Non-critical dimensions for critical problems involving fractional Laplacians

Roberta Musina* and Alexander I. Nazarov†

Abstract

We study the Brezis–Nirenberg effect in two families of noncompact boundary value problems involving Dirichlet-Laplacian of arbitrary real order $m > 0$.

Keywords: Fractional Laplace operators, Sobolev inequality, Hardy inequality, critical dimensions.

1 Introduction

Let $m, s$ be two given real numbers, with $0 \leq s < m < \frac{n}{2}$. Let $\Omega \subset \mathbb{R}^n$ be a bounded and smooth domain in $\mathbb{R}^n$ and put

$$2_s^* = \frac{2n}{n - 2m}.$$ 

We study equations

$$(-\Delta)^m u = \lambda (-\Delta)^s u + |u|^{2s^* - 2} u \quad \text{in } \Omega,$$  \hspace{1cm} (1.1)

$$(-\Delta)^m u = \lambda |x|^{-2s} u + |u|^{2m - 2} u \quad \text{in } \Omega,$$  \hspace{1cm} (1.2)

*Dipartimento di Matematica ed Informatica, Università di Udine, via delle Scienze, 206 – 33100 Udine, Italy. Email: roberta.musina@uniud.it. Partially supported by Miur-PRIN 2009WRJ3W7-001 “Fenomeni di concentrazione e problemi di analisi geometrica”.

†St.Petersburg Department of Steklov Institute, Fontanka 27, St.Petersburg, 191023, Russia, and St.Petersburg State University, Universitetskii pr. 28, St.Petersburg, 198504, Russia. E-mail: al.il.nazarov@gmail.com. Supported by RFBR grant 11-01-00825 and by St.Petersburg University grant 6.38.670.2013.
under suitably defined Dirichlet boundary conditions. In dealing with equation (1.2) we always assume that \( \Omega \) contains the origin. For the definition of fractional Dirichlet–Laplace operators \((-\Delta)^m, (-\Delta)^s\) and for the variational approach to (1.1), (1.2) we refer to the next section.

The celebrated paper [3] by Brezis and Nirenberg was the inspiration for a fruitful line of research about the effect of lower order perturbations in noncompact variational problems. They took as model the case \( n > 2, m = 1, s = 0 \), that is,

\[
-\Delta u = \lambda u + |u|^{\frac{4}{n-2}} u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega.
\] (1.3)

Brezis and Nirenberg pointed out a remarkable phenomenon that appears for positive values of the parameter \( \lambda \): they proved existence of a nontrivial solution for any small \( \lambda > 0 \) if \( n \geq 4 \); in contrast, in the lowest dimension \( n = 3 \) non-existence phenomena for sufficiently small \( \lambda > 0 \) can be observed. For this reason, the dimension \( n = 3 \) has been named critical for problem (1.3).

Clearly, as larger \( s \) is, as stronger the effects of the lower order perturbations are expected in equations (1.1), (1.2). We are interested in the following question: Given \( m < \frac{n}{2} \), how large must be \( s \) in order to have the existence of a ground state solution, for any arbitrarily small \( \lambda > 0 \)? In case of an affirmative answer, we say that \( n \) is not a critical dimension.

We present our main result, that holds for any dimension \( n \geq 1 \) (see Section 4 for a more precise statement).

**THEOREM.** If \( s \geq 2m - \frac{n}{2} \) then \( n \) is not a critical dimension for the Dirichlet boundary value problems associated to equations (1.1) and (1.2).

We point out some particular cases that are included in this result.

- If \( m \) is an integer and \( s = m - 1 \), then at most the lowest dimension \( n = 2m + 1 \) is critical.
- For any \( n > 2m \) there always exist lower order perturbations of the type \( |x|^{-2s} u \) and of the type \((-\Delta)^s u \) such that \( n \) is not a critical dimension.
- If \( m < 1/4 \) then no dimension is critical, for any choice of \( s \in [0, m) \).

