A COMPLETE SOLUTION OF MARKOV'S PROBLEM ON CONNECTED GROUP TOPOLOGIES

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Abstract. Every proper closed subgroup of a connected Hausdorff group must have index at least $\mathfrak{c}$, the cardinality of the continuum. 70 years ago Markov conjectured that a group G can be equipped with a connected Hausdorff group topology provided that every subgroup of G which is closed in *all* Hausdorff group topologies on G has index at least $\mathfrak{c}$. Counterexamples in the non-abelian case were provided 25 years ago by Pestov and Remus, yet the problem whether Markov's Conjecture holds for abelian groups G remained open. We resolve this problem in the positive.

As usual, $\mathbb{Z}$ denotes the group of integers, $\mathbb{Z}(n)$ denotes the cyclic group of order n, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{P}$ denotes the set of all prime numbers and $\mathfrak{c}$ denotes the cardinality of the continuum.

Let G be a group. For a cardinal σ, we use $G^{(\sigma)}$ to denote the direct sum of σ many copies of the group G. For $m \in \mathbb{N}$, we let

$$mG = \{mg : g \in G\} \quad \text{and} \quad G[m] = \{g \in G : mg = e\},$$

where e is the identity element of G. When G is abelian, we use 0 instead of e. A group G is *bounded* (or has *finite exponent*) if $mG = \{e\}$ for some integer $m \geq 1$; otherwise, G is said to be *unbounded*. We denote by

$$t(G) = \bigcup_{m \in \mathbb{N}} G[m] \tag{1}$$

the *torsion part* of G. The group G is *torsion* if $t(G) = G$. When G is abelian, the torsion part $t(G)$ of G is a subgroup of G called the *torsion subgroup* of G.

As usual, we write $G \cong H$ when groups G and H are isomorphic.

All topological groups and all group topologies are assumed to be Hausdorff.

1. Markov's problem for abelian groups

Markov [21, 22] says that subset X of a group G is *unconditionally closed* in G if X is closed in every Hausdorff group topology on G.

Every proper closed subgroup of a connected group must have index at least $\mathfrak{c}$.[1] Therefore, if a group admits a connected group topology, then all its proper unconditionally closed

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2010 *Mathematics Subject Classification.* Primary: 22A05; Secondary: 20K25, 20K45, 54A25, 54D05, 54H11.

Key words and phrases. (locally) connected group, (locally) pathwise connected group, unconditionally closed subgroup, Markov's problem, Hartman-Mycielski construction, Ulm-Kaplanski invariants.

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[1]Indeed, if H is a proper closed subgroup of a connected group G, the the quotient space G/H is non-trivial, connected and completely regular. Since completely regular spaces of size less than $\mathfrak{c}$ are disconnected, this shows that $|G/H| \geq \mathfrak{c}$.

subgroups necessarily have index at least $\mathfrak{c}$; see [23]. Markov [23, Problem 5, p. 271] asked if the converse is also true.

Problem 1.1. If all proper unconditionally closed subgroups of a group G have index at least $\mathfrak{c}$, does then G admit a connected group topology?

Definition 1.2. For brevity, a group satisfying Markov's condition, namely having all proper unconditionally closed subgroups of index at least $\mathfrak{c}$, shall be called an *M-group* (the abbreviation for *Markov group*).

Adopting this terminology, Markov's Problem 1.1 reads: *Does every M-group admit a connected group topology?*

A negative answer to Problem 1.1 was provided first by Pestov [24], then a much simpler counter-example was found by Remus [25]. Nevertheless, the question remained open in the abelian case, as explicitly stated in [25] and later in [2, Question 3.5.3] as well as in [1, Question 3G.1 (Question 517)].

Question 1.3. Let G be an abelian group all proper unconditionally closed subgroups of which have index at least $\mathfrak{c}$. Does G have a connected group topology?

Equivalent re-formulation of this question using the notion of an M-group reads: *Does every abelian M-group admit a connected group topology?*

Since the notion of an unconditionally closed subset of a group G involves checking closedness of this set in *every* Hausdorff group topology on G, in practice it is hard to decide if a given subgroup of G is unconditionally closed in G or not. In other words, there is no clear procedure that would allow one to check whether a given group is an M-group. Our next proposition provides an easy algorithm for verifying whether an abelian group is an M-group. Its proof, albeit very short, is based on the fundamental fact that all unconditionally closed subsets of abelian groups are algebraic [10, Corollary 5.7].

Proposition 1.4. *An abelian group G is an M-group if and only if, for every $m \in \mathbb{N}$, either $mG = \{0\}$ or $|mG| \geq \mathfrak{c}$. In particular, an abelian group G of infinite exponent is an M-group precisely when $|mG| \geq \mathfrak{c}$ for all integers $m \geq 1$.*

Proof. According to [10, Corollary 5.7] and [11, Lemma 3.3], a proper unconditionally closed subgroup H of G has the form $H = G[m]$ for some $m > 0$. Since $G/G[m] \cong mG$, the index of H coincides with $|G/H| = |G/G[m]| = |mG|$. □

According to the Prüfer theorem [15, Theorem 17.2], a non-trivial abelian group G of finite exponent is a direct sum of cyclic subgroups

$$G = \bigoplus_{p\in\pi(G)} \bigoplus_{i=1}^{m_p} \mathbb{Z}(p^i)^{(\alpha_{p,i})}, \tag{2}$$

where $\pi(G)$ is a non-empty finite set of primes and the cardinals $\alpha_{p,i}$ are known as *Ulm-Kaplanski invariants* of G. Note that while some of them may be equal to zero, the cardinals α_{p,m_p} must be positive; they are called *leading Ulm-Kaplanski invariants* of G.

Based on Proposition 1.4, one can re-formulate the Markov property for abelian groups of finite exponent in terms of their Ulm-Kaplanski invariants:

Proposition 1.5. *A non-trivial abelian group G of finite exponent is an M-group if and only if all leading Ulm-Kaplanski invariants of G are at least $\mathfrak{c}$.*

Proof. Write the group G as in (2) and let $k = \prod_{q\in\pi(G)} q^{m_q}$ be the exponent of G. In order to compute the leading Ulm-Kaplanski invariant α_{p,m_p} for $p \in \pi(G)$, let

$$k_p = \frac{k}{p} = p^{m_p-1} \cdot \prod_{q\in\pi(G)\setminus\{p\}} q^{m_q}.$$

Then $k_pG \cong \mathbb{Z}(p)^{(\alpha_{p,m_p})}$, so

$$1 < |k_pG| = \begin{cases} p^{\alpha_{p,m_p}} & \text{if } \alpha_{p,m_p} \text{ is finite} \\ \alpha_{p,m_p} & \text{if } \alpha_{p,m_p} \text{ is infinite.} \end{cases} \tag{3}$$

If G is an M-group, then $|k_pG| \geq \mathfrak{c}$ by (3) and Proposition 1.4, so $\alpha_{p,m_p} \geq \mathfrak{c}$ as well.

Now suppose that all leading Ulm-Kaplanski invariants of G are at least $\mathfrak{c}$. Fix $m \in \mathbb{N}$ with $|mG| > 1$. There exists at least one $p \in \pi(G)$ such that p^{m_p} does not divide m. Let d be the greatest common divisor of m and k. Then $mG = dG$, hence from now on we can assume without loss of generality that $m = d$ divides k. As p^{m_p} does not divide m, it follows that m divides k_p. Therefore, k_pG is a subgroup of mG, and so $|mG| \geq |k_pG| = \alpha_{p,m_p} \geq \mathfrak{c}$ by (3). This shows that, for every $m \in \mathbb{N}$, either $mG = \{0\}$ or $|mG| \geq \mathfrak{c}$. Therefore, G is an M-group by Proposition 1.4. □

Kirku [20] characterized the abelian groups of finite exponent admitting connected group topology.

