[dr, bend left, "\omega'_A"] & \\
F(A,A') \arrow[ur, bend left, "F(\mathrm{id}_A,f)"] \arrow[dr, bend right, "F(f,\mathrm{id}_{A'})"'] & \int^{A\in\mathbf{A}} F(A,A) \arrow[r, dotted, "f"] & B \\
& F(A',A') \arrow[u, "\omega_{A'}"'] \arrow[ur, bend right, "\omega'_{A'}"'] &
\end{tikzcd}$$

Remark A.5.2. Cowedges for a functor $F\colon \mathbf{A}^{op}\times\mathbf{A}\to\mathbf{B}$ form a category $\mathsf{cwd}(F)$ in which a morphism between $\omega = \{\omega_A\}_{A\in\mathbf{A}}$ and $\omega' = \{\omega'_A\}_{A\in\mathbf{A}}$ is an arrow $f\colon B\to B'$ sucht that, for every $A\in\mathbf{A}$, the diagram below is commutative.

$$\begin{tikzcd}
& F(A,A) \arrow[dl, "\omega_A"'] \arrow[dr, "\omega'_A"] & \\
B \arrow[rr, "f"'] & & B'
\end{tikzcd}$$

A coend for F is then an initial object in $\mathsf{cwd}(F)$ and thus it is unique up to a unique isomorphisms.

A.5.1 Left Kan extensions

Definition A.5.3. Let $F\colon \mathbf{A}\to\mathbf{B}$ and $G\colon\mathbf{A}\to\mathbf{C}$ be two functors with common domain. A *left Kan extension* of F along G is a pair $(\mathrm{lan}_G(F),\eta_F)$ given by functor $\mathrm{lan}_G(F)\colon\mathbf{C}\to\mathbf{B}$ and a natural transformation $\eta_F\colon F\to\mathrm{lan}_G(F)\circ G$ such that, for every other $H\colon\mathbf{C}\to\mathbf{B}$ and $\lambda\colon F\to H\circ G$, there exists a unique $\overline{\lambda}\colon\mathrm{lan}_G(F)\to H$ such that $\lambda = \left(\overline{\lambda}*F\right)\circ\eta_F$.

Remark A.5.4. The uniqueness clause entails at once that left Kan extensions are unique up to a unique isomorphisms. More precisely, if (L,η_F) and (L',η'_F) enjoy the universal property of $\mathrm{lan}_G(F)$ a left Kan extension then there exists a unique isomorphism $\lambda\colon L\to L'$ such that $\eta'_F = (\lambda * G)\circ\eta_F$.

We can restate the universal property of a left Kan extension $(\mathrm{lan}_G(F),\eta_F)$ requesting, for every functor $H\colon\mathbf{C}\to\mathbf{B}$, the bijectivity of the function

$$\mathbf{B}^{\mathbf{C}}(\mathrm{lan}_G(F),H)\to\mathbf{B}^{\mathbf{A}}(F,H\circ G)\qquad \lambda\mapsto(\lambda*G)\circ\eta_F$$

The previous condition strongly resembles an adjunction. Indeed, if $G\colon\mathbf{A}\to\mathbf{C}$ is a functor, we can consider its associate precomposition functor $(-)\circ G\colon\mathbf{B}^{\mathbf{C}}\to\mathbf{B}^{\mathbf{A}}$. Now for every $F\colon\mathbf{A}\to\mathbf{B}$, the universal property of $(\mathrm{lan}_G(F),\eta_F)$ amounts exactly to $\eta_F\colon F\to\mathrm{lan}_G(F)\circ G$ being the component in F of the unit of an adjunction, therefore we have just proved the following result.

Proposition A.5.5. *Given $G\colon\mathbf{A}\to\mathbf{C}$, let $(-)\circ G\colon\mathbf{B}^{\mathbf{C}}\to\mathbf{B}^{\mathbf{A}}$ be the precomposition functor. Then $(-)\circ G$ has a left adjoint if and only if a left Kan extension $(\mathrm{lan}_G(F),\eta_F)$ exists for every $F\colon\mathbf{A}\to\mathbf{B}$.*

Take now two functors $G\colon\mathbf{A}\to\mathbf{C}$ and $H\colon\mathbf{C}\to\mathbf{D}$ with the property that left Kan extension along them always exists. Since left adjoints compose, by the previous proposition we get that a left Kan extension of $F\colon\mathbf{A}\to\mathbf{B}$ along $H\circ G$ exists and it is given by $\left(\mathrm{lan}_H(\mathrm{lan}_G(F)),\left(\eta_{\mathrm{lan}_G(F)}*G\right)\circ\eta_F\right)$. We can give a slightly more general result.

