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# Generating formulas for finite reflection groups of the infinite series $S_n$ , $A_n$ , $B_n$ and $D_n$ .

Vittorino Talamini<sup>1,\*</sup>

<sup>1</sup>DMIF, Università di Udine, via delle Scienze 206, 33100 Udine, Italy  
and

INFN, Sezione di Trieste, via Valerio 2, 34127 Trieste, Italy

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Let  $W \subset O(n)$  be a finite reflection group,  $p_1(x), \dots, p_n(x)$ ,  $x \in \mathbb{R}^n$ , be a basis of algebraically independent  $W$ -invariant real homogeneous polynomials, and  $\bar{p}: \mathbb{R}^n \rightarrow \mathbb{R}^n: x \rightarrow (p_1(x), \dots, p_n(x))$  the orbit map, whose image  $\mathcal{S} = \bar{p}(\mathbb{R}^n) \subset \mathbb{R}^n$  is diffeomorphic with the orbit space  $\mathbb{R}^n/W$ . With the given basis of invariant polynomials it is possible to build an  $n \times n$  polynomial matrix,  $\hat{P}(p)$ ,  $p \in \mathbb{R}^n$ , such that  $\hat{P}_{ab}(\bar{p}(x)) = \nabla p_a(x) \cdot \nabla p_b(x)$ ,  $\forall a, b = 1, \dots, n$ . It is known that  $\hat{P}(p)$  enables to determine  $\mathcal{S}$ , and that the polynomial  $\det(\hat{P}(p))$  satisfies a system of  $n$  differential equations that depends on an  $n$ -dimensional polynomial vector  $\lambda(p)$ . If  $n$  is large, the explicit determination of  $\hat{P}(p)$  and  $\lambda(p)$  are in general impossible to calculate from their definitions, because of computing time and computer memory limits. In this article, when  $W$  is one of the finite reflection groups of type  $S_n$ ,  $A_n$ ,  $B_n$ ,  $D_n$ ,  $\forall n \in \mathbb{N}$ , for given choices of the basis of  $W$ -invariant polynomials  $p_1(x), \dots, p_n(x)$ , generating formulas for  $\hat{P}(p)$  and  $\lambda(p)$  are established. Proofs are based on induction principle and elementary algebra. Transformation formulas allow then to determine both the matrices  $\hat{P}(p')$  and the vectors  $\lambda(p')$ , corresponding to any other basis  $p'_1(x), \dots, p'_n(x)$ , of  $W$ -invariant polynomials.

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## I. INTRODUCTION

When a finite reflection group  $W$  acts in  $\mathbb{R}^n$ , one may consider  $W \subset O(n)$ , and there exist a basis of  $n$  algebraically independent  $W$ -invariant real homogeneous polynomial functions  $p_1(x), \dots, p_n(x)$ ,  $x \in \mathbb{R}^n$ , called *basic invariant polynomials*, such that all  $W$ -invariant polynomial (or  $C^\infty$ ) functions  $q(x)$  can be uniquely written as polynomial (or  $C^\infty$ ) functions of the basic set:  $q(x) = \hat{q}(p_1(x), \dots, p_n(x))$ , with  $\hat{q}(p)$  a polynomial (or a  $C^\infty$ ) function of  $p = (p_1, \dots, p_n) \in \mathbb{R}^n$  [4, 19]. By saying that  $q(x)$  is a *W-invariant function*, one means that  $q(gx) = q(x)$ ,  $\forall g \in W$ ,  $\forall x$  in the domain of  $q$ , and by saying that the polynomials  $p_1(x), \dots, p_n(x)$  are algebraically independent one means that a polynomial equation  $\hat{q}(p_1(x), \dots, p_n(x)) = 0$  cannot be identically satisfied in  $\mathbb{R}^n$ , unless  $\hat{q}$  is the null polynomial, that one with all vanishing coefficients. The *orbit* containing  $x \in \mathbb{R}^n$  is the set  $Wx = \{gx \mid g \in W\}$ , and the *orbit space* is the quotient space  $\mathbb{R}^n/W$ , whose points represent the orbits. With the basic invariant polynomials  $p_1(x), \dots, p_n(x)$ , one defines the map  $\bar{p}: \mathbb{R}^n \rightarrow \mathbb{R}^n: x \rightarrow (p_1(x), \dots, p_n(x))$ , called the *orbit map*, that is constant along the orbits, and such that  $\bar{p}(x) \neq \bar{p}(x')$  if the orbits  $Wx$  and  $Wx'$  are different. The image  $\mathcal{S} = \bar{p}(\mathbb{R}^n) \subset \mathbb{R}^n$  of the orbit map is then diffeomorphic with the orbit space  $\mathbb{R}^n/W$ , because  $\bar{p}$  is a polynomial map.

