



UNIVERSITÀ  
DEGLI STUDI  
DI UDINE

## Università degli studi di Udine

### Approximation of eigenvalues of evolution operators for linear renewal equations

*Original*

*Availability:*

This version is available <http://hdl.handle.net/11390/1123175> since 2021-03-14T18:21:21Z

*Publisher:*

*Published*

DOI:10.1137/17M1140534

*Terms of use:*

The institutional repository of the University of Udine (<http://air.uniud.it>) is provided by ARIC services. The aim is to enable open access to all the world.

*Publisher copyright*

(Article begins on next page)

1                   **APPROXIMATION OF EIGENVALUES OF EVOLUTION**  
2                   **OPERATORS FOR LINEAR RENEWAL EQUATIONS\***

3                   DIMITRI BREDA<sup>†</sup> AND DAVIDE LIESSI<sup>†</sup>

4       **Abstract.** A numerical method based on pseudospectral collocation is proposed to approximate  
5 the eigenvalues of evolution operators for linear renewal equations, which are retarded functional  
6 equations of Volterra type. Rigorous error and convergence analyses are provided, together with  
7 numerical tests. The outcome is an efficient and reliable tool which can be used, for instance, to  
8 study the local asymptotic stability of equilibria and periodic solutions of nonlinear autonomous  
9 renewal equations. Fundamental applications can be found in population dynamics, where renewal  
10 equations play a central role.

11       **Key words.** renewal equations, Volterra integral equations, retarded functional equations, evo-  
12 lution operators, eigenvalue approximation, pseudospectral collocation, stability, equilibria, periodic  
13 solutions

14       **AMS subject classifications.** 45C05, 45D05, 47D99, 65L07, 65L15, 65R20

15       **1. Introduction.** Delay equations of renewal or differential type are often used  
16 in different fields of science to model complex phenomena in a more realistic way,  
17 thanks to the presence of delayed terms which relate the current evolution to the past  
18 history. Examples of broad areas where delays arise naturally are control theory in  
19 engineering [37, 39, 53, 59] and population dynamics or epidemics in mathematical  
20 biology [36, 41, 47, 51, 52, 58].

21       In many applications there is a strong interest in determining the asymptotic sta-  
22 bility of particular invariants of the associated dynamical systems, mainly equilibria  
23 and periodic solutions. Notable instances are network consensus, mechanical vibra-  
24 tions, endemic states and seasonal fluctuations. The problem is nontrivial since the  
25 introduction of delays notoriously requires an infinite-dimensional state space [24].

26       A common tool to investigate local stability is the principle of linearized stability  
27 which, generically, links the stability of a solution of a nonlinear system to that of  
28 the null solution of the system linearized around the chosen solution. This linearized  
29 system is autonomous in the case of equilibria and has periodic coefficients in the case  
30 of periodic solutions.

31       As far as renewal equations (REs) and retarded functional differential equations  
32 (RFDEs) are concerned, the stability of the null solution of a linear autonomous  
33 system is determined by the spectrum of the semigroup of solution operators or,  
34 equivalently, by that of its infinitesimal generator [25, 31, 40].

35       For RFDEs, as for ordinary differential equations, the Floquet theory relates  
36 the stability of the null solution of a linear periodic system to the characteristic  
37 multipliers. These are the eigenvalues of the monodromy operator, i.e., the evolution  
38 operator that shifts the state along the solution by one period (see [31, chapter XIV]  
39 and [40, chapter 8]). An analogous formal theory lacks for REs. A possible extension  
40 is still an ongoing effort of the authors and colleagues, in view of the application

---

\*Submitted to the editors July 25, 2017.

**Funding:** This work was supported by the INdAM GNCS projects “Analisi numerica di certi tipi non classici di equazioni di evoluzione” (2016) and “Analisi e sviluppo di metodologie numeriche per certi tipi non classici di sistemi dinamici” (2017) and the Ph.D. Course in Computer Science, Mathematics and Physics of the University of Udine.

<sup>†</sup>Department of Mathematics, Computer Science and Physics, University of Udine, I-33100 Udine, Italy (dimitri.breda@uniud.it, liessi.davide@spes.uniud.it).

41 of sun-star calculus to REs in [25] for equilibria. A preliminary study reveals the  
 42 above a promising approach, with difficulties restricted to the validation of technical  
 43 hypotheses. Thus we retain reasonable to assume here the validity of a Floquet  
 44 theory, as well as that of a corresponding principle of linearized stability (more on  
 45 this is postponed to [section 6](#)).

46 Given the infinite-dimensional nature of delay equations, numerical methods to  
 47 approximate the spectrum of the operators mentioned above characterize part of the  
 48 recent literature (to start see [14] and the references therein). They are based on  
 49 the reduction to finite dimension, in order to exploit the eigenvalues of the obtained  
 50 matrices as approximations to (part of) the exact ones.

51 About equilibria of RFDEs, see [12] for the discretization of the infinitesimal  
 52 generator via pseudospectral collocation and [34] for the discretization of the solution  
 53 operator via linear multistep methods. For equilibria of REs and coupled systems of  
 54 REs and RFDEs, see instead the more recent collocation techniques of [10, 11].

55 Concerning periodic solutions of RFDEs, perhaps the most (indirectly) used tech-  
 56 nique is that behind DDE-BIFTOOL [1, 57], the widespread bifurcation package  
 57 for delay problems (namely delay differential algebraic equations with constant or  
 58 state-dependent discrete delays). There, a discretization of the monodromy opera-  
 59 tor is obtained as a byproduct of the piecewise collocation used to compute periodic  
 60 solutions [33]. Other approaches are the semi-discretization method [43] and the  
 61 Chebyshev-based collocations [19, 20, 21], and [44] contains an interesting account  
 62 of this piece of literature. The most general collocation approach is perhaps [13],  
 63 targeted to the discretization of generic evolution operators, including both solution  
 64 operators (for equilibria) and monodromy operators (for periodic solutions, with any  
 65 ratio between delay and period, even irrational) and any (finite) combination of dis-  
 66 crete and distributed delay terms.

67 From an overall glimpse of the existing works, it emerges clearly that there are  
 68 no currently available methods to approximate the spectrum of evolution operators  
 69 of REs. Given their importance in population dynamics [7, 17, 28, 29, 30, 41, 42, 45,  
 70 48, 52, 61], this lack of tools deserves consideration, especially when the interest is in  
 71 the stability of periodic solutions. Indeed, inspired by the ideas of the pseudospectral  
 72 collocation approach for RFDEs of [13], the present work is a first attempt to fill this  
 73 gap. With respect to [13], in reformulating the evolution operators we introduce an  
 74 essential modification, in order to accommodate for the different kind of equations.  
 75 Namely, RFDEs provide the value of the derivative of the unknown function, while  
 76 REs provide directly the value of the unknown function. Moreover, the state space is  
 77 a space of  $L^1$  functions, instead of continuous functions as in the RFDE case; this is a  
 78 natural choice for REs [25], since in general the initial functions can be discontinuous  
 79 and the solution itself can be discontinuous at the initial time. Finally, provided that  
 80 some hypotheses on the integration kernel are satisfied, the right-hand side of REs  
 81 exhibits a regularizing effect (in the sense that applying the right-hand side to an  $L^1$   
 82 function produces a continuous function), which is not present in general in RFDEs.  
 83 These differences motivate a complete revisit of [13] rather than a mere adaptation.

84 A preliminary algorithm implementing the method we propose is adopted for the  
 85 first time in the recent work [9] for a special class of REs. There it is just marginally  
 86 summarized, as it is only used in the background simply to support the analysis of the  
 87 approach for nonlinear problems described in [8]. In this work, instead, the method  
 88 is central, and we elaborate a full treatment including a rigorous error analysis and

89 proof of convergence, as well as numerical tests for experimental confirmation and  
90 relevant codes.

91 The main practical outcome is the construction of an approximating matrix whose  
92 eigenvalues are demonstrated to converge to the exact ones, possibly with infinite or-  
93 der, under reasonable regularity assumptions on the model coefficients. This infinite  
94 order of convergence, typical of pseudospectral methods [60], represents a key com-  
95 putational feature, especially in case of robust analyses (as for, e.g., stability charts  
96 and bifurcations). Indeed, a good accuracy is ensured in general with low matrix  
97 dimension and, consequently, low computational cost and time.

98 For completeness, let us notice that the literature on Volterra integral and func-  
99 tional equations abounds of numerical methods for initial and boundary value prob-  
100 lems. The monograph [16] and the references therein may serve as a starting point.  
101 However, all these methods deal with time integration to approximate a solution  
102 rather than with spectral approximation to detect stability.

103 The paper is structured as follows. In section 2 we define the problem and reform-  
104 ulate the evolution operators, an essential step hereinafter. In section 3 we define  
105 the discretizations of the relevant function spaces and of the generic evolution op-  
106 erator. In section 4 we prove that the discretized evolution operator is well-defined  
107 and that its eigenvalues approximate those of the infinite-dimensional evolution op-  
108 erator. In section 5 we present two numerical tests. Concluding comments follow in  
109 section 6. Eventually, a matrix representation of the discretized evolution operator  
110 is constructed in Appendix A for the sake of implementation and relevant MATLAB  
111 codes are available at the authors.

112 **2. Formulation of the problem.** For  $d \in \mathbb{N}$  and  $\tau \in \mathbb{R}$  both positive, consider  
113 the function space  $X := L^1([-\tau, 0], \mathbb{R}^d)$  equipped with the usual  $L^1$  norm, denoted by  
114  $\|\cdot\|_X$ . For  $s \in \mathbb{R}$  and a function  $x$  defined on  $[s - \tau, +\infty)$  let

$$115 \quad (2.1) \quad x_t(\theta) := x(t + \theta), \quad t \geq s, \theta \in [-\tau, 0].$$

116 Given a measurable function  $C: [s, +\infty) \times [-\tau, 0] \rightarrow \mathbb{R}^{d \times d}$  and  $\varphi \in X$ , define the  
117 initial value problem for the RE

$$118 \quad (2.2) \quad x(t) = \int_{-\tau}^0 C(t, \theta) x_t(\theta) d\theta, \quad t > s,$$

119 by imposing  $x_s = \varphi$ . As long as  $t \in [0, \tau]$ , this corresponds to the Volterra integral  
120 equation (VIE) of the second kind

$$121 \quad x(t) = \int_0^t K(t, \sigma) x(\sigma) d\sigma + f(t)$$

122 for

$$123 \quad (2.3) \quad K(t, \sigma) := C(s + t, \sigma - t)$$

124 and  $f(t) := \int_{t-\tau}^0 K(t, \sigma) \varphi(\sigma) d\sigma$ . With standard regularity assumptions on the kernel  
125  $C$ , the solution exists unique and bounded in  $L^1$  (see Theorem 2.2 below). Moreover,  
126 a reasoning on the lines of Bellman's method of steps [3, 5] allows to extend well-  
127 posedness to any  $t > s$ , by working successively on  $[\tau, 2\tau]$ ,  $[2\tau, 3\tau]$  and so on (see also

128 [2, 4] for similar arguments, and [16, section 4.1.2] for VIEs). Denote this solution by  
 129  $x(t)$ , or  $x(t; s, \varphi)$  when emphasis on  $s$  and  $\varphi$  is required.

130 Let  $\{T(t, s)\}_{t \geq s}$  be the family of linear and bounded evolution operators [23, 31]  
 131 associated to (2.2), i.e.,

$$132 \quad T(t, s): X \rightarrow X, \quad T(t, s)\varphi = x_t(\cdot; s, \varphi).$$

133 The aim of this work is to approximate the dominant part of the spectrum of the  
 134 infinite-dimensional operator  $T(t, s)$  for the sake of studying stability. This is pur-  
 135 sued by reducing to finite dimension via the pseudospectral collocation described in  
 136 section 3 and by using the eigenvalues of the obtained matrix, computed via standard  
 137 techniques, as approximations to the exact ones.

138 Let, e.g.,  $C(t, \theta)$  be  $\Omega$ -periodic in  $t$ . As anticipated in section 1, we assume the  
 139 validity of a Floquet theory and of a corresponding principle of linearized stability.  
 140 Thus, the eigenvalues of the monodromy operator  $T(\Omega, 0)$ , called characteristic mul-  
 141 tipliers, provide information on the stability of the null solution of (2.2). Moreover,  
 142 if (2.2) comes from the linearization of a nonlinear RE around a periodic solution,  
 143 the multipliers reveal also the local stability of the latter. More precisely, except for  
 144 the trivial multiplier 1, which is always present due to linearization but does not af-  
 145 fect stability, the original periodic solution is locally asymptotically stable if all the  
 146 multipliers are inside the unit circle. Otherwise, a multiplier outside the unit circle is  
 147 enough to declare instability.

148 The same reasoning can be applied equally to  $T(h, 0)$ , independently of  $h > 0$ ,  
 149 to study the stability of the null solution of (2.2) in the autonomous case, i.e., when  
 150  $C(t, \theta)$  is independent of  $t$ . By linearization, again, this is valid also for equilibria  
 151 of nonlinear systems. Here the evolution family reduces to a classic one-parameter  
 152 semigroup, whose generator can be discretized as in [10] or [11], as already mentioned,  
 153 providing alternatives to the method described in this work.

154 One can use the discretization we propose in the framework of [15] also to compute  
 155 Lyapunov exponents for the generic nonautonomous case. Preliminary results appear  
 156 already in [9] and are confirmed by the ones obtained therein for equilibria and periodic  
 157 solutions, with reference to negative and zero exponents, respectively. For further  
 158 comments on this topic see section 6.

159 To keep this level of generality, embracing autonomous, periodic and generic non-  
 160 autonomous problems altogether, let  $h \in \mathbb{R}$  be positive and define for brevity

$$161 \quad (2.4) \quad T := T(s + h, s).$$

162 From now on this is the generic evolution operator that we aim at discretizing. We  
 163 remark that any relation between  $h$  and  $\tau$ , even irrational, is allowed.

164 The following reformulation of  $T$  is inspired by the one used in [13] for RFDEs.  
 165 It is convenient for discretizing  $T$  and approximating its eigenvalues. With respect  
 166 to [13], an essential modification of the operator  $V$  below is introduced to take into  
 167 account the different way by which the equation describes the solution, i.e., directly  
 168 (REs) or through its derivative (RFDEs).

169 Define the function spaces  $X^+ := L^1([0, h], \mathbb{R}^d)$  and  $X^\pm := L^1([-\tau, h], \mathbb{R}^d)$ ,  
 170 equipped with the corresponding  $L^1$  norms denoted, respectively, by  $\|\cdot\|_{X^+}$  and  $\|\cdot\|_{X^\pm}$ .

171 Define the operator  $V: X \times X^+ \rightarrow X^\pm$  as

$$172 \quad (2.5) \quad V(\varphi, w)(t) := \begin{cases} w(t), & t \in (0, h], \\ \varphi(t), & t \in [-\tau, 0]. \end{cases}$$

173 Let also  $V^-: X \rightarrow X^\pm$  and  $V^+: X^+ \rightarrow X^\pm$  be given, respectively, by  $V^-\varphi :=$   
 174  $V(\varphi, 0_{X^+})$  and  $V^+w := V(0_X, w)$ , where  $0_Y$  denotes the null element of a linear  
 175 space  $Y$  (similarly,  $I_Y$  in the sequel stands for the identity operator in  $Y$ ). Observe  
 176 that

$$177 \quad (2.6) \quad V(\varphi, w) = V^-\varphi + V^+w.$$

178 Note as much that  $V(\varphi, w)$  can have a discontinuity in 0 even when  $\varphi$  and  $w$  are  
 179 continuous but  $\varphi(0) \neq w(0)$ . This is an important difference with respect to [13],  
 180 which calls later on for special attention to discontinuities and to the role of 0, both  
 181 in the theoretical treatment of the numerical method and in its implementation.

