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# About the Algebraic Yuzvinski Formula

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**Abstract:** The Algebraic Yuzvinski Formula expresses the algebraic entropy of an endomorphism of a finite-dimensional rational vector space as the Mahler measure of its characteristic polynomial. In a recent paper, we have proved this formula, independently from its counterpart – the Yuzvinski Formula – for the topological entropy proved by Yuzvinski in 1968.
In this paper we first compare the proof of the Algebraic Yuzvinski Formula with a proof of the Yuzvinski Formula given by Lind and Ward in 1988, underlying the common ideas and the differences in the main steps. Then we describe several known applications of the Algebraic Yuzvinski Formula, and some related open problems are discussed.
Finally, we give a new and purely algebraic proof of the Algebraic Yuzvinski Formula for the intrinsic algebraic entropy.

**Keywords:** algebraic entropy; topological entropy; endomorphism; Yuzvinski Formula; intrinsic entropy

**MSC:** Primary: 20K15, 20K30; Secondary: 20K45, 08A35, 28D20, 22B05, 11R06.

**Dedicated to** Alberto Facchini for his 60th birthday.

## 1 Introduction

The Algebraic Yuzvinski Formula is a formula that relates two quantities attached to an endomorphism of a finite dimensional rational vector space: its *algebraic entropy* and the *Mahler measure* of its characteristic polynomial. Let us start introducing these two quantities.

We give immediately the definition of algebraic entropy in the general case of locally compact Abelian (briefly, LCA) groups, following [31]. By endomorphism of a topological group $G$ we always mean continuous endomorphism and with automorphism we intend a group automorphism which is also a homeomorphism. We denote by $\mathrm{End}(G)$ and $\mathrm{Aut}(G)$ respectively the endomorphisms and the automorphisms of $G$.

Given an LCA group $G$, we denote by $\mathcal{C}(G)$ the family of compact neighborhoods of 0 in $G$ ordered by inclusion. Let $\phi : G \to G$ be an endomorphism, let $C \in \mathcal{C}(G)$ and let $n$ be a positive integer; the *$n$-th $\phi$-trajectory of $C$* is

$$T_n(\phi, C) = C + \phi(C) + \ldots + \phi^{n-1}(C).$$

Given a Haar measure $\mu$ on $G$, the *algebraic entropy of $\phi$ with respect to $C \in \mathcal{C}(G)$* is

$$H_A(\phi, C) = \limsup_{n\to\infty} \frac{\log \mu(T_n(\phi, C))}{n}; \tag{1.1}$$

the *algebraic entropy of $\phi$* is

$$h_A(\phi) = \sup\{H_A(\phi, C) : C \in \mathcal{C}(G)\}.$$

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Let $N$ be a positive integer, let $f(X) = sX^N + a_1X^{N-1} + \ldots + a_N \in \mathbb{C}[X]$ be a non-constant polynomial with complex coefficients and let $\{\lambda_i : i = 1, \ldots, N\} \subseteq \mathbb{C}$ be the set of all roots of $f(X)$ (we always assume the roots of a polynomial to be counted with their multiplicity); in particular, $f(X) = s \cdot \prod_{i=1}^{N}(X - \lambda_i)$. Following Lehmer [22] (see also [14]), the *(logarithmic) Mahler measure* of $f(X)$ is

$$m(f(X)) = \log|s| + \sum_{|\lambda_i|>1} \log|\lambda_i|.$$

The Mahler measure was defined independently also by Mahler [24]. It plays an important role in number theory and arithmetic geometry; in particular, it is involved in the famous Lehmer Problem asking whether $\inf\{m(f(X)) : f(X) \in \mathbb{Z}[X] \text{ primitive}, m(f(X)) > 0\}$ is strictly positive (for example see [14], [17] and [25], and for a survey on the Mahler measure of algebraic numbers see [29]).

In [15] we provided a completely self-contained proof of the following algebraic counterpart of the Yuzvinski Formula (see Theorem 2.2 in §2.1 below) proved in [36] (see also [23]), computing the algebraic entropy of endomorphisms of finite dimensional rational vector spaces in terms of the Mahler measure of their characteristic polynomials:

**Theorem 1.1** (Algebraic Yuzvinski Formula). *Let N be a positive integer and let $\phi : \mathbb{Q}^N \to \mathbb{Q}^N$ be an endomorphism. Then*

$$h_A(\phi) = m(p_\phi(X)),$$

*where $p_\phi(X)$ is the characteristic polynomial of $\phi$ over $\mathbb{Z}$.*

Given an endomorphism $\phi : \mathbb{Q}^N \to \mathbb{Q}^N$, its *characteristic polynomial over* $\mathbb{Z}$ mentioned in the above theorem, is defined as follows. Let $M_\phi$ be the $N \times N$ rational matrix representing the action of $\phi$ on $\mathbb{Q}^N$ with respect to the canonical base of $\mathbb{Q}^N$ over $\mathbb{Q}$. The matrix $M_\phi$ has its monic characteristic polynomial $f_\phi(X) \in \mathbb{Q}[X]$; we say that $f_\phi(X)$ is the characteristic polynomial of $\phi$ over $\mathbb{Q}$. Let $s$ be the minimum positive integer such that $sf_\phi(X) \in \mathbb{Z}[X]$; we say that $p_\phi(X) = sf_\phi(X)$ is the characteristic polynomial of $\phi$ over $\mathbb{Z}$. Moreover, we call eigenvalues of $\phi$ the eigenvalues of $M_\phi$.

It is worth mentioning that a first attempt to prove the Algebraic Yuzvinski Formula was done in [37], where several partial results were obtained; some of them were recently used to prove the "case zero" of the Algebraic Yuzvinski Formula in [11] with arguments exclusively of linear algebra. Indeed, under the same notations and hypotheses as above, [11, Corollary 1.4] shows that $h_A(\phi) = 0$ if and only if $m(p_\phi(X)) = 0$.

Moreover, the algebraic counterpart of the Kolmogorov-Sinai Formula (see §2.1) was established in the paper [31], where the algebraic entropy for endomorphisms of LCA groups was introduced. Indeed, in [31] the algebraic entropy of an endomorphism $\phi : \mathbb{Z}^N \to \mathbb{Z}^N$ was computed, proving the Algebraic Kolmogorov-Sinai Formula, that is, $h_A(\phi) = \sum_{|\lambda_i|>1} \log|\lambda_i|$, where $\{\lambda_i : i = 1, \ldots, N\}$ are the eigenvalues of $\phi$. The main idea for the proof of this result was to extend $\phi$ to an endomorphism $\Phi$ of $\mathbb{R}^N$ with the same algebraic entropy, and to use there the Haar measure to estimate the growth of the trajectories of $\Phi$ in order to compute its algebraic entropy. These methods inspired the ones used in [15] for the proof of the Algebraic Yuzvinski Formula. We see in Section 4 that it is possible to deduce the Algebraic Kolmogorov-Sinai Formula from the Algebraic Yuzvinski Formula. Finally, notice that the proof of the Algebraic Yuzvinski Formula in [15] is independent from the results of [11], [31] and [37].

The paper is organized as follows.

In Section 2 we give some historical background on the algebraic entropy, also mentioning several similarities with the topological entropy.

In Section 3, first we describe the fundamental steps in the proof of the Algebraic Yuzvinski Formula given in [15] and then we compare this proof with the one of the Yuzvinski Formula for the topological entropy from [23].

In Section 4 we list several applications of the Algebraic Yuzvinski Formula. In particular, we describe the fundamental results from [3] and [8] about the algebraic entropy of endomorphisms of discrete Abelian

groups, where the Algebraic Yuzvinski Formula applies. As the proof of the Algebraic Yuzvinski Formula is now self-contained and does not depend on its topological counterpart, also the treatment of the algebraic entropy in [3] and [8] becomes independent from the results on topological entropy. On the other hand, it is again the Algebraic Yuzvinski Formula that allows one to prove the fundamental connection between the algebraic entropy of endomorphisms of discrete Abelian groups and the topological entropy of endomorphisms of compact Abelian groups, namely, a general version of the Bridge Theorem proved in [7].
We formulate also several open problems related to the applications, with the aim to understand whether and how the fundamental results for the algebraic entropy of endomorphisms of discrete Abelian groups can be extended to the general case of endomorphisms of LCA groups.

Finally, Section 5 is devoted to a formula which is strictly related to the Algebraic Yuzvinski Formula. Indeed, in [9] the new notion of intrinsic algebraic entropy $\widetilde{\mathrm{ent}}$ is introduced and investigated for endomorphisms of discrete Abelian groups. In particular, the intrinsic algebraic entropy is always smaller than the algebraic entropy, and they coincide on endomorphisms of torsion Abelian groups. The intrinsic algebraic entropy explains in some sense the somewhat mysterious term $\log s$ that appears in the Algebraic Yuzvinski Formula. In fact, it is proved in [9, Theorem 4.2] that, if $N$ is a positive integer, $\phi : \mathbb{Q}^N \to \mathbb{Q}^N$ is an endomorphism and $s$ is the leading coefficient of the characteristic polynomial $p_\phi(X)$ of $\phi$ over $\mathbb{Z}$, then

$$\widetilde{\mathrm{ent}}(\phi) = \log s.$$

We give in Section 5 a self-contained and purely algebraic proof of this fact. To do this, we give first a direct proof of the additivity of the intrinsic algebraic entropy for endomorphisms of finite dimensional rational vector spaces with respect to invariant subspaces.

# 2 Background and definitions

## 2.1 Topological entropy

The topological entropy was introduced in 1965 by Adler, Konheim and McAndrew [1] and it was successively extended by Bowen [2], Hood [18] and others. Let us remark that, even if there are different definitions of topological entropy of a continuous self-map $T$ of a topological space $X$, all of them coincide when $X$ is compact.

Consider now an LCA group $G$ with a Haar measure $\mu$, and an endomorphism $\phi : G \to G$. In particular, $G$ is a locally compact uniform space when endowed with its canonical left uniformity $\mathcal{U}$; furthermore, $\phi : (G, \mathcal{U}) \to (G, \mathcal{U})$ is uniformly continuous, and $\mu$ is $\phi$-homogeneous (in the sense of Bowen and Hood). In this context, Hood's definition of topological entropy applies, and it can be given in the following way (see [12]). For every $K \in \mathcal{C}(G)$ and every positive integer $n$,

$$C_n(\phi, K) = K \cap \phi^{-1}K \cap \ldots \cap \phi^{-n+1}K$$

is the *$n$-th $\phi$-cotrajectory of $K$*. The *topological entropy* of $\phi$ is

$$h_T(\phi) = \sup\left\{\limsup_{n\to\infty} \frac{-\log \mu(C_n(\phi, K))}{n} : K \in \mathcal{C}(G)\right\}.$$

Notice that this definition is correct by [31, Lemma 2.1]. The topological entropy is very well-understood on compact Abelian groups but only few results are known in the setting of LCA groups. For a comprehensive treatment of these aspects we refer to [12], [13] and [32].