With a given set of basic invariant polynomials  $p_a(x)$ ,  $a = 1, \dots, n$ , it is possible to define a symmetric  $n \times n$  matrix  $P(x)$ , with elements  $P_{ab}(x)$ ,  $a, b = 1, \dots, n$ , that are obtained by making the scalar products among the gradients of the basic invariant polynomials:

$$P_{ab}(x) = \nabla p_a(x) \cdot \nabla p_b(x) = \sum_{i=1}^n \frac{\partial p_a(x)}{\partial x_i} \frac{\partial p_b(x)}{\partial x_i} \quad \forall a, b = 1, \dots, n, \quad \forall x \in \mathbb{R}^n. \quad (1)$$

As  $W \subset O(n)$ , the matrix elements of  $P(x)$  are  $W$ -invariant homogeneous polynomials, so it is well defined the  $n \times n$

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\*vittorino.talamini@uniud.it

matrix  $\widehat{P}(p)$ , with matrix elements that are polynomial functions of  $p \in \mathbb{R}^n$ , such that

$$P_{ab}(x) = \widehat{P}_{ab}(\overline{p}(x)), \quad \forall a, b = 1, \dots, n, \quad \forall x \in \mathbb{R}^n. \quad (2)$$

Following [1], let us call the matrix  $\widehat{P}(p)$  with the name  $\widehat{P}$ -matrix, but the matrix  $\widehat{P}(p)$  has also been used with other notations and called “convolution of invariants” matrix [3], “complete displacement of invariants” matrix [7] or it was used without giving it a specific name in [13, 15, 22]. The  $\widehat{P}$ -matrix is strongly related with the set  $\mathcal{S}$ : the boundary of  $\mathcal{S}$ , sometimes called *discriminant*, is contained in the algebraic surface determined by the equation  $\det(\widehat{P}(p)) = 0$ , and the interior of  $\mathcal{S}$  can be characterized by the definite positiveness of  $\widehat{P}(p)$  [1, 13]:  $\text{Int}(\mathcal{S}) = \{p \in \mathbb{R}^n \mid \widehat{P}(p) > 0\}$ . In addition, the polynomial  $\det(\widehat{P}(p))$  satisfies a system of differential equations [16, 22], Eq. (9) below, called *boundary equation*, that depends on an  $n$ -dimensional polynomial vector  $\lambda^{(\det(\widehat{P}))}(p)$ , called  $\lambda$ -vector (of  $\det(\widehat{P}(p))$ ). Both  $\widehat{P}(p)$  and  $\lambda^{(\det(\widehat{P}))}(p)$ , as well as the set  $\mathcal{S}$ , depend on the choice of the basis  $p_1(x), \dots, p_n(x)$ . The first appearance of the  $\widehat{P}$ -matrices has been in [2], just for the groups  $A_n$ , and their use for other finite reflection group has been first made in [3, 14, 15, 22, 23]. The  $\lambda$ -vectors first appeared in [22].

The explicit determination of the matrix  $\widehat{P}(p)$  from Eqs. (1) and (2), requires to calculate the matrix elements  $P_{ab}(x)$ ,  $\forall a \leq b = 1, \dots, n$ , and to express them as polynomials of the basic invariant polynomials  $p_1(x), \dots, p_n(x)$ . When  $n$  is large, these calculations require an enormous effort, even for a powerful computer, and are often impossible to take them to the end in a reasonable time, primarily because of computer memory limits. The possibility to use generating formulas to determine the matrix elements of the  $\widehat{P}$ -matrices are then particularly welcome. In [2], Remark 3.9, Arnold writes a generating formula for the  $\widehat{P}$ -matrices corresponding to the groups of type  $A_n$ , saying that that formula was communicated to him by D. B. Fuks, and gives a hint for its proof (with a typographical error: a definition  $\sigma_0 = 0$  that should be  $\sigma_0 = 1$ ). The formula is correct but its proof, to my knowledge, has never been published. In [7], Givental writes generating formulas for the  $\widehat{P}$ -matrices corresponding to groups of type  $B_n$  and  $D_n$ . His formulas are correct but their proofs are only sketched and based on differential geometry and singularity theory techniques that are difficult to follow by a non-specialist. Without taking into account the results in [2] and [7], the author was looking for bases of invariant polynomials in which the  $\widehat{P}$ -matrices look particularly simple. This allowed him to discover, for particular bases of invariant polynomials, generating formulas to calculate the matrix elements of the corresponding  $\widehat{P}$ -matrices. These generating formulas are reported in Eqs. (20), (26), (29), and (33), for the groups of type  $S_n$ ,  $A_n$ ,  $B_n$ , and  $D_n$ , respectively, and are valid for all values of  $n \in \mathbb{N}$ . Unexpectedly, the bases chosen for the groups of type  $B_n$  and  $D_n$  are the same as those chosen by Givental while those chosen for the groups of type  $A_n$  differ only for real multiples from those chosen by Arnold. The proofs of the generating formulas for  $A_n$ ,  $B_n$  and  $D_n$  here given are based only on elementary algebraic calculations and induction principle on  $n$ , so they are very different than those sketched in both [2] and [7]. For those particular bases, generating formulas for the  $\lambda$ -vectors of  $\det(\widehat{P}(p))$  are also determined, for all the groups of type  $S_n$ ,  $A_n$ ,  $B_n$ ,  $D_n$ , and for all  $n \in \mathbb{N}$ . These results are given in Eqs. (21), (27), (31) and (34), respectively. The proofs are based on Theorem 1 below, which requires to calculate sums over the positive roots. Almost the same  $\lambda$ -vectors, for the groups of type  $A_n$ ,  $B_n$  and  $D_n$ , were obtained in [23] by Yano and Sekiguchi, using induction on  $n$  and the fundamental anti-invariants, products of the linear forms vanishing on the reflection hyperplanes.

The proofs of these generating formulas, both for the  $\widehat{P}$ -matrices and for the  $\lambda$ -vectors, are the main results of this article. Theorems 1 and 2, that are used in the proofs, are also new.