Theorem 1.6. [20, Theorem on p. 71] *If all leading Ulm-Kaplanski invariants of a non-trivial abelian group G of finite exponent are at least $\mathfrak{c}$, then G admits a pathwise connected, locally pathwise connected group topology.*

In view of Proposition 1.5, Kirku's theorem can be considered as a partial positive answer to Question 1.3.

Corollary 1.7. *Each abelian M-group of finite exponent admits a pathwise connected, locally pathwise connected group topology.*

Proof. Obviously, the only group topology on the trivial group is both pathwise connected and locally pathwise connected. For a non-trivial abelian group of finite exponent, the conclusion follows from Proposition 1.5 and Theorem 1.6. □

The main goal of this paper is to offer a positive solution to Markov's problem in the case complementary to Corollary 1.7, namely, for the class of abelian groups of infinite exponent. In order to formulate our main result, we shall need to introduce an intermediate property between pathwise connectedness and connectedness.

Definition 1.8. We shall say that a space X is *densely pathwise connected* (abbreviated to *dp-connected*) provided that X has a dense pathwise connected subspace.

Clearly,

$$\text{pathwise connected} \rightarrow \text{dp-connected} \rightarrow \text{connected.} \tag{4}$$

Obviously, a topological group G is dp-connected if and only if the arc component of the identity of G is dense in G.

As usual, given a topological property $\mathscr{P}$, we say that a space X is *locally* $\mathscr{P}$ provided that for every point $x \in X$ and each open neighbourhood U of x one can find an open neighbourhood V of x such that the closure of V in X is contained in U and has property $\mathscr{P}$. It follows from this definition and (4) that

$$\text{locally pathwise connected} \ \to \ \text{locally dp-connected} \ \to \ \text{locally connected}.$$

Theorem 1.9. *Every abelian M-group G of infinite exponent admits a dp-connected, locally dp-connected group topology.*

The proof of this theorem is postponed until Section 6. It is worth pointing out that not only the setting of Theorem 1.9 is complimentary to that of Corollary 1.7 but also the proof of Theorem 1.9 itself makes no recourse to Corollary 1.7.

The equivalence of items (i) and (ii) in our next corollary offers a complete solution of Question 1.3.

Corollary 1.10. *For an abelian group G, the following conditions are equivalent:*

(i) *G is an M-group;*
(ii) *G admits a connected group topology;*
(iii) *G admits a dp-connected, locally dp-connected group topology.*

A classical theorem of Eilenberg and Pontryagin says that every connected locally compact group has a dense arc component [26, Theorem 39.4(d)], so it is dp-connected. In [16], one can find examples of pseudocompact connected abelian groups without non-trivial convergent sequences, which obviously implies that their arc components are trivial (and so such groups are very far from being dp-connected and locally dp-connected). Under CH, one can find even a countably compact group with the same property [28]. (A rich supply of such groups can also be found in [9].) Our next corollary is interesting in light of these results.

Corollary 1.11. *If an abelian group admits a connected group topology, then it can be equipped with a group topology which is both dp-connected and locally dp-connected.*

Our last corollary characterizes subgroups H of a given abelian group G which can be realized as the connected component of some group topology on G.

Corollary 1.12. *For a subgroup H of an abelian group G the following conditions are equivalent:*

(i) *G admits a group topology $\mathscr{T}$ such that H coincides with the connected component of $(G, \mathscr{T})$;*
(ii) *H admits a connected group topology;*
(iii) *H is an M-group.*

Proof. The implications (i) $\to$ (ii) and (ii) $\to$ (iii) are clear. To check the implication (iii) $\to$ (i), assume that H is an M-group. Applying the implication (i) $\to$ (ii) of Corollary 1.10, we can find a connected group topology $\mathscr{T}_c$ on H. Let $\mathscr{T}$ be the group topology on G obtained by declaring $(H, \mathscr{T}_c)$ to be an open subgroup of $(G, \mathscr{T})$. Then H is a clopen connected subgroup of $(G, \mathscr{T})$, which implies that H coincides with the connected component of $(G, \mathscr{T})$. □

The paper is organized as follows. In Section 2 we recall the necessary background on w-divisible groups, including a criterion for dense embeddings into powers of $\mathbb{T}$. In Section 3 we show that uncountable w-divisible groups can be characterized by a property involving large direct sums of unbounded groups; see Theorem 3.6. In Section 4 we use this theorem to show that every w-divisible group of size at least $\mathfrak{c}$ contains a $\mathfrak{c}$-homogeneous w-divisible subgroup of the same size; see Lemma 4.5. (A group G is $\mathfrak{c}$-homogeneous if $G \cong G^{(\mathfrak{c})}$.) The importance of the notion of $\mathfrak{c}$-homogeneity becomes evident in Section 5, where we recall the classical construction, due to Hartman and Mycielski [18], assigning to every group a pathwise connected and locally pathwise connected topological group. We establish some additional useful properties of this construction and use them to prove that every $\mathfrak{c}$-homogeneous group admits a pathwise connected, locally pathwise connected group topology; see Corollary 5.3. In Section 6, we combine Lemma 4.5, Corollary 5.3 and dense embeddings of w-divisible groups into powers of $\mathbb{T}$ to obtain the proof of Theorem 1.9. Finally, in Section 7 we introduce a new cardinal invariant useful in obtaining a characterization of countable groups that contain infinite direct sums of unbounded groups, covering the case left open in Theorem 3.6.

2. Background on w-divisible groups

We recall here two fundamental notions from [4].

Definition 2.1. [4] An abelian group G is called *w-divisible* if $|mG| = |G|$ for all integers $m \geq 1$.

Definition 2.2. [4] For an abelian group G, the cardinal

$$w_d(G) = \min\{|nG| : n \in \mathbb{N} \setminus \{0\}\} \tag{5}$$

is called the *divisible weight* of G.[2]

Obviously, $w_d(G) \leq |G|$, and G is w-divisible precisely when $w_d(G) = |G|$. Furthermore, we mention here the following easy fact for future reference:

Fact 2.3. [4, p. 255] *If m is a positive integer such that $w_d(G) = |mG|$, then mG is a w-divisible subgroup of G.*

Using the cardinal $w_d(G)$, the last part of Proposition 1.4 can be re-stated as follows:

Fact 2.4. *An abelian group G of infinite exponent is an M-group if and only if $w_d(G) \geq \mathfrak{c}$.*

Fact 2.5. [4, Claim 3.6] *Let n be a positive integer, let $G_1, G_2, \ldots, G_n$ be abelian groups, and let $G = \bigoplus_{i=1}^n G_i$.*

(i) $w_d(G) = \max\{w_d(G_i) : i = 1, 2, \ldots, n\}$.
(ii) *G is w-divisible if and only if there exists index $i = 1, 2, \ldots, n$ such that $|G| = |G_i|$ and G_i is w-divisible.*

The abundance of w-divisible groups is witnessed by the following

Fact 2.6. [16, Lemma 4.1] *Every abelian group G admits a decomposition $G = K \oplus M$, where K is a bounded torsion group and M is a w-divisible group.*

An alternative proof of this fact can be found in [5]; a particular case is contained already in [4].

The next fact describes dense subgroups of $\mathbb{T}^\kappa$ in terms of the invariant $w_d(G)$. (Recall that, for every cardinal κ, $\log \kappa = \min\{\sigma : \kappa \leq 2^\sigma\}$.)

[2] This is a "discrete case" of a more general definition given in [4] involving the weight of a topological group, which explains the appearance of the word "weight" in the term.