In the context of compact Abelian groups, one can always reduce the computation of the topological entropy to two prototypical cases: left Bernoulli shifts and endomorphisms of full solenoids (see [30] and [12]). The former is described in the following example, while the latter is where the classical Yuzvinski Formula is needed.

**Example 2.1.** *Let $K$ be a compact finite Abelian group and let $G = \prod_{n\in\mathbb{N}} K_n$, where each $K_n = K$. The* left Bernoulli shift *is the endomorphism of $G$ defined by*

$$\lambda_K : G \to G \text{ such that } (x_0, x_1, \ldots, x_n, \ldots) \mapsto (x_1, \ldots, x_{n+1}, \ldots).$$

*Then $h_T(\lambda_K) = \log|K|$.*

A solenoid is a finite-dimensional connected compact Abelian group; so its dual group is a finite rank torsion-free discrete Abelian group, that is, a subgroup of $\mathbb{Q}^N$ for some positive integer $N$. Moreover, $\widehat{\mathbb{Q}}^N$ is said to be a full solenoid. With (full) solenoidal endomorphism we mean an endomorphism of a (full) solenoid.

**Theorem 2.2** (Yuzvinski Formula). *Let $N$ be a positive integer and $\psi : \widehat{\mathbb{Q}}^N \to \widehat{\mathbb{Q}}^N$ an endomorphism. Then*

$$h_T(\psi) = m(p_{\widehat{\psi}}(X)),$$

*where $p_{\widehat{\psi}}(X)$ is the characteristic polynomial over $\mathbb{Z}$ of the dual endomorphism $\widehat{\psi} : \mathbb{Q}^N \to \mathbb{Q}^N$ of $\psi$.*

This nice formula was obtained by Yuzvinski in [36]. A different and more conceptual approach to the same result was given by Lind and Ward in [23], where they also described in detail the history of the Yuzvinski Formula and related results.

The Yuzvinski Formula has a wide range of applications, as it allows the computation of the topological entropy of solenoidal automorphisms. As a consequence of the results of [23], one can also obtain the already known Kolmogorov-Sinai Formula, stating that the topological entropy of a toral automorphism $\phi : \mathbb{T}^N \to \mathbb{T}^N$, which is described by an $N \times N$ matrix with integer coefficients, is $h_T(\phi) = \sum_{|\lambda_i|>1} \log|\lambda_i|$, where $\{\lambda_i : i = 1, \ldots, N\}$ are the eigenvalues of $\widehat{\phi}$.

Let us conclude this subsection recalling a powerful result of Bowen, which will be needed later on. Indeed, following [2], a subgroup $K$ of an LCA group $G$, is said to be *uniform discrete* if it is discrete and $G/K$ is compact.

**Example 2.3.** (a) *First, $\mathbb{Z}$ is uniform discrete in $\mathbb{R}$.*
(b) *If $K_i$ is uniform discrete in $G_i$ for $i = 1, \ldots, N$, then $K = K_1 \times \ldots \times K_N$ is uniform discrete in $G = G_1 \times \ldots \times G_N$ (so $\mathbb{Z}^N$ is uniform discrete in $\mathbb{R}^N$).*
(c) *Moreover, $\mathbb{Z}$ diagonally embedded in $\mathbb{R} \times \prod_{p\in\mathbb{P}, p<\infty} \mathbb{Z}_p$ is uniform discrete.*
(d) *Finally, $\mathbb{Q}$ diagonally embedded in the adele ring $\mathbb{Q}_{\mathbb{A}}$ is uniform discrete (see [23, Lemma 4.1]).*

A consequence of [2, Theorem 20] is the following

**Theorem 2.4.** *If $G$ is a metrizable LCA group, $\phi$ is an endomorphism of $G$, and $K$ is a $\phi$-invariant uniform discrete subgroup of $G$, then*

$$h_T(\phi) = h_T(\overline{\phi}),$$

*where $\overline{\phi} : G/K \to G/K$ is the endomorphism induced by $\phi$.*

## 2.2 Algebraic entropy

In the final part of the paper [1], where the topological entropy was defined, also a notion of entropy for endomorphisms $\phi$ of discrete Abelian groups $G$ appears, and it is based on the previously introduced concept of trajectory. Notice that, while the cotrajectories, used in the definition of $h_T(-)$, make sense in arbitrary spaces, the concept of trajectory strongly depends on the algebraic operation of the group. This is the reason why we refer to the notions of entropy based on trajectories as *algebraic entropies*.

The definition of algebraic entropy of $\phi$ given in [1] is

$$\mathrm{ent}(\phi) = \sup\left\{\lim_{n\to\infty} \frac{\log|T_n(\phi, C)|}{n} : C \text{ finite subgroup of } G\right\}. \tag{2.1}$$

Since a torsion-free Abelian group has no finite subgroups but the trivial one, ent(–) is always zero on endomorphisms of torsion-free discrete Abelian groups. So it is natural to consider ent(–) for endomorphisms of torsion discrete Abelian groups (see [10] for more details on ent(–)).

In 1979 Peters [26] proposed an alternative notion of algebraic entropy for an automorphism $\phi$ of a discrete Abelian group $G$, defining

$$h_\infty(\phi) = \sup\left\{\lim_{n\to\infty} \frac{\log |T_n(\phi^{-1}, C)|}{n} : C \text{ finite subset of } G\right\}. \tag{2.2}$$

This entropy takes the same values as ent(–) on automorphisms of torsion discrete Abelian groups but it may be non-zero also in the torsion-free case. The same paper [26] contains some general properties of $h_\infty(-)$.

In 1981 Peters [27] gave a further generalization of the entropy $h_\infty(-)$. In fact, using the Haar measure, he introduced a notion of entropy for automorphisms of LCA groups as follows. Let $G$ be an LCA group with a Haar measure $\mu$, and let $\phi : G \to G$ be an automorphism. Then

$$h_\infty(\phi) = \sup\left\{\limsup_{n\to\infty} \frac{\log \mu(T_n(\phi^{-1}, C))}{n} : C \in \mathcal{C}(G)\right\}. \tag{2.3}$$

Recently Peters' definitions of algebraic entropy were appropriately modified in [3] and [31], respectively, by replacing in (2.2) and (2.3) the trajectories of $\phi^{-1}$ with the trajectories of $\phi$. In this way it was obtained the notion of algebraic entropy recalled in the Introduction, that holds for all the endomorphisms of an LCA group; as we said, this algebraic entropy is denoted by $h_A(-)$.

Notice that $h_A(-)$ has the same values of $h_\infty(-)$ for automorphisms of compact Abelian groups, while they do not coincide in general on automorphisms of LCA groups. Furthermore, when $\phi$ is an endomorphism of a torsion discrete Abelian group, we also have that $h_A(\phi) = \mathrm{ent}(\phi)$.

Let $G$ be an LCA group and let $\phi : G \to G$ be an endomorphism. In practice, usually one does not need to compute $H_A(\phi, C)$ for all $C \in \mathcal{C}(G)$ to obtain $h_A(\phi)$, as the following easy lemma from [31] shows:

**Lemma 2.5.** *Let $G$ be an LCA group and let $\phi : G \to G$ be an endomorphism. If $\mathcal{D}$ is a cofinal subset of $\mathcal{C}(G)$, then $h_A(\phi) = \sup\{H_A(\phi, D) : D \in \mathcal{D}\}$.*

Dually to what occurs for the topological entropy on compact Abelian groups, when trying to compute the algebraic entropy of an endomorphism of a discrete Abelian group, one would like to reduce to two prototypical cases: right Bernoulli shifts and endomorphisms of finite dimensional rational vector spaces (see §4.4). The former is described in the following example, while the latter is covered by the Algebraic Yuzvinski Formula.

**Example 2.6.** *Let $K$ be a discrete Abelian group and let $G = \bigoplus_{n\in\mathbb{N}} K_n$, where each $K_n = K$. The* right Bernoulli shift *is the endomorphism of $G$ defined by*

$$\beta_K : G \to G \text{ such that } (x_0, x_1, \dots, x_n, \dots) \mapsto (0, x_0, \dots, x_n, \dots).$$

*Then $h_A(\beta_K) = \log |K|$, with the usual convention that $\log |K| = \infty$ if $K$ is infinite.*

# 3 Remarks on the proof of the Algebraic Yuzvinski Formula

In the first part of this section we give a brief description of the proof of the Algebraic Yuzvinski Formula given in [15]. In the second subsection we compare our proof with the proof of the classical Yuzvinski Formula given in [23].

## 3.1 Steps of the proof

Let us begin with some notation and background. Denote by $\mathbb{P}$ the set of all prime numbers plus the symbol $\infty$. For every prime $p$ we denote by $\mathbb{Q}_p$ the field of $p$-adic numbers and by $|-|_p$ the $p$-adic norm on $\mathbb{Q}_p$; moreover, we let $\mathbb{Q}_\infty = \mathbb{R}$ and $|-|_\infty$ the usual absolute value on $\mathbb{R}$. If $K_p$ is a finite extension of $\mathbb{Q}_p$, then we denote still by $|-|_p$ the unique extension of the $p$-adic norm to $K_p$.

Let $N$ be a positive integer and $\phi : \mathbb{Q}^N \to \mathbb{Q}^N$ an endomorphism. For every $p \in \mathbb{P}$, $\mathbb{Q}$ can be identified with a subfield of $\mathbb{Q}_p$ and so $\phi$ induces an endomorphism $\phi_p : \mathbb{Q}_p^N \to \mathbb{Q}_p^N$ just extending the scalars, that is,

$$\phi_p = \phi \otimes_{\mathbb{Q}} id_{\mathbb{Q}_p}.$$

It is clear that $\phi_p$ is $\mathbb{Q}_p$-linear (and therefore continuous) and so its action on $\mathbb{Q}_p^N$, with respect to the canonical base of $\mathbb{Q}_p^N$, is represented by an $N \times N$ matrix $M_{\phi_p}$ with coefficients in $\mathbb{Q}_p$. We call characteristic polynomial and eigenvalues of $\phi_p$ the characteristic polynomial and the eigenvalues of $M_{\phi_p}$.