With the  $\widehat{P}$ -matrices  $\widehat{P}(p)$  and the  $\lambda$ -vectors  $\lambda^{(\det(\widehat{P}))}(p)$ , obtained from the generating formulas here reported, corresponding to particular bases of invariant polynomials, using the transformation formulas in Eqs. (15) and (16), one may obtain both the  $\widehat{P}$ -matrices and the  $\lambda$ -vectors of their determinants, in correspondence to any other basis of invariant polynomials of the groups of type  $S_n$ ,  $A_n$ ,  $B_n$  and  $D_n$ , for all  $n \in \mathbb{N}$ .

To keep simpler the reading of the article, the generating formulas of the  $\widehat{P}$ -matrices and of the  $\lambda$ -vectors for the groups of type  $S_n$ ,  $A_n$ ,  $B_n$ ,  $D_n$ , are reported without proofs in Sections III-VI, respectively. Their proofs are collected in Section VIII, and two examples are shown in Section VII. In Section II are reviewed some known results and established some new ones (Theorems 1 and 2), that are used in the proofs of Section VIII. The Appendix lists alternative generating formulas that are somewhat simpler to use.

Orbit spaces and  $\widehat{P}$ -matrices have interesting applications in physics and mathematics, especially in the study of phase transitions and of symmetry breaking [1, 8, 12, 18], in the study of singularities [2, 3, 7, 14], and in constructive invariant theory [20]. No comments are made here about the applications of the  $\widehat{P}$ -matrices in these fields. A short review of these applications can be found in a preliminary version of this article available in ArXiv [21].

## II. BACKGROUND MATERIALS AND PRELIMINARY RESULTS

This review section aims to set up the notation and to recall some known arguments needed in later proofs. It contains also new results in Theorems 1 and 2. We shall always consider  $W$  a finite reflection group of real orthogonal matrices. Unless otherwise stated, the matrices of  $W$  will be considered of order  $n$ , and acting in  $\mathbb{R}^n$ .

Given a root  $\alpha$ , all the vectors in  $W\alpha$  are roots, and form a *root system* ([9], Section 1.2). If  $W$  is irreducible, there is only one root system, except if  $W$  is one of the reflection groups of types  $B_n$ ,  $\forall n \geq 2$ ,  $F_4$ ,  $I_2(m)$ ,  $\forall m \geq 6$ ,  $m$  even, in which cases there are two different root systems. Let us use the symbols  $\mathcal{R}$  and  $\mathcal{R}_+$  for the set of roots and the set of positive roots of  $W$ . There is a one to one correspondence between the set of positive roots  $\mathcal{R}_+$  and the set of reflections of  $W$ . We shall indicate with  $l_\alpha(x)$  the linear form vanishing in the reflection hyperplane of a root  $\alpha \in \mathbb{R}^n$ , such that:

$$l_\alpha(x) = \alpha \cdot x, \quad \text{and} \quad \alpha = \nabla l_\alpha(x). \quad (3)$$

The degrees  $d_1, \dots, d_n$ , of the basic invariant polynomials of all the irreducible finite reflection groups, were determined by Coxeter [5], and for a given irreducible finite reflection group they are all different, except in the case of the groups of type  $D_n$ , with even  $n$ , in which case there are two basic invariant polynomials of degree  $n$ . Usually, the basic invariant polynomials are labeled according to their degrees, for example by taking

$$d_1 \leq d_2 \leq \dots \leq d_n. \quad (4)$$

In the following we shall adopt this choice. If  $W$  is irreducible,  $d_1 = 2$ , and one may choose

$$p_1(x) = x \cdot x = \sum_{i=1}^n x_i^2. \quad (5)$$

From the the definition (1) of the matrix  $P(x)$ , it follows that  $\deg(P_{ab}(x)) = d_a + d_b - 2$ ,  $\forall a, b = 1, \dots, n$ .

The matrix  $P(x)$  can also be written in the following way:

$$P(x) = j(x) j^\top(x), \quad (6)$$

in which the exponent  $\top$  means transposition and  $j(x)$  is the jacobian matrix with elements

$$j_{ai}(x) = \frac{\partial p_a(x)}{\partial x_i}, \quad \forall a, i = 1, \dots, n.$$

A classic result by Coxeter, [5], Theorem 6.2, claims that, when the basic invariant polynomials are algebraically independent,  $\det(j(x))$  is proportional to the product of the  $N = \text{Card}(\mathcal{R}_+)$  linear forms  $l_r(x)$ ,  $r \in \mathcal{R}_+$ , defined in Eq. (3), that is:

$$\det(j(x)) = c \prod_{r \in \mathcal{R}_+} l_r(x), \quad c \in \mathbb{R}, \quad c \neq 0. \quad (7)$$

Then, from Eq. (6), one has:

$$\det(P(x)) = c^2 \prod_{r \in \mathcal{R}_+} l_r^2(x). \quad (8)$$

This implies that  $\det(P(x)) = 0$  if  $x$  belongs to the set of the reflecting hyperplanes and that  $\det(P(x)) > 0$  otherwise. The number  $c$  depends both on the normalization of the roots and on the choice of the basic invariant polynomials.