Fact 2.7. *For every cardinal $\kappa \geq \mathfrak{c}$, an abelian group G is isomorphic to a dense subgroup of $\mathbb{T}^\kappa$ if and only if $\log \kappa \leq w_d(G) \leq |G| \leq 2^\kappa$.*

Proof. For $\kappa = \mathfrak{c}$, this is [13, Corollary 3.3], and for $\kappa > \mathfrak{c}$ this follows from the equivalence of items (i) and (ii) in [12, Theorem 2.6] and (5). □

Corollary 2.8. *Let τ be an infinite cardinal. For every abelian group G satisfying $\tau \leq w_d(G) \leq |G| \leq 2^{2^\tau}$, there exists a monomorphism $\pi : G \to \mathbb{T}^{2^\tau}$ such that $\pi(G)$ is dense in $\mathbb{T}^{2^\tau}$.*

Proof. Let $\kappa = 2^\tau$. Then $\kappa \geq \mathfrak{c}$ and $\log \kappa \leq \tau \leq w_d(G) \leq |G| \leq 2^{2^\tau} = 2^\kappa$, so the conclusion follows from Fact 2.7. □

3. Finding large direct sums in uncountable w-divisible groups

For every $p \in \mathbb{P}$, we use $r_p(G)$ to denote the p-rank of an abelian group G [15, Section 16], while $r_0(G)$ denotes the free rank of G [15, Section 14].

Definition 3.1. Call an abelian group G *strongly unbounded* if G contains a direct sum $\bigoplus_{i \in I} A_i$ such that $|I| = |G|$ and all groups A_i are unbounded.

Remark 3.2. (i) If H is a subgroup of an abelian group G such that $|H| = |G|$ and H is strongly unbounded, then G is strongly unbounded as well.

(ii) An abelian group G satisfying $r_0(G) = |G|$ is strongly unbounded.

Lemma 3.3. *A strongly unbounded abelian group is w-divisible.*

Proof. Let G be a strongly unbounded abelian group. Then G contains a direct sum $\bigoplus_{i \in I} A_i$ as in Definition 3.1.

Let m be a positive integer. Clearly, each mA_i is non-trivial, and $mA = \bigoplus_{i \in I} mA_i$, so $|mG| \geq |mA| \geq |I| = |G|$. The converse inequality $|mG| \leq |G|$ is trivial. Therefore, G is w-divisible by Definition 2.1. □

In the rest of this section we shall prove the converse of Lemma 3.3 for uncountable groups.

Lemma 3.4. *Let p be a prime number and τ be an infinite cardinal. For every abelian p-group G of cardinality τ, the following conditions are equivalent:*

(i) *G contains a subgroup algebraically isomorphic to $L_p^{(\tau)}$, where $L_p = \bigoplus_{n \in \mathbb{N}} \mathbb{Z}(p^n)$;*
(ii) *G is strongly unbounded;*
(iii) *$r_p(p^n G) \geq \tau$ for every $n \in \mathbb{N}$;*
(iv) *$r_p(S_n) \geq \tau$ for each $n \in \mathbb{N}$, where $S_n = p^n G[p^{n+1}]$.*

Proof. Since L_p is unbounded and $|G| = \tau$, the implication (i) → (ii) follows from Definition 3.1.

(ii) → (iii) Let $n \in \mathbb{N}$ be arbitrary. Since G is strongly unbounded, G contains a direct sum $\bigoplus_{i \in I} A_i$ such that $|I| = |G| = \tau$ and all groups A_i are unbounded. Thus, the subgroup $p^n G$ of G contains the direct sum $\bigoplus_{i \in I} p^n A_i$ of non-trivial p-groups $p^n A_i$. Since $r_p(p^n A_i) \geq 1$ for every $i \in I$, it follows that $r_p(p^n G) \geq r_p(\bigoplus_{i \in I} p^n A_i) \geq |I| = \tau$.

(iii) → (iv) Note that $(p^n G)[p] = S_n$, so $r_p(p^n G) = r_p(S_n)$ for every $n \in \mathbb{N}$.

(iv) → (i) Note that each S_{n+1} is a subgroup of the group S_n, and therefore, the quotient group S_n / S_{n+1} is well-defined. We consider two cases.

Case 1. There exists a sequence $0 < m_1 < m_2 < \ldots < m_k < \ldots$ *of natural numbers such that* $|S_{m_k-1}/S_{m_k}| \geq \tau$ *for every* $k \in \mathbb{N}$. In this case for every $k \in \mathbb{N}$ one can find a subgroup V_k of S_{m_k-1} of size τ with $V_k \cap S_{m_k} = \{0\}$. Then the family $\{V_k : k \in \mathbb{N}\}$ is independent; that is, the sum $\sum_{k\in\mathbb{N}} V_k$ is direct. Since V_k is a subgroup of S_{m_k-1}, one can find a subgroup N_k of G such that $N_k \cong \mathbb{Z}(p^{m_k})^{(\tau)}$ and $N_k[p] = V_k$. Now the family $\{N_k : k \in \mathbb{N}\}$ is independent; see [9, Lemma 3.12]. The subgroup $N = \bigoplus_{k\in\mathbb{N}} N_k$ of G is isomorphic to $\bigoplus_{k\in\mathbb{N}} \mathbb{Z}(p^{m_k})^{(\tau)}$, so N contains a copy of $L_p^{(\tau)}$.

Case 2. There exists $m_0 \in \mathbb{N}$ *such that* $|S_m/S_{m+1}| < \tau$ *for all* $m \geq m_0$. Note that in this case $|S_{m_0}/S_{m_0+k}| < \tau$ for every $k \in \mathbb{N}$. Since τ is infinite and $r_p(S_{m_0}) \geq \tau$ by (iv), we can fix an independent family $\mathscr{V} = \{V_k : k \in \mathbb{N}\}$ in S_{m_0} such that $V_k \cong \mathbb{Z}(p)^{(\tau)}$ for all $k \in \mathbb{N}$. For every $k \in \mathbb{N}$ consider the subgroup $W_k = V_k \cap S_{m_0+k}$ of V_k. Since the family $\mathscr{V}$ is independent, so is the family $\mathscr{W} = \{W_k : k \in \mathbb{N}\}$.

Let $k \in \mathbb{N}$. The quotient group V_k/W_k is naturally isomorphic to a subgroup of S_{m_0}/S_{m_0+k}, and thus $|V_k/W_k| \leq |S_{m_0}/S_{m_0+k}| < \tau$. This yields $W_k \cong V_k \cong \mathbb{Z}(p)^{(\tau)}$. Since W_k is a subgroup of S_{m_0+k}, there exists a subgroup N_k of G such that $N_k \cong \mathbb{Z}(p^{m_0+k+1})^{(\tau)}$ and $N_k[p] = W_k$.

Since the family $\mathscr{W}$ is independent, the family $\mathscr{N} = \{N_k : k \in \mathbb{N}\}$ is independent too; see again [9, Lemma 3.12]. Therefore, $\mathscr{N}$ generates a subgroup $\bigoplus_{k\in\mathbb{N}} N_k$ of G isomorphic to $\bigoplus_{k\in\mathbb{N}} \mathbb{Z}(p^{m_0+k+1})^{(\tau)}$. It remains only to note that the latter group contains an isomorphic copy of the group $L_p^{(\tau)}$. □

Lemma 3.5. *An uncountable torsion w-divisible group H is strongly unbounded.*

Proof. Recall that $H = \bigoplus_{p\in\mathbb{P}} H_p$, where each H_p is a p-group [15, Theorem 8.4]. Let $\tau = |H|$ and $\sigma_p = r_p(H) = r_p(H_p)$ for all $p \in \mathbb{P}$. Since H is uncountable, $\tau = \sup_{p\in\mathbb{P}} \sigma_p$. We consider two cases.

Case 1. There exists a finite set $F \subseteq \mathbb{P}$ *such that* $\sup_{p\in\mathbb{P}\setminus F} \sigma_p < \tau$. In this case, $H = G_1 \oplus G_2$, where $G_1 = \bigoplus_{p\in F} H_p$ and $G_2 = \bigoplus_{p\in\mathbb{P}\setminus F} H_p$. Note that $|G_2| \leq \omega + \sup_{p\in\mathbb{P}\setminus F} \sigma_p < \tau = |H|$, as H is uncountable. Since $H = G_1 \oplus G_2$ is w-divisible, from Fact 2.5 (ii) we conclude that G_1 is w-divisible and $|G_1| = |H|$. This implies that the finite set F is non-empty. Applying Fact 2.5 (ii) once again, we can find $p \in F$ such that H_p is w-divisible and $|H_p| = |G_1| = |H| = \tau$.