Lemma A.5.6. *Let $G\colon \mathbf{A} \to \mathbf{C}$, $H\colon \mathbf{C} \to \mathbf{D}$ and $F\colon \mathbf{A} \to \mathbf{B}$ be three functors such that both the left Kan extensions $(\mathrm{lan}_G(F), \eta_F)$ and $\big(\mathrm{lan}_H(\mathrm{lan}_G(F)), \eta_{\mathrm{lan}_G(F)}\big)$ exist. Then $\big(\mathrm{lan}_H(\mathrm{lan}_G(F)), \big(\eta_{\mathrm{lan}_G(F)} * G\big) \circ \eta_F\big)$ is a left Kan extension of F along $H \circ G$.*

Proof. Given $K\colon \mathbf{D} \to \mathbf{B}$, by hypothesis we have a bijection

$$\mathbf{B}^{\mathbf{D}}(\mathrm{lan}_H(\mathrm{lan}_G(F)), K) \to \mathbf{B}^{\mathbf{C}}(\mathrm{lan}_G(F), K \circ H) \qquad \mu \mapsto (\mu * H) \circ \eta_{\mathrm{lan}_G(F)}$$

On the other hand, we also have another bijection

$$\mathbf{B}^{\mathbf{C}}(\mathrm{lan}_G(F), K \circ H) \to \mathbf{B}^{\mathbf{A}}(F, H \circ (H \circ G)) \qquad \nu \mapsto (\nu * G) \circ \eta_F$$

Composing then we get a third bijection

$$\mathbf{B}^{\mathbf{D}}(\mathrm{lan}_H(\mathrm{lan}_G(F)), K) \to \mathbf{B}^{\mathbf{A}}(F, H \circ (H \circ G)) \qquad \lambda \mapsto (\lambda * (H \circ G)) \circ \big(\eta_{\mathrm{lan}_G(F)} * G\big) \circ \eta_F$$

which proves the thesis. $\square$

We are now going to show how to compute left Kan extensions via colimits.

Definition A.5.7. Let $G\colon \mathbf{A} \to \mathbf{C}$ be a functor. For every $C \in \mathbf{C}$, the category G/C has as objects pairs (A, g) made by $A \in \mathbf{A}$ and $g\colon G(A) \to C$, while an arrow $h\colon (A, g) \to (A', g')$ is an arrow $h\colon A \to A'$ in $\mathbf{A}$ such that the triangle below commutes.

$$\begin{tikzcd} G(A) \arrow[rr, "G(h)"] \arrow[dr, "g"'] & & G(A') \arrow[dl, "g'"] \\ & C & \end{tikzcd}$$

Remark A.5.8. Let $\delta_C\colon \mathbf{1} \to \mathbf{C}$ the functor picking the object C, as in Proposition A.3.5, we can define functors $F\colon G{\downarrow}\delta_C \to G/C$ and $G\colon G/C \to G{\downarrow}\delta_C$ as follows.

$$\begin{array}{ccc} (Y, *, f) & \longmapsto & f \\ {\scriptstyle (h,\mathsf{id}_*)}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ (Y', *, g) & \longmapsto & g \end{array} \qquad \begin{array}{ccc} f & \longmapsto & (\mathsf{dom}(f), *, f) \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle (h,\mathsf{id}_*)} \\ g & \longmapsto & (\mathsf{dom}(g), *, g) \end{array}$$

giving us an isomorphism between $G{\downarrow}\delta_C$ and G/C.

Now, we have a forgetful functor $V_C\colon G/C \to \mathbf{A}$ defined by

$$\begin{array}{ccc} (A, g) & \longmapsto & A \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ (A', g') & \longmapsto & A' \end{array}$$

If $F\colon \mathbf{A} \to \mathbf{B}$ is any other functor, we will denote by $V_{C,F}$ the composition $F \circ V$.