182 *Remark 2.1.* The choice of including  $t = 0$  in the past in (2.5), as well as in (2.2), is  
 183 common for REs modeling, e.g., structured populations [25, 27]. From the theoretical  
 184 point of view, it does not make any difference, since  $X$  consists of equivalence classes  
 185 of functions coinciding almost everywhere. From the interpretative point of view,  
 186 it can be motivated by the consideration that although the actual value  $\varphi(0)$  is not  
 187 well-defined, being  $\varphi$  in  $L^1$ , it is reasonable to define the solution as coinciding with  
 188 the initial function  $\varphi$  of the problem on the whole domain of  $\varphi$ . Moreover, from the  
 189 implementation point of view, numerical tests performed including  $t = 0$  in the past  
 190 or in the future show that either choice gives the same results, with the only (obvious)  
 191 requirement to be consistent throughout the code.

192 Now define also the operator  $\mathcal{F}_s: X^\pm \rightarrow X^+$  as

$$193 \quad (2.7) \quad \mathcal{F}_s u(t) := \int_{-\tau}^0 C(s+t, \theta) u(t+\theta) d\theta, \quad t \in [0, h].$$

194 Eventually, the evolution operator  $T$  can be reformulated as

$$195 \quad (2.8) \quad T\varphi = V(\varphi, w^*)_h,$$

196 where  $w^* \in X^+$  is the solution of the fixed point equation

$$197 \quad (2.9) \quad w = \mathcal{F}_s V(\varphi, w),$$

198 which exists unique and bounded thanks to [Theorem 2.2](#) below (where in (2.10), and  
 199 also in the sequel,  $|\cdot|$  denotes any finite-dimensional norm). Recall that in (2.8) the  
 200 subscript  $h$  is used according to (2.1), hence  $V(\varphi, w^*)_h(\theta) = V(\varphi, w^*)(h+\theta)$  for  
 201  $\theta \in [-\tau, 0]$ .

202 **THEOREM 2.2.** *If the interval  $[0, \tau]$  can be partitioned into finitely many subin-*  
 203 *tervals  $J_1, \dots, J_n$  such that, for any  $s \in \mathbb{R}$ ,*

$$204 \quad (2.10) \quad \operatorname{ess\,sup}_{\sigma \in J_i} \int_{J_i} |C(s+t, \sigma-t)| dt < 1, \quad i \in \{1, \dots, n\},$$

205 *then the operator  $I_{X^+} - \mathcal{F}_s V^+$  is invertible with bounded inverse and (2.9) admits a*  
 206 *unique solution in  $X^+$ .*

207 *Proof.* Given  $f \in X^+$  the equation  $(I_{X^+} - \mathcal{F}_s V^+)w = f$  has a unique solution  
 208  $w \in X^+$  if and only if the initial value problem

$$209 \quad \begin{cases} w(t) = \int_{-\tau}^0 C(s+t, \theta)w(t+\theta) d\theta + f(t), & t \in [0, h], \\ w_0 = 0 \in X, \end{cases}$$

210 has a unique solution in  $X^\pm$ , with the two solutions coinciding on  $[0, h]$ . If  $h \leq \tau$ ,  
 211 this follows directly from standard theory on VIEs, see, e.g., [38, Corollary 9.3.14 and  
 212 Theorem 9.3.6], whose validity is ensured via (2.3) by the hypothesis on  $C$ . Otherwise,  
 213 the same argument can be repeated on  $[\tau, 2\tau]$ ,  $[2\tau, 3\tau]$  and so on. So  $I_{X^+} - \mathcal{F}_s V^+$  is  
 214 invertible and bounded and the bounded inverse theorem completes the proof.  $\square$

215 We conclude this section by comparing the choice of (2.2) as a prototype equation  
 216 to that of the general linear nonautonomous RFDE [13, (2.1)] (or, equivalently, [14,  
 217 (2.4)]), i.e.,  $x'(t) = L(t)x_t$  for linear bounded operators  $L(t): X \rightarrow \mathbb{R}^d$ ,  $t \geq s$ . Thanks  
 218 to the Riesz representation theorem for  $L^1$  (see, e.g., [56, page 400]), every linear non-  
 219 autonomous retarded functional equation of the type  $x(t) = L(t)x_t$  can be written  
 220 in the form (2.2), although not all of them satisfy the assumptions of Theorem 2.2.  
 221 Think, e.g., of the difference equation  $x(t) = a(t)x(t - \tau)$ , i.e.,  $C(t, \theta) = a(t)\delta_{-\tau}(\theta)$   
 222 for  $\delta_{-\tau}$  the Dirac delta at  $-\tau$ . Here we exclude these equations because, first and as  
 223 already noted, they might not be well-posed. Second, they do not ensure the regular-  
 224 ization of solutions as it happens for the analogous RFDEs, and this is fundamental  
 225 for the convergence of the numerical method. Third and last, they might be of neutral  
 226 type, a case out of the scope of the present work and about which we comment further  
 227 in section 6.

228 Also with reference to [13, (2.4)], in many applications the function  $C(t, \theta)$  (is  
 229 continuous in  $t$  and) has a finite number of discontinuities in  $\theta$ . Hence (2.2) may often  
 230 be written in the form

$$231 \quad (2.11) \quad x(t) = \sum_{k=1}^p \int_{-\tau_k}^{-\tau_{k-1}} C_k(t, \theta)x(t+\theta) d\theta$$

232 with  $\tau_0 := 0 < \tau_1 < \dots < \tau_p := \tau$  and  $C_k(t, \theta)$  continuous in  $\theta$ . In section 5 we refer  
 233 to this choice, which agrees, for instance, with the literature on physiologically- and  
 234 age-structured populations (where discontinuities are due, e.g., to different behavior  
 235 of juveniles and adults) [29, 41, 52].

236 **3. Discretization.** In order to approximate the eigenvalues of the infinite-di-  
 237 mensional operator  $T: X \rightarrow X$  defined in (2.4), we discretize the function spaces and  
 238 the operator itself by revisiting the pseudospectral collocation method used in [13],  
 239 with the necessary modifications due to the new definition of  $V$  and those anticipated  
 240 in section 1.

241 In the sequel let  $M$  and  $N$  be positive integers, referred to as discretization indices.

242 **3.1. Partition of time intervals.** If  $h \geq \tau$ , let  $\Omega_M := \{\theta_{M,0}, \dots, \theta_{M,M}\}$  be a  
 243 partition of  $[-\tau, 0]$  with  $-\tau = \theta_{M,M} < \dots < \theta_{M,0} = 0$ . If  $h < \tau$ , instead, let  $Q$  be the  
 244 minimum positive integer  $q$  such that  $qh \geq \tau$ . Note that  $Q > 1$ . Let  $\theta^{(q)} := -qh$  for  
 245  $q \in \{0, \dots, Q-1\}$  and  $\theta^{(Q)} := -\tau$ . For  $q \in \{1, \dots, Q\}$ , let  $\Omega_M^{(q)} := \{\theta_{M,0}^{(q)}, \dots, \theta_{M,M}^{(q)}\}$

246 be a partition of  $[\theta^{(q)}, \theta^{(q-1)}]$  with

$$\begin{aligned}
247 \quad & \theta^{(1)} = \theta_{M,M}^{(1)} < \dots < \theta_{M,0}^{(1)} = \theta^{(0)} = 0, \\
248 \quad & \theta^{(q)} = \theta_{M,M}^{(q)} < \dots < \theta_{M,0}^{(q)} = \theta^{(q-1)}, \quad q \in \{2, \dots, Q-1\}, \\
249 \quad & -\tau = \theta^{(Q)} = \theta_{M,M}^{(Q)} < \dots < \theta_{M,0}^{(Q)} = \theta^{(Q-1)}.
\end{aligned}$$

251 Define also the partition  $\Omega_M := \Omega_M^{(1)} \cup \dots \cup \Omega_M^{(Q)}$  of  $[-\tau, 0]$ . Note in particular that  
252 for  $q \in \{1, \dots, Q-1\}$

$$253 \quad (3.1) \quad \theta_{M,M}^{(q)} = -qh = \theta_{M,0}^{(q+1)}.$$

254 In principle, one can use more general meshes in  $[-\tau, 0]$ , e.g., not including the  
255 endpoints or using different families of nodes in the piecewise case. The forthcoming  
256 results can be generalized straightforwardly, but we avoid this choice in favor of a  
257 lighter notation and to reduce technicalities.

258 Finally, let  $\Omega_N^+ := \{t_{N,1}, \dots, t_{N,N}\}$  be a partition of  $[0, h]$  with  $0 \leq t_{N,1} < \dots <$   
259  $t_{N,N} \leq h$ .

260 **3.2. Discretization of function spaces.** If  $h \geq \tau$ , the discretization of  $X$  of  
261 index  $M$  is  $X_M := \mathbb{R}^{d(M+1)}$ . An element  $\Phi \in X_M$  is written as  $\Phi = (\Phi_0, \dots, \Phi_M)^\ddagger$ ,  
262 where  $\Phi_m \in \mathbb{R}^d$  for  $m \in \{0, \dots, M\}$ . The restriction operator  $R_M: \tilde{X} \rightarrow X_M$  is given  
263 by  $R_M\varphi := (\varphi(\theta_{M,0}), \dots, \varphi(\theta_{M,M}))$  for  $\tilde{X}$  any subspace of  $X$  regular enough to make  
264 point-wise evaluation meaningful. The same holds below and see also the comment  
265 concluding this section. The prolongation operator  $P_M: X_M \rightarrow X$  is the discrete  
266 Lagrange interpolation operator  $P_M\Phi(\theta) := \sum_{m=0}^M \ell_{M,m}(\theta)\Phi_m$ ,  $\theta \in [-\tau, 0]$ , where  
267  $\ell_{M,0}, \dots, \ell_{M,M}$  are the Lagrange coefficients relevant to the nodes of  $\Omega_M$ . Observe that  
268 that

$$269 \quad (3.2) \quad R_M P_M = I_{X_M}, \quad P_M R_M = \mathcal{L}_M,$$

270 where  $\mathcal{L}_M: \tilde{X} \rightarrow X$  is the Lagrange interpolation operator that associates to a func-  
271 tion  $\varphi \in \tilde{X}$  the  $M$ -degree  $\mathbb{R}^d$ -valued polynomial  $\mathcal{L}_M\varphi$  such that  $\mathcal{L}_M\varphi(\theta_{M,m}) =$   
272  $\varphi(\theta_{M,m})$  for  $m \in \{0, \dots, M\}$ .

273 If  $h < \tau$ , proceed similarly but in a piecewise fashion. The discretization of  $X$  of  
274 index  $M$  is  $X_M := \mathbb{R}^{d(QM+1)}$ . An element  $\Phi \in X_M$  is written as

$$275 \quad (3.3) \quad \Phi = (\Phi_0^{(1)}, \dots, \Phi_{M-1}^{(1)}, \dots, \Phi_0^{(Q)}, \dots, \Phi_{M-1}^{(Q)}, \Phi_M^{(Q)}),$$

276 where  $\Phi_m^{(q)} \in \mathbb{R}^d$  for  $q \in \{1, \dots, Q\}$  and  $m \in \{0, \dots, M-1\}$  and  $\Phi_M^{(Q)} \in \mathbb{R}^d$ . In  
277 view of (3.1), let also  $\Phi_M^{(q)} := \Phi_0^{(q+1)}$  for  $q \in \{1, \dots, Q-1\}$ . The restriction operator  
278  $R_M: \tilde{X} \rightarrow X_M$  is given by

$$279 \quad R_M\varphi := (\varphi(\theta_{M,0}^{(1)}), \dots, \varphi(\theta_{M,M-1}^{(1)}), \dots, \varphi(\theta_{M,0}^{(Q)}), \dots, \varphi(\theta_{M,M-1}^{(Q)}), \varphi(\theta_{M,M}^{(Q)})).$$

280 The prolongation operator  $P_M: X_M \rightarrow X$  is the discrete piecewise Lagrange inter-  
281 polation operator  $P_M\Phi(\theta) := \sum_{m=0}^M \ell_{M,m}^{(q)}(\theta)\Phi_m^{(q)}$ ,  $\theta \in [\theta^{(q)}, \theta^{(q-1)}]$ ,  $q \in \{1, \dots, Q\}$ ,  
282 where  $\ell_{M,0}^{(q)}, \dots, \ell_{M,M}^{(q)}$  are the Lagrange coefficients relevant to the nodes of  $\Omega_M^{(q)}$  for

<sup>‡</sup>Throughout the text we use this simpler notation to denote a concatenation of column vectors in place of the more formal  $\Phi = (\Phi_0^T, \dots, \Phi_M^T)^T$ .

283  $q \in \{1, \dots, Q\}$ . Observe that the equalities (3.2) hold again, with  $\mathcal{L}_M: \tilde{X} \rightarrow X$   
 284 the piecewise Lagrange interpolation operator that associates to a function  $\varphi \in \tilde{X}$  the  
 285 piecewise polynomial  $\mathcal{L}_M \varphi$  such that  $\mathcal{L}_M \varphi|_{[\theta^{(q)}, \theta^{(q-1)})}$  is the  $M$ -degree  $\mathbb{R}^d$ -valued poly-  
 286 nomial with values  $\varphi(\theta_{M,m}^{(q)})$  at the nodes  $\theta_{M,m}^{(q)}$  for  $q \in \{1, \dots, Q\}$  and  $m = 0, \dots, M$ .  
 287 Notice that to avoid a cumbersome notation the same symbols for  $X_M$ ,  $R_M$ ,  $P_M$  and  
 288  $\mathcal{L}_M$  are used.

289 Finally, the discretization of  $X^+$  of index  $N$  is  $X_N^+ := \mathbb{R}^{dN}$ . An element  $W \in X_N^+$   
 290 is written as  $W = (W_1, \dots, W_N)$ , where  $W_n \in \mathbb{R}^d$  for  $n \in \{1, \dots, N\}$ . The re-  
 291 striction operator  $R_N^+: \tilde{X}^+ \rightarrow X_N^+$  is given by  $R_N^+ w := (w(t_{N,1}), \dots, w(t_{N,N}))$ . The  
 292 prolongation operator  $P_N^+: X_N^+ \rightarrow X^+$  is the discrete Lagrange interpolation oper-  
 293 ator  $P_N^+ W(t) := \sum_{n=1}^N \ell_{N,n}^+(t) W_n$ ,  $t \in [0, h]$ , where  $\ell_{N,1}^+, \dots, \ell_{N,N}^+$  are the Lagrange  
 294 coefficients relevant to the nodes of  $\Omega_N^+$ . Observe again that

$$295 \quad (3.4) \quad R_N^+ P_N^+ = I_{X_N^+}, \quad P_N^+ R_N^+ = \mathcal{L}_N^+,$$

296 where  $\mathcal{L}_N^+: \tilde{X}^+ \rightarrow X^+$  is the Lagrange interpolation operator that associates to a func-  
 297 tion  $w \in \tilde{X}^+$  the  $(N-1)$ -degree  $\mathbb{R}^d$ -valued polynomial  $\mathcal{L}_N^+ w$  such that  $\mathcal{L}_N^+ w(t_{N,n}) =$   
 298  $w(t_{N,n})$  for  $n \in \{1, \dots, N\}$ .

