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Some properties of the growth and of the algebraic entropy of group endomorphisms

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Abstract

We study the growth of group endomorphisms, a generalization of the classical notion of growth of finitely generated groups, which is strictly related to the algebraic entropy. We prove that the inner automorphisms of a group have the same growth type and the same algebraic entropy as the identity automorphism. Moreover, we show that endomorphisms of locally finite groups cannot have intermediate growth. We also find an example showing that the Addition Theorem for the algebraic entropy does not hold for endomorphisms of arbitrary groups.

1 Introduction

The notion of growth for finitely generated groups was introduced by Milnor in the 60s and since then it has become a modern prominent field of research. In particular, the famous Milnor Problem on group growth (see [11]) had a great impact in this context:

(i) Are there finitely generated groups of intermediate growth (that is, between polynomial and exponential)?

(ii) What are the finitely generated groups of polynomial growth?

Part (i) was solved by Grigorchuk (see [8]) by constructing his famous examples of finitely generated groups of intermediate growth. Part (ii) was solved by Gromov in [9] by proving that a finitely generated group G has polynomial growth if and only if G is virtually nilpotent; recall that a group G is *virtually nilpotent* if it contains a nilpotent subgroup having finite index (equivalently, it admits a normal nilpotent subgroup having finite index). The fact that a virtually nilpotent finitely generated group has polynomial growth was already proved by Wolf in [14]. Moreover, Milnor-Wolf's Theorem states that a soluble finitely generated group has either polynomial or exponential growth (see [10, 14]).

In [3], the classical notion of growth is extended to arbitrary groups (that is, not necessarily finitely generated), and also in a fairly natural way to group endomorphisms (see Section 2): these generalizations use the language of algebraic entropy. The first definition of algebraic entropy was given

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for endomorphisms of torsion abelian groups in [1], later studied by Weiss in [13] and more recently in [5]. Then, Peters in [12] extended the notion of algebraic entropy to automorphisms of abelian groups; more recently, in [3, 4] this definition is appropriately modified and extended to arbitrary group endomorphisms (see Section 2).

In Section 2, first we show that the Addition Theorem for the algebraic entropy does not hold for endomorphisms of arbitrary groups, even for metabelian groups (see Example 2.7). This provides a counterexample and gives a negative answer to [3, Question 5.2.12(b)] and also to the more general [3, Problem 5.2.10].

Then, we extend Gromov's Theorem to arbitrary groups G, by showing that G has polynomial growth precisely when G is *locally virtually nilpotent* (i.e., every finitely generated subgroup of G is virtually nilpotent) (see Theorem 2.4).

We extend to this setting also Milnor-Wolf's Theorem, by proving that if a group G is *locally virtually soluble* (i.e., every finitely generated subgroup of G has a soluble subgroup of finite index) then G has either polynomial or exponential growth.

In Section 3, we show that the inner automorphisms of an arbitrary group G have the same growth type and the same algebraic entropy as the identity automorphism of G (see Theorem 3.2).

In Section 4, we prove that if $\phi : G \to G$ is a group endomorphism of zero entropy, then every element of G is contained in a finitely generated ϕ-invariant subgroup of G. When G is locally finite, this means that every element of G belongs to a finite ϕ-invariant subgroup of G, and this answers [3, Problem 5.2.3]; we recall that a group G is *locally finite* if every finite subset of G generates a finite subgroup (i.e., every finite subset of G is contained in a finite subgroup of G).

Finally, in the spirit of Milnor Problem, we consider the following problem on growth of group endomorphisms.

Problem 1.1. *Characterize the groups G admitting no endomorphism of intermediate growth.*

A motivation and a first insight to this problem is given by the abelian case; indeed, it is known from [2] that endomorphisms of abelian groups cannot have intermediate growth, that is, every endomorphism of an abelian group has either polynomial or exponential growth. Here we prove that exactly the same result holds for locally finite groups (see Corollary 4.7).