Proposition A.5.9. *Let $F\colon \mathbf{A} \to \mathbf{B}$ and $G\colon \mathbf{A} \to \mathbf{C}$ be two functors such that $V_{C,F}$ has a colimiting cocone $\Big(B_C, \big\{j_{(A,g)}\big\}_{(A,g)\in G/C}\Big)$ for every $C \in \mathbf{C}$. Then F has a left Kan extension along G, such that*

$$\mathrm{lan}_G(F)(C) = B_C \qquad \eta_{F,A} = j_{(A,\mathsf{id}_{G(A)})}$$

Proof. Let $f\colon C \to C'$ be an arrow of $\mathbf{C}$. Then we can define $\mathrm{lan}_G(F)(f)\colon B_C \to B_{C'}$ as the unique arrow fitting in the diagram below

$$\begin{tikzcd} & F(A) \arrow[dl, "j_{(A,g)}"'] \arrow[dr, "j_{(A,f\circ g)}"] & \\ B_C \arrow[rr, dotted, "\mathrm{lan}_G(F)(f)"'] & & B_{C'} \end{tikzcd}$$

Clearly $\mathrm{lan}_G(F)(f\mathsf{id}_C) = \mathsf{id}_{B_C}$, moreover, if $f'\colon C' \to C''$

$$\begin{aligned} \mathrm{lan}_G(F)(f' \circ f) \circ j_{(A,g)} &= j_{(A,f'\circ f\circ g)} \\ &= \mathrm{lan}_G(F)(f') \circ j_{(A,f\circ g)} \\ &= \mathrm{lan}_G(F)(f') \circ \mathrm{lan}_G(F)(f) \circ j_{(A,g)} \end{aligned}$$

showing that we have built a functor $\mathrm{lan}_G(F)\colon \mathbf{C} \to \mathbf{B}$. Moreover, given $f\colon A \to A'$, if we take $\eta_{F,A}$ to be $j_{(A,\mathsf{id}_{G(A)})}$, then we have

$$\begin{aligned} \mathrm{lan}_G(F)(G(f)) \circ \eta_{F,A} &= \mathrm{lan}_G(F)(G(f)) \circ j_{(A,\mathsf{id}_{G(A)})} \\ &= j_{A,G(f)\circ\mathsf{id}_{G(A)}} \\ &= j_{(A,G(f))} \\ &= j_{(A',\mathsf{id}_{G(A')})} \circ F(f) \\ &= \eta_{F,A'} \circ F(f) \end{aligned}$$

showing the existence of $\eta_F\colon F \to \mathrm{lan}_G(F) \circ G$. Now let λ be any other natural transformation $F \to H \circ G$. For every (A,g) in G/C we can define an arrow $F(A) \to H(C)$ taking the composition $H(g)\circ\lambda_A$. Given $h\colon (A,g) \to (A',g')$ we have

$$\begin{aligned} H(g') \circ \lambda_{A'} \circ F(h) &= H(g') \circ H(G(h)) \circ \lambda_A \\ &= H(g' \circ G(h)) \circ \lambda_A \\ &= H(g) \circ \lambda_A \end{aligned}$$

showing that $\left(H(C), \{H(g) \circ \lambda_A\}_{(A,g)\in G/C}\right)$ is a cocone on $V_{C,F}$. Let $\overline{\lambda}_C$ be the induced arrow $B_C \to H(C)$. Given $f\colon C \to C'$, for every $(A,g) \in G/C$ we have

$$\begin{aligned} H(f) \circ \overline{\lambda}_C \circ j_{(A,g)} &= H(f) \circ H(g) \circ \lambda_A \\ &= H(f \circ g) \circ \lambda_A \\ &= \overline{\lambda}_{C'} \circ j_{(A,f\circ g)} \\ &= \overline{\lambda}_{C'} \circ \mathrm{lan}_G(F)(f) j_{(A,g)} \end{aligned}$$

and thus we have a natural transformation $\overline{\lambda}\colon \mathrm{lan}_G(F) \to H$. By construction, we also have

$$\begin{aligned} \overline{\lambda}_{G(A)} \circ j_{A,\mathsf{id}_{G(A)}} &= H(\mathsf{id}_{G(A)}) \circ \lambda_A \\ &= \lambda_A \end{aligned}$$