\(^1\) compare with [13, 8].
After [4], a large number of papers have been focused on studying the effect of linear perturbations in noncompact variational problems of the type (1.1). Most of these papers deal with \( s = 0 \), when the problems (1.1) and (1.2) coincide. Moreover, as far as we know, all of them consider either polyharmonic case \( 2 \leq m \in \mathbb{N} \), see for instance [13], [6], [2], [10], [7], or the case \( m \in (0, 1) \), see [14], [15]. We cite also [4], where equation (1.1) is studied in case \( m = 2 \), \( s = 1 \). Thus, our Theorem 4.2 covers all earlier existence results.

Finally, we mention [1] (see also [16]) where equation (1.1) for the so-called Navier-Laplacian is studied in case \( m \in (0, 1), s = 0 \). For a comparison between the Dirichlet and Navier Laplacians we refer to [12].

The paper is organized as follows. After introducing some notation and preliminary facts in Section 2, we provide the main estimates in Section 3. In Section 4 we prove Theorem 1 and point out an existence result for the case \( s < 2m - \frac{n}{2} \).

2 Preliminaries

The fractional Laplacian \((-\Delta)^m u\) of a function \( u \in C_0^\infty (\mathbb{R}^n)\) is defined via the Fourier transform

\[
\mathcal{F}[u](\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi \cdot x} u(x) \, dx
\]

by the identity

\[
\mathcal{F} [(-\Delta)^m u] (\xi) = |\xi|^{2m} \mathcal{F}[u](\xi). \tag{2.1}
\]

In particular, Parseval’s formula gives

\[
\int_{\mathbb{R}^n} (-\Delta)^m u \cdot u \, dx = \int_{\mathbb{R}^n} |(-\Delta)^{\frac{m}{2}} u|^2 \, dx = \int_{\mathbb{R}^n} |\xi|^{2m} |\mathcal{F}[u]|^2 \, d\xi.
\]

We recall the well known Sobolev inequality

\[
\int_{\mathbb{R}^n} |(-\Delta)^{\frac{m}{2}} u|^2 \, dx \geq S_m \left( \int_{\mathbb{R}^n} |u|^{2m} \, dx \right)^{2/2m}, \tag{2.2}
\]

that holds for any \( u \in C_0^\infty (\mathbb{R}^n) \) and \( m < \frac{n}{2} \), see for example [17] 2.8.1/15.
Let \( \mathcal{D}^m(\mathbb{R}^n) \) be the Hilbert space obtained by completing \( C_0^\infty(\mathbb{R}^n) \) with respect to the Gagliardo norm
\[
\|u\|_m^2 = \int_{\mathbb{R}^n} |(−\Delta)^{\frac{m}{2}} u|^2 \, dx.
\]

Thanks to (2.2), the space \( \mathcal{D}^m(\mathbb{R}^n) \) is continuously embedded into \( L^{2m}_*(\mathbb{R}^n) \). The best Sobolev constant \( S_m \) was explicitly computed in [5]. Moreover, it has been proved in [5] that \( S_m \) is attained in \( \mathcal{D}^m(\mathbb{R}^n) \) by a unique family of functions, all of them being obtained from
\[
\phi(x) = (1 + |x|^2)^{\frac{2m-n}{2}}
\]
by translations, dilations in \( \mathbb{R}^n \) and multiplication by constants.

Dilations play a crucial role in the problems under consideration. Notice that for any \( \omega \in C_0^\infty(\mathbb{R}^n) \), \( R > 0 \) it turns out that
\[
\int_{\mathbb{R}^n} |\xi|^{2m} |\mathcal{F}[\omega](\xi)|^2 d\xi = R^{n-2m} \int_{\mathbb{R}^n} |\xi|^{2m} |\mathcal{F}[\omega(R\cdot)](\xi)|^2 d\xi
\]
\[
\int_{\mathbb{R}^n} |\omega|^{2m} \, dx = R^m \int_{\mathbb{R}^n} |\omega(R\cdot)|^{2m} \, dx.
\]

Finally, we point out that the Hardy inequality
\[
\int_{\mathbb{R}^n} |(−\Delta)^{\frac{m}{2}} u|^2 \, dx \geq \mathcal{H}_m \int_{\mathbb{R}^n} |x|^{-2m} |u|^2 \, dx
\]
holds for any function \( u \in \mathcal{D}^m(\mathbb{R}^n) \). The best Hardy constant \( \mathcal{H}_m \) was explicitly computed in [11].