299 When not ambiguous (e.g., when applied to an element) the restrictions to sub-  
 300 spaces of the above prolongation, restriction and Lagrange interpolation operators are  
 301 denoted in the same way as the operators themselves.

302 Observe that since an  $L^1$  function is an equivalence class of functions equal almost  
 303 everywhere, values in specific points are not well-defined. Thus, it does not seem  
 304 reasonable to define the restriction operator on the whole space  $X$  (respectively,  $X^+$ ),  
 305 motivating the above use of  $\tilde{X}$  (respectively,  $\tilde{X}^+$ ). Indeed, this is amply justified.  
 306 First of all, it is clear from the following sections that the restriction and interpolation  
 307 operators are actually applied only to continuous functions or polynomials (or their  
 308 piecewise counterparts if  $h < \tau$ ). Moreover, the interest of the present work is in the  
 309 eigenfunctions of the evolution operator (see [Theorem 4.10](#) below), which are expected  
 310 to be sufficiently regular (see relevant comments in [section 6](#)). As a last argument,  
 311 ultimately, the numerical method is applied to finite-dimensional vectors, which bear  
 312 no notion of the function from which they are derived.

313 **3.3. Discretization of  $T$ .** Following (2.8) and (2.9), the discretization of indices  
 314  $M$  and  $N$  of the evolution operator  $T$  in (2.4) is the finite-dimensional operator  
 315  $T_{M,N}: X_M \rightarrow X_M$  defined as

$$316 \quad T_{M,N} \Phi := R_M V (P_M \Phi, P_N^+ W^*)_h,$$

317 where  $W^* \in X_N^+$  is a solution of the fixed point equation

$$318 \quad (3.5) \quad W = R_N^+ \mathcal{F}_s V (P_M \Phi, P_N^+ W)$$

319 for the given  $\Phi \in X_M$ . We establish that (3.5) is well-posed in [subsection 4.2](#).

320 By virtue of (2.6), the operator  $T_{M,N}$  can be rewritten as

$$321 \quad T_{M,N} \Phi = T_M^{(1)} \Phi + T_{M,N}^{(2)} W^*,$$

322 with  $T_M^{(1)}: X_M \rightarrow X_M$  and  $T_{M,N}^{(2)}: X_N^+ \rightarrow X_M$  defined as

$$323 \quad T_M^{(1)} \Phi := R_M (V^- P_M \Phi)_h, \quad T_{M,N}^{(2)} W := R_M (V^+ P_N^+ W)_h.$$

324 Similarly, the fixed point equation (3.5) can be rewritten as

$$325 \quad (I_{X_N^+} - U_N^{(2)})W = U_{M,N}^{(1)}\Phi,$$

326 with  $U_{M,N}^{(1)}: X_M \rightarrow X_N^+$  and  $U_N^{(2)}: X_N^+ \rightarrow X_N^+$  defined as

$$327 \quad U_{M,N}^{(1)}\Phi := R_N^+ \mathcal{F}_s V^- P_M \Phi, \quad U_N^{(2)}W := R_N^+ \mathcal{F}_s V^+ P_N^+ W.$$

328 Since  $I_{X_N^+} - U_N^{(2)}$  is invertible, the operator  $T_{M,N}: X_M \rightarrow X_M$  can be eventually  
329 reformulated as

$$330 \quad (3.6) \quad T_{M,N} = T_M^{(1)} + T_{M,N}^{(2)}(I_{X_N^+} - U_N^{(2)})^{-1}U_{M,N}^{(1)}.$$

331 This reformulation simplifies the construction of the matrix representation of  $T_{M,N}$   
332 given in [Appendix A](#).

333 **4. Convergence analysis.** After introducing some additional spaces and as-  
334 sumptions in [subsection 4.1](#), we first prove that the discretized problem (viz. (3.5))  
335 is well-posed in [subsection 4.2](#). Then, in [subsection 4.3](#), we present the proof of the  
336 convergence of the eigenvalues of the finite-dimensional operator  $T_{M,N}$  to those of the  
337 infinite-dimensional operator  $T$ .

338 **4.1. Additional spaces and assumptions.** Consider the space of continuous  
339 functions  $X_C^+ := C([0, h], \mathbb{R}^d) \subset X^+$  equipped with the uniform norm, denoted by  
340  $\|\cdot\|_{X_C^+}$ . If  $h \geq \tau$  consider also  $X_C := C([- \tau, 0], \mathbb{R}^d) \subset X$  equipped with the uniform  
341 norm, denoted by  $\|\cdot\|_{X_C}$ . If  $h < \tau$ , instead, define

$$342 \quad X_C := \{\varphi \in X \mid \varphi|_{(\theta^{(q+1)}, \theta^{(q)})} \in C((\theta^{(q+1)}, \theta^{(q)}), \mathbb{R}^d), q \in \{0, \dots, Q-1\}$$

and the one-sided limits at  $\theta^{(q)}$  exist finite,  $q \in \{0, \dots, Q\}\} \subset X,$

343 equipped with the same norm  $\|\cdot\|_{X_C}$ . With these choices, all these function spaces  
344 are Banach spaces.

345 *Remark 4.1.* Observe that  $X_C$  and  $X_C^+$  are identified with their projections on the  
346 spaces  $X$  and  $X^+$ , respectively, hence their elements may be seen as equivalence classes  
347 of functions coinciding almost everywhere. In particular, the values of a function in  $X$   
348 or  $X^+$  at the endpoints of the domain interval are not relevant to that function being  
349 an element of  $X_C$  or  $X_C^+$ , respectively. The same is true for the endpoints of domain  
350 pieces for elements of  $X_C$  if  $h < \tau$ .

351 In the following sections, some hypotheses on the discretization nodes in  $[0, h]$  and  
352 on  $\mathcal{F}_s$  and  $V$  are needed beyond the assumption of [Theorem 2.2](#), in order to attain the  
353 regularity required to ensure the convergence of the method. They are all referenced  
354 individually from the following list where needed:

355 (H1) the meshes  $\{\Omega_N^+\}_{N>0}$  are the Chebyshev zeros

$$356 \quad t_{N,n} := \frac{h}{2} \left( 1 - \cos \left( \frac{(2n-1)\pi}{2N} \right) \right), \quad n \in \{1, \dots, N\};$$

357 (H2) the hypothesis of [Theorem 2.2](#) holds;

358 (H3)  $\mathcal{F}_s V^+ : X^+ \rightarrow X^+$  has range contained in  $X_C^+$  and  $\mathcal{F}_s V^+ : X^+ \rightarrow X_C^+$  is  
359 bounded;

360 (H4)  $\mathcal{F}_s V^- : X \rightarrow X^+$  has range contained in  $X_C^+$  and  $\mathcal{F}_s V^- : X \rightarrow X_C^+$  is bounded.

361 With respect to (2.5) and (2.7), hypotheses (H3) and (H4) are fulfilled if the  
362 following two conditions on the kernel  $C$  of (2.2) are satisfied:

363 (C1) there exists  $\gamma > 0$  such that  $|C(t, \theta)| \leq \gamma$  for all  $t \in [0, h]$  and almost all  
364  $\theta \in [-\tau, 0]$ ;

365 (C2)  $t \mapsto C(t, \theta)$  is continuous for almost all  $\theta \in [-\tau, 0]$ , uniformly with respect  
366 to  $\theta$ .

367 Indeed, let  $u \in X^\pm \setminus \{0\}$ ,  $t \in [0, h]$  and  $\epsilon > 0$ . From the continuity of translation  
368 in  $L^1$  there exists  $\delta' > 0$  such that for all  $t' \in [0, h]$  if  $|t' - t| < \delta'$  then  $\int_{-\tau}^0 |u(t' + \theta) -$   
369  $u(t + \theta)| d\theta < \frac{\epsilon}{2\gamma}$ . From condition (C2) there exists  $\delta'' > 0$  such that for all  $t' \in [0, h]$   
370 and almost all  $\theta \in [-\tau, 0]$  if  $|t' - t| < \delta''$  then  $|C(t', \theta) - C(t, \theta)| < \frac{\epsilon}{2\|u\|_{X^\pm}}$ . Hence, for  
371 all  $t' \in [0, h]$  if  $|t' - t| < \delta := \min\{\delta', \delta''\}$  then

$$\begin{aligned} & \left| \int_{-\tau}^0 C(t', \theta) u(t' + \theta) d\theta - \int_{-\tau}^0 C(t, \theta) u(t + \theta) d\theta \right| \\ 372 & \leq \int_{-\tau}^0 |C(t', \theta)| |u(t' + \theta) - u(t + \theta)| d\theta + \int_{-\tau}^0 |C(t', \theta) - C(t, \theta)| |u(t + \theta)| d\theta \\ & < \gamma \frac{\epsilon}{2\gamma} + \frac{\epsilon}{2\|u\|_{X^\pm}} \int_{-\tau}^0 |u(t + \theta)| d\theta \leq \epsilon. \end{aligned}$$

373 Since  $\mathcal{F}_s 0_{X^\pm} = 0_{X^+}$ , this shows that  $\mathcal{F}_s(X^\pm) \subset X_C^+$ , which implies the first part of  
374 hypotheses (H3) and (H4). Boundedness follows immediately. Eventually, observe  
375 that condition (C1) implies also hypothesis (H2). Indeed, the interval  $[0, \tau]$  can be  
376 partitioned into finitely many subintervals  $J_1, \dots, J_n$ , each of length less than  $\frac{1}{\gamma}$ , such  
377 that, for any  $s \in \mathbb{R}$  and all  $i \in \{1, \dots, n\}$ ,

$$378 \quad \operatorname{ess\,sup}_{\sigma \in J_i} \int_{J_i} |C(s+t, \sigma-t)| dt \leq \gamma \int_{J_i} dt < 1.$$

379 Anyway, in the sequel we base the proofs on hypotheses (H2) to (H4) in the case one  
380 uses operators  $V$  and  $\mathcal{F}_s$  more general than or different from (2.5) and (2.7).

381 **4.2. Well-posedness of the collocation equation.** With reference to (3.5),  
382 let  $\varphi \in X$  and consider the collocation equation

$$383 \quad (4.1) \quad W = R_N^+ \mathcal{F}_s V(\varphi, P_N^+ W)$$

384 in  $W \in X_N^+$ . The aim of this section is to show that (4.1) has a unique solution  
385 and to study its relation to the unique solution  $w^* \in X^+$  of (2.9). Using (2.6), the  
386 equations (2.9) and (4.1) can be rewritten, respectively, as  $(I_{X^+} - \mathcal{F}_s V^+)w = \mathcal{F}_s V^- \varphi$   
387 and

$$388 \quad (4.2) \quad (I_{X_N^+} - R_N^+ \mathcal{F}_s V^+ P_N^+) W = R_N^+ \mathcal{F}_s V^- \varphi.$$

389 The following preliminary result concerns the operators

$$390 \quad (4.3) \quad I_{X^+} - \mathcal{L}_N^+ \mathcal{F}_s V^+ : X^+ \rightarrow X^+,$$

391 and

$$392 \quad (4.4) \quad I_{X_N^+} - R_N^+ \mathcal{F}_s V^+ P_N^+ : X_N^+ \rightarrow X_N^+.$$

393 **PROPOSITION 4.2.** *If the operator (4.3) is invertible, then the operator (4.4) is*  
 394 *invertible. Moreover, given  $\bar{W} \in X_N^+$ , the unique solution  $\hat{w} \in X^+$  of*

$$395 \quad (4.5) \quad (I_{X^+} - \mathcal{L}_N^+ \mathcal{F}_s V^+) w = P_N^+ \bar{W}$$

396 *and the unique solution  $\hat{W} \in X_N^+$  of*

$$397 \quad (4.6) \quad (I_{X_N^+} - R_N^+ \mathcal{F}_s V^+ P_N^+) W = \bar{W}$$

398 *are related by  $\hat{W} = R_N^+ \hat{w}$  and  $\hat{w} = P_N^+ \hat{W}$ .*

399 *Proof.* If (4.3) is invertible, then, given  $\bar{W} \in X_N^+$ , (4.5) has a unique solution, say  
 400  $\hat{w} \in X^+$ . Then, by (3.4),

$$401 \quad (4.7) \quad \hat{w} = P_N^+ (R_N^+ \mathcal{F}_s V^+ \hat{w} + \bar{W})$$

402 and

$$403 \quad (4.8) \quad R_N^+ \hat{w} = R_N^+ \mathcal{F}_s V^+ \hat{w} + \bar{W}$$

404 hold. Hence, by substituting (4.8) in (4.7),

$$405 \quad (4.9) \quad \hat{w} = P_N^+ R_N^+ \hat{w}$$

406 and, by substituting (4.9) in (4.8),  $R_N^+ \hat{w} = R_N^+ \mathcal{F}_s V^+ P_N^+ R_N^+ \hat{w} + \bar{W}$ , i.e.,  $R_N^+ \hat{w}$  is a  
 407 solution of (4.6).

408 Vice versa, if  $\hat{W} \in X_N^+$  is a solution of (4.6), then  $P_N^+ \hat{W} = \mathcal{L}_N^+ \mathcal{F}_s V^+ P_N^+ \hat{W} + P_N^+ \bar{W}$   
 409 holds again by (3.4), i.e.,  $P_N^+ \hat{W}$  is a solution of (4.5). Hence, by uniqueness,  $\hat{w} = P_N^+ \hat{W}$   
 410 holds.

411 Finally, if  $\hat{W}_1, \hat{W}_2 \in X_N^+$  are solutions of (4.6), then  $P_N^+ \hat{W}_1 = \hat{w} = P_N^+ \hat{W}_2$  and,  
 412 once again by (3.4),  $\hat{W}_1 = R_N^+ P_N^+ \hat{W}_1 = R_N^+ P_N^+ \hat{W}_2 = \hat{W}_2$ . Therefore  $\hat{W} := R_N^+ \hat{w}$  is  
 413 the unique solution of (4.6) and the operator (4.4) is invertible.  $\square$

414 As observed above, the equation (4.1) is equivalent to (4.2), hence, by choosing

$$415 \quad (4.10) \quad \bar{W} = R_N^+ \mathcal{F}_s V^- \varphi,$$

416 it is equivalent to (4.6). Observe also that thanks to (3.4) the equation

$$417 \quad (4.11) \quad w = \mathcal{L}_N^+ \mathcal{F}_s V(\varphi, w)$$

418 can be rewritten as  $(I_{X^+} - \mathcal{L}_N^+ \mathcal{F}_s V^+) w = \mathcal{L}_N^+ \mathcal{F}_s V^- \varphi = P_N^+ R_N^+ \mathcal{F}_s V^- \varphi$ , which is  
 419 equivalent to (4.5) with the choice (4.10). Thus, by **Proposition 4.2**, if the opera-  
 420 tor (4.3) is invertible, then the equation (4.1) has a unique solution  $W^* \in X_N^+$  such  
 421 that

$$422 \quad (4.12) \quad W^* = R_N^+ w_N^*, \quad w_N^* = P_N^+ W^*,$$

423 where  $w_N^* \in X^+$  is the unique solution of (4.11). Note for clarity that (4.10) implies  
 424  $w_N^* = \hat{w}$  for  $\hat{w}$  in **Proposition 4.2**. So, now we show that (4.3) is invertible under due  
 425 assumptions.