We can divide our proof in four steps that we are now going to describe.

Let $\{e_i : i = 1, \dots, N\}$ be the canonical base of $\mathbb{Q}^N$ and, for any $m \in \mathbb{N}_+$, let

$$E_m = \left\{ \sum_{i=1}^{N} c_i e_i : c_i = 0, \pm 1/m, \pm 2/m, \dots, \pm m/m \right\}.$$

The **first step** of our proof is to prove the following formula

$$h_A(\phi) = \sup\{H_A(\phi, E_m) : m \in \mathbb{N}_+\}. \tag{3.1}$$

(see [15, Proposition 4.2]). Notice that, by Lemma 2.5, we can substitute the family $\{E_m : m \in \mathbb{N}_+\}$ in the above formula, by $\{E_m : m \in \mathcal{N}\}$ for any infinite subset $\mathcal{N}$ of $\mathbb{N}_+$. In fact, one of the subtle parts of the proof is to find a suitable $\mathcal{N} \subseteq \mathbb{N}_+$ for which we are able to compute $H_A(\phi, E_m)$, for all $m \in \mathcal{N}$.

The **second step** of the proof consists in computing what we have called the *p-adic contributions* to $H_A(\phi, E_m)$, that is,

$$H^p(\phi, E_m) = \limsup_{n \to \infty} \frac{\log \mu_p(T_n(\phi, E_m) + D_p)}{n},$$

for any $p \in \mathcal{P}$, where $\mu_p$ is a Haar measure on $\mathbb{Q}_p^N$, identifying $T_n(\phi, E_m)$, and letting $D_p$ be a suitable ball in $\mathbb{Q}_p^N$. We can prove in [15, Proposition 4.13] that there exists an infinite set $\mathcal{N}_1(\phi)$ of $\mathbb{N}_+$ such that

$$H^p(\phi, E_m) = h_A(\phi_p) \tag{3.2}$$

holds for all $p \in \mathbb{P}$ and $m \in \mathcal{N}_1(\phi)$. Furthermore, by the results contained in [15, Section 3], we know that

$$h_A(\phi_p) = \sum_{|\lambda_i|_p > 1} \log |\lambda_i|_p, \tag{3.3}$$

where $\{\lambda_i : i = 1, \dots, N\}$ are the eigenvalues of $\phi_p$ in some finite extension $K_p$ of $\mathbb{Q}_p$. Putting together (3.2) and (3.3) we can compute the entropy of the $p$-adic contributions.

In the **third step** of the proof we express $h_A(\phi)$ as sum of $p$-adic contributions. This is probably the most technical part of our proof and it also marks the main difference between our approach to the Algebraic Yuzvinski Formula and the methods used in the proof of the classical Yuzvinski Formula given by Lind and Ward in [23] (see §3.2). Indeed, for any $m \in \mathcal{N}_1(\phi)$, we consider the following subset of $\mathbb{P}$

$$\mathcal{P}(\phi, m) = \{p \in \mathbb{P} : p \text{ divides } a_{ij} \text{ for some } 1 \le i, j \le N\} \cup \{p \in \mathbb{P} : p \text{ divides } m\} \cup \{\infty\}$$

where $M_\phi = (a_{ij})_{i,j}$ is the matrix of $\phi$. It follows essentially from the definitions that $H^p(\phi, E_m) = 0$ for every $p \in \mathbb{P} \setminus \mathcal{P}(\phi, 1)$. Then, we embed diagonally $\mathbb{Q}^N$ in the finite product $\prod_{p \in \mathcal{P}(\phi,m)} \mathbb{Q}_p^N$, which is an LCA group, and we identify $T_n(\phi, E_m)$ with a subset of $\prod_{p \in \mathcal{P}(\phi,m)} \mathbb{Q}_p^N$. Furthermore, we can choose the balls $D_p \subseteq \mathbb{Q}_p^N$ in the definition of the $p$-adic contributions in such a way that

$$|T_n(\phi, E_m)| = \mu^\star(T_n^\star(\phi, E_m)),$$

where $T_n^\star(\phi, E_m) = T_n(\phi, E_m) + D^\star$, with $D^\star = \prod_{p\in\mathcal{P}(\phi,m)} D_p$, and where $\mu^\star$ is the Haar measure in $\prod_{p\in\mathcal{P}(\phi,m)} \mathbb{Q}_p^N$ such that $\mu^\star(D^\star) = 1$. This allows us to prove that

$$H_A(\phi, E_m) \leq \sum_{p\in\mathcal{P}(\phi,m)} H^p(\phi, E_m) = \sum_{p\in\mathbb{P}} H^p(\phi, E_m) \quad \text{for all } m \in \mathcal{N}_1(\phi).$$

On the other hand, we can find an infinite subset $\mathcal{N}(\phi)$ of $\mathcal{N}_1(\phi)$ such that $\sum_{p\in\mathcal{P}(\phi,m)} H^p(\phi, E_m) \leq H_A(\phi, E_m)$. So we get the equality

$$H_A(\phi, E_m) = \sum_{p\in\mathcal{P}(\phi,m)} H^p(\phi, E_m) = \sum_{p\in\mathbb{P}} H^p(\phi, E_m) = \sum_{p\in\mathbb{P}} h_A(\phi_p). \tag{3.4}$$

Since the quantity on the right hand side does not depend on $m$, we conclude that

$$h_A(\phi) = \sum_{p\in\mathbb{P}} h_A(\phi_p) = \sum_{p\in\mathbb{P}} H^p(\phi, E_m) = \sum_{p\in\mathbb{P}} \left( \sum_{|\lambda_i^{(p)}|_p > 1} \log |\lambda_i^{(p)}|_p \right). \tag{3.5}$$

The **fourth** and last **step** is to decompose the Mahler measure $m(\phi)$ as a sum of “$p$-adic contributions”, similar to the decomposition of the algebraic entropy $h_A(\phi)$ obtained in the third step. In fact, given a primitive polynomial $f(X) = sX^N + a_1X^{N-1} + \ldots + a_N \in \mathbb{Z}[X]$, the a priori “mysterious" term $\log|s|$ appearing in the definition of the Mahler measure, can be rewritten as

$$\log|s| = \sum_{p\in\mathbb{P}\setminus\{\infty\}} \sum_{|\lambda_i^{(p)}|_p > 1} \log |\lambda_i^{(p)}|_p,$$

where, for every $p \in \mathbb{P}$, $\{\lambda_i^{(p)} : i = 1, \ldots, N\}$ are the roots of $f(X)$ in some finite extension $K_p$ of $\mathbb{Q}_p$, when $f(X)$ is considered as an element of $\mathbb{Q}_p[X]$. Consequently,

$$m(f(X)) = \sum_{p\in\mathbb{P}} \left( \sum_{|\lambda_i^{(p)}|_p > 1} \log |\lambda_i^{(p)}|_p \right). \tag{3.6}$$

This decomposition can be deduced, for example, from the main results of [23].

Combining (3.5) and (3.6) we obtain immediately the following theorem, covering the Algebraic Yuzvinski Formula.

**Theorem 3.1.** *Let $\phi : \mathbb{Q}^N \to \mathbb{Q}^N$ be an endomorphism. Then there exists an infinite subset $\mathcal{N}(\phi)$ of $\mathbb{N}$ such that the following formula holds for all $m \in \mathcal{N}(\phi)$:*

$$h_A(\phi) = H_A(\phi, E_m) = m(p_\phi(X)),$$

*where $p_\phi(X)$ is the characteristic polynomial of $\phi$ over $\mathbb{Z}$.*

## 3.2 Comparison with the topological case

Let $N$ be a positive integer and consider an automorphism $\psi : \widehat{\mathbb{Q}}^N \to \widehat{\mathbb{Q}}^N$. Recall that the action of $\psi$ on $\widehat{\mathbb{Q}}^N$ is given by an $N \times N$ matrix with coefficients in $\mathbb{Q}$.

In the proof of the classical Yuzvinski Formula given in [23], a fundamental step is to decompose the topological entropy $h_T(\psi)$ as the sum of the entropies $h_T(\psi_p)$ (with $p \in \mathbb{P}$), where $\psi_p : {\mathbb{Q}_p}^N \to {\mathbb{Q}_p}^N$ is an automorphism with the same matrix of $\psi$ (see (3.7) below and [23, Theorem 1]). The formula obtained by Lind and Ward is analogous to our decomposition (3.5) for the algebraic entropy, which is the crucial point in the proof of the Algebraic Yuzvinski Formula.

In this subsection we recall some of the arguments used in [23] to stress the difference with our methods.

Following [23] and [34], consider the *adele ring* $\mathbb{Q}_{\mathbb{A}}$ of $\mathbb{Q}$, that is the restricted product

$$\mathbb{Q}_{\mathbb{A}} = \left\{ x \in \prod_{p \in \mathbb{P}} \mathbb{Q}_p : |x_p|_p \leq 1 \text{ for all but a finite number of } p \right\}.$$

For a finite $\mathcal{P} \subseteq \mathbb{P}$, let

$$\mathbb{Q}_{\mathbb{A}}(\mathcal{P}) = \{ x \in \mathbb{Q}_{\mathbb{A}} : |x_p|_p \leq 1 \text{ if } p \notin \mathcal{P} \}.$$

Then $\mathbb{Q}_{\mathbb{A}} = \bigcup_{\mathcal{P} \subseteq \mathbb{P} \text{ finite}} \mathbb{Q}_{\mathbb{A}}(\mathcal{P})$. Notice that each $\mathbb{Q}_{\mathbb{A}}(\mathcal{P})$ endowed with the product topology inherited from $\prod_{p \in \mathbb{P}} \mathbb{Q}_p$ is locally compact. In particular, if $\mathcal{P}$ contains $\infty$, then $\mathbb{Q}_{\mathbb{A}}(\mathcal{P})$ is an LCA group isomorphic to $\prod_{p \in \mathcal{P}} \mathbb{Q}_p \times \prod_{p \in \mathbb{P} \setminus \mathcal{P}} \mathbb{Z}_p$. Furthermore, $\mathbb{Q}_{\mathbb{A}}$ with the coarsest topology making each of the $\mathbb{Q}_{\mathbb{A}}(\mathcal{P})$ open is locally compact as well. Let $\alpha : \mathbb{Q} \to \mathbb{Q}_{\mathbb{A}}$ be the diagonal embedding of $\mathbb{Q}$ in $\mathbb{Q}_{\mathbb{A}}$. Then $\alpha(\mathbb{Q})$ is discrete in $\mathbb{Q}_{\mathbb{A}}$ and $\mathbb{Q}_{\mathbb{A}}/\alpha(\mathbb{Q})$ is topologically isomorphic to $\widehat{\mathbb{Q}}$ (see [23, Lemma 4.1]). Consequently, $\alpha^N(\mathbb{Q}^N)$ is discrete in $\mathbb{Q}_{\mathbb{A}}^N$ and $\mathbb{Q}_{\mathbb{A}}^N/\alpha^N(\mathbb{Q}^N)$ is topologically isomorphic to $\widehat{\mathbb{Q}}^N$, thus $\alpha^N(\mathbb{Q}^N)$ is a uniform discrete subset of $\mathbb{Q}_{\mathbb{A}}^N$.