The natural ambient space to study the Dirichlet boundary value problems for (1.1), (1.2) is
\[
\tilde{H}^m(\Omega) = \{ u \in \mathcal{D}^m(\mathbb{R}^n) : \text{supp } u \subset \Omega \},
\]
endowed with the norm \( \| u \|_m \). By Theorem 4.3.2/1 [17], for \( m + \frac{1}{2} \notin \mathbb{N} \) this space coincides with \( H_0^m(\Omega) \) (that is the closure of \( C_0^\infty(\Omega) \)) in \( H^m(\Omega) \), while for \( m + \frac{1}{2} \in \mathbb{N} \) one has \( \tilde{H}^m(\Omega) \subset H_0^m(\Omega) \). Moreover, \( C_0^\infty(\Omega) \) is dense in \( \tilde{H}^m(\Omega) \). Clearly, if \( m \) is an integer then \( \tilde{H}^m(\Omega) \) is the standard Sobolev space of functions \( u \in H^m(\Omega) \) such that \( D^\alpha u = 0 \) for every multiindex \( \alpha \in \mathbb{N}^n \) with \( 0 \leq |\alpha| < m \).
We agree that \((-\Delta)^0 u = u, \tilde{H}^0(\Omega) = L^2(\Omega)\), since \(2.3\) reduces to the standard \(L^2\) norm in case \(m = 0\).

We define (weak) solutions of the Dirichlet problems for \((1.1), (1.2)\) as suitably normalized critical points of the functionals

\[
\mathcal{R}^\Omega_{\lambda,m,s}[u] = \frac{\int_\Omega |(-\Delta)^m u|^2 \, dx - \lambda \int_\Omega |(-\Delta)^s u|^2 \, dx}{\left(\int_\Omega |u|^{2m} \, dx\right)^{2/2m}} \quad (2.7)
\]

\[
\tilde{\mathcal{R}}^\Omega_{\lambda,m,s}[u] = \frac{\int_\Omega |(-\Delta)^m u|^2 \, dx - \lambda \int_\Omega |x|^{-2s}|u|^2 \, dx}{\left(\int_\Omega |u|^{2m} \, dx\right)^{2/2m}} \quad , \quad (2.8)
\]

respectively. It is easy to see that both functionals \((2.7), (2.8)\) are well defined on \(\tilde{H}^m(\Omega) \setminus \{0\}\).

We conclude this preliminary section with some embedding results.

**Proposition 2.1** Let \(m, s\) be given, with \(0 \leq s < m < n/2\).

i) The space \(\tilde{H}^m(\Omega)\) is compactly embedded into \(\tilde{H}^s(\Omega)\). In particular the infima

\[
\Lambda_1(m, s) := \inf_{\substack{u \in \tilde{H}^m(\Omega) \setminus \{0\}}} \frac{\|u\|_{2m}^2}{\|u\|_s^2} , \quad \tilde{\Lambda}_1(m, s) := \inf_{\substack{u \in \tilde{H}^m(\Omega) \setminus \{0\}}} \frac{\|u\|_{2m}^2}{\|x|^{-s}u\|_0^2} \quad (2.9)
\]

are positive and achieved.

\[
\inf_{u \in \tilde{H}^m(\Omega) \setminus \{0\}} \frac{\|u\|_{2m}^2}{\|u\|_{L^{2m}}^2} = S_m.
\]

Statement i) is well known for \(\Lambda_1(m, s)\) and follows from \(2.6\) for \(\tilde{\Lambda}_1(m, s)\). To check ii), use the inclusion \(\tilde{H}^m(\Omega) \hookrightarrow D^m(\mathbb{R}^n)\) and a rescaling argument. Clearly, the Sobolev constant \(S_m\) is never achieved on \(\tilde{H}^m(\Omega)\).
3 Main estimates