426 PROPOSITION 4.3. *If hypotheses (H1) to (H3) hold, then there exists a positive*  
 427 *integer  $N_0$  such that, for any  $N \geq N_0$ , the operator (4.3) is invertible and*

$$428 \quad \|(I_{X^+} - \mathcal{L}_N^+ \mathcal{F}_s V^+)^{-1}\|_{X^+ \leftarrow X^+} \leq 2 \|(I_{X^+} - \mathcal{F}_s V^+)^{-1}\|_{X^+ \leftarrow X^+}.$$

429 *Moreover, for each  $\varphi \in X$ , (4.11) has a unique solution  $w_N^* \in X^+$  and*

$$430 \quad \|w_N^* - w^*\|_{X^+} \leq 2 \|(I_{X^+} - \mathcal{F}_s V^+)^{-1}\|_{X^+ \leftarrow X^+} \|\mathcal{L}_N^+ w^* - w^*\|_{X^+},$$

431 *where  $w^* \in X^+$  is the unique solution of (2.9).*

432 *Proof.* In this proof, let  $I := I_{X^+}$ . By [35, Corollary of Theorem Ia], assuming  
 433 hypothesis (H1), if  $w \in X_C^+$ , then  $\|(\mathcal{L}_N^+ - I)w\|_{X^+} \rightarrow 0$  for  $N \rightarrow \infty$ . By the Banach-  
 434 Steinhaus theorem, the sequence  $\|(\mathcal{L}_N^+ - I)\downarrow_{X_C^+}\|_{X^+ \leftarrow X_C^+}$  is bounded, hence

$$435 \quad (4.13) \quad \|(\mathcal{L}_N^+ - I)\downarrow_{X_C^+}\|_{X^+ \leftarrow X_C^+} \xrightarrow{N \rightarrow \infty} 0.$$

436 Assuming hypothesis (H3), this implies

$$437 \quad \|(\mathcal{L}_N^+ - I)\mathcal{F}_s V^+\|_{X^+ \leftarrow X^+} \leq \|(\mathcal{L}_N^+ - I)\downarrow_{X_C^+}\|_{X^+ \leftarrow X_C^+} \|\mathcal{F}_s V^+\|_{X_C^+ \leftarrow X^+} \xrightarrow{N \rightarrow \infty} 0.$$

438 In particular, there exists a positive integer  $N_0$  such that, for each integer  $N \geq N_0$ ,

$$439 \quad \|(\mathcal{L}_N^+ - I)\mathcal{F}_s V^+\|_{X^+ \leftarrow X^+} \leq \frac{1}{2 \|(I - \mathcal{F}_s V^+)^{-1}\|_{X^+ \leftarrow X^+}},$$

440 i.e.,  $\|(\mathcal{L}_N^+ - I)\mathcal{F}_s V^+\|_{X^+ \leftarrow X^+} \|(I - \mathcal{F}_s V^+)^{-1}\|_{X^+ \leftarrow X^+} \leq \frac{1}{2}$ , which holds since  $I - \mathcal{F}_s V^+$   
 441 is invertible with bounded inverse by virtue of hypothesis (H2) and Theorem 2.2.  
 442 Considering the operator  $I - \mathcal{L}_N^+ \mathcal{F}_s V^+$  as a perturbed version of  $I - \mathcal{F}_s V^+$  and writing  
 443  $I - \mathcal{L}_N^+ \mathcal{F}_s V^+ = I - \mathcal{F}_s V^+ - (\mathcal{L}_N^+ - I)\mathcal{F}_s V^+$ , by the Banach perturbation lemma [46,  
 444 Theorem 10.1], there exists a positive integer  $N_0$  such that, for each integer  $N \geq N_0$ ,  
 445 the operator  $I - \mathcal{L}_N^+ \mathcal{F}_s V^+$  is invertible and

$$446 \quad \|(I - \mathcal{L}_N^+ \mathcal{F}_s V^+)^{-1}\|_{X^+ \leftarrow X^+} \leq \frac{\|(I - \mathcal{F}_s V^+)^{-1}\|_{X^+ \leftarrow X^+}}{1 - \|(I - \mathcal{F}_s V^+)^{-1}((\mathcal{L}_N^+ - I)\mathcal{F}_s V^+)\|_{X^+ \leftarrow X^+}} \\ \leq 2 \|(I - \mathcal{F}_s V^+)^{-1}\|_{X^+ \leftarrow X^+}.$$

447 Hence, fixed  $\varphi \in X$ , (4.11) has a unique solution  $w_N^* \in X^+$ . For the same  $\varphi$ , let  $e_N^* \in$   
 448  $X^+$  such that  $w_N^* = w^* + e_N^*$ , where  $w^* \in X^+$  is the unique solution of (2.9). Then  
 449  $w^* + e_N^* = \mathcal{L}_N^+ \mathcal{F}_s V(\varphi, w^* + e_N^*) = \mathcal{L}_N^+ \mathcal{F}_s V(\varphi, w^*) + \mathcal{L}_N^+ \mathcal{F}_s V^+ e_N^* = \mathcal{L}_N^+ w^* + \mathcal{L}_N^+ \mathcal{F}_s V^+ e_N^*$   
 450 and  $(I - \mathcal{L}_N^+ \mathcal{F}_s V^+)e_N^* = (\mathcal{L}_N^+ - I)w^*$ , completing the proof.  $\square$

451 **4.3. Convergence of the eigenvalues.** The proof that the eigenvalues of  $T_{M,N}$   
 452 approximate those of  $T$  follows the lines of the proof for RFDEs in [13], modulo the  
 453 difference about  $V$  mentioned in section 2 and those due to the change of state space.  
 454 As a consequence, although the proof of the main step (Proposition 4.7) is simplified,  
 455 the outcome is a stronger result than [13, Proposition 4.5]. Indeed, restricting the state  
 456 space to a subspace of more regular functions is no longer necessary. This is basically  
 457 due to the regularizing nature of the right-hand side of (2.2) under hypothesis (H4),  
 458 which is usually satisfied in applications, as remarked at the end of section 2.

459 Observe that  $T$  and  $T_{M,N}$  live on different spaces, which cannot be compared  
 460 directly because of the different dimensions, viz. infinite vs. finite. In view of this,

461 we first translate the problem of studying the eigenvalues of  $T_{M,N}$  on  $X_M$  to that of  
 462 studying the eigenvalues of finite-rank operators  $\hat{T}_{M,N}$  and  $\hat{T}_N$  on  $X$  (**Propositions 4.4**  
 463 and **4.5**). Then, in **Proposition 4.7**, we show that  $\hat{T}_N$  converges in operator norm to  $T$   
 464 and, by applying results from spectral approximation theory [22] (**Lemma 4.8**), we  
 465 obtain the desired convergence of the eigenvalues of  $T_{M,N}$  to the eigenvalues of  $T$   
 466 (**Proposition 4.9** and **Theorem 4.10**), which represents the main result of the work.

467 Under some additional hypotheses on the smoothness of the eigenfunctions of  $T$ ,  
 468 the eigenvalues converge with infinite order. The numerical tests of **section 5** show  
 469 that in practice the infinite order of convergence can be attained. It is reasonable  
 470 to expect that the regularity of the eigenfunctions depends on the regularity of the  
 471 model coefficients. A rigorous investigation is ongoing in parallel to the completion  
 472 of the Floquet theory and more comments are given in **section 6**.

473 Now we introduce the finite-rank operator  $\hat{T}_{M,N}$  associated to  $T_{M,N}$  and show  
 474 the relation between their spectra.

475 **PROPOSITION 4.4.** *The finite-dimensional operator  $T_{M,N}$  has the same nonzero*  
 476 *eigenvalues, with the same geometric and partial multiplicities, of the operator*

$$477 \quad \hat{T}_{M,N} := P_M T_{M,N} R_M \downarrow_{X_C} : X_C \rightarrow X_C.$$

478 *Moreover, if  $\Phi \in X_M$  is an eigenvector of  $T_{M,N}$  associated to a nonzero eigenvalue  $\mu$ ,*  
 479 *then  $P_M \Phi \in X_C$  is an eigenvector of  $\hat{T}_{M,N}$  associated to the same eigenvalue  $\mu$ .*

480 *Proof.* Apply [13, Proposition 4.1], since prolongations are polynomials, hence  
 481 continuous.  $\square$

482 Define the operator  $\hat{T}_N : X \rightarrow X$  as

$$483 \quad \hat{T}_N \varphi := V(\varphi, w_N^*)_h,$$

484 where  $w_N^* \in X^+$  is the solution of the fixed point equation (4.11), which, under  
 485 **hypotheses (H1) to (H3)**, is unique thanks to **Propositions 4.2** and **4.3**. Observe that  
 486  $w_N^*$  is a polynomial, hence, in particular,  $w_N^* \in X_C^+$ . Then, for  $\varphi \in X_C$ , by (4.12),

$$\begin{aligned} 487 \quad \hat{T}_{M,N} \varphi &= P_M T_{M,N} R_M \varphi \\ &= P_M R_M V(P_M R_M \varphi, P_N^+ W^*)_h \\ &= \mathcal{L}_M V(\mathcal{L}_M \varphi, w_N^*)_h \\ &= \mathcal{L}_M \hat{T}_N \mathcal{L}_M \varphi, \end{aligned}$$

488 where  $W^* \in X_N^+$  and  $w_N^* \in X_C^+$  are the solutions, respectively, of (3.5) applied to  
 489  $\Phi = R_M \varphi$  and of (4.11) with  $\mathcal{L}_M \varphi$  replacing  $\varphi$ . These solutions are unique under  
 490 **hypotheses (H1) to (H3)**, thanks again to **Propositions 4.2** and **4.3**.

491 Now we show the relation between the spectra of  $\hat{T}_{M,N}$  and  $\hat{T}_N$ .

492 **PROPOSITION 4.5.** *Assume that **hypotheses (H1) to (H3)** hold and let  $M \geq N \geq$*   
 493  *$N_0$ , with  $N_0$  given by **Proposition 4.3**. Then the operator  $\hat{T}_{M,N}$  has the same nonzero*  
 494 *eigenvalues, with the same geometric and partial multiplicities and associated eigen-*  
 495 *vectors, of the operator  $\hat{T}_N$ .*

496 *Proof.* Denote by  $\Pi_r$  and  $\Pi_r^+$  the subspaces of polynomials of degree  $r$  of  $X$   
 497 and  $X^+$ , respectively, and observe that **Remark 4.1** applies also here. Note that  
 498  $w_N^* \in \Pi_{N-1}^+$ .

499 If  $h \geq \tau$ , for all  $\varphi \in X$ ,  $\hat{T}_N \varphi = V(\varphi, w_N^*)_h \in \Pi_{N-1}$ . Thus both  $\hat{T}_N$  and  $\hat{T}_{M,N} =$   
500  $\mathcal{L}_M \hat{T}_N \mathcal{L}_M$  have range contained in  $\Pi_M$ , being  $M \geq N$ . By [13, Proposition 4.3  
501 and Remark 4.4],  $\hat{T}_N$  and  $\hat{T}_{M,N}$  have the same nonzero eigenvalues, with the same  
502 geometric and partial multiplicities and associated eigenvectors, as their restrictions  
503 to  $\Pi_M$ . Observing that  $\hat{T}_{M,N} \upharpoonright_{\Pi_M} = \mathcal{L}_M \hat{T}_N \mathcal{L}_M \upharpoonright_{\Pi_M} = \hat{T}_N \upharpoonright_{\Pi_M}$ , the thesis follows.

504 Consider now the case  $h < \tau$ . Denote by  $\Pi_M^{\text{pw}}$  the subspace of piecewise poly-  
505 nomials of degree  $r$  of  $X$  on the intervals  $[\theta^{(q+1)}, \theta^{(q)}]$ , for  $q = 0, \dots, Q-1$ . For  
506 all  $\varphi \in \Pi_M^{\text{pw}}$ ,  $\hat{T}_N \varphi = V(\varphi, w_N^*)_h \in \Pi_M^{\text{pw}}$ . Let  $\mu \neq 0$ ,  $\varphi \in X$  and  $\bar{\varphi} \in \Pi_M^{\text{pw}}$  such  
507 that  $(\mu I_X - \hat{T}_N)\varphi = \mu\varphi - V(\varphi, w_N^*)_h = \bar{\varphi}$ . This equation can be rewritten as  
508  $\mu\varphi(\theta) = w_N^*(h+\theta) + \bar{\varphi}(\theta)$  if  $\theta \in (-h, 0]$  and as  $\mu\varphi(\theta) = \varphi(h+\theta) + \bar{\varphi}(\theta)$  if  $\theta \in [-\tau, -h]$ .  
509 From the first equation,  $\varphi$  restricted to  $[-h, 0]$  is a polynomial of degree  $M$ , being  
510  $M \geq N$ . From the second equation it is easy to show that  $\varphi \in \Pi_M^{\text{pw}}$  by induction on the  
511 intervals  $[\theta^{(q+1)}, \theta^{(q)}]$ , for  $q = 1, \dots, Q-1$ . Hence, by [13, Proposition 4.3],  $\hat{T}_N$  has the  
512 same nonzero eigenvalues, with the same geometric and partial multiplicities and as-  
513 sociated eigenvectors, as its restriction to  $\Pi_M^{\text{pw}}$ . The same holds for  $\hat{T}_{M,N} = \mathcal{L}_M \hat{T}_N \mathcal{L}_M$   
514 by [13, Proposition 4.3 and Remark 4.4] since its range is contained in  $\Pi_M^{\text{pw}}$ . The thesis  
515 follows by observing that  $\hat{T}_{M,N} \upharpoonright_{\Pi_M^{\text{pw}}} = \mathcal{L}_M \hat{T}_N \mathcal{L}_M \upharpoonright_{\Pi_M^{\text{pw}}} = \hat{T}_N \upharpoonright_{\Pi_M^{\text{pw}}}$ .  $\square$

516 Below we prove the norm convergence of  $\hat{T}_N$  to  $T$ , which is the key step to obtain  
517 the main result of this work. First we need to extend the results of [Theorem 2.2](#) to  
518  $X_C^+$  in the following lemma.