Letting the matrix of $\psi$ act on $\mathbb{Q}_{\mathbb{A}}^N$, we obtain an endomorphism $\widetilde{\psi} : \mathbb{Q}_{\mathbb{A}}^N \to \mathbb{Q}_{\mathbb{A}}^N$. Using the strong Theorem 2.4, Lind and Ward show that

$$h_T(\psi) = h_T(\widetilde{\psi}).$$

As a second step they verify that the topological entropy of $\widetilde{\psi}$ coincides with the topological entropy of its restriction $\widetilde{\psi}_{\mathcal{P}}$ to $\mathbb{Q}_{\mathbb{A}}(\mathcal{P})^N$, for a suitable finite subset $\mathcal{P}$ of $\mathbb{P}$ containing $\infty$. Notice that $\mathcal{P}$ contains (in general properly) the set $\mathcal{P}(\widehat{\psi}, 1)$ that we used.

Finally, after noticing that the restriction of $\widetilde{\psi}$ to $\prod_{p \in \mathbb{P} \setminus \mathcal{P}} \mathbb{Z}_p^N$ has zero topological entropy, they show that

$$h_T(\widetilde{\psi}) = h_T\left(\widetilde{\psi} \restriction_{\prod_{p \in \mathcal{P}} \mathbb{Q}_p^N}\right).$$

As $\widetilde{\psi} \restriction_{\prod_{p \in \mathcal{P}} \mathbb{Q}_p^N} = \prod_{p \in \mathcal{P}} \psi_p$, one easily concludes that $h_T(\psi) = \sum_{p \in \mathcal{P}} h_T(\psi_p)$. Since $h_T(\psi_p) = 0$ for every $p \in \mathbb{P} \setminus \mathcal{P}$, they can give the desired formula:

$$h_T(\psi) = \sum_{p \in \mathbb{P}} h_T(\psi_p). \tag{3.7}$$

As described above, the differences between our proof of the Algebraic Yuzvinski Formula and the proof of the Yuzvinski Formula given by Lind and Ward are forced by the fact that we do not known whether the counterpart of Theorem 2.4 holds for the algebraic entropy (see Question 4.6 below). For this reason, we had to develop in [15] several ad-hoc techniques to give a completely self-contained proof of Theorem 3.1.

# 4 Applications and open questions

In this section we describe the main results from [3], [7] and [8], which are the fundamental properties of the algebraic entropy of endomorphisms of discrete Abelian groups, underlying where the Algebraic Yuzvinski Formula is used. In general, in the proofs of these results one can separate the torsion and the torsion-free case, and reduce step by step to endomorphisms of divisible torsion-free Abelian groups of finite rank, namely, to endomorphisms of $\mathbb{Q}^N$ for some positive integer $N$; at this stage the Algebraic Yuzvinski Formula applies.

Furthermore, a lot of related open problems are discussed.

## 4.1 The category of flows

We start recalling a subcategory of the category of morphisms, which is useful in working with entropy. For a category $\mathfrak{C}$, the category $\mathrm{Flow}(\mathfrak{C})$ of *flows* of $\mathfrak{C}$ has as objects the pairs $(G, \phi)$, where $G$ is an object of $\mathfrak{C}$

and $\phi \in \mathrm{End}_{\mathfrak{C}}(G)$. Moreover, a morphism in $\mathrm{Flow}(\mathfrak{C})$ between two flows $(G, \phi)$ and $(G', \phi')$ is a morphism $u : G \to G'$ in $\mathfrak{C}$ such that the diagram

$$\begin{tikzcd} G \arrow[r, "\phi"] \arrow[d, "u"'] & G \arrow[d, "u"] \\ G' \arrow[r, "\phi'"'] & G' \end{tikzcd} \tag{4.1}$$

in $\mathfrak{C}$ commutes. When the domain $G$ of $\phi$ is clear and no confusion is possible, we denote a flow $(G, \phi)$ simply by $\phi$.

Let now $\mathfrak{LA}$ be the category of all LCA groups, and $\mathfrak{A}$ the category of all discrete Abelian groups. Usually entropy is defined as a function from the endomorphisms $\mathrm{End}_{\mathfrak{LA}}(G)$ of a given object $G$ of $\mathfrak{LA}$ to $\mathbb{R}_{\geq 0} \cup \{\infty\}$. The category of flows allows for a more precise description of entropy in categorical terms, namely, we can consider $h_A(-)$ (as well as $h_T(-)$ or $h_\infty(-)$) as a function

$$h_A : \mathrm{Flow}(\mathfrak{LA}) \to \mathbb{R}_{\geq 0} \cup \{\infty\} \text{ such that } (G, \phi) \mapsto h_A(\phi).$$

This approach makes it easier to state (and sometimes to understand) many known results.

**Example 4.1.** *It can be deduced from classical results (see for example [19, Chapter 12]) that the category* $\mathrm{Flow}(\mathfrak{A})$ *is isomorphic to the category* $\mathrm{Mod}(\mathbb{Z}[X])$ *of all* $\mathbb{Z}[X]$*-modules. Indeed, a* $\mathbb{Z}[X]$*-module* $M_{\mathbb{Z}[X]}$ *is exactly an Abelian group* $M$*, together with an endomorphism*

$$\phi_X : M \to M \text{ defined by } m \mapsto m \cdot X,$$

*representing the action of* $X$ *on* $M$*.*

A short exact sequence in $\mathrm{Flow}(\mathfrak{LA})$ is a commutative diagram in $\mathfrak{LA}$ of the form

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, equal] & A \arrow[r, "\alpha"] \arrow[d, "\phi_1"] & B \arrow[r, "\beta"] \arrow[d, "\phi_2"] & C \arrow[r] \arrow[d, "\phi_3"] & 0 \arrow[d, equal] \\ 0 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & C \arrow[r] & 0 \end{tikzcd} \tag{4.2}$$

where the two rows are short exact sequences in $\mathfrak{LA}$. In particular, $\alpha$ is a homomorphism which is also a topological embedding, while $\beta$ is a surjective open and continuous homomorphism.

## 4.2 Upper continuity

Consider a flow $(G, \phi) \in \mathrm{Flow}(\mathfrak{LA})$. If there exists a family $\{K_i : i \in I\}$ of $\phi$-invariant closed subgroups of $G$ directed by inclusion such that $G = \bigcup_{i \in I} K_i$, then $(G, \phi)$ is the direct limit of $\{(K_i, \phi\restriction_{K_i}) : i \in I\}$ in $\mathrm{Flow}(\mathfrak{LA})$.

We recall that a function $h : \mathrm{Flow}(\mathfrak{LA}) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ (respectively, $h : \mathrm{Flow}(\mathfrak{A}) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$) is *upper continuous* if, given a flow $(G, \phi)$ in $\mathrm{Flow}(\mathfrak{LA})$ (respectively, in $\mathrm{Flow}(\mathfrak{A})$) that is the direct limit of a directed system of subobjects $\{(K_i, \phi\restriction_{K_i}) : i \in I\}$ as above, $h_A(\phi) = \sup\{h_A(\phi\restriction_{K_i}) : i \in I\}$;

With this new terminology we write the following consequence of [15, Lemma 2.8], which is a result from [3].

**Corollary 4.2.** *The function* $h_A : \mathrm{Flow}(\mathfrak{A}) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ *is upper continuous.*

Notice also that [15, Lemma 2.8] is stronger than Corollary 4.2, as it holds for directed systems of invariant open subgroups of arbitrary LCA groups. Nevertheless we do not know whether it is possible to prove upper continuity of $h_A(-)$ in full generality, that is, the following question remains open:

**Question 4.3.** *Is* $h_A : \mathrm{Flow}(\mathfrak{LA}) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ *upper continuous?*

### 4.3 Addition Theorem

The first application of the Algebraic Yuzvinski Formula we take into account is a deep result from [3], called Addition Theorem, which shows the additivity of $h_A(-)$ when restricted to flows of $\mathfrak{A}$. Indeed, a function $h : \mathrm{Flow}(\mathfrak{L}\mathfrak{A}) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ (respectively, $h : \mathrm{Flow}(\mathfrak{A}) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$) is called *additive* if, given a short exact sequence as in (4.2) in $\mathrm{Flow}(\mathfrak{L}\mathfrak{A})$ (respectively in $\mathrm{Flow}(\mathfrak{A})$), $h(\phi_2) = h(\phi_1) + h(\phi_3)$.

**Theorem 4.4** (Addition Theorem). *The function $h_A : \mathrm{Flow}(\mathfrak{A}) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is additive.*

In the case of endomorphisms of torsion discrete Abelian groups, Theorem 4.4 was proved (for ent(–)) in [10]. In order to extend it to the whole $\mathrm{Flow}(\mathfrak{A})$, the Algebraic Yuzvinski Formula is needed. Notice also that the Addition Theorem for the topological entropy of endomorphisms of compact groups, that inspired its algebraic counterpart, can be obtained as a consequence of [2, Theorem 19]; moreover, it was previously proved in the metric case in [35].

Given a flow $(G, \phi) \in \mathrm{Flow}(\mathfrak{A})$, the following are particular cases of the Addition Theorem:
(1) if $(G', \phi') \cong (G, \phi)$ then $h_A(\phi) = h_A(\phi')$;
(2) if $K \leq G$ is $\phi$-invariant, then $h_A(\phi) \geq h_A(\phi \restriction_K)$;
(3) if $K \leq G$ is $\phi$-invariant, then $h_A(\phi) \geq h_A(\overline{\phi})$, where $\overline{\phi} : G/K \to G/K$ is induced by $\phi$.