We are inclined to believe that Problem 1.1 has no easy answer, moreover it is conceivable that the class of groups arising from this problem might not have a natural algebraic description. Despite this, we dare to conjecture that locally virtually nilpotent groups admit no endomorphism of intermediate growth. If this were true, this would offer in our opinion a beautiful entropy-analogue to the celebrated theorem of Gromov.[1]

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2 Growth and algebraic entropy

2.1 Growth for finitely generated groups

Given two maps $\gamma, \gamma' : \mathbb{N} \to \{z \in \mathbb{R} : z \geq 0\}$, we write $\gamma \preceq \gamma'$ if there exist $n_0, C \in \mathbb{N}$ such that $\gamma(n) \leq \gamma'(Cn)$ for every $n \geq n_0$. Moreover, we say that γ and γ' are *equivalent*, and write $\gamma \sim \gamma'$, if $\gamma \preceq \gamma'$ and $\gamma' \preceq \gamma$; indeed, $\sim$ is an equivalence relation. Routine computations show that, for every

[1] During the refereeing process of this paper, by adapting the proof of Milnor-Wolf's Theorem to our more general situation, we proved ourselves right here. Actually, we have proved something stronger: we have shown that locally virtually soluble groups admit no endomorphism of intermediate growth and, moreover, in some cases we have determined when an endomorphism has polynomial growth, see [7].

$\alpha, \beta \in \{z \in \mathbb{R} : z \geq 0\}$, $n^\alpha \sim n^\beta$ if and only if $\alpha = \beta$; moreover, for every $a, b \in \{z \in \mathbb{R} : z > 1\}$, $a^n \sim b^n$.

A map $\gamma : \mathbb{N} \to \mathbb{N}$ is called:

(a) *polynomial*, if $\gamma(n) \preceq n^d$ for some $d \in \mathbb{N}_+$;

(b) *exponential*, if $\gamma(n) \sim e^n$;

(c) *intermediate*, if $n^d \preceq \gamma(n)$ for every $d \in \mathbb{N}_+$, $\gamma(n) \preceq e^n$ and $e^n \not\preceq \gamma(n)$.

Let G be a finitely generated group and let S be a finite set of generators for G. For every $g \in G$, denote by $\ell_S(g)$ the smallest $\ell \in \mathbb{N}_+$ with

$$g = s_1^{\varepsilon_1} s_2^{\varepsilon_2} \cdots s_\ell^{\varepsilon_\ell},$$

where $s_1, \ldots, s_\ell \in S$ and $\varepsilon_1, \ldots, \varepsilon_\ell \in \{-1, 1\}$. In particular, $\ell_S(g)$ is the length of a shortest word representing g in the alphabet $S \cup S^{-1}$, where $S^{-1} = \{s^{-1} : s \in S\}$. By abuse of notation, we let $\ell_S(e_G) = 0$ where e_G is the identity element of G. The *growth function* of G with respect to S is

$$\begin{aligned} \gamma_S :\ & \mathbb{N} \to \mathbb{N} \\ & n \mapsto |B_S(n)|, \end{aligned}$$

where $B_S(n) = \{g \in G : \ell_S(g) \leq n\}$ is the ball of radius n in the word metric of G. Note that $B_S(0) = \{e_G\}$ and $B_S(1) = S \cup S^{-1} \cup \{e_G\}$.

Routine computations show that $\gamma_S \sim \gamma_{S'}$, for every finite generating set S' for G. This observation allows us to say that G has *polynomial* (respectively, *exponential*, *intermediate*) *growth* if γ_S is polynomial (respectively, exponential, intermediate), and to notice that this definition does not depend upon S.

We recall that the *growth rate of* G *with respect to* S is

$$\lambda_S = \lim_{n \to \infty} \frac{\log \gamma_S(n)}{n}.$$

It is straightforward to prove that G has exponential growth if and only if $\lambda_S > 0$.

2.2 Growth for group endomorphisms

For a group G, denote by $\mathcal{F}(G)$ the family of all finite non-empty subsets of G. If $\phi : G \to G$ is an endomorphism and $F \in \mathcal{F}(G)$, the *growth function* of ϕ with respect to F is

$$\begin{aligned} \gamma_{\phi,F} :\ & \mathbb{N} \to \mathbb{N} \\ & n \mapsto |T_n(\phi, F)|, \end{aligned}$$

where

$$T_n(\phi, F) = F \cdot \phi(F) \cdots \phi^{n-1}(F)$$

is the *n-th ϕ-trajectory* of F. Here, we define $\phi^0(F) = F$ for every $F \in \mathcal{F}(G)$, and, by abuse of notation, we write $T_0(\phi, F) = \{e_G\}$. When $e_G \in F$, we get $T_n(\phi, F) \subseteq T_{n+1}(\phi, F)$ and hence $\{T_n(\phi, F)\}_{n \in \mathbb{N}}$ is an increasing (with respect to inclusion) family of subsets of G.