Let $\phi$ be the extremal of the Sobolev inequality (2.2) given by (2.4). In particular, it holds that

$$M := \int_{\mathbb{R}^n} |(-\Delta)^{m/2} \phi|^2 \, dx = S_m \left( \int_{\mathbb{R}^n} |\phi|^{2^*_m} \, dx \right)^{2/2^*_m}. \quad (3.1)$$

Fix $\delta > 0$ and a cutoff function $\varphi \in C_0^\infty(\Omega)$, such that $\varphi \equiv 1$ on the ball $\{|x| < \delta\}$ and $\varphi \equiv 0$ outside $\{|x| < 2\delta\}$. If $\delta$ is sufficiently small, the function

$$u_\epsilon(x) := \epsilon^{2m-n} \varphi(x) \phi \left( \frac{x}{\epsilon} \right) = \varphi(x) \left( \epsilon^2 + |x|^2 \right)^{\frac{2m-n}{2}}$$

has compact support in $\Omega$. Next we define

$$A_{m}^\epsilon := \int_{\Omega} \left| (-\Delta)^{m/2} u_\epsilon \right|^2 \, dx \quad A_s^\epsilon := \int_{\Omega} \left| (-\Delta)^{s/2} u_\epsilon \right|^2 \, dx$$

$$\tilde{A}_s^\epsilon := \int_{\Omega} |x|^{-2s} |u_\epsilon|^2 \, dx \quad B^\epsilon := \int_{\Omega} |u_\epsilon|^{2m} \, dx$$

and we denote by $c$ any universal positive constant.

**Lemma 3.1** It holds that

$$\begin{cases}
A_{m}^\epsilon \leq \epsilon^{2m-n} \left( M + c\epsilon^{n-2m} \right) & \text{if } s > 2m - \frac{n}{2} \\
A_s^\epsilon, \tilde{A}_s^\epsilon \geq c\epsilon^{4m-n-2s} & \text{if } s > 2m - \frac{n}{2} \\
A_s^\epsilon, \tilde{A}_s^\epsilon \geq c \log \epsilon & \text{if } s = 2m - \frac{n}{2} \\
B^\epsilon \geq \epsilon^{-n} \left( (MS_m^{-1})^{2m/2} - c\epsilon^n \right). & \end{cases} \quad (3.2a)$$

**Proof of (3.2a).** First of all, from (2.5) we get

$$A_{m}^\epsilon = \epsilon^{2m-n} \int_{\mathbb{R}^n} |\xi|^{2m} |\mathcal{F} \{ \varphi(\epsilon \cdot) \phi \}|^2 \, d\xi. \quad (3.3)$$

Thus

$$\Gamma_{m}^\epsilon := \epsilon^{n-2m} A_{m}^\epsilon - M = \int_{\mathbb{R}^n} |\xi|^{2m} |\mathcal{F} \{ \varphi(\epsilon \cdot) \phi \}|^2 \, d\xi - \int_{\mathbb{R}^n} |\xi|^{2m} |\mathcal{F} \{ \phi \}|^2 \, d\xi.$$
We need to prove that
\[ |\Gamma_m^c| \leq c\varepsilon^{n-2m}. \] \hspace{1cm} (3.4)

If \( m \in \mathbb{N} \), the proof of (3.3) has been carried out in [3], [7]. Here we limit ourselves to the more difficult case, namely, when \( m \) is not an integer. We denote by \( k := \lfloor m \rfloor \geq 0 \) the integer part of \( m \), so that \( m - k > 0 \). Then

\[
\Gamma_m^c = \int_{\mathbb{R}^n} |\xi|^{2k} F[U_-] \cdot |\xi|^{2(m-k)} F[U_+] d\xi
\]

\[
= 2^{2(m-k)+\frac{n}{2}} \frac{\Gamma(m-k + \frac{n}{2})}{\Gamma(-(m-k))} \int_{\mathbb{R}^n} (-\Delta)^k U_-(x) \cdot \text{V.P.} \int_{\mathbb{R}^n} \frac{U_+(x) - U_+(y)}{|x-y|^{n+2(m-k)}} dy dx,
\]

where \( U_\pm = \varphi(\varepsilon \cdot ) \phi \pm \phi \) (the last equality follows from [9, Ch. 2, Sec. 3]).