Note that the stability under isomorphisms in (1) is proved for endomorphisms of LCA groups in [15, Proposition 2.6(1)]. Moreover, the monotonicity for invariant subgroups in (2) is verified for endomorphisms of LCA groups in [15, Lemma 2.4(3)], provided the subgroup $K$ is open. We have no counterpart in the general case of the monotonicity for quotients in (3).

**Problem 4.5.** *Study when $h_A : \mathrm{Flow}(\mathfrak{L}\mathfrak{A}) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is additive.*

*Is it possible at least to extend (2) or (3) above to the case when G is an LCA group, $\phi \in \mathrm{End}(G)$ and K is a $\phi$-invariant closed subgroup of G?*

Let us conclude this subsection describing a particular case of the above question. Indeed, let $G$ be an LCA group, let $\phi$ be an endomorphism and let $H \leq G$ be a $\phi$-invariant closed subgroup. If $H$ is uniform discrete, we saw in Theorem 2.4 that $h_T(\phi) = h_T(\bar{\phi})$, where $\bar{\phi} : G/H \to G/H$ is the induced map. Since $h_T(-)$ is trivial on discrete groups, this can be viewed as a particular case of some Addition Theorem for $h_T(-)$ on LCA groups. As described in §3.2, it would be of clear interest to answer the following question, suggested to us by Dikran Dikranjan, which asks for an algebraic counterpart of Theorem 2.4:

**Question 4.6.** *Given $(G, \phi) \in \mathrm{Flow}(\mathfrak{L}\mathfrak{A})$ and a uniform discrete $\phi$-invariant subgroup $K$ of $G$, is it true that $h_A(\phi) = h_A(\phi \restriction_K)$?*

Since $h_A(-)$ trivializes on compact Abelian groups, a positive answer to Problem 4.5, namely, additivity of $h_A(-)$ in the general case of flows of $\mathfrak{L}\mathfrak{A}$, would answer positively to Question 4.6 as well.

### 4.4 Algebraic entropy of torsion-free Abelian groups

As a second application, we show how the Algebraic Yuzvinski Formula can be used to compute the algebraic entropy of any endomorphism of a torsion-free discrete Abelian group. The first step is to extend a given endomorphism to an endomorphism of a rational vector space with the same algebraic entropy:

**Lemma 4.7.** *[3, Proposition 2.12] Let G be a torsion-free discrete Abelian group and $\phi : G \to G$ an endomorphism. Denote by $D(G) = G \otimes_{\mathbb{Z}} \mathbb{Q}$ the divisible hull of G and by $\phi \otimes_{\mathbb{Z}} id_{\mathbb{Q}} = \widetilde{\phi} : D(G) \to D(G)$ the unique extension of $\phi$ to $D(G)$. Then $h_A(\phi) = h_A(\widetilde{\phi})$.*

*Proof.* Note that $D(G)$ is the direct limit of its $\widetilde{\phi}$-invariant subgroups of the form $\frac{1}{n!}G$, with $n$ ranging in $\mathbb{N}$. Hence, by Corollary 4.2 we have that $h_A(\widetilde{\phi})$ is the supremum of the algebraic entropies of the restrictions of $\widetilde{\phi}$ to each $\frac{1}{n!}G$. Furthermore, the groups $\frac{1}{n!}G$ are all isomorphic to $G$ and the action of $\widetilde{\phi}$ on $\frac{1}{n!}G$ is conjugated to the action of $\phi$. Hence

$$h_A(\widetilde{\phi}) = \sup\left\{h_A\left(\phi\restriction_{\frac{1}{n!}G}\right) : n \in \mathbb{N}\right\} = h_A(\phi). \qquad \square$$

Note that the Algebraic Kolmogorov-Sinai Formula stated in Section 2 can be proved using Lemma 4.7 and the Algebraic Yuzvinski Formula.

We remark that Lemma 4.7 is the exact counterpart of [23, Proposition 3.1] that allows the computation of the topological entropy of a solenoidal automorphism extending it to an automorphism of a full solenoid which has the same topological entropy.

Let $G$ be a torsion-free discrete Abelian group and let $\phi$ be an endomorphism of $G$. To compute the algebraic entropy of $\phi$, we can suppose $G$ to be a rational vector space in view of Lemma 4.7. One can realize the group $G$ as the union of a continuous chain of $\phi$-invariant subspaces

$$0 = K_0 \subseteq K_1 \subseteq K_2 \subseteq \ldots \subseteq K_\sigma = G \tag{4.3}$$

for some ordinal $\sigma$; denote by $\phi_\gamma$ the endomorphism of $K_{\gamma+1}/K_\gamma$ induced by $\phi$, for all $\gamma < \sigma$. It is shown in [3] that the chain in (4.3) can be constructed in such a way that either $K_{\gamma+1}/K_\gamma$ is finite dimensional or it is infinite dimensional and $\phi_\gamma$ is conjugated to the right Bernoulli shift $\beta_{\mathbb{Q}}$ described in Example 2.6, for all $\gamma < \sigma$. Additivity and upper continuity of $h_A(-)$ allow one to prove by transfinite induction that

$$h_A(\phi) = \sum_{\gamma<\sigma} h_A(\phi_\gamma),$$

where the algebraic entropy of each $\phi_\gamma$ is either infinite or it can be computed using the Algebraic Yuzvinski Formula.

In the above discussion we could determine the algebraic entropy of an endomorphism of a torsion-free discrete Abelian group $G$ just using additivity, upper continuity and the specific values that $h_A(-)$ takes on the right Bernoulli shifts and on endomorphisms of finite dimensional rational vector spaces (given by the Algebraic Yuzvinski Formula). Using more carefully the same arguments as above, one can prove the following result from [3].

**Theorem 4.8** (Uniqueness Theorem). *The algebraic entropy $h_A : \mathrm{Flow}(\mathfrak{A}) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is the unique function such that:*
*(1) $h_A(-)$ is additive;*
*(2) $h_A(-)$ is upper continuous;*
*(3) $h_A(\beta_K) = \log|K|$ for any finite Abelian group $K$ (where $\beta_K$ is the right Bernoulli shift defined in Example 2.6);*
*(4) the Algebraic Yuzvinski Formula holds for $h_A(-)$.*

The first three axioms in the Uniqueness Theorem are enough to characterize the algebraic entropy in the class of torsion discrete Abelian groups as it was proved in [10, Theorem 6.1]. More precisely, other two axioms appeared in the Uniqueness Theorem from [10], that is, invariance under conjugation and a logarithmic law; nevertheless, it can be proved that these two axioms are not necessary.

Moreover, it is worth to note that the Uniqueness Theorem for the algebraic entropy was inspired by the Uniqueness Theorem proved by Stojanov [30] for the topological entropy in the class of compact (non-necessarily Abelian) groups.

## 4.5 Bridge Theorem

In 1974 Weiss [33] studied the basic properties of ent(–) and connected it with the topological entropy of endomorphisms of profinite Abelian groups via the Pontryagin-Van Kampen duality in the following “Bridge

Theorem". Recall that, for an LCA group $G$, we denote by $\widehat{G}$ the dual group of $G$, endowed with its compact-open topology, while for an endomorphism $\phi : G \to G$, we denote by $\widehat{\phi} : \widehat{G} \to \widehat{G}$ its dual endomorphism.

**Weiss Bridge Theorem.** *Let $G$ be a torsion discrete Abelian group and $\phi : G \to G$ an endomorphism. Then* $\mathrm{ent}(\phi) = h_T(\widehat{\phi})$.

Peters proved the following Bridge Theorem for automorphisms of countable discrete Abelian groups.

**Peters Bridge Theorem.** *Let $G$ be a countable discrete Abelian group and $\phi : G \to G$ an automorphism. Then* $h_\infty(\phi) = h_T(\widehat{\phi})$.

A third application of the Algebraic Yuzvinski Formula is the following general form of the Bridge Theorem proved in [7], that extends to all endomorphisms of discrete Abelian groups both Weiss Bridge Theorem and Peters Bridge Theorem.

**Theorem 4.9** (Bridge Theorem)**.** *Let* $(G, \phi) \in \mathrm{Flow}(\mathfrak{A})$. *Then* $h_A(\phi) = h_T(\widehat{\phi})$.

The torsion case of this theorem is covered by Weiss Bridge Theorem. Then additivity and upper continuity of $h_A(-)$, and additivity and "continuity on inverse limits" of $h_T(-)$, are applied to restrict to the case of automorphisms of $\mathbb{Q}^N$ and of its dual $\widehat{\mathbb{Q}}^N$. At this stage the Algebraic Yuzvinski Formula and the Yuzvinski Formula conclude the proof.

The problem of whether it is possible to extend the Bridge Theorem to the whole $\mathrm{Flow}(\mathfrak{L}\mathfrak{A})$ is open (see also [7]):

**Problem 4.10.** *Study whether the Bridge Theorem holds for every* $(G, \phi) \in \mathrm{Flow}(\mathfrak{L}\mathfrak{A})$. *Classify the flows* $(G, \phi) \in \mathrm{Flow}(\mathfrak{L}\mathfrak{A})$ *such that* $h_A(\phi) = h_T(\widehat{\phi})$.

It was recently proved a partial solution of this problem in [4].

**Theorem 4.11.** *Let $G$ be a totally disconnected locally compact Abelian group and $\phi : G \to G$ a continuous endomorphism. Then*

$$h_T(\phi) = h_A(\widehat{\phi}).$$

## 4.6 Dichotomy Theorem

We conclude with a last application of the Algebraic Yuzvinski Formula coming from [8].

Consider a fixed LCA group $G$, a Haar measure $\mu$ on $G$, and an endomorphism $\phi : G \to G$. For every $C \in \mathcal{C}(G)$ we can define a sequence

$$\tau_C : \mathbb{N}_+ \to \mathbb{R}_{\geq 0} \text{ such that } n \mapsto \tau_C(n) = \mu(T_n(\phi, C)).$$

Following [3], we say that the $\phi$-trajectory of $C$ converges exactly when the sequence

$$\left\{ \frac{\log \tau_C(n)}{n} : n \in \mathbb{N} \right\}$$

is convergent. It is very useful to know whether the $\phi$-trajectory of $C$ converges for every $C \in \mathcal{C}(G)$, indeed, for example in this case the lim sup in (1.1) becomes a limit. This occurs when $G$ is compact (the above sequences converge to 0), discrete (see for example [3, Corollary 2.2]), $G = \mathbb{R}^N$ or $G = \mathbb{Q}_p^N$ with $N$ a positive integer and $p$ a prime. We do not know if this holds in general:

**Question 4.12.** *Let* $(G, \phi) \in \mathrm{Flow}(\mathfrak{L}\mathfrak{A})$ *and* $C \in \mathcal{C}(G)$. *Does the $\phi$-trajectory of $C$ converge?*

*If this is not true in general, classify the flows* $(G, \phi) \in \mathrm{Flow}(\mathfrak{L}\mathfrak{A})$ *such that the $\phi$-trajectory of $C$ converges for every* $C \in \mathcal{C}(G)$.

