A global bifurcation result
for a second order singular equation

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Dedicated, with gratefulness and friendship, to Professor Fabio Zanolin
on the occasion of his 60th birthday

Abstract. We deal with a boundary value problem associated to a
second order singular equation in the open interval $(0, 1)$. We first
study the eigenvalue problem in the linear case and discuss the nodal
properties of the eigenfunctions. We then give a global bifurcation result
for nonlinear problems.

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1. Introduction

We are concerned with a second order ODE of the form

$$-u'' + q(x)u = \lambda u + g(x, u), \quad x \in (0, 1],$$

where $q \in C((0, 1])$ satisfies

$$\lim_{x \to 0^+} \frac{q(x)}{l/x^\alpha} = 1,$$

for some $l > 0$ and $\alpha \in (0, 5/4)$, and $g \in C([0, 1] \times \mathbb{R})$ is such that

$$\lim_{u \to 0^+} g(x, u) = 0, \quad \text{uniformly in } x \in (0, 1].$$

The constant $5/4$ arises in a rather straightforward manner in the study of the
differential operator in the left-hand side of (1) (cf. [17, p. 287-288]); details
are given in Remark 2.3 below.

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We will look for solutions $u$ of (1) such that $u \in H^2_0(0, 1)$.

When the $x$-variable belongs to a compact interval, problems of the form (1) have been very widely studied. A more limited number of contributions is available in the literature when the $x$-variable belongs to a (semi)-open interval, as it is the case in the present paper, or to an unbounded interval [7, 8].

We treat (1) in the framework of bifurcation theory. For this reason, we first discuss in Section 2 the eigenvalue problem

$$-u'' + q(x)u = \lambda u, \quad x \in (0, 1], \quad \lambda \in \mathbb{R}.$$  

(4)

For such singular problems, the well-known embedding of (4) (by an elementary application of the integration by parts rule, together with the boundary condition $u(0) = 0 = u(1)$) in the setting of eigenvalue problems for compact self-adjoint operators cannot be performed. Thus, the questions of the existence of eigenvalues and of the nodal properties of the associated eigenfunctions have various delicate features. For a comprehensive account on the spectral properties of the Schrödinger operator we refer to the books [12] and [10]; for more specific results on singular problems in $(0, 1)$ we refer, among many others, to [5, 14].

However, the linear spectral theory for singular problems is well-established and can be found, among others, in the classical book by Coddington and Levinson [4] and in the (relatively) more recent text by Weidmann [17]. The former monograph focuses on a generalization of the so-called “expansion theorem” valid for functions in $L^2([0, 1])$ and, by doing this, a sort of “generalized shooting method” is performed. On the other hand, in [17] the singular problem is tackled from an abstract point of view; more precisely, it is considered the general question of the existence of a self-adjoint realization of the formal differential expression $\tau u = -u'' + q(x)u$ and the important Weyl alternative theorem [17, Theorem 5.6] is used. It is interesting to observe that the approach in [4] (based on more elementary ODE techniques) and the abstract one in [17] lead in different ways to the important concepts of “limit point case” and “limit circle case”. The knowledge of one (or the other) case is ensured by suitable assumptions on $q$ and leads to information on the boundary conditions to be added to (4) in order to have a self-adjoint realization of $\tau$.

In the setting of the present paper, the operator $\tau$ is regular at $x = 1$; this implies that it is in the limit circle case. Moreover, under assumption (2), from [17, Theorem 6.4] it follows that $\tau$ is in the limit circle case also in $x = 0$. Thus, the differential operator $A : u \mapsto \tau u$ with

$$D(A) = \{ u \in L^2(0, 1) : u, u' \in AC(0, 1), \tau u \in L^2(0, 1), \lim_{x \to 0^+} (xu'(x) - u(x)) = 0 = u(1) \}$$
is a self-adjoint realization of $\tau$ ([17, p. 287-288]). We prove in Proposition 2.2 that in fact $D(A) = H^2_0(0,1)$; to do this, we need some knowledge of the behaviour of the solutions of (4) near zero. These estimates are developed in Proposition 2.1 by means of the classical Levinson theorem [6, Theorem 1.8.1].