We split the interior integral as follows:

\[
\text{V.P.} \int_{\mathbb{R}^n} \Psi dy = \text{V.P.} \int_{|y-x| \leq \frac{1}{2} |x|} \Psi dy + \int_{|y-x| \geq \frac{1}{2} |x|} \Psi dy + \int_{|y-x| \geq |x|} \Psi dy.
\]

We claim that \( |I_j| \leq c|x|^{2k-n} \) for \( j = 1, 2, 3 \). Indeed, the Lagrange formula gives

\[
|I_1| \leq \max_{|y-x| \leq \frac{1}{2} |x|} |D^2 U_+(y)| \cdot \int_{|z| \leq \frac{|x|}{2}} \frac{dz}{|z|^{n+2(m-k)-2}} \leq c|x|^{-(n-2m+2)} \cdot |x|^{2-2(m-k)} = c|x|^{2k-n}.
\]

As concerns the last two integrals we estimate

\[
|I_2| \leq \int_{|y-x| \geq \frac{|x|}{2}} \frac{c|x|^{-(n-2m)}}{|x-y|^{n+2(m-k)}} dy \leq |x|^{-(n+2(m-k))} \cdot c|x|^{2m} = c|x|^{2k-n}
\]

and finally

\[
|I_3| \leq \int_{y-x| \geq |x|} \frac{c|x|^{-(n-2m)}}{|x-y|^{n+2(m-k)}} dy \leq c|x|^{-(n-2m)} \cdot \int_{|z| \geq \frac{|x|}{2}} \frac{dz}{|z|^{n+2(m-k)}} \leq c|x|^{-(n-2m)} \cdot |x|^{2(m-k)} = c|x|^{2k-n},
\]

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and the claim follows. Now, since
\[ |(-\Delta)^kU_-(x)| \leq \frac{c}{|x|^{n-2(m-k)}} \chi_{\{|x|\geq \delta/\varepsilon\}} + \frac{c\varepsilon^{2k}}{|x|^{n-2m}} \chi_{\{|x|\leq 2\delta/\varepsilon\}}, \]
we obtain
\[ |\Gamma^{\varepsilon}_m| \leq c \int_{|x|\geq \delta/\varepsilon} \frac{dx}{|x|^{2n-2m}} + c \int_{\delta/\varepsilon \leq |x| \leq 2\delta/\varepsilon} \frac{\varepsilon^{2k} \, dx}{|x|^{2n-2(m+k)}} \leq c\varepsilon^{n-2m}, \]
that completes the proof of (3.4) and of (3.2a).

Proof of (3.2b) and (3.2c). We use the Hardy inequality (2.6) to get
\[ A^\varepsilon_s \geq cA^\varepsilon_s \geq c\varepsilon^{4m-2s-n} \int_{\mathbb{R}^n} |x|^{-2s} |\varphi(\varepsilon \cdot)\phi|^2 \, dx \]
\[ \geq c\varepsilon^{4m-2s-n} \int_{|x|<\delta/\varepsilon} \frac{dx}{|x|^{2s}(1+|x|^2)^{n-2m}}. \]
The last integral converges as \( \varepsilon \to 0 \) if \( s > 2m - \frac{n}{2} \), and diverges with speed \( |\log \varepsilon| \) if \( s = 2m - \frac{n}{2} \).

Proof of (3.2d). For \( \varepsilon \) small enough we estimate by below
\[ \int_{\mathbb{R}^n} |u^\varepsilon_{\rho}|^2 m = \varepsilon^{-n} \int_{\mathbb{R}^n} |\varphi(\varepsilon \cdot)\phi|^2 m \, dx = \varepsilon^{-n} \left( \int_{|x|>\delta/\varepsilon} |\varphi(\varepsilon \cdot)\phi|^2 m \, dx \right) \]
\[ \geq \varepsilon^{-n} (MS_m^{-1})^{2m/2} - c \int_{|x|>\delta/\varepsilon} |x|^{-2n} \, dx \]
\[ = \varepsilon^{-n} (MS_m^{-1})^{2m/2} - c\varepsilon^n \]
and the Lemma is completely proved. \( \square \)