In general, for every $F \in \mathcal{F}(G)$, we have $|F| \leq \gamma_{\phi,F}(n) \leq |F|^n$ for every $n \in \mathbb{N}_+$, hence the growth of $\gamma_{\phi,F}$ is always at most exponential.

Definition 2.1. [See [2, 3]] Let G be a group and let $\phi : G \to G$ be an endomorphism. Then:

(a) ϕ has *polynomial growth* if $\gamma_{\phi,F}$ is polynomial for every $F \in \mathcal{F}(G)$;

(b) ϕ has *exponential growth* if there exists $F_0 \in \mathcal{F}(G)$ such that γ_{ϕ,F_0} is exponential;

(c) ϕ has *intermediate growth* if $\gamma_{\phi,F}$ is not exponential for every $F \in \mathcal{F}(G)$ and there exists $F_0 \in \mathcal{F}(G)$ such that γ_{ϕ,F_0} is intermediate.

Actually the definition in [3] is sligthly different from Definition 2.1; indeed the set $\mathcal{F}(G)$ here is replaced by the smaller set $\{F \in \mathcal{F}(G) : e_G \in F\}$ in [3]. However, it is straightforward to prove that these definitions are equivalent.

Remark 2.2. The notion of growth for group endomorphisms extends the classical one. For instance, let G be a finitely generated group and let S be a finite set of generators for G. Then, for every $n \in \mathbb{N}$, $B_S(n) = T_n(id_G, F)$ where $F = B_S(1) = S \cup S^{-1} \cup \{e_G\}$; in other words the balls of radius n in the alphabet S are exactly the n-th id_G-trajectories of F. Hence, for every $n \in \mathbb{N}$, $\gamma_S(n) = \gamma_{id_G,F}(n)$, and so the classical definition of growth for G coincides with the definition of growth for the identity automorphism $id_G : G \to G$.

Now, in view of Definition 2.1 and Remark 2.2, one can extend the concept of growth to any group (not necessarily finitely generated):

Definition 2.3. A group G has *polynomial* (respectively, *intermediate*, *exponential*) growth if the identity automorphism id_G of G has polynomial (respectively, intermediate, exponential) growth.

By applying Gromov's Theorem one can extend the characterization of groups of polynomial growth:

Theorem 2.4. *A group G has polynomial growth if and only if G is locally virtually nilpotent.*

Proof. Assume that G has polynomial growth and consider $H = \langle F \rangle$ with $F \in \mathcal{F}(G)$. As G has polynomial growth, so does H. Then H is a finitely generated group of polynomial growth in the classical sense, and hence H is virtually nilpotent by Gromov's Theorem (see [9]). Assume now that every finitely generated subgroup of G is virtually nilpotent and let $F \in \mathcal{F}(G)$. Then $H = \langle F \rangle$ has polynomial growth by Wolf's Theorem, and hence $\gamma_{id_G,F}$ is polynomial. Therefore, G has polynomial growth. □

Also Milnor-Wolf's Theorem can be extended to our more general case:

Theorem 2.5. *A locally virtually soluble group G has either polynomial or exponential growth. Moreover, G has polynomial growth if and only if G is locally virtually nilpotent.*

Proof. Assume that G has not exponential growth and consider $H = \langle F \rangle$ with $F \in \mathcal{F}(G)$. Then H is a finitely generated virtually soluble group and H has not exponential growth. Consequently, H has a soluble subgroup K of finite index in H. Then K is finitely generated and has the same growth type of H, hence K has polynomial growth by Milnor-Wolf's Theorem. Thus, we conclude that H has polynomial growth and so G has polynomial growth as well.