Finally, at the end of Section 2 we focus on the nodal properties of a solution to (4); more precisely, in Proposition 2.4 we prove that (4) is non-oscillatory and conclude in Proposition 2.5 that the spectrum of $A$ is purely discrete and that, for every $n \in \mathbb{N}$, the eigenfunction associated to the eigenvalue $\lambda_n$ has $(n-1)$ simple zeros in $(0,1)$.

Section 3 contains a global bifurcation result (Theorem 3.2) which follows in a rather straightforward manner as an application of the celebrated Rabinowitz theorem in [11].

In order to exclude alternative (2) in Theorem 3.2, we use a technique that we already applied for Hamiltonian systems in $\mathbb{R}^2\mathbb{N}$ in [2] and for planar Dirac-type systems in [3]. More precisely, we introduce a continuous integer-valued functional defined on the set of solutions to (1). Due to the singularity at $x = 0$, some care is necessary in order to prove its continuity; this is the content of Proposition 3.4. We can then state and prove our main result (Theorem 3.5).

In what follows, for a given function $p$ we write $p(x) \sim \frac{m}{x^a}$, $x \to 0^+$, when

$$\lim_{x \to 0^+} \frac{p(x)}{m/x^a} = 1$$

for some $m, a \in \mathbb{R}^+$. Finally, we write

$$H^2_0(0,1) = \{ u \in H^2(0,1) : u(0) = 0 = u(1) \},$$

equipped with the norm defined by

$$||u||^2 = ||u||^2_{L^2(0,1)} + ||u'||^2_{L^2(0,1)}, \quad \forall \ u \in H^2_0(0,1).$$

2. The linear equation

In this section we study a linear second order equation of the form

$$-u'' + q(x)u = \lambda u, \quad x \in (0,1], \ \lambda \in \mathbb{R}.\tag{6}$$

We will assume that $q \in C((0,1])$ and that

$$q(x) \sim \frac{1}{x^a}, \quad x \to 0^+,\tag{7}$$
for some \( l > 0 \) and \( \alpha \in (0, 5/4) \). Without loss of generality we may suppose that
\[
q(x) > 0, \quad \forall x \in (0, 1].
\] (8)

For every \( u : (0, 1] \to \mathbb{R} \) we denote by \( \tau u \) the formal expression
\[
\tau u = -u'' + q(x)u;
\]

First of all, we study the asymptotic behaviour of solutions of (6) when \( x \to 0^+ \); to this aim, let us introduce the change of variables \( t = -\log x \) and let
\[
w(t) = u(e^{-t}), \quad \forall t > 0.
\]
From the relations
\[
w'(t) = -e^{-t}u'(e^{-t})\]
\[
w''(t) = e^{-t}u'(e^{-t}) + e^{-2t}u''(e^{-t}),
\]
we deduce that \( u \) is a solution of (6) on \((0, 1)\) if and only if \( w \) is a solution of
\[
-w'' - w' + e^{-2t}q(e^{-t})w = \lambda e^{-2t}w
\] (10)
on \((0, +\infty)\). Equation (10) can be written in the form
\[
Y' = (C + R(t))Y,
\] (11)
where \( Y = (w, z)^T \) and
\[
C = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, \quad R(t) = \begin{pmatrix} 0 & 0 \\ e^{-2t}q(e^{-t}) - \lambda e^{-2t} & 0 \end{pmatrix}, \quad \forall t > 0.
\] (12)

Now, let us observe that \( C \) has eigenvalues \( \lambda_1 = 0, \lambda_2 = -1 \) and corresponding eigenvectors \( u_1 = (1, 0), u_2 = (1, -1) \) and that \( R \in L^1(0, +\infty) \); therefore, an application of [6, Theorem 1.8.1] implies that (11) has two linearly independent solutions \( Y_1, Y_2 \) such that
\[
Y_1(t) = u_1 + o(1), \quad t \to +\infty,
\]
\[
Y_2(t) = (u_2 + o(1))e^{-t}, \quad t \to +\infty.
\] (13)

As a consequence, we obtain the following result:
Proposition 2.1. For every \( \lambda \in \mathbb{R} \) the equation (6) has two linearly independent solutions \( u_{1, \lambda}, u_{2, \lambda} \) such that

\[
\begin{align*}
  u_{1, \lambda}(x) &= 1 + o(1), \quad u'_{1, \lambda}(x) = o\left(\frac{1}{x}\right), \quad x \to 0^+,
  \\
  u_{2, \lambda}(x) &= x + o(1), \quad u'_{2, \lambda}(x) = 1 + o(1), \quad x \to 0^+,
\end{align*}
\] (14)

and \( u_{2, \lambda} \in H^2(0, 1) \).