4 Two noncompact minimization problems

In this section we deal with the minimization problems
\[ S_\lambda^\Omega(m,s) = \inf_{u \in H^m(\Omega)} \mathcal{R}_\lambda^\Omega m,s[u]; \quad \tilde{S}_\lambda^\Omega(m,s) = \inf_{u \neq 0} \tilde{\mathcal{R}}^\Omega_{\lambda,m,s}[u], \]
where the functionals \( \mathcal{R} \) and \( \tilde{\mathcal{R}} \) are introduced in (2.7) and (2.8), respectively.
Lemma 4.1 The following facts hold for any \( \lambda \in \mathbb{R} \):

i) \( S_\lambda^\Omega(m, s) \leq S_m \);

ii) If \( \lambda \leq 0 \) then \( S_\lambda^\Omega(m, s) = S_m \) and it is not achieved;

iii) If \( 0 < S_\lambda^\Omega(m, s) < S_m \), then \( S_\lambda^\Omega(m, s) \) is achieved.

The same statements hold for \( \tilde{S}_\lambda^\Omega(m, s) \) instead of \( S_\lambda^\Omega(m, s) \).

Proof. The proof is nowadays standard, and is essentially due to Brezis and Nirenberg \[3\]. We sketch it for the infimum \( S_\lambda^\Omega(m, s) \), for the convenience of the reader.

Fix \( \varepsilon > 0 \) and take \( u \in C_0^\infty(\mathbb{R}^n) \setminus \{0\} \) such that

\[
(S_m + \varepsilon) \left( \int_{\mathbb{R}^n} |u|^{2^*_m} \, dx \right)^{2/2^*_m} \geq \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} u|^2 \, dx.
\]

(4.1)

Let \( R > 0 \) be large enough, so that \( u_R(\cdot) := u(R \cdot) \in C_0^\infty(\Omega) \). Using (2.5) we get

\[
S_\lambda^\Omega(m, s) \leq \frac{\|u\|_m^2 - \lambda R^{2(s-m)}\|u\|_s^2}{\|u\|_{L^2^*_m}^2} \leq (S_m + \varepsilon) \left( 1 + c R^{2(s-m)} \right),
\]

where \( c \) depends only on \( u \) and \( \lambda \). Letting \( R \to \infty \) we get \( S_\lambda^\Omega(m, s) \leq (S_m + \varepsilon) \) for any \( \varepsilon > 0 \), and i) is proved.

Next, if \( \lambda \leq 0 \) then clearly \( S_\lambda^\Omega(m, s) = S_m \). If \( \lambda = 0 \) then \( S_m \) is not achieved. The more it is not achieved for \( \lambda < 0 \), and ii) holds.

Finally, to prove iii) take a minimizing sequence \( u_h \). It is convenient to normalize \( u_h \) with respect to the \( L^{2^*_m} \)-norm, so that

\[
\int_{\Omega} |(-\Delta)^{\frac{s}{2}} u_h|^2 \, dx - \lambda \int_{\Omega} |(-\Delta)^{\frac{s}{2}} u_h|^2 \, dx = S_\lambda^\Omega(m, s) + o(1).
\]

We can assume that \( u_h \to u \) weakly in \( H^m(\Omega) \) and strongly in \( \tilde{H}^s(\Omega) \) by Proposition \[2.1\]. Since

\[
\lambda \int_{\Omega} |(-\Delta)^{\frac{s}{2}} u|^2 \, dx = \lambda \int_{\Omega} |(-\Delta)^{\frac{s}{2}} u_h|^2 \, dx + o(1)
\]

\[
= \int_{\Omega} |(-\Delta)^{\frac{s}{2}} u_h|^2 \, dx - S_\lambda^\Omega(m, s) + o(1)
\]

\[
\geq (S_m - S_\lambda^\Omega(m, s)) + o(1),
\]