Now we restrict to the context of discrete Abelian groups. Following [8], fix a flow $(G, \phi) \in \mathrm{Flow}(\mathfrak{A})$; as usual we consider on $G$ the Haar measure given by the cardinality of subsets. Hence, given $F \in \mathcal{C}(G)$, the sequence defined above becomes

$$\tau_F : \mathbb{N}_+ \to \mathbb{R}_{\geq 0} \text{ such that } n \mapsto \tau_F(n) = |T_n(\phi, F)|.$$

For every $n \in \mathbb{N}_+$, $\tau_F(n) \leq |F|^n$, thus the sequence $\{\tau_F(n) : n \in \mathbb{N}_+\}$ has at most exponential growth. This justifies the following definitions given in [8]:

1. $(G, \phi)$ has *exponential growth* at $F$ if there exists $b \in \mathbb{R}$, $b > 1$, such that $\tau_F(n) \geq b^n$ for every $n \in \mathbb{N}_+$;
2. $(G, \phi)$ has *polynomial growth* at $F$ if there exists $P_F(X) \in \mathbb{Z}[X]$ such that $\tau_F(n) \leq P_F(n)$ for every $n \in \mathbb{N}_+$.

If the growth of $(G, \phi)$ at $F$ is polynomial, then $H_A(\phi, F) = 0$. On the other hand, if the growth of $(G, \phi)$ at $F$ is exponential, then $H_A(\phi, F) \neq 0$. Nevertheless, for an arbitrary sequence there are a lot of possible growths between polynomial and exponential. One of the main results from [8] states that this is not the case for sequences of the form $\{\tau_F(n) : n \in \mathbb{N}_+\}$:

**Theorem 4.13** (Dichotomy Theorem). *Let* $(G, \phi) \in \mathrm{Flow}(\mathfrak{A})$ *and* $F \in \mathcal{C}(G)$. *Then:*
*(1)* $H(\phi, F) = 0$ *if and only if* $(G, \phi)$ *has polynomial growth at* $F$*;*
*(2)* $H(\phi, F) > 0$ *if and only if* $(G, \phi)$ *has exponential growth at* $F$.
*In particular,* $(G, \phi)$ *has either exponential or polynomial growth at* $F$.

To prove this theorem the following consequence of the Algebraic Yuzvinski Formula is applied in [8]. In particular, it is used to find non-zero periodic points of an automorphism of $\mathbb{Q}^N$ with zero algebraic entropy.

**Corollary 4.14.** *Let* $N$ *be a positive integer and* $\phi : \mathbb{Q}^N \to \mathbb{Q}^N$ *an automorphism. If* $h_A(\phi) = 0$, *then all the eigenvalues of* $\phi$ *are roots of unity.*

Indeed the Algebraic Yuzvinski Formula implies that the characteristic polynomial $p_\phi(X)$ of such $\phi$ over $\mathbb{Z}$ is monic and all the roots $\{\lambda_i : i = 1, \dots, N\} \subseteq \mathbb{C}$ of $p_\phi(X)$ have $|\lambda_i| \leq 1$. Now Kronecker Theorem [21] implies that all $|\lambda_i| = 1$.

For more details about this topic see [8]. We conclude with the following general problem.

**Problem 4.15.** *Extend (when it is possible) the results from [8] to the general case of* $\mathrm{Flow}(\mathfrak{L}\mathfrak{A})$.
*Does the dichotomy of growths hold also in this general case or is it possible to find flows of intermediate growth in* $\mathrm{Flow}(\mathfrak{L}\mathfrak{A})$*?*

Let us remark that the growth of flows is strictly related to the classical notion of growth rate of finitely generated groups that was introduced in geometric group theory by Milnor. In fact, as explained in [6], the notion of algebraic entropy can be extended to (locally compact) groups that are not necessarily abelian. Then, also the notion of growth of flows recalled above is extended to the non-abelian case, and this general notion extends Milnor's growth rate to flows.

Answering Milnor's famous problem, Gromov proved that a finitely generated group has polynomial growth if and only if it has a nilpotent subgroup of finite index. On the other hand, Grigorchuck furnished his famous example of a finitely generated group of intermediate growth. In this context, Theorem 4.13 appears as a natural extension of Gromov's result in the commutative setting.

# 5 Algebraic Yuzvinski Formula for the intrinsic algebraic entropy

Let us start recalling the definition of intrinsic algebraic entropy given in [9]. Indeed, let $G$ be an Abelian group and let $\phi : G \to G$ an endomorphism. A subgroup $H$ of $G$ is $\phi$*-inert* if $(H + \phi(H))/H$ is finite. Recall that, by [9, Lemma 2.4], if $G$ is a torsion-free Abelian group of finite rank and if $H$ is a finitely generated subgroup of $G$ of maximal rank, then $H$ is $\phi$-inert.

Given a $\phi$-inert subgroup $H$ of $G$, the *intrinsic algebraic entropy of $\phi$ with respect to $H$* is

$$\widetilde{\mathrm{ent}}(\phi, H) = \lim_{n\to\infty} \frac{\log |T_n(\phi, H)/H|}{n}.$$

The *intrinsic algebraic entropy* of $\phi$ is

$$\widetilde{\mathrm{ent}}(\phi) = \sup\{\widetilde{\mathrm{ent}}(\phi, H) : H \text{ is } \phi\text{-inert}\}.$$

As proved in [9, Proposition 3.6], the inequalities $\mathrm{ent}(\phi) \le \widetilde{\mathrm{ent}}(\phi) \le h_A(\phi)$ hold true, and the three notions of algebraic entropy coincide in case $G$ is torsion.

The following result proved in [9, Theorem 4.2] is an Algebraic Yuzvinski Formula for the intrinsic algebraic entropy. The aim of this section is to give a completely algebraic and self-contained proof of this fact, using a different approach to the one adopted in [9].

**Theorem 5.1.** *If $N$ is a positive integer, $\phi : \mathbb{Q}^N \to \mathbb{Q}^N$ is an endomorphism and $s$ is the leading coefficient of the characteristic polynomial $p_\phi(X)$ of $\phi$ over $\mathbb{Z}$, then*

$$\widetilde{\mathrm{ent}}(\phi) = \log s.$$

The following properties of the intrinsic algebraic entropy are proved in [9]:

**Fact 5.2.** *Let $G$ be an Abelian group and let $\phi : G \to G$ be an endomorphism. The following statements hold true:*
*(1) if $H \le G$ is finitely generated, $\phi$-inert and such that $G = T(\phi, H)$, then $\widetilde{\mathrm{ent}}(\phi) = \widetilde{\mathrm{ent}}(\phi, H)$;*
*(2) if $\{G_i\}_{i\in I}$ is a directed family of $\phi$-invariant subgroups of $G$ such that $\bigcup_{i\in I} G_i = G$, then $\widetilde{\mathrm{ent}}(\phi) = \sup_{i\in I} \widetilde{\mathrm{ent}}(\phi\restriction_{G_i})$.*

Notice that the second fact above is equivalent to say that the function $\widetilde{\mathrm{ent}}$ is upper continuous on $\mathrm{Flow}(\mathfrak{A})$. Furthermore, a useful consequence of Fact 5.2 is the following.

**Corollary 5.3.** *Let $N$ be a a positive integer, let $V = \mathbb{Q}^N$ and let $\phi : V \to V$ be an endomorphism. If $H$ is a finitely generated subgroup of $V$ of maximal rank, then $\widetilde{\mathrm{ent}}(\phi) = \widetilde{\mathrm{ent}}(\phi, H)$.*

*Proof.* By property (1) in Fact 5.2, $\widetilde{\mathrm{ent}}(\phi, H) = \widetilde{\mathrm{ent}}(\phi\restriction_{T(\phi,H)})$. Furthermore, since $H$ has maximal rank, $\mathbb{Q} \otimes_{\mathbb{Z}} T(\phi, H) = V$. Using property (2) in Fact 5.2 and arguing as in Lemma 4.7, this shows that $\widetilde{\mathrm{ent}}(\phi\restriction_{T(\phi,H)}) = \widetilde{\mathrm{ent}}(\phi)$. □

The following is a limit-free formula for the computation of the intrinsic algebraic entropy of automorphisms of Abelian groups. It generalizes its counterpart for the algebraic entropy of automorphisms of torsion Abelian groups stated in [35] (see also [5] for a proof).

**Lemma 5.4.** *Let $G$ be an Abelian group and let $\phi : G \to G$ be an automorphism. Let $H$ be a finitely generated $\phi$-inert subgroup of $G$. Then*

$$\widetilde{\mathrm{ent}}(\phi, H) = \log |\phi(T(\phi^{-1}, H)/T(\phi^{-1}, H))|.$$

*Proof.* By [9, Lemma 3.2],

$$\widetilde{\mathrm{ent}}(\phi, H) = \lim_{n\to\infty} \log \left| \frac{T_{n+1}(\phi, H)}{T_n(\phi, H)} \right|. \tag{5.1}$$

Furthermore, for every $n \in \mathbb{N}_+$,

$$\frac{T_{n+1}(\phi, H)}{T_n(\phi, H)} \cong \frac{\phi^n(H)}{\phi^n(H) \cap T_n(\phi, H)} \cong \frac{\phi(H)}{\phi(H) \cap T_n(\phi^{-1}, H)}.$$

Since $\phi(H) \cap T(\phi^{-1}, H)$ is a subgroup of a finitely generated (i.e., Noetherian) group, it follows that the increasing sequence of subgroups $\{\phi(H) \cap T_n(\phi^{-1}, H)\}_{n\in\mathbb{N}}$ stabilizes. Hence, $\phi(H) \cap T(\phi^{-1}, H) = \phi(H) \cap T_n(\phi^{-1}, H)$ for all $n \in \mathbb{N}_+$ large enough. Therefore, for all $n \in \mathbb{N}_+$ large enough,

$$\frac{\phi(H)}{\phi(H) \cap T_n(\phi^{-1}, H)} \cong \frac{\phi(H)}{\phi(H) \cap T(\phi^{-1}, H)} \cong \frac{\phi(T(\phi^{-1}, H))}{T(\phi^{-1}, H)}.$$

Now (5.1) gives the thesis. □

In the following proposition we show the additivity of $\widetilde{\mathrm{ent}}$ on flows of divisible Abelian groups of finite rank. We use this result to give a direct proof of Theorem 5.1 deducing it from Proposition 5.6 below. Notice that Proposition 5.5 also follows from a more general Addition Theorem in [9], but the proof in [9] depends on Theorem 5.1.