For every \( f \in L^2(0, 1) \) the solutions of \( \tau u = f \) are given by

\[
  u(x) = c_1 u_{1,0}(x) + c_2 u_{2,0}(x) + u_f(x), \quad \forall \ x \in (0, 1), \ c_1, c_2 \in \mathbb{R},
\] (15)

where

\[
  u_f(x) = \int_0^x G(x, t)f(t) \, dt, \quad \forall \ x \in (0, 1),
\] (16)

\[
  G(x, t) = u_{1,0}(t)u_{2,0}(x) - u_{2,0}(t)u_{1,0}(x), \quad \forall \ x \in (0, 1), \ t \in (0, 1)
\]

fulfill \( G \in L^\infty((0, 1)^2) \), \( u_f(0) = 0 = u'_f(0) \) and \( u_f \in H^2(0, 1) \).

Proof. The estimates in (14) follow from (9) and (13), while (16) is the usual variation of constants formula. Moreover, from (14) we obtain that \( u_{2, \lambda}, u'_{2, \lambda} \in L^2(0, 1) \). On the other hand we have

\[
  q(x)u_{2, \lambda}(x) \sim x^{1-\alpha}, \quad x \to 0^+,
\] (17)

which implies that \( qu_{2, \lambda} \in L^2(0, 1) \), since \( \alpha < 5/4 \) (cf. Remark 2.3 for comments on this restriction); using the fact that \( \tau u_{2, \lambda} = \lambda u_{2, \lambda} \), we deduce that

\[
  u''_{2, \lambda} = \lambda u_{2, \lambda} - qu_{2, \lambda} \in L^2(0, 1).
\]

From now on, we will indicate \( u_i = u_{i,0}, \ i = 1, 2 \). The fact that the function \( G \) defined in (16) belongs to the space \( L^\infty((0, 1)^2) \) is a consequence of the asymptotic estimates (14). Moreover, from (16) we also deduce that \( u_f(0) = 0 \) and that

\[
  u'_f(x) = \int_0^x (u_1(t)u'_2(x) - u_2(t)u'_1(x))f(t) \, dt, \quad \forall \ x \in (0, 1),
\] (18)

which implies \( u'_f(0) = 0 \).

Finally, the condition \( u_f(0) = 0 = u'_f(0) \) guarantees that \( u_f, u'_f \in L^2(0, 1) \); as far as the second derivative of \( u_f \) is concerned, let us observe that we have

\[
  \tau u_f = f
\]

and so

\[
  u''_f = f - qu_f.
\] (19)

Using the fact that \( u_f(0) = 0 = u'_f(0) \) and (7), it follows that \( qu_f \in L^2(0, 1) \); hence \( u_f \in H^2(0, 1) \). \( \square \)
In what follows, we study the spectral properties of suitable self-adjoint realizations of $\tau$; to this aim, let us first observe that the differential operator $\tau$ is regular at $x = 1$. As a consequence, it is in the limit circle case at $x = 1$; moreover, from (7), according to [17, Theorem 6.4], $\tau$ is in the limit circle case also in $x = 0$.

The differential operator $A$ defined by

$$D(A) = \{u \in L^2(0,1): u, u' \in AC(0,1), \tau u \in L^2(0,1), \lim_{x \to 0^+} (xu'(x) - u(x)) = 0 = u(1)\}$$

$$Au = \tau u, \quad \forall u \in D(A),$$

is then a self-adjoint realization of $\tau$ ([17, p. 287-288]). We can show the validity of the following Proposition:

**Proposition 2.2.** The relation

$$D(A) = H^2_0(0,1)$$

holds true. Moreover, $A$ has a bounded inverse $A^{-1} : L^2(0,1) \to H^2_0(0,1)$.