As the basic invariant polynomials  $p_1(x), \dots, p_n(x)$  are homogeneous, it is natural to consider the coordinates  $p_1, \dots, p_n$ , in the image space of the orbit map  $\bar{p}$ , as a set of  $n$  graded, or weighted, coordinates, whose *weights* coincide with the degrees of the basic invariant polynomials, that is:  $w(p_a) = \deg(p_a(x)) = d_a$ ,  $\forall a = 1, \dots, n$ . The variables  $p$  and  $x$  are here used to label points in different spaces  $\mathbb{R}^n$ , the  $x$  are used for the  $\mathbb{R}^n$  in the domain of  $\bar{p}$ , while the  $p$  are used for the  $\mathbb{R}^n$  containing the image of  $\bar{p}$ , the variables  $p$  are weighted, while the variables  $x$  are not. We shall use indices  $a, b, c, \dots$ , for the variables in  $p \in \mathbb{R}^n$ , while indices  $i, j, k, \dots$ , for the variables in  $x \in \mathbb{R}^n$ . A polynomial  $\hat{q}(p)$  is said to be of *weight*  $d$  if the  $W$ -invariant polynomial  $q(x) = \hat{q}(\bar{p}(x))$  is of degree  $d$ . We shall write this as  $w(\hat{q}) = d$ . The polynomial  $\hat{q}(p)$  is said to be *weighted homogeneous*, or *w-homogeneous* if the  $W$ -invariant polynomial  $q(x) = \hat{q}(\bar{p}(x))$  is homogeneous. Hence, the matrix elements of the  $\hat{P}$ -matrix  $\hat{P}(p)$  are  $w$ -homogeneous polynomials of weights  $w(\hat{P}_{ab}) = d_a + d_b - 2$ , and  $\det(\hat{P}(p))$  is a  $w$ -homogeneous polynomial of weight  $w(\det(\hat{P})) = \sum_{a=1}^n (2d_a - 2)$ .

Suppose the basic invariant polynomials algebraically independent. A  $w$ -homogeneous polynomial  $a(p)$  is called *active* if it satisfies the following system of differential equations [16, 22], called the *boundary equation*:

$$\sum_{c=1}^n \widehat{P}_{bc}(p) \frac{\partial a(p)}{\partial p_c} = \lambda_b^{(a)}(p) a(p), \quad \forall b = 1, \dots, n, \quad (9)$$

where  $\lambda^{(a)}(p) = (\lambda_1^{(a)}(p), \dots, \lambda_n^{(a)}(p))$  is a polynomial vector function of  $p \in \mathbb{R}^n$ , dependent on  $a(p)$ , called  $\lambda$ -vector (of  $a$ ). Active polynomials differing only by real multiplicative factors satisfy Eq. (9) with the same  $\lambda$ -vector. From Eq. (9), it follows that the  $\lambda_b^{(a)}(p)$ ,  $b = 1, \dots, n$ , are  $w$ -homogeneous polynomials of weights:

$$w(\lambda_b^{(a)}) = d_b - 2, \quad \forall b = 1, \dots, n.$$

If  $p_1(x)$  is as in Eq. (5), one has ([17], Theorem 4.1, item i)):  $\lambda_1^{(a)}(p) = 2w(a)$ . Eq. (9) was first given in [22], and was further studied in [23], just for the case in which  $a(\bar{p}(x)) = \det(P(x))$ . Eq. (9) was rediscovered in [16], and was further studied in [17]. Among other things, Theorem 4.1 of [17] states that:

P1. The irreducible active polynomials coincide with the irreducible factors of  $\det(\widehat{P}(p))$ ;

P2. All products  $a(p) = \prod_{i=1}^r (a_i(p))^{m_i}$ ,  $m_i \in \mathbb{N}$ , of irreducible active polynomials  $a_i(p)$ ,  $i = 1, \dots, r$ , are active polynomials too, for which  $\lambda^{(a)}(p) = \sum_{i=1}^r m_i \lambda^{(a_i)}(p)$ , where  $\lambda^{(a_i)}(p)$ ,  $\forall i = 1, \dots, r$ , are the  $\lambda$ -vectors of the active irreducible polynomials  $a_i(p)$ ,  $i = 1, \dots, r$ .

From Properties P1 and P2 above, it follows that  $\det(\widehat{P}(p))$  is an active polynomial. The following theorem establishes general formulas for the  $\lambda$ -vectors of  $\det(\widehat{P}(p))$ .

**Theorem 1** 1. Let  $W$  be an irreducible finite reflection group, and  $\mathcal{R}_+$  be a set of positive roots of  $W$ . The  $\lambda$ -vector  $\lambda^{(\det(\widehat{P}))}(p)$  can be calculated from the following formula:

$$\lambda_a^{(\det(\widehat{P}))}(p) \Big|_{p=\bar{p}(x)} = 2 \sum_{r \in \mathcal{R}_+} \frac{\nabla p_a(x) \cdot r}{l_r(x)}, \quad \forall a = 1, \dots, n, \quad \forall x \in \mathbb{R}^n, \quad (10)$$

where  $l_r(x)$  is defined in Eq. (3).

2. In addition, if  $W$  has two root systems, and  $\mathcal{R}_+ = \mathcal{R}_{+,s} \cup \mathcal{R}_{+,l}$  is the partition of the set of positive roots according to those belonging to a same root system, then  $\det(\widehat{P}(p))$  has two irreducible active factors,  $\widehat{s}(p)$  and  $\widehat{l}(p)$ :

$$\det(\widehat{P}(p)) = c^2 \widehat{s}(p) \widehat{l}(p), \quad (11)$$

where  $c$  is the real number in Eq. (7), whose  $\lambda$ -vectors  $\lambda^{(s)}(p)$  and  $\lambda^{(l)}(p)$  can be calculated from the following formulas:

$$\lambda_a^{(s)}(p) \Big|_{p=\bar{p}(x)} = 2 \sum_{r \in \mathcal{R}_{+,s}} \frac{\nabla p_a(x) \cdot r}{l_r(x)}, \quad \lambda_a^{(l)}(p) \Big|_{p=\bar{p}(x)} = 2 \sum_{r \in \mathcal{R}_{+,l}} \frac{\nabla p_a(x) \cdot r}{l_r(x)}, \quad \forall a = 1, \dots, n, \quad \forall x \in \mathbb{R}^n, \quad (12)$$

and satisfy the equation:

$$\lambda_a^{(s)}(p) + \lambda_a^{(l)}(p) = \lambda_a^{(\det(\widehat{P}))}(p), \quad \forall a = 1, \dots, n. \quad (13)$$