519 **LEMMA 4.6.** *If [hypotheses \(H2\)](#) and [\(H3\)](#) hold, then  $(I_{X^+} - \mathcal{F}_s V^+) \upharpoonright_{X_C^+}$  is invert-*  
520 *ible with bounded inverse.*

521 *Proof.* Since  $I_{X^+} - \mathcal{F}_s V^+$  is invertible with bounded inverse by virtue of [hypoth-](#)  
522 [esis \(H2\)](#) and [Theorem 2.2](#), given  $f \in X_C^+$  the equation  $(I_{X^+} - \mathcal{F}_s V^+)w = f$  has a  
523 unique solution  $w \in X^+$ , which by [hypothesis \(H3\)](#) is in  $X_C^+$ . Hence, the operator  
524  $(I_{X^+} - \mathcal{F}_s V^+) \upharpoonright_{X_C^+}$  is invertible. It is also bounded, since  $\|\cdot\|_{X^+} \leq h\|\cdot\|_{X_C^+}$ , which  
525 implies  $\|\mathcal{F}_s V^+ \upharpoonright_{X_C^+}\|_{X_C^+ \leftarrow X^+} \leq h\|\mathcal{F}_s V^+ \upharpoonright_{X_C^+ \leftarrow X^+}$ . The bounded inverse theorem com-  
526 pletes the proof.  $\square$

527 **PROPOSITION 4.7.** *If [hypotheses \(H1\)](#) to [\(H4\)](#) hold, then  $\|\hat{T}_N - T\|_{X \leftarrow X} \rightarrow 0$  for*  
528  *$N \rightarrow \infty$ .*

529 *Proof.* Let  $\varphi \in X$  and let  $w^*$  and  $w_N^*$  be the solutions of the fixed point equa-  
530 tions (2.9) and (4.11), respectively. Recall that  $w_N^*$  is a polynomial. Assuming  
531 [hypotheses \(H3\)](#) and [\(H4\)](#) and recalling that  $w^* = \mathcal{F}_s V^+ w^* + \mathcal{F}_s V^- \varphi$ , it is clear  
532 that  $w^* \in X_C^+$ . Hence it follows that  $V(\varphi, w^*)_h \in X_C$  (recall [Remark 4.1](#) and that  
533 for  $h < \tau$  the space  $X_C$  is piecewise defined, [subsection 4.1](#)). Then  $(\hat{T}_N - T)\varphi =$   
534  $V(\varphi, w_N^*)_h - V(\varphi, w^*)_h = V^+(w_N^* - w^*)_h$ . Assuming also [hypotheses \(H1\)](#) and [\(H2\)](#),  
535 by [Proposition 4.3](#), there exists a positive integer  $N_0$  such that, for any  $N \geq N_0$ ,

$$\begin{aligned} \|(\hat{T}_N - T)\varphi\|_X &= \|V^+(w_N^* - w^*)_h\|_X \\ &\leq \|w_N^* - w^*\|_{X^+} \\ &\leq 2\|(I_{X^+} - \mathcal{F}_s V^+)^{-1}\|_{X^+ \leftarrow X^+} \|\mathcal{L}_N^+ w^* - w^*\|_{X^+} \\ &\leq 2\|(I_{X^+} - \mathcal{F}_s V^+)^{-1}\|_{X^+ \leftarrow X^+} \|(\mathcal{L}_N^+ - I_{X^+}) \upharpoonright_{X_C^+}\|_{X^+ \leftarrow X_C^+} \|w^*\|_{X_C^+} \end{aligned}$$

537 holds by virtue of (4.13). Eventually,

$$538 \quad \|w^*\|_{X_C^+} \leq \|((I_{X^+} - \mathcal{F}_s V^+) \upharpoonright_{X_C^+})^{-1}\|_{X_C^+ \leftarrow X_C^+} \|\mathcal{F}_s V^-\|_{X_C^+ \leftarrow X} \|\varphi\|_X$$

539 completes the proof thanks to Lemma 4.6 and hypothesis (H4).  $\square$

540 The final convergence results rely on a combination of tools from [22], as summa-  
541 rized in the following lemma.

542 LEMMA 4.8. *Let  $U$  be a Banach space,  $A$  a linear and bounded operator on  $U$  and*  
543  *$\{A_N\}_{N \in \mathbb{N}}$  a sequence of linear and bounded operators on  $U$  such that  $\|A_N - A\|_{U \leftarrow U} \rightarrow$*   
544 *0 for  $N \rightarrow \infty$ . If  $\mu \in \mathbb{C}$  is an eigenvalue of  $A$  with finite algebraic multiplicity  $\nu$  and*  
545 *ascent  $l$ , and  $\Delta$  is a neighborhood of  $\mu$  such that  $\mu$  is the only eigenvalue of  $A$  in  $\Delta$ ,*  
546 *then there exists a positive integer  $\bar{N}$  such that, for any  $N \geq \bar{N}$ ,  $A_N$  has in  $\Delta$  exactly*  
547  *$\nu$  eigenvalues  $\mu_{N,j}$ ,  $j \in \{1, \dots, \nu\}$ , counting their multiplicities. Moreover, by setting*  
548  *$\epsilon_N := \|(A_N - A) \upharpoonright_{\mathcal{E}_\mu}\|_{U \leftarrow \mathcal{E}_\mu}$ , where  $\mathcal{E}_\mu$  is the generalized eigenspace of  $\mu$  equipped with*  
549 *the norm  $\|\cdot\|_U$  restricted to  $\mathcal{E}_\mu$ , the following holds:*

$$550 \quad (4.14) \quad \max_{j \in \{1, \dots, \nu\}} |\mu_{N,j} - \mu| = O(\epsilon_N^{1/l}).$$

551 *Proof.* By [22, Example 3.8 and Theorem 5.22], the norm convergence of  $A_N$  to  
552  $A$  implies the strongly stable convergence  $A_N - \mu I_U \xrightarrow{ss} A - \mu I_U$  for all  $\mu$  in the  
553 resolvent set of  $A$  and all isolated eigenvalues  $\mu$  of finite multiplicity of  $A$ . The thesis  
554 follows then by [22, Proposition 5.6 and Theorem 6.7].  $\square$

555 PROPOSITION 4.9. *Assume that hypotheses (H1) to (H4) hold. If  $\mu \in \mathbb{C} \setminus \{0\}$*   
556 *is an eigenvalue of  $T$  with finite algebraic multiplicity  $\nu$  and ascent  $l$ , and  $\Delta$  is a*  
557 *neighborhood of  $\mu$  such that  $\mu$  is the only eigenvalue of  $T$  in  $\Delta$ , then there exists*  
558 *a positive integer  $N_1 \geq N_0$ , with  $N_0$  given by Proposition 4.3, such that, for any*  
559  *$N \geq N_1$ ,  $\hat{T}_N$  has in  $\Delta$  exactly  $\nu$  eigenvalues  $\mu_{N,j}$ ,  $j \in \{1, \dots, \nu\}$ , counting their*  
560 *multiplicities. Moreover, if for each  $\varphi \in \mathcal{E}_\mu$ , where  $\mathcal{E}_\mu$  is the generalized eigenspace*  
561 *of  $T$  associated to  $\mu$ , the function  $w^*$  that solves (2.9) is of class  $C^p$ , with  $p \geq 1$ , then*

$$562 \quad \max_{j \in \{1, \dots, \nu\}} |\mu_{N,j} - \mu| = o\left(N^{\frac{1-p}{l}}\right).$$

563 *Proof.* By Proposition 4.7,  $\|\hat{T}_N - T\|_{X \leftarrow X} \rightarrow 0$  for  $N \rightarrow \infty$ . The first part of the  
564 thesis is obtained by applying Lemma 4.8. From the same Lemma 4.8, (4.14) follows  
565 with  $\epsilon_N := \|(\hat{T}_N - T) \upharpoonright_{\mathcal{E}_\mu}\|_{X \leftarrow \mathcal{E}_\mu}$  and  $\mathcal{E}_\mu$  the generalized eigenspace of  $\mu$  equipped with  
566 the norm of  $X$  restricted to  $\mathcal{E}_\mu$ .

567 Let  $\varphi_1, \dots, \varphi_\nu$  be a basis of  $\mathcal{E}_\mu$ . An element  $\varphi$  of  $\mathcal{E}_\mu$  can be written as  $\varphi =$   
568  $\sum_{j=1}^\nu \alpha_j(\varphi) \varphi_j$ , with  $\alpha_j(\varphi) \in \mathbb{C}$ , for  $j \in \{1, \dots, \nu\}$ , hence

$$569 \quad \|(\hat{T}_N - T)\varphi\|_X \leq \max_{j \in \{1, \dots, \nu\}} |\alpha_j(\varphi)| \sum_{j=1}^\nu \|(\hat{T}_N - T)\varphi_j\|_X.$$

570 The function  $\varphi \mapsto \max_{j \in \{1, \dots, \nu\}} |\alpha_j(\varphi)|$  is a norm on  $\mathcal{E}_\mu$ , so it is equivalent to the norm  
571 of  $X$  restricted to  $\mathcal{E}_\mu$ . Thus, there exists a positive constant  $c$  independent of  $\varphi$  such  
572 that  $\max_{j \in \{1, \dots, \nu\}} |\alpha_j(\varphi)| \leq c \|\varphi\|_X$  and

$$573 \quad \epsilon_N = \|(\hat{T}_N - T) \upharpoonright_{\mathcal{E}_\mu}\|_{X \leftarrow \mathcal{E}_\mu} \leq c \sum_{j=1}^\nu \|(\hat{T}_N - T)\varphi_j\|_X.$$

574 Let  $j \in \{1, \dots, \nu\}$ . As seen in [Proposition 4.7](#),

$$575 \quad \|(\hat{T}_N - T)\varphi_j\|_X \leq 2\|(I_{X^+} - \mathcal{F}_s V^+)^{-1}\|_{X^+ \leftarrow X^+} \|(\mathcal{L}_N^+ - I_{X^+})w_j^*\|_{X^+},$$

576 where  $w_j^*$  is the solution of [\(2.9\)](#) associated to  $\varphi_j$ . Now, by well-known results in  
577 interpolation theory (see, e.g., [\[55, Theorems 1.5 and 4.1\]](#)), since  $w_j^*$  is of class  $C^p$ ,  
578 the bound

$$579 \quad \begin{aligned} \|(\mathcal{L}_N^+ - I_{X^+})w_j^*\|_{X^+} &\leq h(1 + \Lambda_N)E_{N-1}(w_j^*) \\ &\leq h(1 + \Lambda_N) \frac{6^{p+1}e^p}{1+p} \left(\frac{h}{2}\right)^p \frac{1}{(N-1)^p} \omega\left(\frac{h}{2(N-1-p)}\right) \end{aligned}$$

580 holds, where  $\Lambda_N$  is the Lebesgue constant for  $\Omega_N^+$ ,  $E_{N-1}(\cdot)$  is the best uniform ap-  
581 proximation error and  $\omega(\cdot)$  is the modulus of continuity of  $(w_j^*)^{(p)}$  on  $[0, h]$ . Since  
582 [hypothesis \(H1\)](#) is assumed, by classic results on interpolation (see, e.g., [\[55, Theo-](#)  
583 [rem 4.5\]](#)),  $\Lambda_N = o(N)$ . Hence,  $\epsilon_N = o(N^{1-p})$  and the thesis follows immediately.  $\square$

584 **THEOREM 4.10.** *Assume that [hypotheses \(H1\) to \(H4\)](#) hold. If  $\mu \in \mathbb{C} \setminus \{0\}$  is a  
585 eigenvalue of  $T$  with finite algebraic multiplicity  $\nu$  and ascent  $l$ , and  $\Delta$  is a neighbor-  
586 hood of  $\mu$  such that  $\mu$  is the only eigenvalue of  $T$  in  $\Delta$ , then there exists a positive  
587 integer  $N_1 \geq N_0$ , with  $N_0$  given by [Proposition 4.3](#), such that, for any  $N \geq N_1$  and  
588 any  $M \geq N$ ,  $T_{M,N}$  has in  $\Delta$  exactly  $\nu$  eigenvalues  $\mu_{M,N,j}$ ,  $j \in \{1, \dots, \nu\}$ , count-  
589 ing their multiplicities. Moreover, if for each  $\varphi \in \mathcal{E}_\mu$ , where  $\mathcal{E}_\mu$  is the generalized  
590 eigenspace of  $T$  associated to  $\mu$ , the function  $w^*$  that solves [\(2.9\)](#) is of class  $C^p$ , with  
591  $p \geq 1$ , then*

$$592 \quad \max_{j \in \{1, \dots, \nu\}} |\mu_{M,N,j} - \mu| = o\left(N^{\frac{1-p}{l}}\right).$$

593 *Proof.* If  $M \geq N \geq N_0$ , by [Propositions 4.4 and 4.5](#) the operators  $T_{M,N}$ ,  $\hat{T}_{M,N}$   
594 and  $\hat{T}_N$  have the same nonzero eigenvalues, with the same geometric and partial  
595 multiplicities and associated eigenvectors. The thesis follows by [Proposition 4.9](#).  $\square$

596 We conclude this section with a couple of comments. First, nodes other than those  
597 required by [hypothesis \(H1\)](#) may be used. Indeed, they are only asked to satisfy the  
598 hypotheses of [\[35, Corollary of Theorem Ia\]](#) and  $\Lambda_N = o(N)$ . Let us notice that  
599 both are guaranteed by zeros of other families of classic orthogonal polynomials [\[18\]](#).  
600 Anyway, here we assume [hypothesis \(H1\)](#) since these are the nodes we actually use in  
601 implementing the method.

602 Second, in general, it may not be possible to compute exactly the integral in [\(2.7\)](#).  
603 If this is the case, an approximation  $\tilde{\mathcal{F}}_s$  of  $\mathcal{F}_s$  must be used, leading to a further contri-  
604 bution in the final error. See [\[14, section 6.3.3\]](#) and further comments in [Appendix A](#)  
605 as far as implementation is concerned.

606 **5. Numerical tests.** REs with known solutions and stability properties are  
607 rather rare. A notable difficulty is the lack of a characteristic equation for non-  
608 autonomous equations, which makes it hard to obtain both theoretical and numerical  
609 results to compare with our method. For these reasons, we first compare our method  
610 with that of [\[10\]](#) in the autonomous case, where, instead, a characteristic equation  
611 can be derived. Then we study a nonlinear equation which possesses a branch of  
612 analytically known periodic solutions in a certain range of a varying parameter.

613 In the following tests we use Chebyshev zeros in  $[0, h]$  as  $\Omega_N^+$ , as required by  
614 [hypothesis \(H1\)](#). In  $[-\tau, 0]$  we use Chebyshev extrema as  $\Omega_M$  if  $h \geq \tau$  and as  $\Omega_M^{(q)}$  for  
615  $q \in \{1, \dots, Q\}$  if  $h < \tau$ .

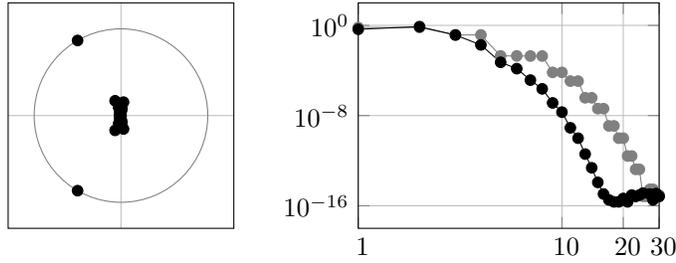


FIGURE 1. Numerical test with (5.1) where  $a = 2$  and  $\beta = \frac{1}{2} \exp(1 + \frac{2\pi}{3\sqrt{3}})$ . Left: eigenvalues of  $T(4, 0)$  for  $M = N = 20$  with respect to the unit circle. Right: error with respect to 1 of the absolute value of the dominant eigenvalues of  $T(4, 0)$  in black and error on the 0 real part of the rightmost characteristic roots obtained with the method of [10] in gray.