**Proof.** 1. Let us start proving that $H^2_0(0,1) \subset D(A)$. It is well known that $H^2_0(0,1) \subset C^1(0,1)$; hence, for every $u \in H^2_0(0,1)$ we have $u, u' \in AC(0,1)$. Moreover, using the fact that $u(0) = 0$ we deduce that

$$u(x) = u'(0)x + o(x), \quad x \to 0^+$$

and

$$q(x)u(x) = u'(0)x^{1-\alpha} + o(x^{1-\alpha}), \quad x \to 0^+;$$

the condition $\alpha < 5/4$ guarantees again that $qu \in L^2(0,1)$ and therefore $\tau u = -u'' + qu \in L^2(0,1)$. Finally, the regularity of $u$ and $u'$ imply that

$$\lim_{x \to 0^+} (xu'(x) - u(x)) = 0$$

and so also the boundary condition in the definition of $D(A)$ is satisfied.

Now, let us prove that $D(A) \subset H^2_0(0,1)$; for every $u \in D(A)$ let $f = \tau u \in L^2(0,1)$. From (15) we deduce that $u$ can be written as

$$u = c_1 u_1 + c_2 u_2 + u_f,$$

for some $c_1, c_2 \in \mathbb{R}$; it is easy to see that the function $u_1$ does not satisfy the boundary condition given in $x = 0$ in the definition of $D(A)$, while $u_2$ and $u_f$ do. Hence $u \in D(A)$ if and only if $c_1 = 0$; the last statement of Proposition 2.1 implies then that $u \in H^2(0,1)$. As in the first part of the proof, the regularity
of $u$ allows to conclude that the boundary condition in $x = 0$ given in $D(A)$ reduces to $u(0) = 0$.

2. Let us study the invertibility of $A$; the existence of a bounded inverse of $A$ is equivalent to the fact that $0 \in \rho_A$, being $\rho_A$ the resolvent of $A$. Since $A$ is self-adjoint on $H^2_0(0,1)$, this follows from the surjectivity of $A$ (cf. [16, Theorem 5.24]); hence, it is sufficient to prove that $A$ is surjective.

To this aim, let us first observe that condition (8) guarantees that 0 cannot be an eigenvalue of $A$. Now, let us fix $f \in L^2(0,1)$ and let us prove that there exists $u \in H^2_0(0,1)$ such that $A u = f$, i.e. $\tau u = f$; by applying Proposition 2.1 we deduce again that (20) holds true and the same argument of the first part of the proof implies that $c_1 = 0$.

Hence we obtain $u = c_2 u_2 + u_f$; from Proposition 2.1 we deduce that this function belongs to $H^2(0,1)$ and satisfies the boundary condition $u(0) = 0$. In order to prove that the missing condition $u(1) = 0$ is fulfilled for every $f \in L^2(0,1)$, let us observe that $u_2(1) \neq 0$, otherwise $u_2$ would be an eigenfunction of $A$ associated to the zero eigenvalue. Therefore, $u(1) = 0$ is satisfied if

$$c_2 = -\frac{u_f(1)}{u_2(1)},$$

for every $f \in L^2(0,1)$.

\[\square\]

**Remark 2.3.** As for the restriction $\alpha < 5/4$, we observe that for the proofs of Proposition 2.1 and Proposition 2.2 it is sufficient to require the milder condition $\alpha < 3/2$. The fact that $\alpha < 5/4$ is used (cf. [17, p. 287-288]) in order to obtain that $D(A)$ is the one described above. Finally, we observe that in the particular case when $\alpha < 1$ the problem is regular (cf., among others, [9]).

The spectral properties of $A$ are related to the oscillatory behaviour of solutions of (6). We first recall the following definition:

**Definition 2.4.** The differential equation (6) is oscillatory if every solution $u$ has infinitely many zeros in $(0,1)$. It is non-oscillatory when it is not oscillatory.

We observe that the regularity assumptions on $q$ imply that solutions of (6) have a finite number of zeros in any interval of the form $[a, 1)$, for every $0 < a < 1$. Moreover, from (7) we infer that for every $\lambda \in \mathbb{R}$ there exists $c(\lambda) \in (0,1]$ such that

$$\lambda - q(x) < 0, \quad \forall x \in (0, c(\lambda)).$$

An application of the Sturm comparison theorem proves that every solution of (6) has at most one zero in $(0, c(\lambda))$; as a consequence, we obtain the following result:
Proposition 2.5. For every $\lambda \in \mathbb{R}$ the differential equation (6) is non-oscillatory.