**Proof.** 1. Eqs. (2) and (6) imply  $\det(\widehat{P}(\bar{p}(x))) = \det(P(x)) = (\det(j(x)))^2$ . Using this result in the first member of Eq. (9), written for  $a(p) = \det(\widehat{P}(p))$ , and taking  $p = \bar{p}(x)$ , one has:

$$\begin{aligned} \sum_{b=1}^n \widehat{P}_{ab}(p) \frac{\partial \det(\widehat{P}(p))}{\partial p_b} \Big|_{p=\bar{p}(x)} &= \sum_{b=1}^n \nabla p_a(x) \cdot \nabla p_b(x) \frac{\partial \det(\widehat{P}(p))}{\partial p_b} \Big|_{p=\bar{p}(x)} = \nabla p_a(x) \cdot \nabla \det(P(x)) \\ &= 2 \det(j(x)) \nabla p_a(x) \cdot \nabla \det(j(x)). \end{aligned}$$

Using Eqs. (7) and (3), one obtains:

$$\nabla \det(j(x)) = c \nabla \prod_{r \in \mathcal{R}_+} l_r(x) = c \sum_{r \in \mathcal{R}_+} \left( \nabla l_r(x) \prod_{\substack{r' \in \mathcal{R}_+ \\ r' \neq r}} l_{r'}(x) \right) = \det(j(x)) \sum_{r \in \mathcal{R}_+} \frac{\nabla l_r(x)}{l_r(x)} = \det(j(x)) \sum_{r \in \mathcal{R}_+} \frac{r}{l_r(x)},$$

and inserting this result in the preceding equation, one obtains:

$$\left( \sum_{b=1}^n \widehat{P}_{ab}(p) \frac{\partial \det(\widehat{P}(p))}{\partial p_b} \right) \Big|_{p=\bar{p}(x)} = 2 (\det(j(x)))^2 \nabla p_a(x) \cdot \sum_{r \in \mathcal{R}_+} \frac{r}{l_r(x)} = \left( 2 \sum_{r \in \mathcal{R}_+} \frac{\nabla p_a(x) \cdot r}{l_r(x)} \right) \det(\widehat{P}(\bar{p}(x))).$$

This last expression must be equal to the second member of Eq. (9), written for  $a(p) = \det(\widehat{P}(p))$ , and with  $p = \bar{p}(x)$ , that is with  $\lambda_a^{(\det(\widehat{P}))}(p) \det(\widehat{P}(p)) \Big|_{p=\bar{p}(x)}$ . By comparing, one finds Eq. (10).

2. We can rewrite Eq. (8) in the following way:

$$\det(P(x)) = c^2 \left( \prod_{r \in \mathcal{R}_{+,l}} l_r(x)^2 \right) \left( \prod_{r \in \mathcal{R}_{+,s}} l_r(x)^2 \right) = c^2 s(x) l(x).$$

Both  $s(x)$  and  $l(x)$  are  $W$ -invariant polynomials, because the roots in the two root systems  $\mathcal{R}_{+,s}$  and  $\mathcal{R}_{+,l}$  are transformed into themselves by  $W$  transformations. Then, both  $s(x)$  and  $l(x)$  can be written in terms of the basic invariant polynomials, that is:  $s(x) = \widehat{s}(\bar{p}(x))$ , and  $l(x) = \widehat{l}(\bar{p}(x))$ , and this implies Eq. (11). This is well known. Properties P1 and P2 imply then that the polynomials  $\widehat{s}(p)$  and  $\widehat{l}(p)$  are active polynomials, satisfying the boundary equation (9) with proper  $\lambda$ -vectors  $\lambda^{(s)}(p)$  and  $\lambda^{(l)}(p)$ , respectively, satisfying Eq. (13). To prove the first of Eq. (12), take  $a(p) = \widehat{s}(p)$ , and  $p = \bar{p}(x)$  in Eq. (9). Using also the second of Eq. (3), the first member of Eq. (9), becomes:

$$\begin{aligned} & \left( \sum_{b=1}^n \widehat{P}_{ab}(p) \frac{\partial \widehat{s}(p)}{\partial p_b} \right) \Big|_{p=\bar{p}(x)} = \sum_{b=1}^n \nabla p_a(x) \cdot \nabla p_b(x) \left( \frac{\partial \widehat{s}(p)}{\partial p_b} \right) \Big|_{p=\bar{p}(x)} = \nabla p_a(x) \cdot \nabla s(x) \\ & = 2 \nabla p_a(x) \cdot \sum_{r \in \mathcal{R}_{+,s}} \left( l_r(x) \nabla l_r(x) \prod_{\substack{r' \in \mathcal{R}_{+,s} \\ r' \neq r}} l_{r'}^2(x) \right) = 2 s(x) \nabla p_a(x) \cdot \sum_{r \in \mathcal{R}_{+,s}} \frac{r}{l_r(x)} = \left( 2 \sum_{r \in \mathcal{R}_{+,s}} \frac{\nabla p_a(x) \cdot r}{l_r(x)} \right) \widehat{s}(\bar{p}(x)), \end{aligned}$$

while the second member of Eq. (9) becomes:  $\lambda_a^{(s)}(\bar{p}(x)) \widehat{s}(\bar{p}(x))$ . Comparing the two expressions, one finds the first of Eq. (12). Repeating the proof with  $l(p)$  in place of  $s(p)$  one finds the second of Eq. (12). Alternatively, one may use Eqs. (13), (10), and the first of Eq. (12).  $\diamond$