The last assertion follows from Theorem 2.4. □

2.3 Algebraic entropy

For G a group, $\phi : G \to G$ an endomorphism and $F \in \mathcal{F}(G)$, the *algebraic entropy of ϕ with respect to F* is

$$H(\phi, F) = \lim_{n\to\infty} \frac{\log \gamma_{\phi,F}(n)}{n}.$$

Observe that this limit exists because the sequence $\{\log \gamma_{\phi,F}(n)\}_{n\in\mathbb{N}}$ is subadditive, and hence Fekete Lemma applies (see [6]). Now, the *algebraic entropy* of ϕ is

$$h(\phi) = \sup_{F\in\mathcal{F}(G)} H(\phi, F).$$

It was proved in [3] that $H(\phi, F) > 0$ if and only if $\gamma_{\phi,F}$ is exponential, and

$$h(\phi) > 0 \text{ if and only if } \phi \text{ has exponential growth.} \tag{2.1}$$

Equivalently, $h(\phi) = 0$ if and only if ϕ has either polynomial or intermediate growth.

Remark 2.6. Let G be a finitely generated group, let S be a finite set of generators for G and let $F = B_S(1) = S \cup S^{-1} \cup \{e_G\}$. Since $\gamma_S(n) = \gamma_{id_G,F}(n)$ for every $n \in \mathbb{N}$ as noted in Remark 2.2, the classical growth rate λ_S of G with respect to S coincides with the algebraic entropy $H(id_G, F)$ of id_G with respect to F.

The main "working" property that one wishes to have for the algebraic entropy is the so-called Addition Theorem: namely, for a group G, an endomorphism $\phi : G \to G$ and a ϕ-invariant normal subgroup H of G, a wishful thinking asks for

$$h(\phi) = h(\phi \restriction_H) + h(\overline{\phi}),$$

where $\overline{\phi} : G/H \to G/H$ is the endomorphism induced by ϕ on the quotient G/H and $\phi \restriction_H$ is the restriction of ϕ to the subgroup H. Remarkably, the Addition Theorem holds true when G is abelian (see [4]).

The next example shows that the Addition Theorem does not hold in general, even for metabelian groups.

Example 2.7. Consider the lamplighter-type group $G = \mathbb{Z}^{(\mathbb{Z})} \rtimes \mathbb{Z}$ and the identity automorphism $id_G : G \to G$. An easy computation shows that G has exponential growth, and hence $h(id_G) > 0$ by (2.1). Actually, $h(id_G) = \infty$ because $h((id_G)^n) = nh(id_G)$ for every $n \in \mathbb{N}_+$ (see [4]). On the other hand, $\mathbb{Z}^{(\mathbb{Z})}$ and $\mathbb{Z} = G/(\mathbb{Z}^{(\mathbb{Z})})$ are abelian groups and hence it is straightforward to prove directly that $h(id_{\mathbb{Z}^{(\mathbb{Z})}}) = 0$ and $h(id_{\mathbb{Z}}) = 0$, otherwise apply Theorem 2.4 and (2.1) (or see [4]).
In particular, the Addition Theorem does not hold for G.

This example answers [3, Question 5.2.12(b)], and so also [3, Problem 5.2.10].

3 Growth and algebraic entropy of inner automorphisms

For a group G and $g \in G$, we denote by $\phi_g : G \to G$ the inner automorphism of G defined by $x \mapsto g^{-1}xg$, for every $x \in G$.

Lemma 3.1. *Let G be a group, let g be in G and let F be in $\mathcal{F}(G)$. Then $\gamma_{\phi_g,F}(n) = \gamma_{id_G,Fg^{-1}}(n)$ for every $n \in \mathbb{N}$.*

Proof. Let $n \in \mathbb{N}$. If $n = 0$, then the result is obvious. Suppose that $n > 0$. Then

$$\begin{aligned} T_n(\phi_g, F) &= F\phi_g(F)\phi_g^2(F)\cdots\phi_g^{n-1}(F) \\ &= F(g^{-1}Fg)(g^{-2}Fg^2)\dots(g^{-(n-1)}Fg^{n-1}) \\ &= (Fg^{-1})(Fg^{-1})\cdots(Fg^{-1})g^{n-1} \\ &= T_n(id_G, Fg^{-1})g^{n-1}. \end{aligned}$$

Therefore,

$$\gamma_{\phi_g,F}(n) = |T_n(\phi_g, F)| = |T_n(id_G, Fg^{-1})g^{n-1}| = |T_n(id_G, Fg^{-1})| = \gamma_{id_G,Fg^{-1}}(n). \quad \square$$

For finitely generated groups, in view of Remark 2.2, item (a) of the next theorem relates the growth for inner automorphisms to the classical definition of growth.

Theorem 3.2. *Let G be a group and let g be in G. Then:*

(a) ϕ_g has the same growth type of G (i.e., of id_G);

(b) $h(\phi_g) = h(id_G)$.