Once Proposition 2.5 is obtained, we can provide in a straightforward way some useful information on the spectral properties of $A$; more precisely, denoting by $\sigma_{\text{ess}}$ the essential spectrum of a given operator, we have:

Proposition 2.6. ([17, Theorem 14.3, Theorem 14.6 and Theorem 14.9], [12, Theorem XIII.1]) The differential operator $A$ is bounded-below and satisfies

$$\sigma_{\text{ess}}(A) = \emptyset.$$ 

Moreover, there exists a sequence $\{\lambda_n\}_{n \in \mathbb{N}}$ of simple eigenvalues of $A$ such that

$$\lim_{n \to +\infty} \lambda_n = +\infty$$

and for every $n \in \mathbb{N}$ the eigenfunction $u_n$ of $A$ associated to the eigenvalue $\lambda_n$ has $(n - 1)$ simple zeros in $(0, 1)$.

Remark 2.7. According to [17], operators of the form $\tau$ (defined on functions whose domain is $(0, +\infty)$) arise when the time independent Schrödinger equation with spherically symmetric potential

$$-\Delta u(x) + V(|x|)u(x) = \lambda u(x), \quad u \in L^2(\mathbb{R}^m)$$

is reduced to an infinite system of eigenvalue problems associated to the ordinary differential operators in $L^2(0, +\infty)$

$$\tau_i = -\frac{d^2}{dr^2} + \frac{1}{r^2} \left[ i(i + m - 2) + \frac{1}{4}(m - 1)(m - 3) \right] + V(r)$$

($i \in \mathbb{N}$). In Appendix 17.F of [17] it is treated the case of a potential $V$ satisfying assumptions (which enable to consider Coulomb potentials) that lead to (7). More precisely, it is shown that for $m = 3, i = 0$ the operator is in the limit circle case at zero and self-adjoint extensions of $\tau_0$ are described.

3. The main result

In this section we are interested in proving a global bifurcation result for a nonlinear eigenvalue problem of the form

$$-u'' + q(x)u = \lambda u + g(x, u)u, \quad \lambda \in \mathbb{R}, \quad x \in (0, 1],$$

(22)

where $q \in C((0, 1])$ satisfies (7) and $g \in C([0, 1] \times \mathbb{R})$ is such that

$$\lim_{u \to 0} g(x, u) = 0, \quad \text{uniformly in } x \in [0, 1].$$

(23)
We will look for solutions $u$ of (22) such that $u \in H^2_0(0,1)$. To this aim, let $\Sigma$ denote the set of nontrivial solutions of (22) in $H^2_0(0,1) \times \mathbb{R}$ and let $\Sigma' = \Sigma \cup \{(0, \lambda) \in H^2_0(0,1) \times \mathbb{R} : \lambda$ is an eigenvalue of $A\}$, where $A$ is as in Section 2.

Let $M$ denote the Nemitskii operator associated to $g$, given by
\[ M(u)(x) = g(x, u(x))u(x), \quad \forall \ x \in [0,1], \]
for every $u \in H^2_0(0,1)$. We can show the validity of the following:

**Proposition 3.1.** Assume $g \in C([0,1] \times \mathbb{R})$ and (23). Then $M : H^2_0(0,1) \rightarrow L^2(0,1)$ is a continuous map and satisfies
\[ M(u) = o(||u||), \quad u \rightarrow 0. \] (24)

**Proof.** 1. We first show that $Mu \in L^2(0,1)$ when $u \in H^2_0(0,1)$. When this condition holds, $u \in L^\infty(0,1)$ and the continuity of $g$ implies that there exists $C_u > 0$ such that
\[ |g(x, u(x))u(x)| \leq C_u, \quad \forall \ x \in [0,1]. \]
As a consequence we obtain $Mu \in L^\infty(0,1) \subset L^2(0,1)$.