Let $n \in \mathbb{N}$. Since H_p is a w-divisible p-group, $|p^n H_p| = |H_p| = \tau$. Since τ is uncountable and $p^n H_p$ is a p-group, $r_p(p^n H_p) = |H_p|$. Applying the implication (iii)→(ii) of Lemma 3.4, we obtain that H_p is strongly unbounded. Since $|H_p| = \tau = |H|$, the group H is strongly unbounded by Remark 3.2 (i).

Case 2. $\sup_{p\in\mathbb{P}\setminus F} \sigma_p = \tau$ *for all finite sets* $F \subseteq \mathbb{P}$. For each $p \in \mathbb{P}$ choose a p-independent subset S_p of $H_p[p]$ with $|S_p| = \sigma_p$. Let $S = \bigcup_{p\in\mathbb{P}} S_p$. One can easily see that the assumption of Case 2 implies the following property:

$$\text{(6)} \qquad \text{If } Y \subseteq S \text{ and } |Y| < \tau, \text{ then the set } \{p \in \mathbb{P} : S_p \setminus Y \neq \emptyset\} \text{ is infinite.}$$

By transfinite induction, we shall construct a family $\{X_\alpha : \alpha < \tau\}$ of pairwise disjoint countable subsets of S such that the set $P_\alpha = \{p \in \mathbb{P} : X_\alpha \cap S_p \neq \emptyset\}$ is infinite for each $\alpha < \tau$.

Basis of induction. By our assumption, infinitely many of sets S_p are non-empty, so we can choose a countable set $X_0 \subseteq S$ such that P_0 is infinite.

Inductive step. Let α be an ordinal such that $0 < \alpha < \tau$, and suppose that we have already defined a family $\{X_\beta : \beta < \alpha\}$ of pairwise disjoint countable subsets of S such that the set P_β is infinite for each $\beta < \alpha$. Since $\alpha < \tau$ and τ is uncountable, the set $Y = \bigcup\{X_\beta : \beta < \alpha\}$ satisfies $|Y| \le \omega + |\alpha| < \tau$. Using (6), we can select a countable subset X_α of $S \setminus Y$ which intersects infinitely many S_p's.

The inductive construction been complete, for every $\alpha < \tau$, let A_α be the subgroup of H generated by X_α. Since P_α is infinite, A_α contains elements of arbitrary large order, so A_α is unbounded. Since the family $\{X_\alpha : \alpha < \tau\}$ is pairwise disjoint, the sum $\sum_{\alpha<\tau} A_\alpha = \bigoplus_{\alpha<\tau} A_\alpha$ is direct. Since $|H| = \tau$, this shows that H is strongly unbounded. □

Theorem 3.6. *An uncountable abelian group is strongly unbounded if and only if it is w-divisible.*

Proof. The "only if" part is proved in Lemma 3.3. To prove the "if" part, assume that G is an uncountable w-divisible group. If $r_0(G) = |G|$, then G is strongly unbounded by Remark 3.2 (ii). Assume now that $r_0(G) < |G|$. Then $|G| = |H|$ and $|G/H| = r_0(G) \cdot \omega < |G|$, where $H = t(G)$ is the torsion subgroup of G; see (1).

Let n be a positive integer. Since $nH = H \cap nG$, the quotient group $nG/nH = nG/(H \cap nG)$ is isomorphic to a subgroup of the quotient group G/H, so $|nG/nH| \le |G/H| < |G|$. Since G is w-divisible, $|G| = |nG|$ by Definition 2.1. Now $|nG/nH| < |nG|$ implies $|nH| = |nG|$, as $|nG| = |nH| \cdot |nG/nH|$. Since $|nG| = |G| = |H|$, this yields $|nH| = |H|$. Since this equation holds for all $n \in \mathbb{N} \setminus \{0\}$, H is w-divisible by Definition 2.1.

Since $|H| = |G|$ and G is uncountable by our assumption, H is an uncountable torsion group. Applying Lemma 3.5, we conclude that H is strongly unbounded. Since $|H| = |G|$, G is also strongly unbounded by Remark 3.2 (i). □

Since every unbounded abelian group contains a countable unbounded subgroup, from Theorem 3.6 we obtain the following

Corollary 3.7. *For every uncountable abelian group G, the following conditions are equivalent:*

(i) *G is w-divisible;*
(ii) *G contains a direct sum $\bigoplus_{i\in I} A_i$ of countable unbounded groups A_i such that $|I| = |G|$.*

Remark 3.8. The assumption that G is uncountable cannot be omitted either in Theorem 3.6 or in its Corollary 3.7; that is, the converse of Lemma 3.3 does not hold for countable groups. Indeed, the group $\mathbb{Z}$ of integer numbers is w-divisible, yet it is not strongly unbounded. A precise description of countable strongly unbounded groups is given in Proposition 7.5 below; see also Theorem 7.6 that unifies both the countable and uncountable cases.

4. σ-homogeneous groups

Definition 4.1. For an infinite cardinal σ, we say that an abelian group G is *σ-homogeneous* if $G \cong G^{(\sigma)}$.

The relevance of this definition to the topic of our paper shall become clear from Corollary 5.3 below.

The straightforward proof of the next lemma is omitted.

Lemma 4.2. (i) *The trivial group is σ-homogeneous for every cardinal σ.*

(ii) *If κ, σ are cardinals with $\kappa \geq \sigma \geq \omega$ and A is an abelian group, then the group $A^{(\kappa)}$ is σ-homogeneous.*

(iii) *Let σ be an infinite cardinal, and let $\{G_i : i \in I\}$ be a non-empty family of σ-homogeneous abelian groups. Then $G = \bigoplus_{i\in I} G_i$ is σ-homogeneous.*

Lemma 4.3. *Let σ be an infinite cardinal. Every abelian bounded torsion group K admits a decomposition $K = L \oplus N$, where $|L| < \sigma$ and N is σ-homogeneous.*

Proof. If K is trivial, then one can take L and N to be the trivial group. (Note that the trivial group is σ-homogeneous for every cardinal σ.) Suppose now that K is non-trivial. Then K admits a decomposition $K = \bigoplus_{i=1}^{n} K_i^{(\alpha_i)}$, where $n \in \mathbb{N} \setminus \{0\}$, $\alpha_1, \alpha_2, \ldots, \alpha_n$ are cardinals and each K_i is isomorphic to $\mathbb{Z}(p_i^{k_i})$ for some $p_i \in \mathbb{P}$ and $k_i \in \mathbb{N}$. Let

$$I = \{i \in \{1, 2, \ldots, n\} : \alpha_i < \sigma\} \quad \text{and} \quad J = \{1, 2, \ldots, n\} \setminus I. \tag{7}$$

If $I = \emptyset$, we define L to be the trivial group. Otherwise, we let $L = \bigoplus_{i\in I} K_i^{(\alpha_i)}$. Since K_i is finite and $\alpha_i < \sigma$ for every $i \in I$, and σ is an infinite cardinal, we conclude that $|L| < \sigma$.

If $J = \emptyset$, we define N to be the trivial group and note that N is σ-homogeneous by Lemma 4.2 (i). If $J \neq \emptyset$, we let $N = \bigoplus_{j\in J} K_j^{(\alpha_j)}$. Let $i \in J$ be arbitrary. Since $\alpha_i \geq \sigma$, Lemma 4.2 (ii) implies that $K_i^{(\alpha_i)}$ is σ-homogeneous. Now Lemma 4.2 (iii) implies that N is σ-homogeneous as well.

The equality $K = L \oplus N$ follows from (7) and our definition of L and N. □

For a cardinal σ, we use σ^+ to denote the smallest cardinal bigger than σ.