616 Consider the egg cannibalism model

$$617 \quad x(t) = \beta \int_{-4}^{-a} x(t + \theta) e^{-x(t+\theta)} d\theta,$$

618 where  $\beta > 0$  and  $0 < a < 4$ , for which some theoretical results are known [10,  
619 section 5.1]. By linearizing it around the nontrivial equilibrium  $\log(\beta(4 - a))$ , we  
620 obtain the linear equation

$$621 \quad (5.1) \quad x(t) = \frac{1 - \log(\beta(4 - a))}{4 - a} \int_{-4}^{-a} x(t + \theta) d\theta.$$

622 It corresponds to (2.2) by setting  $C(t, \theta) := \frac{1 - \log(\beta(4 - a))}{4 - a}$  for  $\theta \in [-\tau, -a]$ ,  $C(t, \theta) := 0$   
623 for  $\theta \in (-a, 0]$  and  $\tau := 4$ . Observe that  $C(t, \theta)$  is independent of  $t$  and piecewise  
624 constant in  $\theta$ , thus making (5.1) an instance of (2.11) with  $p = 2$ ,  $\tau_1 = a$  and  $\tau_2 = 4$ .  
625 By studying the characteristic equation it is known that the equilibrium undergoes a  
626 Hopf bifurcation for  $a = 2$  and  $\beta = \frac{1}{2} \exp(1 + \frac{2\pi}{3\sqrt{3}})$ , hence the operator  $T(h, 0)$  has a  
627 complex conjugate pair on the unit circle as its dominant eigenvalues, independently  
628 of  $h > 0$ . In this test we choose  $h = \tau (= 4)$ . Figure 1 shows the eigenvalues of  $T(4, 0)$   
629 for  $M = N = 20$  and the errors with respect to 1 of the absolute value of the dominant  
630 eigenvalues as  $M = N$  varies from 1 to 30, compared with the errors on the 0 real part  
631 of the characteristic roots obtained with the method of [10]. Observe that the latter  
632 approximates the eigenvalues  $\lambda$  of the infinitesimal generator (characteristic roots),  
633 which are related to the eigenvalues  $\mu$  of  $T$  (characteristic multipliers) by  $\mu = e^{\lambda h}$ .  
634 Notice that both methods experiment the proved convergence of infinite order, with  
635 apparently larger error constants for the method of [10].

636 The second numerical test is based on the nonlinear equation

$$637 \quad (5.2) \quad x(t) = \frac{\gamma}{2} \int_{-3}^{-1} x(t + \theta)(1 - x(t + \theta)) d\theta,$$

638 linearized around the periodic solution

$$639 \quad (5.3) \quad \bar{x}(t) = \frac{1}{2} + \frac{\pi}{4\gamma} + \sqrt{\frac{1}{2} - \frac{1}{\gamma} - \frac{\pi}{2\gamma^2} \left(1 + \frac{\pi}{4}\right)} \sin\left(\frac{\pi}{2}t\right),$$

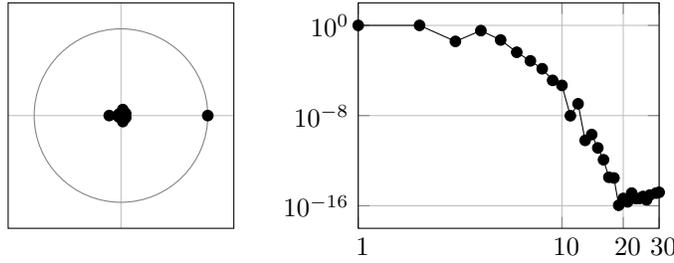


FIGURE 2. Numerical test with (5.2) where  $\gamma = 4$ , linearized around (5.3). Left: eigenvalues of  $T(4, 0)$  for  $M = N = 20$  with respect to the unit circle. Right: error on the known eigenvalue 1 of  $T(4, 0)$ .

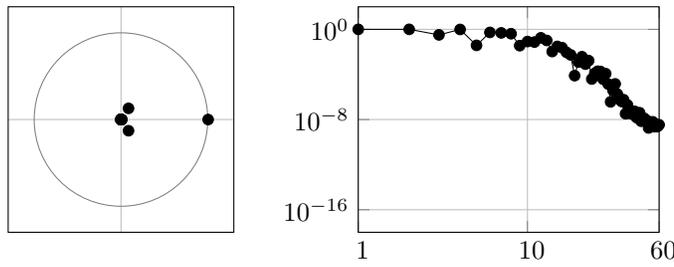


FIGURE 3. Numerical test with (5.2) where  $\gamma = 4.4$ , linearized around a numerically approximated periodic solution of period  $\Omega \approx 8.0189$ . Left: eigenvalues of  $T(\Omega, 0)$  for  $M = N = 20$  with respect to the unit circle. Right: error on the known eigenvalue 1 of  $T(\Omega, 0)$ .

640 which exists for  $\gamma \geq 2 + \frac{\pi}{2}$  and has period 4 [9]. The linearized equation reads

$$641 \quad x(t) = \frac{\gamma}{2} \int_{-3}^{-1} (1 - 2\bar{x}(t + \theta))x(t + \theta) d\theta,$$

642 which corresponds to (2.2) by setting  $C(t, \theta) := \frac{\gamma}{2}(1 - 2\bar{x}(t + \theta))$  for  $\theta \in [-\tau, -1]$ ,  
 643  $C(t, \theta) := 0$  for  $\theta \in (-1, 0]$  and  $\tau := 3$ . Observe that  $C(t, \theta)$  is continuous in  $t$  and for  
 644 each  $t$  it may have a single discontinuity in  $\theta$ , thus adhering to (2.11) with  $p = 2$ ,  $\tau_1 = 1$   
 645 and  $\tau_2 = 3$ . Although not much is known theoretically about stability, the monodromy  
 646 operator  $T(4, 0)$  has always an eigenvalue 1 due to the linearization around the periodic  
 647 solution, which allows us to test the accuracy of the approximation. Figure 2 shows  
 648 the eigenvalues of  $T(4, 0)$  and the errors on the known eigenvalue 1 for  $\gamma = 4$ . By using  
 649 standard zero-finding routines (e.g., MATLAB's `fzero`), we can detect for  $\gamma \approx 4.3247$   
 650 an eigenvalue crossing the unit circle outwards through  $-1$ , which characterizes a  
 651 period doubling bifurcation. The branch of periodic solutions arising from the latter  
 652 is not known analytically. In [9] these periodic solutions are computed numerically by  
 653 adapting the method of [32] for RFDEs or of [49] for differential algebraic equations  
 654 with delays (see relevant comments in section 6). The method is then applied to the  
 655 equation linearized around the numerical solution. Figure 3 shows the eigenvalues of  
 656  $T(\Omega, 0)$  and the errors on the known eigenvalue 1 for  $\gamma = 4.4$ , where  $\Omega \approx 8.0189$  is  
 657 the computed period of the numerically approximated periodic solution. Notice again  
 658 that our method works equally well, independently of the relation between  $\Omega$  and  $\tau$ .

659 It can be seen that to achieve the same accuracy as for the branch of periodic

660 solutions (5.3), a number of nodes more than double than before must be used. This  
 661 fact is in line with usual properties of pseudospectral methods, which exhibit slower  
 662 convergence as the length of the discretization interval increases (although the infinite  
 663 order is preserved). Indeed, by standard results on interpolation, the error depends  
 664 both on the length of the interpolation interval and on bounds on the derivatives  
 665 of the interpolated function: in this case, after the period doubling bifurcation both  
 666 the period of the solution (length of the interpolation interval) and the number of  
 667 oscillations (related to the magnitude of the derivatives) are roughly double than  
 668 before. Observe, however, that here the error takes also into account for the error in  
 669 the computation of the reference solution.

670 **6. Future perspectives.** In this work we propose a numerical method to ap-  
 671 proximate the spectrum of evolution operators for linear REs. This concluding section  
 672 contains diverse comments on open problems and possible future research lines, most  
 673 of which were briefly touched along the text.

674 The numerical experiments suggest that the order of convergence of the approx-  
 675 imated eigenvalues to the exact ones is infinite and [Theorem 4.10](#) guarantees that  
 676 this is the case if the eigenfunctions of the evolution operator are sufficiently smooth.  
 677 Although it is reasonable to expect that any desired regularity of the eigenfunctions  
 678 can be achieved by imposing suitable conditions on  $C(t, \theta)$  (see, e.g., [\[54\]](#) for some  
 679 results in this direction for convolution products), this has not been proved yet and  
 680 remains an open question that the authors are investigating.

681 Regarding the application to the asymptotic stability of periodic solutions of  
 682 nonlinear autonomous REs, another open problem is the validity of a Floquet theory  
 683 for linear periodic REs and of a corresponding principle of linearized stability. In  
 684 view of [\[25\]](#), this would be guaranteed by the validity of assumptions (F), (H) and ( $\Xi$ )  
 685 of [\[31, section XIV.4\]](#). A preliminary study reveals that assumption (F) should be  
 686 guaranteed by suitable regularity assumptions on  $C(t, \theta)$ . On the other hand, some  
 687 results on the regularity of Volterra integrals, similar to the ones mentioned above  
 688 with respect to the regularity of eigenfunctions, seem to be needed for assumptions (H)  
 689 and ( $\Xi$ ). Investigating these details and thus proving the validity of a Floquet theory  
 690 is an ongoing effort by the authors and colleagues.

691 As mentioned in [section 2](#), the discretization proposed in this work can be used  
 692 in principle in the framework of [\[15\]](#) to compute Lyapunov exponents for generic  
 693 solutions of nonautonomous REs. Numerical tests on this approach appear in [\[9\]](#)  
 694 with promising results. Investigating this natural development is in the future plans  
 695 of the authors. Indeed, it goes beyond the scopes of the present paper since it requires  
 696 to work in a Hilbert rather than in a Banach setting. Incidentally, notice how this  
 697 change would require a restriction of the state space, as opposed to RFDEs in [\[15\]](#).

698 In the literature of population dynamics, the recent paper [\[26\]](#) deals with a model  
 699 based on retarded functional equations containing also point evaluation terms, i.e.,  
 700 Volterra integrals with kernel of Dirac type. The presence of these terms may give  
 701 rise to neutral dynamics, adding several difficulties both to the theoretical treatment  
 702 (they are not covered in general by [\[25, 31\]](#)) and to the proof of convergence of the  
 703 numerical method (the regularization effect on the solutions, essential to the current  
 704 proof, is not guaranteed and in general does not take place). Anyway, investigating  
 705 the neutral case remains in the interests of the authors.

706 Finally, in structured population models, REs are often coupled with RFDEs  
 707 (see, e.g., [\[29, 50\]](#)). Extending the method to such coupled equations, as in the case  
 708 of [\[10, 11\]](#) for equilibria, poses additional and nontrivial difficulties in proving the

709 convergence of the approximated eigenvalues, with respect to both the RFDE case  
 710 of [13] and the RE case of the present work. In fact, due to the coupling, there  
 711 is a delicate interplay between the diverse regularization mechanisms, with different  
 712 consequences on the two components of the solution. With respect to the regularity  
 713 of eigenfunctions and to the validity of a Floquet theory, coupled equations retain the  
 714 same difficulties as outlined above for REs and may be addressed by similar solutions,  
 715 as it appears reasonable. The extension of the method to coupled equations, including  
 716 a rigorous convergence proof and error analysis, together with numerical tests, is  
 717 the subject of a distinct paper in preparation by the authors. Nevertheless, in the  
 718 nonlinear context and for practical applications, this approach inevitably relies on the  
 719 computation of the relevant periodic solutions. In this sense, an extension of [32] is  
 720 being developed by the authors and colleagues. The final objective of these research  
 721 lines is the study of the dynamics of the realistic *Daphnia* model of [29], which brings in  
 722 several nontrivial challenges beyond those related to the discretization of the evolution  
 723 operators.

724 **Appendix A. Matrix representation.** In this appendix we describe the ex-  
 725 plicit construction of a matrix representing the discretization of the evolution operator  
 726 (2.4) according to (3.6). The reference is to model (2.11). We start by introducing  
 727 some notations for block matrices.

728 If  $h \geq \tau$ , for  $\Phi \in X_M$  and  $m \in \{0, \dots, M\}$ , denote  $(\Phi_{dm+1}, \dots, \Phi_{d(m+1)})$ , i.e.,  
 729 the  $(m+1)$ -th  $d$ -sized block of components of  $\Phi$ , as  $[\Phi]_m$ . If  $h < \tau$ , instead, for  
 730  $\Phi \in X_M$ ,  $q \in \{1, \dots, Q\}$  and  $m \in \{0, \dots, M-1\}$  and for  $q = Q$  and  $m = M$ ,  
 731 denote  $(\Phi_{d((q-1)M+m)+1}, \dots, \Phi_{d((q-1)M+m+1)})$ , i.e., the  $(m+1)$ -th  $d$ -sized block of  
 732 components of the  $q$ -th block of  $\Phi$ , as  $[\Phi]_{q,m}$ . Finally, for  $W \in X_N^+$  and  $n \in \{1, \dots, N\}$ ,  
 733 denote  $(W_{d(n-1)+1}, \dots, W_{dn})$ , i.e., the  $n$ -th  $d$ -sized block of components of  $W$ , as  $[W]_n$ .

734 In the following,  $0$  denotes the scalar zero or a matrix of zeros of the dimensions  
 735 implied by the context.