2. Let us prove that $M$ is continuous. Let us fix $u_0 \in X$ and let $u_n \in X$ such that $u_n \rightarrow u_0$ when $n \rightarrow +\infty$; the continuous embedding
\[ H^2_0(0,1) \subset L^\infty(0,1) \]
and the uniform continuity of $g$ on compact subsets of $[0,1] \times \mathbb{R}$ ensure that
\[ g(x, u_n(x)) \rightarrow g(x, u_0(x)) \quad \text{in} \quad L^\infty(0,1). \] (25)
This is sufficient to conclude that $Mu_n \rightarrow Mu_0$ in $L^\infty(0,1)$ and hence $Mu_n \rightarrow M u_0$ in $L^2(0,1)$.

3. Finally, let us prove (24): using again the fact that $H^2_0(0,1) \subset L^\infty(0,1)$, we have
\[ ||Mu||_{L^2(0,1)} \leq ||g(x, u(x))||_{L^\infty(0,1)}||u||_{L^2(0,1)} \leq ||g(x, u(x))||_{L^\infty(0,1)}||u||, \]
for all $u \in H^2_0(0,1)$; hence, we deduce that
\[ \frac{||Mu||_{L^2(0,1)}}{||u||} \leq ||g(x, u(x))||_{L^\infty(0,1)}, \quad \forall \ u \in H^2_0(0,1), \ u \neq 0. \]
Therefore the result follows from (23) and (25).
Now, let us observe that the search of solutions \( u \in H^2_0(0,1) \) of (22) is equivalent to the search of solutions of the abstract equation

\[
Au = \lambda u + M(u), \quad (u, \lambda) \in H^2_0(0,1) \times \mathbb{R};
\]

(26)
on the other hand, (26) can be written in the form

\[
w = \lambda Rw + M(Rw), \quad (w, \lambda) \in L^2(0,1) \times \mathbb{R},
\]

(27)
where \( R : L^2(0,1) \to H^2_0(0,1) \) is the inverse of \( A \) (cf. Proposition 2.2).

Now, from [17, Theorem 7.10] we deduce that \( R \) is compact; this fact and the continuity of \( M \) guarantee that the operator \( MR : L^2(0,1) \to H^2_0(0,1) \) is compact. Moreover, the condition

\[
M(Rw) = o(||w||_{L^2(0,1)}), \quad w \to 0,
\]

(28)
is a consequence of (24). From an application of the global bifurcation result of Rabinowitz (cfr. [11]) to (27) we then obtain the following result:

**Theorem 3.2.** Assume (7) and (23). Then, for every eigenvalue \( \lambda_n \) of \( A \) there exists a continuum \( C_n \) of nontrivial solutions of (22) in \( H^2_0(0,1) \times \mathbb{R} \) bifurcating from \((0, \lambda_n)\) and such that one of the following conditions holds true:

1. \( C_n \) is unbounded in \( H^2_0(0,1) \times \mathbb{R} \);
2. \( C_n \) contains \((0, \lambda_n') \in \Sigma' \), with \( n' \neq n \).

Now, let us observe that a more precise description of the bifurcating branch, eventually leading to exclude condition (2), can be obtained when there exists a continuous functional \( j : \Sigma' \to \mathbb{N} \) (cfr. [2, Pr. 2.1]). In order to define such a functional, we will use the fact that nontrivial solutions of (22) have a finite number of zeros in \((0,1)\); this will be a consequence of our next result.

For every \( \lambda \in \mathbb{R} \) and for every nontrivial solution \( u \in H^2_0(0,1) \) of (22) let us define \( q_{u,\lambda} : (0,1] \to \mathbb{R} \) by \( q_{u,\lambda}(x) = q(x) - \lambda - g(x, u(x)) \), for every \( x \in (0,1] \). The following Lemma holds true:

**Lemma 3.3.** For every \( \lambda \in \mathbb{R} \) and for every nontrivial solution \( u \in H^2_0(0,1) \) of (22) there exists a neighborhood \( U \subset H^2_0(0,1) \times \mathbb{R} \) of \((u, \lambda)\) and \( x_{u,\lambda} \in (0,1) \) such that

\[
q_{u,\lambda}(x) > 0, \quad \forall (v, \mu) \in U, \ x \in (0, x_{u,\lambda}].
\]

(29)