As the first members of Eqs. (10) and (12) are  $W$ -invariant homogeneous polynomials, also the second members are, and can then be expressed as  $w$ -homogeneous polynomials of the basic invariant polynomials. This allows one to obtain the explicit expressions of the polynomials  $\lambda_a^{(\det(\widehat{P}))}(p)$ ,  $\lambda_a^{(s)}(p)$ , and  $\lambda_a^{(l)}(p)$ ,  $\forall a = 1, \dots, n$ .

It is convenient to recall how the  $\widehat{P}$ -matrices, the active polynomials, and the  $\lambda$ -vectors transform when the basic invariant polynomials are transformed, because we shall often use these transformations in the proofs of Section VIII. We shall consider two classes of transformations. The first one is related to changes of the canonical basis in  $\mathbb{R}^n$ , that naturally implies, changes of the coordinates of the roots, as vectors of  $\mathbb{R}^n$ , of the matrix representatives in  $O(n)$  of the group elements, and of the explicit expressions of the basic invariant polynomials. The second one is related to changes of the basis of invariant polynomials, performed without changing the basis of  $\mathbb{R}^n$ , that is, obtained through algebraic transformations of the basic invariant polynomials, done in such a way to maintain their degrees and algebraic independence. Both these two classes of transformations are well known.

A change of the canonical basis in  $\mathbb{R}^n$ , implies a change of the coordinates of the vectors of  $\mathbb{R}^n$  realized with a rotation matrix  $R \in SO(n)$ , so that,  $\forall x \in \mathbb{R}^n$ ,  $x' = Rx$ , and  $\forall g \in O(n)$ ,  $g' = RgR^\top$ .

**Proposition 1** Let  $R \in SO(n)$  and  $W \subset O(n)$  a finite reflection group.  $R$  induces a coordinate transformation of  $\mathbb{R}^n$ ,  $x' = Rx$ , for which  $W' = \{g' \mid g' = RgR^\top, g \in W\}$  is the transformed reflection group. One then has:

1.  $p(x)$  is a  $W$ -invariant polynomial if and only if the polynomial  $p'(x') = p(R^\top x')$ , is a  $W'$ -invariant polynomial.
2. The polynomials  $p_a(x)$ ,  $a = 1, \dots, n$ , form a basis of  $W$ -invariant homogeneous polynomials, if and only if the polynomials  $p'_a(x') = p_a(R^\top x')$ ,  $a = 1, \dots, n$ , form a basis of  $W'$ -invariant homogeneous polynomials.
3. If  $p(x)$  is a  $W$ -invariant polynomial, then the  $W'$ -invariant polynomial  $p'(x') = p(R^\top x')$  has the same functional dependence  $\widehat{p}$  on the basis  $p'_1(x'), \dots, p'_n(x')$ , as that one of  $p(x)$  on the basis  $p_1(x), \dots, p_n(x)$ , that is:

$$p(x) = \widehat{p}(p_1(x), \dots, p_n(x)) \iff p'(x') = \widehat{p}(p'_1(x'), \dots, p'_n(x')).$$

4. The  $\widehat{P}$ -matrices  $\widehat{P}(p)$  and  $\widehat{P}(p')$ , calculated using Eqs. (1) and (2) from the bases  $p_a(x)$  and  $p'_a(x') = p_a(R^\top x')$ ,  $a = 1, \dots, n$ , of  $W$  and  $W'$ , respectively, are identical, except for the substitution of the variables  $p_a$  with the variables  $p'_a$ ,  $\forall a = 1, \dots, n$ :

$$\widehat{P}(p') = \widehat{P}(p) \Big|_{p=p'}.$$

5. The active polynomials  $a'(p')$  and their  $\lambda$ -vectors  $\lambda^{(a')}(p')$ , corresponding to the  $\widehat{P}$ -matrix  $\widehat{P}(p')$  of  $W'$ , are identical to those corresponding to the  $\widehat{P}$ -matrix  $\widehat{P}(p)$  of  $W$ , except for the substitution of the variables  $p_a$  with the variables  $p'_a$ ,  $\forall a = 1, \dots, n$ :

$$a'(p') = \widehat{a}(p) \Big|_{p=p'}, \quad \lambda^{(a')}(p') = \lambda^{(a)}(p) \Big|_{p=p'}.$$

**Proof.** The proofs of items 1.–4. are elementary exercises. The proof of item 5. follows then from P1 and P2.  $\diamond$

We stress an important consequence of Proposition 1: the set  $\mathcal{S}$ , describing the orbit space, that is univocally determined by the  $\widehat{P}$ -matrix, does not change under changes of the system of coordinates used in  $\mathbb{R}^n$ .