Lemma 4.4. *Let σ be an infinite cardinal. Every abelian group G satisfying $w_d(G) \geq \sigma$ admits a decomposition $G = N \oplus H$ such that N is a bounded σ^+-homogeneous group and H is a w-divisible group with $|H| = w_d(G)$.*

Proof. Let $G = K \oplus M$ be the decomposition as in Fact 2.6. Use (5) to fix an integer $n \geq 1$ such that $w_d(G) = |nG|$. Since K is a bounded group, $mK = \{0\}$ for some positive integer m. Now $nmG = nmK \oplus nmM = nmM$, so $|nmG| = |mnM| = |M|$, as M is w-divisible. Since $nmG \subseteq nG$, from the choice of n and (5), we get $w_d(G) \leq |nmG| \leq |nG| = w_d(G)$. This shows that $w_d(G) = |M|$.

By Lemma 4.3, there exist a subgroup L of K satisfying $|L| \leq \sigma$ and a σ^+-homogeneous subgroup N of K such that $K = L \oplus N$. Since K is bounded, so is N. Clearly, $G = K \oplus M = L \oplus N \oplus M = N \oplus H$, where $H = L \oplus M$. Since $|M| = w_d(G) \geq \sigma \geq |L|$ and σ is infinite, $|H| = |M|$. In particular, $|H| = w_d(G)$. Since $H = L \oplus M$, $|H| = |M|$ and M is w-divisible, H is w-divisible by Fact 2.5 (ii). □

Lemma 4.5. *Every w-divisible abelian group G of cardinality at least $\mathfrak{c}$ contains a $\mathfrak{c}$-homogeneous w-divisible subgroup H such that $|H| = |G|$.*

Proof. By Corollary 3.7, G contains a subgroup

$$A = \bigoplus_{i\in I} A_i, \tag{8}$$

where $|I| = |G|$ and all A_i are countable and unbounded. In particular, $|A| = |I| = |G|$. By the same corollary, A itself is w-divisible.

Case 1. $|G| > \mathfrak{c}$. Note that there are at most $\mathfrak{c}$-many countable groups, so the sum (8) can be rewritten as

$$A = \bigoplus_{j \in J} A_j^{(\kappa_j)} \tag{9}$$

for a subset J of I with $|J| \leq \mathfrak{c}$ and a suitable family $\{\kappa_j : j \in J\}$ of cardinals such that $|A| = \sup_{j \in J} \kappa_j$. Define $S = \{j \in J : \kappa_j < \mathfrak{c}\}$. Then (9) becomes

$$A = \bigoplus_{j \in J} A_j^{(\kappa_j)} = \left(\bigoplus_{j \in S} A_j^{(\kappa_j)} \right) \oplus \left(\bigoplus_{j \in J \setminus S} A_j^{(\kappa_j)} \right). \tag{10}$$

Since $\kappa_j < \mathfrak{c}$ for all $j \in S$ and $|S| \leq |J| \leq \mathfrak{c}$, we have $\left| \bigoplus_{j \in S} A_j^{(\kappa_j)} \right| \leq \mathfrak{c}$. Since $|A| = |G| > \mathfrak{c}$, equation (10) implies that $|G| = |H|$, where

$$H = \bigoplus_{j \in J \setminus S} A_j^{(\kappa_j)}. \tag{11}$$

Let $j \in J \setminus S$ be arbitrary. Since $\kappa_j \geq \mathfrak{c}$ by the choice of S, $A_j^{(\kappa_j)}$ is $\mathfrak{c}$-homogeneous by Lemma 4.2 (ii). From this, (11) and Lemma 4.2 (iii), we conclude that H is $\mathfrak{c}$-homogeneous as well.

From (11), we conclude that H is a direct sum of $|H|$-many unbounded groups, so H is w-divisible by Corollary 3.7.

Case 2. $|G| = \mathfrak{c}$. For any non-empty set $P \subseteq \mathbb{P}$ let

$$\mathrm{Soc}_P(\mathbb{T}) = \bigoplus_{p \in P} \mathbb{Z}(p). \tag{12}$$

For every prime $p \in \mathbb{P}$ let $L_p = \bigoplus_{n \in \mathbb{N}} \mathbb{Z}(p^n)$.

It is not hard to realize that every unbounded abelian group contains a subgroup isomorphic to one of the groups from the fixed family $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ of unbounded groups, where

$$\mathcal{S}_1 = \{\mathbb{Z}\} \cup \{\mathbb{Z}(p^\infty) : p \in \mathbb{P}\} \cup \{L_p : p \in \mathbb{P}\} \quad \text{and} \quad \mathcal{S}_2 = \{\mathrm{Soc}_P(\mathbb{T}) : P \in [\mathbb{P}]^\omega\}. \tag{13}$$

(Here $[\mathbb{P}]^\omega$ denotes the set of all infinite subsets of $\mathbb{P}$.)

It is not restrictive to assume that, in the decomposition (8), $A_i \in \mathcal{S}$ for each $i \in I$. For $l = 1, 2$ define $I_l = \{i \in I : A_i \in \mathcal{S}_l\}$. We consider two subcases.

Subcase A. $|I_1| = \mathfrak{c}$. For every $N \in \mathcal{S}_1$, define $E_N = \{i \in I_1 : A_i \cong N\}$. Then $I_1 = \bigcup_{N \in \mathcal{S}_1} E_N$. Since the family $\mathcal{S}_1$ is countable, $|I_1| = \mathfrak{c}$ and $\mathrm{cf}(\mathfrak{c}) > \omega$, there exists $N \in \mathcal{S}_1$ such that $|E_N| = \mathfrak{c}$. Then A (and thus, G) contains the direct sum

$$H = \bigoplus_{i \in E_N} N \cong N^{(|E_N|)} = N^{(\mathfrak{c})}. \tag{14}$$

Clearly, H is $\mathfrak{c}$-homogeneous. Since $|H| = \mathfrak{c}$ and H is a direct sum of $\mathfrak{c}$-many unbounded groups, H is w-divisible by Corollary 3.7.

Subcase B. $|I_1| < \mathfrak{c}$. Since $I = I_1 \cup I_2$ and $|I| = \mathfrak{c}$, this implies that $|I_2| = \mathfrak{c}$. By discarding A_i with $i \in I_1$, we may assume, without loss of generality, that $I = I_2$; that is, $A_i \in \mathcal{S}_2$ for all $i \in I$. From this, (12) and (13), we conclude that the direct sum (8) can be re-written as

$$A = \bigoplus_{p \in \mathbb{P}} \mathbb{Z}(p)^{(\sigma_p)} \tag{15}$$

for a suitable family $\{\sigma_p : p \in \mathbb{P}\}$ of cardinals.

Claim 1. The set $C = \{p \in \mathbb{P} : \sigma_p = \mathfrak{c}\}$ is infinite.