**Proof.** Let \((u, \lambda) \in H^2_0(0,1) \times \mathbb{R}, u \neq 0, \) be fixed and let \( U \) be the neighborhood of radius 1 of \((u, \lambda)\) in \( H^2_0(0,1) \times \mathbb{R} \); from the continuous embedding \( L^\infty(0,1) \subset H^2_0(0,1) \) we deduce that if \((w, \mu) \in \Sigma \cap U_1 \) then

\[
||w||_{L^\infty(0,1)} \leq 1 + ||u||_{L^\infty(0,1)}, \quad |\mu| \leq 1 + |\lambda|
\]
and

\[ q(x) - \mu - g(x, w(x)) \geq q(x) - |\lambda| - 1 - \max_{x \in [0,1], |s| \leq 1 + ||u||_{L^\infty}} |g(x, s)|, \quad \forall x \in (0, 1). \]

From (7) we then deduce that there exists \( x_{(u, \lambda)} \in (0, 1) \), depending only on \((u, \lambda)\), such that

\[ q(x) - \mu - g(x, w(x)) > 0, \quad \forall x \in (0, x_{(u, \lambda)}]. \]

Now, let us observe that for every \( \lambda \in \mathbb{R} \) and for every nontrivial solution \( u \in H^2_0(0, 1) \) of (22) the function \( u \) is a nontrivial solution of the linear equation

\[ -w'' + (q(x) - g(x, u(x)) - \lambda)w = 0. \]

From Lemma 3.3, with an argument similar to the one which led to Proposition 2.5, we deduce that all the nontrivial solutions of (30) (in particular \( u \)) have a finite number of zeros in \((0, 1)\). We denote by \( n(u) \) this number.

We are then allowed to define the functional \( j \) by setting

\[ j(u, \lambda) = \begin{cases} n(u) & \text{if } u \neq 0 \\ n - 1 & \text{if } u \equiv 0 \text{ and } \lambda = \lambda_n, \end{cases} \]

for every \((u, \lambda) \in \Sigma'\). Let us observe that the definition \( j(0, \lambda_n) = n - 1 \) is suggested by Proposition 2.6.

**Proposition 3.4.** The function \( j : \Sigma' \to \mathbb{N} \) is continuous.

*Proof.* 1. As for the continuity of \( j \) in every point of the form \((0, \lambda_n), n \in \mathbb{N}\), we refer to [15, Lemma 2.5].

2. Let us now fix \((u_0, \lambda_0) \in \Sigma\) and let \((u, \lambda) \in U\), with \( U \) as in Lemma 3.3; this Lemma guarantees that both \( u \) and \( u_0 \) have no zeros in \((0, x_{u_0, \lambda_0})\).

On the other hand, in the interval \([x_{u_0, \lambda_0}, 1]\) a standard continuous dependence argument (cf. also [11]) ensures that \( u \) and \( u_0 \) have the same numbers of zeros if \((u, \lambda)\) is in a sufficiently small neighborhood of \((u_0, \lambda_0)\). As a consequence, we obtain that there exists a neighborhood \( U_0 \) of \((u_0, \lambda_0)\) such that

\[ j(u, \lambda) = j(u_0, \lambda_0), \quad \forall (u, \lambda) \in U_0. \]

As a consequence, from Theorem 3.2 and Proposition 3.4 we deduce the final result:
Theorem 3.5. Assume (7) and (23). Then, for every eigenvalue $\lambda_n$ of $A$ there exists a continuum $C_n$ of nontrivial solutions of (22) in $H^2_0(0,1) \times \mathbb{R}$ bifurcating from $(0, \lambda_n)$ and such that condition (1) of Theorem 3.2 holds true and

$$j(u, \lambda) = n - 1, \quad \forall (u, \lambda) \in C_n.$$  \hspace{1cm} (32)

Remark 3.6. Theorem 3.2 can be proved as an application of Stuart’s result \cite[Theorem 1.2]{15} as well. However, since in the situation considered in this paper the singularity at zero does not affect the compactness of the operator $R$ defined after (27), we chose to apply Rabinowitz theorem \cite{11}. We finally mention the interesting paper \cite{1}, where global branches of solutions, with prescribed nodal properties, are obtained for a second order degenerate problem in $(0,1)$.

References

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