Let us now consider transformations of the basis of invariant polynomials obtained without changes of the coordinate basis in  $\mathbb{R}^n$ . One is allowed to transform a basic set of invariant polynomials  $p_a(x)$ ,  $a = 1, \dots, n$ , to a new one  $p'_a(x)$ ,  $a = 1, \dots, n$ , only in such a way to maintain the homogeneity, the degrees  $d_a$ ,  $a = 1, \dots, n$ , and the property of being a basis. As  $\forall a = 1, \dots, n$ , the  $p'_a(x)$  are  $W$ -invariant homogeneous polynomials of degrees  $d_a$ , it is possible to write them as  $w$ -homogeneous polynomials of weights  $d_a$  of the basic invariant polynomials  $p_a(x)$ ,  $a = 1, \dots, n$ . Avoiding to write the  $x$  dependence, we have thus  $n$  equations:

$$p'_a = \widehat{p}'_a(p_1, \dots, p_n) = \widehat{p}'_a(p), \quad a = 1, \dots, n,$$

in which,  $\widehat{p}'_a(p)$ ,  $\forall a = 1, \dots, n$ , is a  $w$ -homogeneous polynomial of weight  $d_a$ . The jacobian matrix of this transformation,  $J(p)$ , has elements:

$$J_{ab}(p) = \frac{\partial \widehat{p}'_a(p)}{\partial p_b}, \quad \forall a, b = 1, \dots, n, \quad (14)$$

that are  $w$ -homogeneous polynomials of weights  $d_a - d_b$ , so they are vanishing if  $d_a < d_b$ , and constant if  $d_a = d_b$ . The requirement that the new invariant polynomials form a basis gives is obtained by requiring  $\det(J(p)) \neq 0$ . For example, one can require that,  $\forall a = 1, \dots, n$ , the explicit expression of  $\widehat{p}'_a(p)$  necessarily contains (linearly) the variable  $p_a$ , and if  $d \geq 2$  degrees are equal, one can require, in addition, that the  $d \times d$  sub-matrix of  $J(p)$ , located in correspondence with the rows and columns corresponding to the  $d$  variables of equal weight, be non-singular. This implies that  $\det(J(p))$  is a non-vanishing real number, and that the inverse transformation is everywhere well defined. This inverse transformation is  $p_a = \widehat{p}_a(p')$ ,  $a = 1, \dots, n$ , with the  $\widehat{p}_a(p')$ ,  $w$ -homogeneous polynomial functions of the weighted variables  $p'_1, \dots, p'_n$ , of weights  $w(\widehat{p}_a) = d_a$ .

In the following we shall only consider *basis transformations* with the above properties.

Let  $p' = \widehat{p}'(p)$  a basis transformation, and let  $\widehat{P}(p')$  and  $\widehat{P}(p)$  be the  $\widehat{P}$ -matrices, calculated with Eqs. (1) and (2), from to the bases  $p'_1(x), \dots, p'_n(x)$ , and  $p_1(x), \dots, p_n(x)$ , respectively. The relations between the matrices  $\widehat{P}(p')$  and  $\widehat{P}(p)$ , the active polynomials  $a(p)$  and  $a'(p')$ , and their  $\lambda$ -vectors, are given by the following proposition.

**Proposition 2** Let  $J(p)$  be the jacobian matrix (14) of a basis transformation  $p' = \widehat{p}'(p)$ , and let  $p = \widehat{p}(p')$  be the inverse transformation.

1. The  $\widehat{P}$ -matrices  $\widehat{P}(p')$  and  $\widehat{P}(p)$ , corresponding to the bases  $p'_1(x), \dots, p'_n(x)$ , and  $p_1(x), \dots, p_n(x)$ , respectively, are related by the following transformation formula:

$$\widehat{P}(p') = J(p) \widehat{P}(p) J^\top(p) \Big|_{p=\widehat{p}(p')} . \quad (15)$$

2. The active polynomials and their  $\lambda$ -vectors, corresponding to the basis  $p_1(x), \dots, p_n(x)$ , and those corresponding to the basis  $p'_1(x), \dots, p'_n(x)$ , are related by the following transformation formulas:

$$a'(p') = a(p) \Big|_{p=\widehat{p}(p')} , \quad \lambda^{(a')}(p') = J(p) \lambda^{(a)}(p) \Big|_{p=\widehat{p}(p')} . \quad (16)$$

**Proof.** The proof is sketched in [17], while proving Eq. (2.11), and in the proof of Theorem 4.1 i).  $\diamond$

A basis transformation causes a transformation of the  $\widehat{P}$ -matrix and this implies also a transformation of the set  $\mathcal{S}$  describing the orbit space. Because the maps  $p = \widehat{p}(p')$  and  $p' = \widehat{p}'(p)$  are one to one and polynomial, this transformation of  $\mathcal{S}$  is a diffeomorphism.

In Example 2., Section VII, it is shown an explicit calculation making use of Eqs. (15) and (16).

Both Propositions 1 and 2 are valid for a general compact linear group  $G \subseteq O(n)$ . In that case, by Hilbert's and Nöther's theorems,  $G$  possess a finite basis of  $m \geq n$   $G$ -invariant homogeneous polynomials:  $p_1(x), \dots, p_m(x)$ . It is then possible to determine univocally an  $m \times m$   $\widehat{P}$ -matrix  $\widehat{P}(p)$ , through Eqs. (1) and (2), and when the basic invariant polynomials are algebraically independent, there exist active polynomials satisfying Eq. (9) [16, 17].