On the other hand, if γ is another natural transformation $\mathrm{lan}_G(F) \to H$ such that

$$\lambda_A = \gamma_{G(A)} \circ j_{A,\mathrm{id}_{G(A)}}$$

then, for every object (A, g) of G/C, we must have

$$\begin{aligned}\gamma_C \circ j_{(A,g)} &= \gamma_C \circ \mathrm{lan}_G(F)(g) \circ j_{(A,\mathrm{id}_{G(A)})} \\ &= H(g) \circ \gamma_{G(A)} \circ j_{A,\mathrm{id}_{G(A)}} \\ &= H(g) \circ \lambda_A \\ &= \overline{\lambda}_C \circ j_{(A,g)}\end{aligned}$$

Therefore we can conclude that $\gamma = \overline{\lambda}$, from which the thesis follows. □

Remark A.5.4 and Proposition A.5.9 now yield at once the following result.

Corollary A.5.10. *Let $F\colon \mathbf{A} \to \mathbf{B}$ and $G\colon \mathbf{A} \to \mathbf{C}$ be two functors. If $\mathbf{A}$ is essentially small and $\mathbf{B}$ is cocomplete, then for every object C of $\mathbf{C}$, $\left(\mathrm{lan}_G(F)(C), \{\mathrm{lan}_G(F)(g) \circ \eta_{F,A}\}_{(A,g)\in G/C}\right)$ is a colimiting cocone for the functor $V_{C,F}\colon G/C \to \mathbf{B}$.*

Let $F\colon \mathbf{A} \to \mathbf{B}$ be a functor with a cocomplete codomain, and suppose that $G\colon \mathbf{A} \to \mathbf{C}$ is another functor such that a left Kan extension $(\mathrm{lan}_G(F), \eta_F)$ exists. For every $C \in \mathbf{C}$ we can define a functor $T_C\colon \mathbf{A}^{op} \times \mathbf{A} \to \mathbf{B}$ in the following way. A pair (A, A') is sent to $\mathbf{C}(G(A), C) \bullet F(A')$, while the image of $f_1\colon A_1' \to A_1$ and $f_2\colon A_2 \to A_2'$ is the unique arrow fitting in the diagram below.

$$\begin{tikzcd} F(A_2) \arrow[r, "F(f_2)"] \arrow[d, "\iota_g"'] & F(A_2') \arrow[d, "\iota'_{g\circ G(f_1)}"] \\ \mathbf{C}(G(A_1), C) \bullet F(A_2) \arrow[r, dotted, "T_C(f_1,f_2)"'] & \mathbf{C}(G(A_1'), C) \bullet F(A_2') \end{tikzcd}$$

where $\iota_g\colon F(A_2) \to \mathbf{C}(G(A_1), C) \bullet F(A_2)$ and $\iota'_{g\circ G(f_1)}\colon F(A_2') \to \mathbf{C}(G(A_1'), C) \bullet F(A_2')$ are the coprojections corresponding to, respectively, $g\colon G(A_1) \to C$ and $g \circ G(f_1)\colon G(A_1') \to C$.

Using Proposition A.5.9 we can now establish a link between left Kan extension and coends.

Theorem A.5.11. *Let $F\colon \mathbf{A} \to \mathbf{B}$ and $G\colon \mathbf{A} \to \mathbf{C}$ be two functors and suppose that $\mathbf{B}$ is cocomplete. If a coend ω for T_C exists, then $\left(\int^{A\in\mathbf{A}} T_C(A, A), \{\omega_A \circ k_{(A,g)}\}_{(A,g)\in G/C}\right)$ is a colimiting cocone for $V_{C,F}$, where $k_{(A,g)}$ is the coprojection $F(A) \to T_C(A, A)$. In particular, there is a left Kan extension of F along G such that*

$$\mathrm{lan}_G(F)(C) = \int^{A\in\mathbf{A}} \mathbf{C}(G(A), C) \bullet F(A) \qquad \eta_{F,A} = \omega_A \circ k_{(A,\mathrm{id}_{G(A)})}$$