736 **A.1. The matrix  $T_M^{(1)}$ .** Let  $\Phi \in X_M$ . If  $h > \tau$ , for  $m \in \{0, \dots, M\}$   $[T_M^{(1)}\Phi]_m =$   
 737  $(V^- P_M \Phi)_h(\theta_{M,m}) = V^- P_M \Phi(h + \theta_{M,m}) = 0$ , hence  $T_M^{(1)} = 0 \in \mathbb{R}^{d(M+1) \times d(M+1)}$ .  
 738 If  $h = \tau$ , instead, for  $m \in \{0, \dots, M-1\}$ ,  $[T_M^{(1)}\Phi]_m = 0$  as above. For  $m = M$ ,  
 739  $[T_M^{(1)}\Phi]_M = V^- P_M \Phi(h + \theta_{M,M}) = P_M \Phi(\theta_{M,0}) = \Phi_0$ . Thus

$$740 \quad T_M^{(1)} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix} \otimes I_d \in \mathbb{R}^{d(M+1) \times d(M+1)}.$$

741 Finally, if  $h < \tau$ , for  $m \in \{0, \dots, M-1\}$  and  $q \in \{1, \dots, Q-1\}$ ,

$$742 \quad [T_M^{(1)}\Phi]_{q,m} = V^- P_M \Phi(h + \theta_{M,m}^{(q)}) = \begin{cases} 0, & q = 1, \\ P_M \Phi(\theta_{M,m}^{(q-1)}) = \Phi_m^{(q-1)}, & q \in \{2, \dots, Q-1\}, \end{cases}$$

743 while for  $m \in \{0, \dots, M\}$  and  $q = Q$ ,

$$744 \quad [T_M^{(1)}\Phi]_{Q,m} = P_M \Phi(h + \theta_{M,m}^{(Q)}) = \sum_{j=0}^M \ell_{M,j}^{(Q-1)}(h + \theta_{M,m}^{(Q)}) \Phi_j^{(Q-1)}.$$



769 while for  $m = M$ ,  $[T_{M,N}^{(2)}W]_M = V^+P_N^+W(h + \theta_{M,M}) = V^+P_N^+W(0) = 0$ . Thus

$$770 \quad T_{M,N}^{(2)} = \begin{pmatrix} \ell_{N,1}^+(h + \theta_{M,0}) & \cdots & \ell_{N,N}^+(h + \theta_{M,0}) \\ \vdots & \ddots & \vdots \\ \ell_{N,1}^+(h + \theta_{M,M-1}) & \cdots & \ell_{N,N}^+(h + \theta_{M,M-1}) \\ 0 & \cdots & 0 \end{pmatrix} \otimes I_d \in \mathbb{R}^{d(M+1) \times dN}.$$

771

772 Finally, if  $h < \tau$ , for  $m \in \{0, \dots, M-1\}$  and  $q \in \{1, \dots, Q\}$ ,

$$773 \quad [T_{M,N}^{(2)}W]_{q,m} = V^+P_N^+W(h + \theta_{M,m}^{(q)}) = \begin{cases} \sum_{n=1}^N \ell_{N,n}^+(h + \theta_{M,m}^{(q)})W_n, & q = 1, \\ 0, & q \in \{2, \dots, Q\}, \end{cases}$$

774 and  $[T_{M,N}^{(2)}W]_{Q,M} = V^+P_N^+W(h + \theta_{M,M}^{(Q)}) = V^+P_N^+W(h - \tau) = 0$ . Then

$$775 \quad T_{M,N}^{(2)} = \begin{pmatrix} \ell_{N,1}^+(h + \theta_{M,0}^{(1)}) & \cdots & \ell_{N,N}^+(h + \theta_{M,0}^{(1)}) \\ \vdots & \ddots & \vdots \\ \ell_{N,1}^+(h + \theta_{M,M-1}^{(1)}) & \cdots & \ell_{N,N}^+(h + \theta_{M,M-1}^{(1)}) \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \otimes I_d \in \mathbb{R}^{d(QM+1) \times dN}.$$

776 **A.3. The matrix  $U_{M,N}^{(1)}$ .** Let  $\Phi \in X_M$  and, for  $t > 0$ , define

$$777 \quad (\text{A.1}) \quad \kappa(t) := \max_{k \in \{0, \dots, p\}} \{\tau_k < t\}.$$

778 Note that  $\kappa$  is nondecreasing. For  $n \in \{1, \dots, N\}$ ,

$$779 \quad [U_{M,N}^{(1)}\Phi]_n = \mathcal{F}_s V^- P_M \Phi(t_{N,n}) = \sum_{k=1}^p \int_{-\tau_k}^{-\tau_{k-1}} C_k(s + t_{N,n}, \theta) V^- P_M \Phi(t_{N,n} + \theta) d\theta.$$

780 If  $h \geq \tau$ , define also

$$781 \quad \hat{N} := \begin{cases} 0, & t_{N,n} > \tau \text{ for all } n \in \{1, \dots, N\}, \\ \max_{n \in \{1, \dots, N\}} \{t_{N,n} \leq \tau\}, & \text{otherwise.} \end{cases}$$

782 Hence, for  $n \in \{1, \dots, \hat{N}\}$  (if  $\hat{N} \neq 0$ ),

$$783 \quad (\text{A.2}) \quad [U_{M,N}^{(1)}\Phi]_n = \int_{-\tau_{\kappa(t_{N,n})+1}}^{-t_{N,n}} C_{\kappa(t_{N,n})+1}(s + t_{N,n}, \theta) \sum_{m=0}^M \ell_{M,m}(t_{N,n} + \theta) \Phi_m d\theta \\ + \sum_{k=\kappa(t_{N,n})+2}^p \int_{-\tau_k}^{-\tau_{k-1}} C_k(s + t_{N,n}, \theta) \sum_{m=0}^M \ell_{M,m}(t_{N,n} + \theta) \Phi_m d\theta,$$

784 and, for  $n \in \{\hat{N} + 1, \dots, N\}$ ,  $[U_{M,N}^{(1)} \Phi]_n = 0$ . Observe that the first integral in (A.2)  
 785 may be zero. For  $m \in \{0, \dots, M\}$  and  $n \in \{1, \dots, \hat{N}\}$  (if  $\hat{N} \neq 0$ ), let

$$786 \quad \mathbb{R}^{d \times d} \ni \Theta_{n,m} := \int_{-\tau_{\kappa(t_{N,n})+1}}^{-t_{N,n}} C_{\kappa(t_{N,n})+1}(s + t_{N,n}, \theta) \ell_{M,m}(t_{N,n} + \theta) d\theta$$

$$787 \quad + \sum_{k=\kappa(t_{N,n})+2}^p \int_{-\tau_k}^{-\tau_{k-1}} C_k(s + t_{N,n}, \theta) \ell_{M,m}(t_{N,n} + \theta) d\theta.$$

789 Then

$$790 \quad U_{M,N}^{(1)} = \begin{pmatrix} \Theta_{1,0} & \cdots & \Theta_{1,M} \\ \vdots & \ddots & \vdots \\ \Theta_{\hat{N},0} & \cdots & \Theta_{\hat{N},M} \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}^{dN \times d(M+1)},$$

791 which is the zero matrix if  $\hat{N} = 0$ .

792 If  $h < \tau$ , instead, for  $n \in \{1, \dots, N\}$  and  $q \in \{0, \dots, Q-1\}$ , define  $t_{N,n}^{(q)} = qh +$   
 793  $t_{N,n}$ . Observe that, for  $q \in \{1, \dots, Q-1\}$ ,  $[t_{N,n} - \tau_k, t_{N,n} - \tau_{k-1}] \cap (-qh, -(q-1)h] \neq \emptyset$   
 794 if and only if  $\kappa(t_{N,n}^{(q-1)}) + 1 \leq k \leq \kappa(t_{N,n}^{(q)}) + 1$  and  $[t_{N,n} - \tau_k, t_{N,n} - \tau_{k-1}] \cap [-\tau, -(Q-1)h] \neq \emptyset$   
 795 if and only if  $k \geq \kappa(t_{N,n}^{(Q-1)}) + 1$ . Observe also that  $\kappa(t_{N,n}^{(q-1)})$  and  $\kappa(t_{N,n}^{(q)})$  may  
 796 be equal. For  $n \in \{1, \dots, N\}$ ,  $k \in \{1, \dots, p\}$  and  $q \in \{1, \dots, Q-1\}$ , define

$$797 \quad a_{k,q} := \max\{-\tau_k, -t_{N,n}^{(q)}\}, \quad a_{k,Q} := -\tau_k,$$

$$798 \quad b_{k,q} := \min\{-\tau_{k-1}, -t_{N,n}^{(q-1)}\}, \quad b_{k,Q} := \min\{-\tau_{k-1}, -t_{N,n}^{(Q-1)}\},$$

$$799 \quad \kappa_{n,q} := \min\{\kappa(t_{N,n}^{(q)}) + 1, p\}, \quad \kappa_{n,Q} := p.$$

801 Then, for  $n \in \{1, \dots, N\}$ ,

$$802 \quad [U_{M,N}^{(1)} \Phi]_n = \sum_{q=1}^Q \sum_{k=\kappa(t_{N,n}^{(q-1)})+1}^{\kappa_{n,q}} \int_{a_{k,q}}^{b_{k,q}} C_k(s + t_{N,n}, \theta) \sum_{m=0}^M \ell_{M,m}^{(q)}(t_{N,n} + \theta) \Phi_m^{(q)} d\theta,$$

803 with the convention that  $\sum_{k=k_1}^{k_2} a_k = 0$  if  $k_2 < k_1$ . Observe that some of the integrals  
 804 may be zero. For  $n \in \{1, \dots, N\}$ ,  $m \in \{0, \dots, M\}$  and  $q \in \{1, \dots, Q\}$ , define

$$805 \quad \mathbb{R}^{d \times d} \ni \Theta_{n,m}^{(q)} := \sum_{k=\kappa(t_{N,n}^{(q-1)})+1}^{\kappa_{n,q}} \int_{a_{k,q}}^{b_{k,q}} C_k(s + t_{N,n}, \theta) \ell_{M,m}^{(q)}(t_{N,n} + \theta) d\theta$$

806 and recall that, for  $q \in \{1, \dots, Q-1\}$ ,  $\Phi_M^{(q)} = \Phi_0^{(q+1)}$ . Then  $U_{M,N}^{(1)} \in \mathbb{R}^{dN \times d(QM+1)}$  is  
 807 given by

$$808 \quad U_{M,N}^{(1)} = \begin{pmatrix} \Theta_{1,0}^{(1)} & \cdots & \Theta_{1,M-1}^{(1)} & \Theta_{1,M}^{(1)} + \Theta_{1,0}^{(2)} & \Theta_{1,1}^{(2)} & \cdots & \Theta_{1,M-1}^{(2)} & \Theta_{1,M}^{(Q-1)} + \Theta_{1,0}^{(Q)} & \Theta_{1,1}^{(Q)} & \cdots & \Theta_{1,M-1}^{(Q)} & \Theta_{1,M}^{(Q)} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \Theta_{N,0}^{(1)} & \cdots & \Theta_{N,M-1}^{(1)} & \Theta_{N,M}^{(1)} + \Theta_{N,0}^{(2)} & \Theta_{N,1}^{(2)} & \cdots & \Theta_{N,M-1}^{(2)} & \Theta_{N,M}^{(Q-1)} + \Theta_{N,0}^{(Q)} & \Theta_{N,1}^{(Q)} & \cdots & \Theta_{N,M-1}^{(Q)} & \Theta_{N,M}^{(Q)} \end{pmatrix}.$$

810 Eventually, with reference to the last comment of [section 4](#), the various integrals  
 811 appearing in the construction of the elements of  $U_{M,N}^{(1)}$  should be computed with a  
 812 quadrature formula that, in presence of sufficient regularity of the model coefficients,  
 813 preserves the infinite order of convergence of [Theorem 4.10](#). The same remark holds  
 814 for the elements of  $U_N^{(2)}$  in [Appendix A.4](#). Specifically, in the MATLAB codes we  
 815 resort to Clenshaw–Curtis quadrature [\[60\]](#).

816 **A.4. The matrix  $U_N^{(2)}$ .** Let  $W \in X_N^+$ . Define  $\kappa(t)$  as in [\(A.1\)](#), for  $t > 0$ . For  
 817  $n \in \{1, \dots, N\}$ ,

$$\begin{aligned} [U_N^{(2)}W]_n &= \mathcal{F}_s V^+ P_N^+ W(t_{N,n}) \\ &= \sum_{k=1}^p \int_{-\tau_k}^{-\tau_{k-1}} C_k(s + t_{N,n}, \theta) V^+ P_N^+ W(t_{N,n} + \theta) d\theta \\ 818 &= \sum_{k=1}^{\kappa(t_{N,n})} \int_{-\tau_k}^{-\tau_{k-1}} C_k(s + t_{N,n}, \theta) \sum_{i=1}^N \ell_{N,i}^+(t_{N,n} + \theta) W_i d\theta \\ &\quad + \int_{-\min\{t_{N,n}, \tau\}}^{-\tau_{\kappa(t_{N,n})}} C_{\min\{\kappa(t_{N,n})+1, p\}}(s + t_{N,n}, \theta) \sum_{i=1}^N \ell_{N,i}^+(t_{N,n} + \theta) W_i d\theta, \end{aligned}$$

819 with the convention that  $\sum_{k=k_1}^{k_2} a_k = 0$  if  $k_2 < k_1$ . Observe that the last integral may  
 820 be zero. For  $n \in \{1, \dots, N\}$  and  $i \in \{1, \dots, N\}$ , let

$$\begin{aligned} \mathbb{R}^{d \times d} \ni \Gamma_{n,i} &:= \sum_{k=1}^{\kappa(t_{N,n})} \int_{-\tau_k}^{-\tau_{k-1}} C_k(s + t_{N,n}, \theta) \ell_{N,i}^+(t_{N,n} + \theta) d\theta \\ 821 &\quad + \int_{-\min\{t_{N,n}, \tau\}}^{-\tau_{\kappa(t_{N,n})}} C_{\min\{\kappa(t_{N,n})+1, p\}}(s + t_{N,n}, \theta) \ell_{N,i}^+(t_{N,n} + \theta) d\theta. \end{aligned}$$

822 Then

$$823 \quad U_N^{(2)} = \begin{pmatrix} \Gamma_{1,1} & \cdots & \Gamma_{1,N} \\ \vdots & \ddots & \vdots \\ \Gamma_{N,1} & \cdots & \Gamma_{N,N} \end{pmatrix} \in \mathbb{R}^{dN \times dN}.$$

824

#### REFERENCES

- 825 [1] *DDE-BIFTOOL*, <http://ddebiftool.sourceforge.net/>.  
 826 [2] A. BELLEN AND M. ZENNARO, *Numerical Methods for Delay Differential Equations*, Numer.  
 827 Math. Sci. Comput., Clarendon Press, 2003.  
 828 [3] R. E. BELLMAN, *On the computational solution of differential-difference equations*, J. Math.  
 829 Anal. Appl., 2 (1961), pp. 108–110, [https://doi.org/10.1016/0022-247X\(61\)90049-X](https://doi.org/10.1016/0022-247X(61)90049-X).  
 830 [4] R. E. BELLMAN AND K. L. COOKE, *Differential-Difference Equations*, Academic Press, 1963.  
 831 [5] R. E. BELLMAN AND K. L. COOKE, *On the computational solution of differential-difference*  
 832 *equations*, J. Math. Anal. Appl., 12 (1965), pp. 495–500, [https://doi.org/10.1016/](https://doi.org/10.1016/0022-247X(65)90017-X)  
 833 [0022-247X\(65\)90017-X](https://doi.org/10.1016/0022-247X(65)90017-X).  
 834 [6] J.-P. BERRUT AND L. N. TREFETHEN, *Barycentric Lagrange interpolation*, SIAM Rev., 46  
 835 (2004), pp. 501–517, <https://doi.org/10.1137/S0036144502417715>.