Proof. Let $P = \{p_1, p_2, \dots, p_k\}$ be an arbitrary finite set. Define $m = p_1 p_2 \dots p_k$. From (15),

$$mA = \bigoplus_{p \in \mathbb{P} \setminus P} \mathbb{Z}(p)^{(\sigma_p)},$$

so

$$\sup\{\sigma_p : p \in \mathbb{P} \setminus P\} = |mA| = |A| = \mathfrak{c} \tag{16}$$

as A is w-divisible. Note that (15) implies that $\sigma_p \leq |A| = \mathfrak{c}$ for every $p \in \mathbb{P}$. Since $\mathrm{cf}(\mathfrak{c}) > \omega$ and the set $\mathbb{P} \setminus P$ is countable, from (16) we conclude that $\sigma_p = \mathfrak{c}$ for some $p \in \mathbb{P} \setminus P$. Therefore, $C \setminus P \neq \emptyset$. Since P is an arbitrary finite subset of $\mathbb{P}$, this shows that the set $C = \{p \in \mathbb{P} : \sigma_p = \mathfrak{c}\}$ is infinite. □

By this claim, A (and thus, G) contains the direct sum

$$H = \bigoplus_{p \in C} \mathbb{Z}(p)^{(\mathfrak{c})} \cong \left(\bigoplus_{p \in C} \mathbb{Z}(p) \right)^{(\mathfrak{c})} = \mathrm{Soc}_C(\mathbb{T})^{(\mathfrak{c})}.$$

Clearly, $|H| = \mathfrak{c}$. Finally, since $\mathrm{Soc}_C(\mathbb{T})^{(\mathfrak{c})}$ is unbounded, H is w-divisible by Corollary 3.7. □

5. The Hartman-Mycielski construction

Let G be an abelian group, and let I be the unit interval $[0,1]$. As usual, G^I denotes the set of all functions from I to G. Clearly, G^I is a group under the coordinate-wise operations. For $g \in G$ and $t \in (0,1]$ let $g_t \in G^I$ be the function defined by

$$g_t(x) = \begin{cases} g & \text{if } x < t \\ e & \text{if } x \geq t, \end{cases}$$

where e is the identity element of G. For each $t \in (0,1]$, $G_t = \{g_t : g \in G\}$ is a subgroup of G^I isomorphic to G. Therefore, $\mathsf{HM}(G) = \sum_{t \in (0,1]} G_t$ is a subgroup of G^I. It is straightforward to check that this sum is direct, so that

$$\mathsf{HM}(G) = \bigoplus_{t \in (0,1]} G_t. \tag{17}$$

Since $G_t \cong G$ for each $t \in (0,1]$, from (17) we conclude that $\mathsf{HM}(G) \cong G^{(\mathfrak{c})}$. It follows from this that $\mathsf{HM}(G)$ is divisible and abelian whenever G is. Thus, we have proved the following:

Lemma 5.1. *For every group G, the group $\mathsf{HM}(G)$ is algebraically isomorphic to $G^{(\mathfrak{c})}$. Furthermore, if G is divisible and abelian, then so is $\mathsf{HM}(G)$.*

In general even a group G with $|G| \geq \mathfrak{c}$ need not be isomorphic to $\mathsf{HM}(G)$. Indeed, as shown in [18], G is contained in $\mathsf{HM}(G)$ and splits as a semidirect addend. So any group G indecomposable into a non-trivial semi-direct product fails to be algebraically isomorphic to $\mathsf{HM}(G)$.

When G is a topological group, Hartman and Mycielski [18] equip $\mathsf{HM}(G)$ with a topology making it pathwise connected and locally pathwise connected. Let μ be the standard probability measure on I. The *Hartman-Mycielski topology* on the group $\mathsf{HM}(G)$ is the topology generated by taking the family of all sets of the form

$$O(U,\varepsilon) = \{g \in G^I : \mu(\{t \in I : g(t) \notin U\}) < \varepsilon\}, \tag{18}$$

where U is an open neighbourhood U of the identity e in G and $\varepsilon > 0$, as the base at the identity function of $\mathsf{HM}(G)$.

The next lemma lists two properties of the functor $G \mapsto \mathsf{HM}(G)$ that are needed for our proofs.

Lemma 5.2. *Let G be a topological group.*

(i) $\mathsf{HM}(G)$ *is pathwise connected and locally pathwise connected.*
(ii) *If D is a dense subgroup of G, then* $\mathsf{HM}(D)$ *is a dense subgroup of* $\mathsf{HM}(G)$.

Proof. (i) For every $g \in G$ and $t \in (0,1]$, the map $f : [0,t] \to \mathsf{HM}(G)$ defined by $f(s) = g_s$ for $s \in [0,t]$ is a continuous injection, so $f([0,t])$ is a path in $\mathsf{HM}(G)$ connecting $f(0) = e$ and $f(t) = g_t$. Combining this with (17), one concludes that $\mathsf{HM}(G)$ is pathwise connected.

To check that $\mathsf{HM}(G)$ is locally pathwise connected, it suffices to show that every set of the form $O(U,\varepsilon)$ as in (18) is pathwise connected. Let U be an open neighbourhood of the identity e in G and let $\varepsilon > 0$. Fix an arbitrary $h \in O(U,\varepsilon) \setminus \{e\}$. For each $s \in [0,1]$, define

$$h_s(x) = \begin{cases} h(x) & \text{if } x < s \\ e & \text{if } x \geq s, \end{cases}$$

and note that $h_s \in O(U,\varepsilon)$. Therefore, the map $\varphi : [0,1] \to \mathsf{HM}(G)$ defined by $\varphi(s) = h_s$ for $s \in [0,t]$ is continuous, so $\varphi([0,1])$ is a path in $O(U,\varepsilon)$ connecting $f(0) = e$ and $f(1) = h$.

(ii) Let D be a dense subgroup of G. First, from the definition of the topology of $\mathsf{HM}(G)$, one easily sees that D_t is a dense subgroup of G_t for each $t \in (0,1]$. Let $g \in \mathsf{HM}(G)$ be arbitrary, and let W be an open subset of $\mathsf{HM}(G)$ containing g. By (17), $g = \sum_{i=1}^{n} (g_i)_{t_i}$ for some $n \in \mathbb{N}$, $g_1, g_2, \ldots, g_n \in G$ and $t_1, t_2, \ldots, t_n \in (0,1]$. Since the group operation in $\mathsf{HM}(G)$ is continuous, for every $i = 1,2,\ldots,n$ we can fix an open neighbourhood V_i of $(g_i)_{t_i}$ in $\mathsf{HM}(G)$ such that $\sum_{i=1}^{n} V_i \subseteq W$. For each $i = 1,2,\ldots,n$, we use the fact that D_{t_i} is dense in G_{t_i} to select $d_i \in V_i \cap D_{t_i}$. Clearly, $d = \sum_{i=1}^{n} d_i \in \sum_{i=1}^{n} V_i \subseteq W$. Since $d_i \in D_{t_i} \subseteq \mathsf{HM}(D)$ for $i = 1,2,\ldots,n$, it follows that $d \in \mathsf{HM}(D)$. This shows that $d \in \mathsf{HM}(D) \cap W \neq \emptyset$. □

Corollary 5.3. *Every $\mathfrak{c}$-homogeneous group G admits a pathwise connected, locally pathwise connected group topology.*

Proof. Consider G with the discrete topology. By Lemma 5.2 (i), $\mathsf{HM}(G)$ is pathwise connected and locally pathwise connected. Since $\mathsf{HM}(G) \cong G^{(\mathfrak{c})}$ by Lemma 5.1, and $G^{(\mathfrak{c})} \cong G$ by our assumption, we can transfer the topology from $\mathsf{HM}(G)$ to G by means of the isomorphism $\mathsf{HM}(G) \cong G$. Therefore, G admits a pathwise connected, locally pathwise connected group topology. □

6. Proof of Theorem 1.9

Lemma 6.1. *Let H be a subgroup of an abelian group G and K be a divisible abelian group such that*

$$r_p(H) < r_p(K) \text{ and } r_p(G) \leq r_p(K) \text{ for all } p \in \{0\} \cup \mathbb{P}. \tag{19}$$

Then every monomorphism $j : H \to K$ can be extended to a monomorphism $j' : G \to K$.

Proof. Let us recall some notions used in this proof. Let G be an abelian group. A subgroup H of G is is said to be *essential in* G provided that $\langle x \rangle \cap H \neq \{0\}$ for every non-trivial cyclic subgroup $\langle x \rangle$ of G. As usual, $\mathrm{Soc}(G) = \bigoplus_{p\in\mathbb{P}} G[p]$ denotes the *socle* of G.

By means of Zorn's lemma, we can find a maximal subgroup M of G with the property

$$M \cap H = \{0\}. \tag{20}$$

Then the subgroup $H + M = H \oplus M$ is essential in G. Indeed, if $0 \neq x \in G$ with $(H + M) \cap \langle x \rangle = \{0\}$, then also $(\langle x \rangle + M) \cap H = \{0\}$. Indeed, if

$$H \ni h = kx + m \in \langle x \rangle + M$$

for some $k \in \mathbb{Z}$ and $m \in M$, then

$$h - m = kx \in (H + M) \cap \langle x \rangle = \{0\};$$

so $kx = 0$ and $h = m$, which yields $h = 0$ in view of (20).