Proof. Let us start showing that $\left(\int^{A\in\mathbf{A}} T_C(A, A), \{\omega_A \circ k_{(A,g)}\}_{(A,g)\in G/C}\right)$ is a cocone on $V_{C,F}$. This follows at once noticing that, for every $h\colon (A', g') \to (A', g)$, we have a commutative diagram

$$\begin{tikzcd}
& \mathbf{C}(G(A'),C) \bullet F(A') \arrow[dr, bend left, "\omega_{A'}"] & \\
F(A') \arrow[ur, bend left, "k_{(A',g')}"] \arrow[r, "\iota_g"'] \arrow[d, "F(h)"'] & \mathbf{C}(G(A),C) \bullet F(A') \arrow[u, "T_C(h,\mathrm{id}_{A'})"'] \arrow[d, "T_C(\mathrm{id}_A,h)"'] & \int^{A\in\mathbf{A}} \mathbf{C}(G(A),C) \bullet F(A) \\
F(A) \arrow[r, "k_g"'] & \mathbf{C}(G(A),C) \bullet F(A) \arrow[ur, bend right, "\omega_A"'] &
\end{tikzcd}$$

Now let $\left(B, \left\{b_{(A,g)}\right\}_{(A,g)\in G(C)}\right)$ be another cocone, then for every $A \in \mathbf{A}$ we can define $\omega'_A\colon \mathbf{C}(G(A),C)\bullet F(A)$ as the unique arrow such that

$$b_{(A,g)} = \omega'_A \circ k_{(A,g)}$$

To see that $\{\omega'_A\}_{A\in\mathbf{A}}$ is indeed a cowedge ω' for T_C it is enough to notice that the diagram below commutes

$$\begin{tikzcd}
& F(A') \arrow[dl, bend right, "\iota_g"'] \arrow[ddl, "k_{(A',g\circ G(f))}"'] \arrow[dd, bend right, "b_{A',g\circ G(f)}"'] \arrow[d, "F(f)"] \arrow[dr, bend left, "\iota_g"] & \\
\mathbf{C}(G(A),C) \bullet F(A') \arrow[d, "T_C(f,\mathrm{id}_{A'})"'] & F(A) \arrow[d, "b_{(A,g)}"] \arrow[dr, "k(A,g)"] & \mathbf{C}(G(A),C) \bullet F(A') \arrow[d, "T_C(\mathrm{id}_A,f)"] \\
\mathbf{C}(G(A'),C) \bullet F(A') \arrow[r, "\omega'_{A'}"'] & B & \mathbf{C}(G(A),C) \bullet F(A) \arrow[l, "\omega'_A"]
\end{tikzcd}$$

Then we know that there exists a unique $f\colon \int^{A\in\mathbf{A}} T_C(A,A) \to B$ such that

$$\begin{aligned} f \circ \omega_A \circ k_{(A,g)} &= \omega'_A \circ k_{(A,g)} \\ &= b_{(A,g)} \end{aligned}$$

which is precisely the thesis. $\square$

We want to proceed in the other direction. Take F and G as before and suppose that a left Kan extension $(\mathrm{lan}_G(F), \eta_F)$ of F along G exists. Using η_F we can build a cowedge on T_C: for every $A \in \mathbf{A}$, define $\omega_A\colon T_C(A,A) \to \mathrm{lan}_G(F)(C)$ as the unique arrow filling the diagram

$$\begin{tikzcd}
F(A) \arrow[r, "k_{(A,g)}"] \arrow[d, "\eta_{F,A}"'] & \mathbf{C}(G(A),C) \bullet F(A) \arrow[d, dotted, "\omega_A"] \\
\mathrm{lan}_G(F)(G(A)) \arrow[r, "\mathrm{lan}_G(F)(g)"'] & \mathrm{lan}_G(F)(C)
\end{tikzcd}$$

To see that in this way we get a cowedge, let f be an arrow $A' \to A$ in $\mathbf{A}$, then it is enough to notice that, for every $g\colon G(A) \to C$ the following diagram commutes.