**Proposition 5.5.** *Let $N$ be a positive integer, let $V = \mathbb{Q}^N$ and let $\phi : V \to V$ be an endomorphism. If $W \leq V$ is a $\phi$-invariant subspace, then*

$$\widetilde{\mathrm{ent}}(\phi) = \widetilde{\mathrm{ent}}(\phi \restriction_W) + \widetilde{\mathrm{ent}}(\bar{\phi}),$$

*where $\bar{\phi} : V/W \to V/W$ is the map induced on the quotient.*

*Proof.* Let us start noticing that, since $\ker(\phi^n) \subseteq \ker(\phi^{n+1})$ for every natural number $n$, there exists $k \leq N$ such that $\ker(\phi^k) = \ker(\phi^{k+1}) = K$. Denote by

$$\psi : V/K \to V/K \quad \psi_W : (W+K)/K \to (W+K)/K \quad \bar{\psi} : V/(W+K) \to V/(W+K)$$

the maps induced by $\phi$; these three maps are automorphisms (they are injective endomorphisms of finite dimensional vector spaces). We claim that

$$\widetilde{\mathrm{ent}}(\phi) = \widetilde{\mathrm{ent}}(\psi) \quad \widetilde{\mathrm{ent}}(\phi \restriction_W) = \widetilde{\mathrm{ent}}(\psi_W) \quad \widetilde{\mathrm{ent}}(\bar{\phi}) = \widetilde{\mathrm{ent}}(\bar{\psi}),$$

Let us prove that $\widetilde{\mathrm{ent}}(\phi) = \widetilde{\mathrm{ent}}(\psi)$, the other two equalities follow similarly. Indeed, let $H \leq V$ be a finitely generated subgroup of $V$ of maximal rank; then $(H+K)/K$ is a subgroup of $V/K$ of maximal rank (indeed, $\mathrm{rk}(V/K) - \mathrm{rk}((H+K)/K) = \mathrm{rk}(V/(H+K)) \leq \mathrm{rk}(V/H) = 0$). Let also $K' = K \cap T(\phi, H)$. Using the Noetherianity of $\mathbb{Z}[X]$, one can find a finitely generated subgroup $F \subseteq K'$ such that $K' = T(\phi, F)$. Since by construction $F \subseteq K$, $T(\phi, F) = T_k(\phi, F)$ and so $K'$ is a finitely generated group. Thus, $H' = H + K'$ is a finitely generated subgroup of $V$ of maximal rank, and so

$$\widetilde{\mathrm{ent}}(\phi) = \widetilde{\mathrm{ent}}(\phi, H) = \widetilde{\mathrm{ent}}(\phi, H') \quad \text{and} \quad \widetilde{\mathrm{ent}}(\psi) = \widetilde{\mathrm{ent}}(\psi, (H+K)/K)$$

by Corollary 5.3. It remains to show that $\widetilde{\mathrm{ent}}(\phi, H') = \widetilde{\mathrm{ent}}(\psi, (H+K)/K)$, but this follows easily by the following sequence of isomorphisms, that holds for all $n \in \mathbb{N}_+$:

$$\begin{aligned}
\frac{T_n(\phi, H')}{H'} &\cong \frac{T_n(\phi, H) + (K \cap T(\phi, H))}{H + (K \cap T(\phi, H))} \\
&\cong \frac{(T_n(\phi, H) + (K \cap T(\phi, H)))/(K \cap T(\phi, H))}{(H + (K \cap T(\phi, H)))/(K \cap T(\phi, H))} \\
&\cong \frac{T_n(\phi, H)/(T_n(\phi, H) \cap K \cap T(\phi, H))}{H/(H \cap K \cap T(\phi, H))} \\
&= \frac{T_n(\phi, H)/(T_n(\phi, H) \cap K)}{H/(H \cap K)} \\
&\cong \frac{T_n(\psi, (H+K)/K)}{(H+K)/K}.
\end{aligned}$$

Suppose now that $\phi$ is an automorphism, this can be done by the above discussion. In this case also $\phi \restriction_W$ and $\bar{\phi}$ are automorphisms. Choose two finitely generated subgroups $H_1 \leq W$ and $\overline{H}_2 \leq V/W$ of maximal rank and let $H \leq V$ be a finitely generated subgroup of maximal rank such that $H_1 \leq H$ and $H + W/W = \overline{H}_2$. By Corollary 5.3,

$$\widetilde{\mathrm{ent}}(\phi) = \widetilde{\mathrm{ent}}(\phi, H),\ \widetilde{\mathrm{ent}}(\phi\restriction_W) = \widetilde{\mathrm{ent}}(\phi, H \cap W) \text{ and } \widetilde{\mathrm{ent}}(\bar{\phi}) = \widetilde{\mathrm{ent}}(\bar{\phi}, H + W/W)\,.$$

Let also $H' = T(\phi^{-1}, H)$; by Proposition 5.4 we have $\widetilde{\mathrm{ent}}(\phi, H) = \log|\phi(H')/H'|$. Similarly, one has

$$\widetilde{\mathrm{ent}}(\phi\restriction_W, H \cap W) = \log|(\phi(H') \cap W)/(H' \cap W)|$$

and

$$\widetilde{\mathrm{ent}}(\bar{\phi}, H/(H \cap W)) = \log|(\phi(H') + W)/(H' + W)|.$$

Consider now the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & W \cap H' \arrow[r] \arrow[d] & H' \arrow[r] \arrow[d] & (H' + W)/W \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \phi(W \cap H') \arrow[r] & \phi(H') \arrow[r] & (\phi(H') + W)/W \arrow[r] & 0
\end{tikzcd}$$

where the columns are the canonical inclusions. Using the Snake Lemma one obtains directly that

$$\widetilde{\mathrm{ent}}(\phi) = \log|\phi(H')/H'| = \log|\phi(W \cap H')/W \cap H'| + \log|(\phi(H') + W)/H' + W| = \mathrm{ent}(\phi\restriction_W) + \mathrm{ent}(\bar{\phi})\,,$$

so the thesis. □

In the following proposition we prove a particular case of Theorem 5.1, from which we will deduce the theorem using Proposition 5.5.

In what follows we use the well-known fact that, if $\phi : \mathbb{Q}^N \to \mathbb{Q}^N$ is an endomorphism, the characteristic polynomial $p_\phi(X)$ of $\phi$ over $\mathbb{Z}$ is irreducible if and only if the only proper $\phi$-invariant subspace of $V$ is 0.

**Proposition 5.6.** *Let N be a positive integer and let $\phi : \mathbb{Q}^N \to \mathbb{Q}^N$ be an endomorphism whose characteristic polynomial over $\mathbb{Z}$*

$$p_\phi(X) = sX^N + a_1X^{N-1} + \cdots + a_{N-1}X + a_N$$

*is irreducible of degree $N$. Then $\widetilde{\mathrm{ent}}(\phi) = \log s$.*

*Proof.* Let $Z \leq \mathbb{Q}^N$ be a free subgroup of rank 1. We prove first that

$$\mathrm{rk}(T_N(\phi, Z)) = N,$$

that is, $F = T_N(\phi, Z)$ has maximal rank. In fact, since $p_\phi(X)$ is irreducible, the only proper $\phi$-invariant subspace of $\mathbb{Q}^N$ is the trivial one. Hence, the $\phi$-invariant subspace $T(\phi, D(Z)) = D(T(\phi, Z))$ is the whole $\mathbb{Q}^N$, so $\mathrm{rk}(T(\phi, Z)) = N$. If, looking for a contradiction, $\mathrm{rk}(T_N(\phi, Z)) < N$, then there exists $k < N$ such that $\mathrm{rk}(T_k(\phi, Z)) = \mathrm{rk}(T_{k+1}(\phi, Z))$, that is, $\phi^k Z \subseteq D(T_k(\phi, Z))$. But then, $\phi^{k+1}Z \subseteq \phi D(T_k(\phi, Z)) \subseteq D(T_{k+1}(\phi, Z)) = D(T_k(\phi, Z))$; proceeding by induction this way, one shows that $T(\phi, Z) \leq D(T_k(\phi, Z))$ which contradicts the fact that $\mathrm{rk}(T(\phi, Z)) = N$. Therefore, $F$ has maximal rank. Since $F$ is also finitely generated, Corollary 5.3 yields

$$\widetilde{\mathrm{ent}}(\phi) = \widetilde{\mathrm{ent}}(\phi, F).$$

Notice that

$$\frac{F + \phi(F)}{F} = \frac{T_{N+1}(\phi, Z)}{T_N(\phi, Z)} \cong \frac{\phi^N(Z)}{T_N(\phi, Z) \cap \phi^N(Z)}\,,$$

that is, $(F + \phi(F))/F$ is a quotient of $Z$. On the other hand, using the explicit form of the characteristic polynomial, we have that $s\phi^N(Z) \subseteq T_N(\phi, Z)$. Thus, the multiplication by $s$ is the trivial morphism on $(F + \phi(F))/F$. In particular, $(F + \phi(F))/F$ is a quotient of $Z/sZ$. This implies that

$$\widetilde{\mathrm{ent}}(\phi, F) \leq \log|(F + \phi(F))/F| \leq \log|Z/sZ| = \log|\mathbb{Z}/s\mathbb{Z}| = \log s.$$

It remains to verify the converse inequality. By the irreducibility of $p_\phi(X)$, one can say that, given a polynomial $p(X) \in \mathbb{Q}[X]$ such that $p(\phi) = 0$, then $p_\phi(X)$ divides $p(X)$ in $\mathbb{Q}[X]$. Furthermore, fixed a generator $z$ of $Z$ and given an element $x \in \mathbb{Q}^N$, since $D(F) = \mathbb{Q}^N$ by the maximality of the rank of $F$ proved above, there exists $0 \neq \alpha \in \mathbb{Z}$ such that $\alpha x \in F$. So, one can find non-trivial integers $\alpha_1, \dots, \alpha_N$ such that $\alpha x = \alpha_1 z + \dots + \alpha_N \phi^{N-1}(z)$. Using this fact, one shows that

$$p(\phi) = 0 \text{ if and only if } p(\phi)(z) = 0.$$

We have obtained that, for any given $p(X) \in \mathbb{Q}[X]$

$$\text{if } p(\phi)(z) = 0, \text{ then } p_\phi(X) \text{ divides } p(X) \text{ in } \mathbb{Q}[X].$$

Since $p_\phi(X) \in \mathbb{Z}[X]$, if also $p(X) \in \mathbb{Z}[X]$ and $p(X)$ is primitive, as a consequence of Gauss Lemma we have that $p_\phi(X)$ divides $p(X)$ in $\mathbb{Z}[X]$.