Proof. Observe that the mapping $F \mapsto Fg^{-1}$ defines a permutation of $\mathcal{F}(G)$. By Lemma 3.1 we have $\gamma_{\phi_g,F}(n) = \gamma_{id_G,Fg^{-1}}(n)$, so part (a) follows immediately from Lemma 3.1.

Moreover, for $F \in \mathcal{F}(G)$, we have $\gamma_{\phi_g,F}(n) = \gamma_{id_G,Fg^{-1}}(n)$ for every $n \in \mathbb{N}$, and hence

$$H(\phi_g, F) = \lim_{n\to\infty} \frac{\log \gamma_{\phi_g,F}(n)}{n} = \lim_{n\to\infty} \frac{\log \gamma_{id_G,Fg^{-1}}(n)}{n} = H(id_G, Fg^{-1}).$$

Therefore,

$$h(\phi_g) = \sup_{F\in\mathcal{F}(G)} H(\phi_g, F) = \sup_{F\in\mathcal{F}(G)} H(id_G, Fg^{-1}) = \sup_{F\in\mathcal{F}(G)} H(id_G, F) = h(id_G). \quad \square$$

4 Dichotomy Theorem for locally finite groups

Let G be a group and let $\phi : G \to G$ be an endomorphism. For $F \in \mathcal{F}(G)$ and $n \in \mathbb{N}$, we let

$$V_n(\phi, F) = \langle \phi^i(F) : i \in \{0, \dots, n\}\rangle,$$
$$V(\phi, F) = \langle \phi^n(F) : n \in \mathbb{N}\rangle = \bigcup_{n\in\mathbb{N}} V_n(\phi, F).$$

Observe that $V(\phi, F)$ is the smallest ϕ-invariant subgroup of G containing F. Similarly, if $g \in G$ and $n \in \mathbb{N}$, we let $V_n(\phi, g) = V_n(\phi, \{g\})$ and $V(\phi, g) = V(\phi, \{g\})$.

Note that $V_0(\phi, F) = \langle F\rangle$ and $T_{n+1}(\phi, F) \subseteq V_n(\phi, F)$ for every $n \in \mathbb{N}$. Moreover, if $e_G \in F$, then $V_{n+1}(\phi, F) = \langle T_{n+1}(\phi, F)\rangle$.

Lemma 4.1. *Let G be a group and let $\phi : G \to G$ be an endomorphism. The subgroup $V(\phi, F)$ is finitely generated if and only if $V(\phi, F) = V_n(\phi, F)$ for some $n \in \mathbb{N}$.*

Proof. If $V(\phi, F) = V_n(\phi, F)$ for some $n \in \mathbb{N}$, then $V(\phi, F)$ is generated by $(n + 1)|F|$ elements and hence it is finitely generated.

Assume that $V(\phi, F)$ is finitely generated. In particular, $V(\phi, F) = \langle S\rangle$, for some $S \in \mathcal{F}(V(\phi, F))$. Observe that, by definition,

$$V(\phi, F) = \bigcup_{n\in\mathbb{N}} V_n(\phi, F).$$

In particular, as S is finite, there exists $n_0 \in \mathbb{N}$ with $S \subseteq V_{n_0}(\phi, F)$. Therefore, $V_{n_0}(\phi, F) \le V(\phi, F) = \langle S\rangle \le V_{n_0}(\phi, F)$ and the lemma follows. $\square$

Lemma 4.2. *Let G be a group and let $\phi : G \to G$ be an endomorphism. If $g \in G$ and $V(\phi, g)$ is not finitely generated, then $H(\phi, \{e_G, g\}) > 0$.*

Proof. By Lemma 4.1, $V_n(\phi, g) \subsetneq V_{n+1}(\phi, g)$ for every $n \in \mathbb{N}$. Set $F = \{e_G, g\}$. We claim that $\gamma_{\phi,F}(n) = 2^n$ for every $n \in \mathbb{N}$. We argue by induction on n. If $n = 0$, then $\gamma_{\phi,F}(0) = |T_0(\phi, F)| =$