Fix a free subgroup F of M such that M/F is torsion. Then $\mathrm{Soc}(M) \oplus F$ is an essential subgroup of M. Let us see first that j can be extended to a monomorphism

$$j_1 : H \oplus \mathrm{Soc}(M) \oplus F \to K. \tag{21}$$

Since $r_0(H) < r_0(K)$ and $r_0(G) \leq r_0(K)$, we can find a free subgroup F' of K such that $F' \cap j(H) = \{0\}$ and $r_0(F') = r_0(F)$. On the other hand, for every prime $p \in \mathbb{P}$, $r_p(H) < r_p(K)$ and $r_p(G) \leq r_p(K)$. Hence, we can find in $K[p]$ a subgroup T_p of p-rank $r_p(M)$ with $T_p \cap j(H)[p] = \{0\}$. Since $F' \cong F$ and $T_p \cong M[p]$ for every prime p, we obtain the desired monomorphism j_1 as in (21) extending j. Since the subgroup $H \oplus \mathrm{Soc}(M) \oplus F$ of G is essential in G, any extension $j' : G \to K$ of j_1 will be a monomorphism as j_1 is a monomorphism. Such an extension j' exists since K is divisible [15, Theorem 21.1]. □

Lemma 6.2. *Every w-divisible abelian group of size at least $\mathfrak{c}$ admits a dp-connected, locally dp-connected group topology.*

Proof. Let G be a w-divisible abelian group such that $|G| = \tau \geq \mathfrak{c}$. Use Lemma 4.5 to find a w-divisible subgroup H of G such that $|H| = \tau$ and $H \cong H^{(\mathfrak{c})}$. Since H is w-divisible, $w_d(H) = |H| = \tau$. By Corollary 2.8 (applied to H), there exists a monomorphism $\pi : H \to \mathbb{T}^\kappa$ such that $D = \pi(H)$ is dense in $\mathbb{T}^\kappa$, where $\kappa = 2^\tau$. By Lemma 5.2 (ii), $N = \mathsf{HM}(D)$ is a dense subgroup of $K = \mathsf{HM}(\mathbb{T}^\kappa)$. By Lemma 5.2 (i), N is pathwise connected and locally pathwise connected.

By our choice of H, $H \cong H^{(\mathfrak{c})}$. Since π is a monomorphism, $H \cong \pi(H) = D$, so $H^{(\mathfrak{c})} \cong D^{(\mathfrak{c})}$. Finally, $D^{(\mathfrak{c})} \cong \mathsf{HM}(D) = N$ by Lemma 5.1. This allows us to fix a monomorphism $j : H \to K$ such that $j(H) = N$.

Since $\mathbb{T}^\kappa$ is a subgroup of K,

$$|G| = \tau < 2^\tau = \kappa \leq r_p(\mathbb{T}^\kappa) \leq r_p(K) \text{ for all } p \in \{0\} \cup \mathbb{P}.$$

Clearly, $r_p(H) \leq |H| \leq |G|$ and $r_p(G) \leq |G|$ for all $p \in \{0\} \cup \mathbb{P}$. Therefore, all assumptions of Lemma 6.1 are satisfied, and its conclusion allows us to find a monomorphism $j' : G \to K$ extending j. Since $G' = j'(G)$ contains $j'(H) = j(H) = N$ and N is dense in K, N is dense also in G'. Since N is pathwise connected and locally pathwise connected, we conclude that G' is dp-connected and locally dp-connected.

Finally, since j' is an isomorphism between G and G', we can use it to transfer the subspace topology which G' inherits from K onto G. □

Proof of Theorem 1.9. Let G be an M-group of infinite exponent. Then $w_d(G) \geq \mathfrak{c}$ by Fact 2.4. Therefore, we can apply Lemma 4.4 with $\sigma = \mathfrak{c}$ to obtain a decomposition $G = N \oplus H$, where N is a bounded $\mathfrak{c}^+$-homogeneous group and H is a w-divisible group such that $|H| = w_d(G) \geq \mathfrak{c}$. By Lemma 4.2 (ii), N is $\mathfrak{c}$-homogeneous as well. Corollary 5.3 guarantees the existence of a pathwise connected, locally pathwise connected group topology $\mathscr{T}_N$ on N, and Lemma 6.2 guarantees the existence of a dp-connected, locally dp-connected group topology $\mathscr{T}_H$ on H. The topology $\mathscr{T}$ of the direct product $(N, \mathscr{T}_N) \times (H, \mathscr{T}_H)$ is dp-connected and locally dp-connected. Since $G = N \oplus H$ and $N \times H$ are isomorphic, $\mathscr{T}$ is the desired topology on G. □

7. Characterization of countable strongly unbounded groups

The description of uncountable strongly unbounded groups obtained in Theorem 3.6 fails for countable groups, as mentioned in Remark 3.8. For the sake of completeness, we provide here a description of countable strongly unbounded groups.

Following [15], for an abelian group G, we let

$$r(G) = \max\left\{r_0(G), \sum\{r_p(G) : p \in \mathbb{P}\}\right\}.$$

The sum in this definition may differ from the supremum $\sup\{r_p(G) : p \in \mathbb{P}\}$; the latter may be strictly less than the sum, in case all p-ranks are finite and uniformly bounded.

Remark 7.1. (i) $r(G) > 0$ if and only if G is non-trivial.
(ii) If $r(G) \geq \omega$, then $r(G) = |G|$.
(iii) If G is uncountable, then $r(G) = |G|$.

Replacing the cardinality $|G|$ with the rank $r(G)$ in Definition 2.2, one gets the following "rank analogue" of the divisible weight:

Definition 7.2. For an abelian group G, call the cardinal

$$r_d(G) = \min\{r(nG) : n \in \mathbb{N} \setminus \{0\}\} \tag{22}$$

the *divisible rank* of G. We say that G is *r-divisible* if $r_d(G) = r(G)$.

The notion of divisible rank was defined, under the name of *final rank*, by Szele [27] for (discrete) p-groups. The relevance of this notion to the class of strongly unbounded groups will become clear from Proposition 7.5 below.

Let G be an abelian group. Observe that $r(nG) \leq |nG| \leq |G|$ for every positive integer n. Combining this with (5) and (22), we get

$$r_d(G) \leq w_d(G) \leq |G|. \tag{23}$$

It turns out that the first inequality in (23) is strict precisely when G has finite divisible rank. The algebraic structure of such groups is described in Proposition 7.7 below.

Lemma 7.3. *An abelian group G satisfies $r_d(G) < w_d(G)$ if and only if $r_d(G)$ is finite.*

Proof. Assume first that $r_d(G)$ is finite. We shall show that $r_d(G) < w_d(G)$. If G is bounded, then $nG = \{0\}$ for some $n \in \mathbb{N} \setminus \{0\}$. Therefore, $r_d(G) = 0$ by (22), while $w_d(G) \geq 1$ by (5). If G is unbounded, then nG must be infinite for every $n \in \mathbb{N} \setminus \{0\}$, and so $w_d(G) \geq \omega$ by (5). Since $r_d(G) < \omega$, we get $r_d(G) < w_d(G)$ in this case as well.

Next, suppose that $r_d(G)$ is infinite. Let $n \in \mathbb{N}\setminus\{0\}$ be arbitrary. Then $r(nG) \geq r_d(G) \geq \omega$ by (22). Therefore, $r(nG) = |nG|$ by Remark 7.1 (ii). Since this equality holds for every $n \in \mathbb{N} \setminus \{0\}$, from (5) and (22) we get $r_d(G) = w_d(G)$. □

The next lemma outlines the most important connections between the classes of strongly unbounded, r-divisible and w-divisible groups.