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then \( u \neq 0 \). By the Brezis–Lieb lemma we get
\[
1 = \|u_h\|_{L^{2m}_m}^{2m} = \|u_h - u\|_{L^{2m}_m}^{2m} + \|u\|_{L^{2m}_m}^{2m} + o(1).
\]
Thus
\[
S_{\lambda}^{\Omega}(m, s) = \|u_h\|_m^2 - \lambda \|u_h\|_s^2 + o(1)
\]
\[
= \left( \|u_h - u\|_m^2 + \|u\|_m^2 \right) - \lambda \left( \|u_h - u\|_s^2 + \|u\|_s^2 \right) + o(1)
\]
\[
= \left( \|u_h - u\|_m^2 - \lambda \|u_h - u\|_s^2 \right) + \left( \|u\|_m^2 - \lambda \|u\|_s^2 \right)
\]
\[
\geq S_{\lambda}^{\Omega}(m, s) \cdot \frac{\xi_h^2 + 1}{(\xi_h^m + 1)^{2/2m}} + o(1),
\]
where we have set
\[
\xi_h := \frac{\|u_h - u\|_{L^{2m}_m}}{\|u\|_{L^{2m}_m}}.
\]
Since \( 2^* > 2 \), this implies that \( \xi_h \to 0 \), that is, \( u_h \to u \) in \( L^{2^*}_m \) and hence \( u \) achieves \( S_{\lambda}^{\Omega}(m, s) \).

We are in position to prove our existence result, that includes the theorem already stated in the introduction.

**Theorem 4.2** Assume \( s \geq 2m - \frac{n}{2} \).

i) If \( 0 < \lambda < \Lambda_1(m, s) \) then \( S_{\lambda}^{\Omega}(m, s) \) is achieved and (1.7) has a nontrivial solution in \( \tilde{H}^m(\Omega) \).

ii) If \( 0 < \lambda < \tilde{\Lambda}_1(m, s) \) then \( \tilde{S}_{\lambda}^{\Omega}(m, s) \) is achieved and (1.2) has a nontrivial solution in \( \tilde{H}^m(\Omega) \).

**Proof.** Since \( 0 < \lambda < \Lambda_1(m, s) \) then \( S_{\lambda}^{\Omega}(m, s) \) is positive, by Proposition 2.1. The main estimates in Lemma 3.1 readily imply \( S_{\lambda}^{\Omega}(m, s) < \mathcal{S}_m \). By Lemma 4.1, \( S_{\lambda}^{\Omega}(m, s) \) is achieved by a nontrivial \( u \in \tilde{H}^m(\Omega) \), that solves (1.1) after multiplication by a suitable constant. Thus i) is proved. For ii) argue in the same way. \( \square \)
In the case \( s < 2m - \frac{n}{2} \) the situation is more complicated. We limit ourselves to point out the next simple existence result.

**Theorem 4.3** Assume \( s < 2m - \frac{n}{2} \).

i) There exists \( \lambda^* \in [0, \Lambda_1(m,s)) \) such that the infimum \( S_\lambda^0(m,s) \) is attained for any \( \lambda \in (\lambda^*, \Lambda_1(m,s)) \), and hence (1.1) has a nontrivial solution.

ii) There exists \( \tilde{\lambda}^* \in [0, \tilde{\Lambda}_1(m,s)) \) such that the infimum \( \tilde{S}_\lambda^0(m,s) \) is attained for any \( \lambda \in (\tilde{\lambda}^*, \tilde{\Lambda}_1(m,s)) \), and hence (1.2) has a nontrivial solution.

**Proof.** Use Proposition 2.1 to find \( \varphi_1 \in \tilde{H}^m(\Omega), \varphi_1 \neq 0 \), such that

\[
\int_{\Omega} |(-\Delta)^{s/2} \varphi_1|^2 \, dx = \Lambda_1(m,s) \int_{\Omega} |(-\Delta)^{s/2} \varphi_1|^2 \, dx.
\]

Then test \( S_\lambda^0(m,s) \) with \( \varphi_1 \) to get the strict inequality \( S_\lambda^0(m,s) < S_m \). The first conclusion follows by Proposition 2.1 and Lemma 4.1. For (1.2) argue similarly. \( \square \)

**References**