$|\{e_G\}| = 1$; if $n = 1$, then $\gamma_{\phi,F}(1) = |T_1(\phi, F)| = |F| = 2$. Assume that $n \in \mathbb{N}_+$ and $\gamma_{\phi,F}(n) = 2^n$. Note that

$$T_{n+1}(\phi, F) = F\phi(F)\cdots\phi^n(F) = T_n(\phi, F)\phi^n(F) =$$
$$= T_n(\phi, F)\{e_G, \phi^n(g)\} = T_n(\phi, F) \cup T_n(\phi, F)\phi^n(g).$$

As $V_n(\phi, g) = \langle V_{n-1}(\phi, g), \phi^n(g)\rangle$ and $V_{n-1}(\phi, g) \subsetneq V_n(\phi, g)$, we see that $\phi^n(g) \in V_n(\phi, g) \setminus V_{n-1}(\phi, g)$. Therefore,

$$T_n(\phi, F) \subseteq V_{n-1}(\phi, g) \quad \text{and} \quad T_n(\phi, F)\phi^n(g) \subseteq V_n(\phi, g) \setminus V_{n-1}(\phi, g).$$

This shows that $T_n(\phi, F) \cap T_n(\phi, F)\phi^n(g) = \emptyset$, and hence $\gamma_{\phi,F}(n+1) = 2\gamma_{\phi,F}(n) = 2^{n+1}$. □

Corollary 4.4 is a direct consequence of Lemma 4.2 in view of the following:

Remark 4.3. Let G be a group and let $\phi : G \to G$ be an endomorphism. The following conditions are equivalent:

(a) $V(\phi, F)$ is finitely generated for every $F \in \mathcal{F}(G)$;

(b) $V(\phi, g)$ is finitely generated for every $g \in G$.

Corollary 4.4. *Let G be a group and let $\phi : G \to G$ be an endomorphism. If $h(\phi) = 0$, then $V(\phi, F)$ is finitely generated for every $F \in \mathcal{F}(G)$.*

The converse implication of Lemma 4.2 does not hold true; indeed it is possible that each $V(\phi, g)$ is finitely generated while $h(\phi) > 0$: consider a group G of exponential growth and the identity automorphism; in this case, $V(id_G, F) = \langle F\rangle$ is finitely generated for every $F \in \mathcal{F}(G)$, while $h(id_G) = \infty$ by (2.1).

On the other hand, the converse implication of Lemma 4.2 holds true assuming that G is locally finite:

Proposition 4.5. *Let G be a locally finite group, let $\phi : G \to G$ be an endomorphism and let F be in $\mathcal{F}(G)$. Then the following conditions are equivalent:*

(a) $H(\phi, F) = 0$*;*

(b) $\gamma_{\phi,F}$ *is bounded (in particular, polynomial);*

(c) $V(\phi, F)$ *is finite (i.e., finitely generated).*

Proof. (a)⇒(c) Assume that $V(\phi, F)$ is infinite. By Remark 4.3, there exists $g \in F$ such that $V(\phi, g)$ is infinite, that is, not finitely generated. By Lemma 4.2 we conclude that $H(\phi, F) > 0$.

(c)⇒(b) Suppose that $V(\phi, F)$ is finite. Then $\gamma_{\phi,F}(n) \leq |V(\phi, F)|$ for every $n \in \mathbb{N}$. In particular, $\gamma_{\phi,F}$ is bounded.

(b)⇒(a) is clear. □

The following result is a consequence of Proposition 4.5 and gives an entirely complete solution to [3, Problem 5.2.3].

Theorem 4.6. *Let G be a locally finite group and let $\phi : G \to G$ be an endomorphism. Then the following conditions are equivalent:*

(a) ϕ *has polynomial growth;*

(b) $h(\phi) = 0$*;*

(c) $V(\phi, F)$ *is finite for every* $F \in \mathcal{F}(G)$.

Theorem 4.6 shows that, if ϕ is an endomorphism of a locally finite group G of zero entropy, then G is a direct limit of finite ϕ-invariant subgroups.

As a consequence of Theorem 4.6 and (2.1), we get that locally finite groups satisfy the condition of Problem 1.1 and indeed are in line with our conjecture: locally virtually nilpotent groups admit no endomorphism of intermediate growth.

Corollary 4.7 (Dichotomy Theorem)**.** *Let* G *be a locally finite group and let* $\phi : G \to G$ *be an endomorphism. Then* ϕ *has either polynomial or exponential growth.*

This solves [3, Problem 5.4.5] for locally finite groups.

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