Lemma 7.4. (i) *If an abelian group G is strongly unbounded, then $r_d(G) = w_d(G) = |G|$, so G is r-divisible and $r_d(G) \geq \omega$.*
(ii) *Every r-divisible group G satisfying $r_d(G) \geq \omega$ is w-divisible.*
(iii) *If G is an uncountable w-divisible group, then G is r-divisible.*

Proof. (i) If G is strongly unbounded, then G contains a direct sum $\bigoplus_{i\in I} A_i$ of unbounded groups A_i such that $|I| = |G|$; see Definition 3.1. Since for every integer $n > 0$ the group nA_i remains unbounded, $r(nA_i) \geq 1$, so $r(nG) = r(\bigoplus_{i\in I} nA_i) \geq |I| = |G|$. This proves the inequality $r_d(G) \geq |G|$. The rest follows from (23).

(ii) Since G is r-divisible, $r(G) = r_d(G)$ by Definition 7.2. Since $r_d(G) \geq \omega$, $|G| = r(G)$ by Remark 7.1 (ii) and $r_d(G) = w_d(G)$ by Lemma 7.3. Therefore, $|G| = w_d(G)$, which implies that G is w-divisible.

(iii) If G is an uncountable w-divisible group, then for every integer $n > 0$, $|nG| = |G|$ is uncountable, and so $r(nG) = |nG|$ by Remark 7.1 (iii). So $r(G) \geq r(nG) = |nG| = |G| \geq r(G)$, and consequently, $r(nG) = r(G)$. This proves that G is r-divisible. □

Clearly, a strongly unbounded group must be infinite. Our next proposition describes countable strongly unbounded groups in terms of their divisible rank r_d.

Proposition 7.5. *A countable abelian group G is strongly unbounded if and only if $r_d(G) \geq \omega$.*

Proof. If G is strongly unbounded, then $r_d(G) \geq \omega$ by Lemma 7.4 (i).

Assume now that $r_d(G) \geq \omega$. We shall prove that G is strongly unbounded.

If $r_0(G)$ is infinite, then $r_0(G) = |G| = \omega$, so G is strongly unbounded by Remark 3.2 (ii).

If $\pi = \{p \in \mathbb{P} : r_p(G) > 0\}$ is infinite, then G contains a subgroup isomorphic to the strongly unbounded group $H = \bigoplus_{p\in\pi} \mathbb{Z}(p)$. (Indeed, if $\pi = \bigcup\{S_i : i \in \mathbb{N}\}$ is a partition of π into pairwise disjoint infinite sets S_i, then $H = \bigoplus_{i\in\mathbb{N}} A_i$, where each $A_i = \bigoplus_{p\in S_i} \mathbb{Z}(p)$ is unbounded.) Since $|G| = |H| = \omega$, Remark 3.2 (i) implies that G is strongly unbounded as well.

From now on we shall assume that both $r_0(G)$ and π are finite. The former assumption implies that $r(mt(G)) = \omega$ for every integer $m > 0$. The latter assumption entails that there exists $p \in \pi$ such that $r(mt_p(G)) = r_p(mt_p(G))$ is infinite for all integers $m > 0$, where $t_p(G) = \bigcup\{G[p^n] : n \in \mathbb{N}\}$ is the p-torsion part of G. By the implication (iii) → (ii) of Lemma 3.4, we conclude that $t_p(G)$ is strongly unbounded. As $|G| = |t_p(G)| = \omega$, Remark 3.2 (i) yields that G is strongly unbounded. □

Finally, we can unify the countable case considered above with Theorem 3.6 to obtain a description of *all* strongly unbounded groups.

Theorem 7.6. *For an abelian group G the following are equivalent:*

(i) *G is strongly unbounded;*
(ii) *G is r-divisible and $r_d(G) \geq \omega$;*
(iii) *G is w-divisible and $r_d(G) \geq \omega$.*

Proof. The implication (i) → (ii) follows from Lemma 7.4 (i). The implication (ii) → (iii) follows from Lemma 7.4 (ii). It remains to check the remaining implication (iii) → (i). For a countable group G, it follows from Proposition 7.5. For an uncountable group G, Theorem 3.6 applies. □

Our last proposition completely describes abelian groups of finite divisible rank.

Proposition 7.7. *If G is an abelian group satisfying $r_d(G) < \omega$, then $G = G_0 \times D \times B$, where G_0 is a torsion-free group, D is a divisible torsion group and B is a bounded group such that $r_d(G) = r_0(G_0) + r(D)$.*

Proof. Since $r_d(G) < \omega$, it follows from (22) that $r(nG) < \omega$ for some integer $n > 0$. Let $t(nG) = F \times D$, where D is a divisible group and F is a reduced group. Since $r(F) \leq r(t(nG)) \leq r(nG) < \omega$, the reduced torsion group F must be finite. Since D is divisible, it splits as a direct summand of nG [15, Theorem 21.2]; that is, we can find a subgroup A of nG containing F such that $nG = A \times D$. Since $t(A) \cong t(nG/D) \cong t(nG)/D \cong F$ is finite, we deduce from [15, Theorem 27.5] that F splits in A; that is, $A = F \times H$ for some torsion-free (abelian) group H. Now $nG = F \times H \times D$.

Let m be the exponent of the finite group F and let $k = mn$. Then $kG = N \times D$, where $N = mH$ is a torsion-free group and $D = mD$ is a divisible torsion group. In particular, $N \cong kG/D$. Since divisible subgroups split, $G = D \times G_1$ for an appropriate subgroup G_1 of G. Multiplying by k and taking into account that $kD = D$, we get $kG = D \times kG_1$. This implies that the group $kG_1 \cong kG/D \cong N$ is torsion-free. Hence, the torsion subgroup $t(G_1) = G_1[k]$ of G_1 is bounded, so it splits by [15, Theorem 27.5]; that is, $G_1 = t(G_1) \times G_0$, where G_0 is a torsion-free group.

We now have obtained the decomposition $G = G_0 \times D \times B$, where $B = t(G_1)$ is a bounded torsion group. Recalling (22), one easily sees that $r_d(G) = r_0(G_0) + r(D)$. □

Remark 7.8. Let φ be a cardinal invariant of topological abelian groups. In analogy with the divisible weight and the divisible rank, for every topological abelian group G, one can define the cardinal

$$\varphi_d(G) = \min\{\varphi(nG) : n \in \mathbb{N} \setminus \{0\}\} \tag{24}$$

and call G *φ-divisible* provided that $\varphi_d(G) = \varphi(G)$. (This terminology is motivated by the fact that divisible groups are obviously φ-divisible.) We will not pursue here a study of this general cardinal invariant.

8. Open questions

We finish this paper with a couple of questions. Now that Question 1.3 is completely resolved, one may try to look at variations of this question.

One such possible variation could be obtained by asking about the existence of a connected group topology on abelian groups having some additional compactness-like properties. Going in this direction, abelian groups which admit a pseudocompact connected group topology were characterized in [7, 8]; the necessary condition was independently established in [3]. Recently, the authors obtained complete characterizations of abelian groups which admit a maximally almost periodic connected group topology, as well as abelian groups which admit a minimal connected group topology.

Another possible variation of Question 1.3 is obtained by trying to ensure the existence of a group topology with some stronger connectedness properties.

Question 8.1. Can every abelian M-group be equipped with a pathwise connected (and locally pathwise connected) group topology?

One may wonder if Theorem 1.9 remains valid for groups which are close to being abelian.

Question 8.2. Can Theorem 1.9 be extended to all nilpotent groups?

Acknowledgments. The authors are grateful to the referee for careful reading of the manuscript and helpful suggestions. The first named author gratefully acknowledges a FY2013 Long-term visitor grant L13710 by the Japanese Society for the Promotion of Science (JSPS). The second named author was partially supported by the Grant-in-Aid for Scientific Research (C) No. 26400091 by the Japan Society for the Promotion of Science (JSPS).

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