- 836 [7] D. BREDA, O. DIEKMANN, W. F. DE GRAAF, A. PUGLIESE, AND R. VERMIGLIO, *On the for-*  
 837 *mulation of epidemic models (an appraisal of Kermack and McKendrick)*, *J. Biol. Dyn.*,  
 838 6 (2012), pp. 103–117, <https://doi.org/10.1080/17513758.2012.716454>.
- 839 [8] D. BREDA, O. DIEKMANN, M. GYLLENBERG, F. SCARABEL, AND R. VERMIGLIO, *Pseudospectral*  
 840 *discretization of nonlinear delay equations: New prospects for numerical bifurcation anal-*  
 841 *ysis*, *SIAM J. Appl. Dyn. Syst.*, 15 (2016), pp. 1–23, <https://doi.org/10.1137/15M1040931>.
- 842 [9] D. BREDA, O. DIEKMANN, D. LIESSI, AND F. SCARABEL, *Numerical bifurcation analysis of a*  
 843 *class of nonlinear renewal equations*, *Electron. J. Qual. Theory Differ. Equ.*, 65 (2016),  
 844 pp. 1–24, <https://doi.org/10.14232/ejqtde.2016.1.65>.
- 845 [10] D. BREDA, O. DIEKMANN, S. MASET, AND R. VERMIGLIO, *A numerical approach for investi-*  
 846 *gating the stability of equilibria for structured population models*, *J. Biol. Dyn.*, 7 (2013),  
 847 pp. 4–20, <https://doi.org/10.1080/17513758.2013.789562>.
- 848 [11] D. BREDA, P. GETTO, J. SÁNCHEZ SANZ, AND R. VERMIGLIO, *Computing the eigenvalues of*  
 849 *realistic Daphnia models by pseudospectral methods*, *SIAM J. Sci. Comput.*, 37 (2015),  
 850 pp. A2607–A2629, <https://doi.org/10.1137/15M1016710>.
- 851 [12] D. BREDA, S. MASET, AND R. VERMIGLIO, *Pseudospectral differencing methods for character-*  
 852 *istic roots of delay differential equations*, *SIAM J. Sci. Comput.*, 27 (2005), pp. 482–495,  
 853 <https://doi.org/10.1137/030601600>.
- 854 [13] D. BREDA, S. MASET, AND R. VERMIGLIO, *Approximation of eigenvalues of evolution operators*  
 855 *for linear retarded functional differential equations*, *SIAM J. Numer. Anal.*, 50 (2012),  
 856 pp. 1456–1483, <https://doi.org/10.1137/100815505>.
- 857 [14] D. BREDA, S. MASET, AND R. VERMIGLIO, *Stability of Linear Delay Differential Equations: A*  
 858 *Numerical Approach with MATLAB*, SpringerBriefs in Control, Automation and Robotics,  
 859 Springer-Verlag, New York, 2015, <https://doi.org/10.1007/978-1-4939-2107-2>.
- 860 [15] D. BREDA AND E. VAN VLECK, *Approximating Lyapunov exponents and Sacker–Sell spectrum*  
 861 *for retarded functional differential equations*, *Numer. Math.*, 126 (2014), pp. 225–257,  
 862 <https://doi.org/10.1007/s00211-013-0565-1>.
- 863 [16] H. BRUNNER, *Collocation Methods for Volterra Integral and Related Functional Differential*  
 864 *Equations*, no. 15 in Cambridge Monogr. Appl. Comput. Math., Cambridge University  
 865 Press, Cambridge, 2004.
- 866 [17] H. BRUNNER, *Volterra Integral Equations: An Introduction to Theory and Applications*, no. 30  
 867 in Cambridge Monogr. Appl. Comput. Math., Cambridge University Press, Cambridge,  
 868 2017.
- 869 [18] L. BRUTMAN, *Lebesgue functions for polynomial interpolation – a survey*, *Annals of Numerical*  
 870 *Mathematics*, 4 (1997), pp. 111–127.
- 871 [19] E. BUELER, *Error bounds for approximate eigenvalues of periodic-coefficient linear delay dif-*  
 872 *ferential equations*, *SIAM J. Numer. Anal.*, 45 (2007), pp. 2510–2536, [https://doi.org/10.](https://doi.org/10.1137/050633330)  
 873 [1137/050633330](https://doi.org/10.1137/050633330).
- 874 [20] E. A. BUTCHER AND O. A. BOBRENKOV, *On the Chebyshev spectral continuous time approx-*  
 875 *imation for constant and periodic delay differential equations*, *Commun. Nonlinear Sci.*  
 876 *Numer. Simul.*, 16 (2011), pp. 1541–1554, <https://doi.org/10.1016/j.cnsns.2010.05.037>.
- 877 [21] E. A. BUTCHER, H. MA, E. BUELER, V. AVERINA, AND Z. SZABÓ, *Stability of linear time-*  
 878 *periodic delay-differential equations via Chebyshev polynomials*, *Internat. J. Numer. Meth-*  
 879 *ods Engrg.*, 59 (2004), pp. 895–922, <https://doi.org/10.1002/nme.894>.
- 880 [22] F. CHATELIN, *Spectral Approximation of Linear Operators*, no. 65 in Classics Appl. Math.,  
 881 SIAM, Philadelphia, 2011, <https://doi.org/10.1137/1.9781611970678>.
- 882 [23] C. CHICONE AND Y. LATUSHKIN, *Evolution Semigroups in Dynamical Systems and Differential*  
 883 *Equations*, no. 70 in Math. Surveys Monogr., American Mathematical Society, Providence,  
 884 1999.
- 885 [24] R. F. CURTAIN AND H. ZWART, *An Introduction to Infinite-Dimensional Linear Systems*  
 886 *Theory*, no. 21 in Texts in Applied Mathematics, Springer-Verlag, New York, 1995,  
 887 <https://doi.org/10.1007/978-1-4612-4224-6>.
- 888 [25] O. DIEKMANN, P. GETTO, AND M. GYLLENBERG, *Stability and bifurcation analysis of Volterra*  
 889 *functional equations in the light of suns and stars*, *SIAM J. Math. Anal.*, 39 (2008),  
 890 pp. 1023–1069, <https://doi.org/10.1137/060659211>.
- 891 [26] O. DIEKMANN, P. GETTO, AND Y. NAKATA, *On the characteristic equation  $\lambda = \alpha_1 + (\alpha_2 +$*   
 892  *$\alpha_3\lambda)e^{-\lambda}$  and its use in the context of a cell population*, *J. Math. Biol.*, 72 (2015), pp. 877–  
 893 908, <https://doi.org/10.1007/s00285-015-0918-8>.
- 894 [27] O. DIEKMANN AND M. GYLLENBERG, *Equations with infinite delay: Blending the abstract and*  
 895 *the concrete*, *J. Differential Equations*, 252 (2012), pp. 819–851, [https://doi.org/10.1016/](https://doi.org/10.1016/j.jde.2011.09.038)  
 896 [j.jde.2011.09.038](https://doi.org/10.1016/j.jde.2011.09.038).
- 897 [28] O. DIEKMANN, M. GYLLENBERG, H. HUANG, M. KIRKILIONIS, J. A. J. METZ, AND H. R.

- 898 THIEME, *On the formulation and analysis of general deterministic structured population*  
 899 *modelsii. nonlinear theory*, J. Math. Biol., 43 (2001), pp. 157–189, [https://doi.org/10.1007/](https://doi.org/10.1007/s002850170002)  
 900 [s002850170002](https://doi.org/10.1007/s002850170002).
- [29] O. DIEKMANN, M. GYLLENBERG, J. A. J. METZ, S. NAKAOKA, AND A. M. DE ROOS, *Daphnia*  
 902 *revisited: local stability and bifurcation theory for physiologically structured population*  
 903 *models explained by way of an example*, J. Math. Biol., 61 (2010), pp. 277–318, <https://doi.org/10.1007/s00285-009-0299-y>.  
 904
- [30] O. DIEKMANN, M. GYLLENBERG, J. A. J. METZ, AND H. R. THIEME, *On the formulation and*  
 905 *analysis of general deterministic structured population models I. Linear theory*, J. Math.  
 906 Biol., 36 (1998), pp. 349–388, <https://doi.org/10.1007/s002850050104>.  
 907
- [31] O. DIEKMANN, S. A. VAN GILS, S. M. VERDUYN LUNEL, AND H.-O. WALTHER, *Delay Equations*  
 908 *– Functional-, Complex- and Nonlinear Analysis*, no. 110 in Appl. Math. Sci., Springer-  
 909 Verlag, New York, 1995, <https://doi.org/10.1007/978-1-4612-4206-2>.  
 910
- [32] K. ENGELBORGHs, T. LUZYANINA, K. J. IN 'T HOUT, AND D. ROOSE, *Collocation methods for*  
 912 *the computation of periodic solutions of delay differential equations*, SIAM J. Sci. Comput.,  
 913 22 (2001), pp. 1593–1609, <https://doi.org/10.1137/S1064827599363381>.  
 914
- [33] K. ENGELBORGHs, T. LUZYANINA, AND D. ROOSE, *Numerical bifurcation analysis of delay*  
 915 *differential equations using DDE-BIFTOOL*, ACM Trans. Math. Softw., 28 (2002), pp. 1–  
 916 21, <https://doi.org/10.1145/513001.513002>.  
 917
- [34] K. ENGELBORGHs AND D. ROOSE, *On stability of LMS methods and characteristic roots of delay*  
 918 *differential equations*, SIAM J. Numer. Anal., 40 (2002), pp. 629–650, <https://doi.org/10.1137/S003614290037472X>.  
 919
- [35] P. ERDŐS AND P. TURÁN, *On interpolation I. Quadrature- and mean-convergence in the*  
 920 *Lagrange-interpolation*, Ann. Math., 38 (1937), pp. 142–155, [https://doi.org/10.2307/](https://doi.org/10.2307/1968516)  
 921 [1968516](https://doi.org/10.2307/1968516).  
 922
- [36] W. FELLER, *On the integral equation of renewal theory*, Ann. Math. Statist., 12 (1941), pp. 243–  
 923 267, <https://doi.org/10.1214/aoms/1177731708>.  
 924
- [37] E. FRIDMAN, *Introduction to Time-Delay Systems: Analysis and Control*, Systems Control  
 925 Found. Appl., Birkhäuser, Basel, 2014, <https://doi.org/10.1007/978-3-319-09393-2>.  
 926
- [38] G. GRIPENBERG, S.-O. LONDEN, AND O. STAFFANS, *Volterra Integral and Functional Equations*,  
 927 no. 34 in Encyclopedia Math. Appl., Cambridge University Press, Cambridge, 1990.  
 928
- [39] K. GU, V. L. KHARITONOV, AND J. CHEN, *Stability of Time-Delay Systems*, Control Eng.,  
 929 Birkhäuser, Basel, 2003, <https://doi.org/10.1007/978-1-4612-0039-0>.  
 930
- [40] J. K. HALE AND S. M. VERDUYN LUNEL, *Introduction to Functional Differential Equations*,  
 931 no. 99 in Appl. Math. Sci., Springer-Verlag, New York, 1993, [https://doi.org/10.1007/](https://doi.org/10.1007/978-1-4612-4342-7)  
 932 [978-1-4612-4342-7](https://doi.org/10.1007/978-1-4612-4342-7).  
 933
- [41] M. IANNELLI, *Mathematical Theory of Age-Structured Population Dynamics*, Giardini Editori  
 934 e Stampatori, Pisa, 1995.  
 935
- [42] H. INABA, *Age-Structured Population Dynamics in Demography and Epidemiology*, Springer-  
 936 Verlag, Singapore, 2017, <https://doi.org/10.1007/978-981-10-0188-8>.  
 937
- [43] T. INSPERGER AND G. STÉPÁN, *Semi-Discretization for Time-Delay Systems*, no. 178 in Appl.  
 938 Math. Sci., Springer-Verlag, New York, 2011, <https://doi.org/10.1007/978-1-4614-0335-7>.  
 939
- [44] E. JARLEBRING, *The spectrum of delay-differential equations: numerical methods, stability and*  
 940 *perturbation*, PhD thesis, Inst. Comp. Math. TU Braunschweig, 2008.  
 941
- [45] W. O. KERMACK AND A. G. MCKENDRICK, *A contribution to the mathematical theory of*  
 942 *epidemics*, P. Roy. Soc. Lond. A Mat., 115 (1927), pp. 700–721, [https://doi.org/10.1098/](https://doi.org/10.1098/rspa.1927.0118)  
 943 [rspa.1927.0118](https://doi.org/10.1098/rspa.1927.0118).  
 944
- [46] R. KRESS, *Linear Integral Equations*, no. 82 in Appl. Math. Sci., Springer, 3 ed., 2014, <https://doi.org/10.1007/978-1-4614-9593-2>.  
 945
- [47] Y. KUANG, *Delay Differential Equations: With Applications in Population Dynamics*, no. 191  
 946 in Math. Sci. Eng., Academic Press, San Diego, 1993.  
 947
- [48] A. J. LOTKA, *On an integral equation in population analysis*, Ann. Math. Statist., 10 (1939),  
 948 pp. 144–161, <https://doi.org/10.1214/aoms/1177732213>.  
 949
- [49] T. LUZYANINA AND D. ROOSE, *Periodic solutions of differential algebraic equations with time*  
 950 *delays: computation and stability analysis*, Int. J. Bifurcat. Chaos, 16 (2006), pp. 67–84,  
 951 <https://doi.org/10.1142/S0218127406014599>.  
 952
- [50] K. A. LYTHGOE, L. PELLIS, AND C. FRASER, *Is HIV short-sighted? Insights from a multistrain*  
 953 *nested model*, Evolution, 67 (2013), pp. 2769–2782, <https://doi.org/10.1111/evo.12166>.  
 954
- [51] N. MACDONALD, *Biological Delay Systems: Linear Stability Theory*, no. 8 in Cambridge Studies  
 955 in Mathematical Biology, Cambridge University Press, Cambridge, 1989.  
 956
- [52] *The Dynamics of Physiologically Structured Populations*, no. 68 in Lecture Notes in  
 957 Biomathematics, Springer-Verlag, Berlin, Heidelberg, 1986, <https://doi.org/10.1007/>  
 958

- 960                   978-3-662-13159-6.
- 961 [53] W. MICHIELS AND S.-I. NICULESCU, *Stability, Control, and Computation of Time-Delay Sys-*  
962 *tems: An Eigenvalue-Based Approach, second edition*, no. 27 in Adv. Des. Control, SIAM,  
963 Philadelphia, 2014, <https://doi.org/10.1137/1.9781611973631>.
- 964 [54] J. MIKUSIŃSKI AND C. RYLL-NARDZEWSKI, *Sur le produit de composition*, *Studia Mathematica*,  
965 12 (1951), pp. 51–57.
- 966 [55] T. J. RIVLIN, *An Introduction to the Approximation of Functions*, Blaisdell Publishing Com-  
967 pany, Waltham, MA, 1969.
- 968 [56] H. L. ROYDEN AND P. M. FITZPATRICK, *Real Analysis*, Prentice Hall, 4th ed., 2010.
- 969 [57] J. SIEBER, K. ENGELBORGH, T. LUZYANINA, G. SAMAËY, AND D. ROOSE, *DDE-BIFTOOL*  
970 *manual: Bifurcation analysis of delay differential equations*, ArXiv e-prints, (2014), <https://arxiv.org/abs/1406.7144>.
- 971 [58] H. SMITH, *An Introduction to Delay Differential Equations with Applications to the Life Sci-*  
972 *ences*, no. 57 in Texts Appl. Math., Springer-Verlag, New York, 2011, [https://doi.org/10.](https://doi.org/10.1007/978-1-4419-7646-8)  
973 [1007/978-1-4419-7646-8](https://doi.org/10.1007/978-1-4419-7646-8).
- 974 [59] G. STÉPÁN, *Retarded Dynamical Systems: Stability and Characteristic Functions*, no. 210 in  
975 Pitman Research Notes in Mathematics Series, Longman Scientific & Technical, Harlow,  
976 1989.
- 977 [60] L. N. TREFETHEN, *Spectral Methods in MATLAB*, Software Environ. Tools, SIAM, Philadel-  
978 phia, 2000, <https://doi.org/10.1137/1.9780898719598>.
- 979 [61] G. F. WEBB, *Theory of Nonlinear Age-Dependent Population Dynamics*, Chapman & Hall  
980 Pure and Applied Mathematics, CRC Press, 1985.
- 981