$$\begin{tikzcd}
 & & F(A') \arrow[dll, bend right, "\iota_g"'] \arrow[ddll, bend right, "k_{(A',g\circ G(f))}"] \arrow[ddl, "\eta_{F,A'}"'] \arrow[ddr, "F(f)"] \arrow[drr, bend left, "\iota_g"] & & \\
\mathbf{C}(G(A),C)\bullet F(A') \arrow[dd, "T_C(f,\mathsf{id}_{A'})"'] & & & & \mathbf{C}(G(A),C)\bullet F(A') \arrow[dd, "T_C(\mathsf{id}_A,f)"] \\
 & \operatorname{lan}_G(F)(G(A')) \arrow[dr, "\operatorname{lan}_G(F)(G(f))"] \arrow[ddr, bend right, "\operatorname{lan}_G(F)(g\circ G(f))"'] & & F(A) \arrow[dl, "\eta_{F,A}"] \arrow[dr, "k(A,g)"] & \\
\mathbf{C}(G(A'),C)\bullet F(A') \arrow[drr, bend right, "\omega_{A'}"'] & & \operatorname{lan}_G(F)(G(A)) \arrow[d, "\operatorname{lan}_G(F)(g)"] & & \mathbf{C}(G(A),C)\bullet F(A) \arrow[dll, bend left, "\omega_A"] \\
 & & \operatorname{lan}_G(F)(C) & &
\end{tikzcd}$$

Theorem A.5.12. *Let $F\colon \mathbf{A} \to \mathbf{B}$ be a functor with a cocomplete codomain, if $G\colon \mathbf{A} \to \mathbf{C}$ is any functor such that a left Kan extension $(\operatorname{lan}_G(F), \eta_F)$ exists, then for every $C \in \mathbf{C}$ the cowedge $\{\omega_A\}_{A\in\mathbf{A}}$ defined above is a coend for the functor T_C.*

Proof. We have to show that $\{\omega_A\}_{A\in\mathbf{A}}$ is initial in $\mathsf{cwd}(T_C)$. Let $\{\omega'_A\}_{A\in\mathbf{A}}$ be a cowedge for T_C and denote by B the common codomain of each ω'_A. Now, given a morphism $h\colon (A', g') \to (A, g)$ in G/C, if $k_{(A,g)}$ is the coprojection $F(A) \to T_C(A, A)$, then we have a diagram

$$\begin{tikzcd}
 & \mathbf{C}(G(A'),C)\bullet F(A') \arrow[dr, bend left, "\omega'_{A'}"] & \\
F(A') \arrow[ur, bend left, "k_{(A',g')}"] \arrow[r, "\iota_g"'] \arrow[d, "F(h)"'] & \mathbf{C}(G(A),C)\bullet F(A') \arrow[u, "T_C(h,\mathsf{id}_{A'})"'] \arrow[d, "T_C(\mathsf{id}_A,h)"'] & B \\
F(A) \arrow[r, "k_g"'] & \mathbf{C}(G(A),C)\bullet F(A) \arrow[ur, bend right, "\omega'_A"'] &
\end{tikzcd}$$

which shows that $\left(B, \{\omega'_A \circ k_{(A,g)}\}_{(A,g)\in G(C)}\right)$ is a cocone on $V_{C,F}$. By Corollary A.5.10, there exists a unique $f\colon \operatorname{lan}_G(F)(C) \to B$ such that

$$\begin{aligned}
\omega'_A \circ k_{(A,g)} &= f \circ \operatorname{lan}_G(F)(g) \circ \eta_{F,A} \\
&= f \circ \omega_A \circ k_{(A,g)}
\end{aligned}$$

and the thesis now follows. □

We can sum up the results contained in Theorem A.5.12 and Theorem A.5.11 to get the following.

Corollary A.5.13. *Let $\mathbf{A}$ be an essentially small category and $\mathbf{B}$ a cocomplete one. Given two functors $F\colon \mathbf{A} \to \mathbf{B}$ and $G\colon \mathbf{A} \to \mathbf{C}$, if $(\operatorname{lan}_G(F), \eta_F)$ is a left Kan extension of F along G, then*

$$\operatorname{lan}_G(F) \simeq \int^{A\in\mathbf{A}} \mathbf{C}(G(A), -) \bullet F(A)$$

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