Now suppose, looking for a contradiction, that there exist a positive integer $k \in \mathbb{N}_+$ and $t \in \mathbb{N}$, such that $|T_{k+1}(\phi, F)/T_k(\phi, F)| = t \lneq s$. This means that $tT_{k+1}(\phi, F) \subseteq T_k(\phi, F)$ and so $t\phi^{N+k}Z \subseteq T_k(\phi, F) = T_{N+k}(\phi, Z)$, that is, $t\phi^{N+k}(z) \in T_{N+k}(\phi, Z)$. Hence, we can find a polynomial $p(X) \in \mathbb{Z}[X]$ of degree $N + k$ and with leading coefficient $t$, such that $p(\phi)(z) = 0$. We can assume without loss of generality that $p(X)$ is primitive. By the above discussion, $p_\phi(X)$ divides $p(X)$ in $\mathbb{Z}[X]$ and so $s|t$ in $\mathbb{Z}$, which is the contradiction we were looking for. Therefore, $|T_{k+1}(\phi, F)/T_k(\phi, F)| = s$ for every $k \in \mathbb{N}_+$, and so $\widetilde{\mathrm{ent}}(\phi, F) \geq \log s$. □

*Proof of Theorem 5.1..* Let $N$ be a positive integer, $V = \mathbb{Q}^N$, and let $\phi : V \to V$ be an endomorphism. Let $p_\phi(X)$ be the characteristic polynomial of $\phi$ over $\mathbb{Z}$ and let $s \in \mathbb{N}_+$ be the leading coefficient of $p_\phi(X)$.

We can consider the flow $(V, \phi)$ as a $\mathbb{Q}[X]$-module, letting $X \cdot v = \phi(v)$ for every $v \in V$. Since it is a $\mathbb{Q}[X]$-module of finite length, there exists a composition series

$$0 = V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_k \subsetneq V_{k+1} = V,$$

where each $V_i$ is a $\mathbb{Q}[X]$-submodule of $V$. In particular, each $V_i$ is a $\phi$-invariant subspace of $V$. For every $i \in \{0, \dots, k\}$, denote by $\phi_i$ the endomorphism induced by $\phi$ on $V_{i+1}/V_i$ and let $p_{\phi_i}(X)$ be the characteristic polynomial of $\phi_i$ over $\mathbb{Z}$.

By Proposition 5.5 we have that $\widetilde{\mathrm{ent}}(\phi \restriction_{V_{i+1}}) = \widetilde{\mathrm{ent}}(\phi \restriction_{V_i}) + \widetilde{\mathrm{ent}}(\phi_{i+1})$ for every $i \in \{1, \dots, k\}$. Then

$$\widetilde{\mathrm{ent}}(\phi) = \widetilde{\mathrm{ent}}(\phi_1) + \dots + \widetilde{\mathrm{ent}}(\phi_{k+1}). \tag{5.2}$$

Moreover, $p_{\phi\restriction_{V_{i+1}}}(X) = p_{\phi\restriction_{V_i}}(X) p_{\phi_i}(X)$ for every $i \in \{0, \dots, k\}$. Hence, $p_\phi(X) = p_{\phi_1}(X) \cdot \dots \cdot p_{\phi_{k+1}}(X)$. Let $s_i \in \mathbb{N}_+$ be the leading coefficient of $p_{\phi_i}$ for every $i \in \{1, \dots, k\}$. Then

$$s = s_1 \cdot \dots \cdot s_k. \tag{5.3}$$

Since the quotient $V_{i+1}/V_i$ is a simple $\mathbb{Q}[X]$-module, so $p_{\phi_i}(X)$ is irreducible. Hence, by Proposition 5.6 we have that $\widetilde{\mathrm{ent}}(\phi_i) = \log s_i$ for every $i \in \{1, \dots, k\}$. Therefore, (5.2) and (5.3) yield $\widetilde{\mathrm{ent}}(\phi) = \log s$. □

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## References

[1] R. L. Adler, A. G. Konheim and M. H. McAndrew, *Topological entropy*, Trans. Amer. Math. Soc. **114** (1965), 309–319.
[2] R. Bowen, *Entropy for group endomorphisms and homogeneous spaces*, Trans. Amer. Math. Soc. **153** (1971), 401–414.
[3] D. Dikranjan and A. Giordano Bruno, *Entropy on Abelian groups*, submitted ; arXiv:1007.0533.

[4] D. Dikranjan and A. Giordano Bruno, *The Bridge Theorem for totally disconnected LCA groups*, Topology Appl. **169** (2014) no. 1, 21–32.
[5] D. Dikranjan and A. Giordano Bruno, *Limit free computation of entropy*, Rend. Istit. Mat. Univ. Trieste **44** (2012), 297–312.
[6] D. Dikranjan and A. Giordano Bruno, *Topological entropy and algebraic entropy for group endomorphisms*, Proceedings ICTA2011 Islamabad, Pakistan July 4-10 2011 Cambridge Scientific Publishers, 133–214.
[7] D. Dikranjan and A. Giordano Bruno, *The connection between topological and algebraic entropy*, Topology Appl. **159** (2012) no.13, 2980–2989.
[8] D. Dikranjan and A. Giordano Bruno, *The Pinsker subgroup of an algebraic flow*, J. Pure Appl. Algebra **216** (2012) no.2, 364–376.
[9] D. Dikranjan, A. Giordano Bruno, L. Salce and S.Virili, *Intrinsic algebraic entropy*, J. Pure Appl. Algebra 219 (2015) 2933–2961.
[10] D. Dikranjan, B. Goldsmith, L. Salce and P. Zanardo, *Algebraic entropy for Abelian groups*, Trans. Amer. Math. Soc. **361** (2009), 3401–3434.
[11] D. Dikranjan, K. Gong and P. Zanardo, *Endomorphisms of Abelian groups with small algebraic entropy*, Linear Algebra Appl. **439** (2013) no.7, 1894–1904.
[12] D. Dikranjan, M. Sanchis and S. Virili, *New and old facts about entropy on uniform spaces and topological groups*, Topology Appl. **159** (2012) no.7, 1916–1942.
[13] M. Einsiedler and T. Ward, *Ergodic Theory (with a view towards Number Theory)*, Graduate Texts in Mathematics Volume **259**, 2011.
[14] G. Everest and T. Ward, *Heights of Polynomials and Entropy in Algebraic Dynamics*, Springer Verlag, 1999.
[15] A. Giordano Bruno and S. Virili, *Algebraic Yuzvinski Formula*, J. Algebra 423 (2015) 114–147.
[16] E. Hewitt and K. A. Ross, *Abstract harmonic analysis I*, Springer-Verlag, Berlin-Heidelberg-New York, 1963.
[17] E. Hironaka, *What is... Lehmer's number?*, Not. Amer. Math. Soc. **56** (2009) no. 3, 374–375.
[18] B. M. Hood, *Topological entropy and uniform spaces*, J. London Math. Soc. (2) **8** (1974), 633–641.
[19] I. Kaplansky, *Infinite Abelian groups*, University of Michigan Publications in Mathematics, no. 2, Ann Arbor, University of Michigan Press, 1954.
[20] N. Koblitz, *$p$-adic Numbers, $p$-adic Analysis, and Zeta-Functions, Second Edition*, Graduate Texts in Mathematics **58**, Springer-Verlag New York, Berlin, Heidelberg, Tokyo, 1984.
[21] L. Kronecker, *Zwei Sätze über Gleichungen mit ganzzahligen Coefficienten*, Jour. Reine Angew. Math. **53** (1857), 173–175.
[22] D. H. Lehmer, *Factorization of certain cyclotomic functions*, Ann. of Math. (2) **34** (1933), 461-479.
[23] D. A. Lind and T. Ward, *Automorphisms of solenoids and $p$-adic entropy*, Ergod. Th. & Dynam. Sys. **8** (1988), 411–419.
[24] K. Mahler, *On some inequalities for polynomials in several variables*, J. London Math. Soc. **37** (1962), 341–344.
[25] M. J. Mossinghoff, *Lehmer's Problem web page*, http://www.cecm.sfu.ca/ mjm/Lehmer/lc.html.
[26] J. Peters, *Entropy on discrete Abelian groups*, Adv. Math. **33** (1979), 1–13.
[27] J. Peters, *Entropy of automorphisms on L.C.A. groups*, Pacific J. Math. **96** (1981) no.2, 475–488.
[28] F. Quadros Gouvêa, *$p$-adic numbers: an introduction*, Springer-Verlag, Berlin-Heidelberg-New York, 1997.
[29] C. Smyth, *The Mahler measure of algebraic numbers: a survey*. Number theory and polynomials, 322–349, London Math. Soc. Lecture Note Ser., 352, Cambridge Univ. Press, Cambridge, 2008.
[30] L. N. Stojanov, *Uniqueness of topological entropy for endomorphisms on compact groups*, Boll. Un. Mat. Ital. B **7** (1987) no. 3, 829–847.
[31] S. Virili, *Entropy for endomorphisms of LCA groups*, Topology Appl. **159** (2012) no.9, 2546–2556.
[32] P. Walters, *An Introduction to Ergodic Theory*, Springer-Verlag, New-York, 1982.
[33] M. D. Weiss, *Algebraic and other entropies of group endomorphisms*, Math. Systems Theory **8** (1974/75) no.3, 243–248.
[34] A. Weil, *Basic Number Theory*, third edition, Springer Verlag, New York (1974).
[35] S. A. Yuzvinski, *Metric properties of endomorphisms of compact groups*, Izv. Acad. Nauk SSSR, Ser. Mat. **29** (1965), 1295–1328 (in Russian). English Translation: Amer. Math. Soc. Transl. (2) **66** (1968), 63–98.
[36] S. A. Yuzvinski, *Computing the entropy of a group endomorphism*, Sibirsk. Mat. Z. **8** (1967), 230–239 (in Russian). English Translation: Siberian Math. J. **8** (1968), 172–178.
[37] P. Zanardo, *Yuzvinski's Formula*, unpublished notes.