For some reflection groups it is sometimes convenient to define the reflections in a real vector space of dimension  $m$ , greater than the rank  $n$  of the group. This may avoid square roots in the coordinates of the roots and in the coefficients of the explicit forms of the basic invariant polynomials. Also the calculations to determine the  $\widehat{P}$ -matrices are sometimes easier using these basic polynomials in  $m > n$  variables. This approach will be here used for the groups of type  $A_n$ , that will be first defined in  $\mathbb{R}^{n+1}$ . Other examples can be found in [11], where the exceptional groups of types  $E_6$ ,  $E_7$  and  $E_8$ , are at first defined in  $\mathbb{R}^8$ ,  $\mathbb{R}^8$ , and  $\mathbb{R}^9$ , respectively. In all these cases the representation in  $\mathbb{R}^m$  of a rank  $n < m$  reflection group  $W \subset O(m)$  is completely reducible, and there are, one or more, linear invariant polynomials. Let  $p_1(x), \dots, p_m(x)$ ,  $x \in \mathbb{R}^m$ , be a basis of  $W$ -invariant polynomials, such that  $\deg(p_1(x)) = 1$ , and let  $H \subset \mathbb{R}^m$  be the hyperplane  $p_1(x) = 0$ . The orbits of  $W$  are either contained in  $H$  or lie outside  $H$ . The basis of invariant polynomials of  $W$  induces a basis of invariant polynomials for its representation  $W_H$  in  $H$ , obtained by discarding the invariant polynomial  $p_1(x)$  from the original basis. Often, one wants to express these basic polynomials with only  $m - 1$  variables, corresponding to the coordinates with respect to a system of reference spanning  $H$ . To do this, one can make a rotation  $x' = Rx$ ,  $R \in SO(m)$  in such a way that  $\nabla p_1(x)$  be transformed into a vector parallel to the canonical unit vector  $e_m \in \mathbb{R}^m$ . After this rotation, the basic  $W$ -invariant polynomials are  $p'_a(x') = p_a(R^\top x')$ ,  $a = 1, \dots, m$ ,  $x' \in \mathbb{R}^m$ , with  $p'_1(x') \propto x'_m$ ,  $H$  is spanned by the canonical unit vectors  $e_1, \dots, e_{m-1}$ , and all transformed matrices  $RgR^\top$ ,  $g \in W$ , are block diagonal, with blocks of order  $m-1$  and 1, with those of the  $(m-1)$ -dimensional block forming the matrices of the reflection group  $W_H$ . By putting  $x'_m = 0$  in the explicit expressions of  $p'_2(x'), \dots, p'_m(x')$ , one obtains a basis of  $m - 1$   $W_H$ -invariant polynomials, written in terms of  $m - 1$  variables, only. Both  $W_H$  and the basic  $W_H$ -invariant polynomials so obtained depend on  $R$ . We shall say that they are  $R$ -induced by  $W$  and the basis  $p_1(x), \dots, p_m(x)$ .

Let us give some further notations and definitions. Given a point  $x' \in \mathbb{R}^m$ , its projection in  $\mathbb{R}^{m-1}$ , obtained by discarding its last coordinate will be indicated with a bar above the symbol. The set of polynomials  $q_{a-1}(\bar{x}') = p_a(R^\top x') \Big|_{x'_m=0}$ ,  $a = 2, \dots, m$ , are thus the  $R$ -induced basis of invariant polynomials of the  $R$ -induced group  $W_H$ . For any  $R \in SO(m)$  such that  $p_1(R^\top x') \propto x'_m$ , one can define an  $R$ -induced group  $W_H$ , and an  $R$ -induced basis of  $W_H$ -invariant polynomials. If  $R_1, R_2 \in SO(m)$ , are two of such matrices, there exists a matrix  $S \in SO(m)$  mixing the first  $m - 1$  coordinates, only (this means that  $S_{a,m} = S_{m,a} = \delta_{a,m}$ ,  $\forall a = 1, \dots, m$ ), such that  $R_1 = SR_2S^\top$ . Then, for Proposition 1, any two  $R$ -induced bases generate  $\widehat{P}$ -matrices differing at most for a renaming of the variables.

If  $a(p)$  is a scalar,  $v(p)$  an  $m$ -dimensional vector, and  $M(p)$  an  $m \times m$  matrix, dependent on the variables  $p_1, \dots, p_m$ , we define with  $[a(p)]^q$ ,  $[v(p)]^q$ , and  $[M(p)]^q$ , the scalar, the  $(m - 1)$ -dimensional vector, and the  $(m - 1) \times (m - 1)$  matrix, dependent on the variables  $q_1, \dots, q_{m-1}$ , obtained from the scalar  $a(p)$ , the vector  $v(p)$ , and the matrix  $M(p)$ , respectively, by applying the following three rules (of which the second one has no effect on the scalar  $a(p)$ ):

1. substitute  $p_1 = 0$  in the scalar  $a(p)$ , in the vector  $v(p)$ , or in the matrix  $M(p)$ ;



2. remove the first component of the resulting vector, or the first row and column of the resulting matrix;
3. rename the variables in the following way:  $p_a = q_{a-1}, \forall a = 2, \dots, m$ .

We prove the following theorem that relates  $\widehat{P}$ -matrices, active factors, and  $\lambda$ -vectors, corresponding to a basis  $p_a(x), a = 1, \dots, m$ , with  $\deg(p_1(x)) = 1$ , with those corresponding to an  $R$ -induced basis  $q_a(\bar{x}'), a = 1, \dots, m-1$ .

**Theorem 2** *Let  $W \subset O(m)$  be a reducible reflection group acting in  $\mathbb{R}^m$ ,  $p_a(x), a = 1, \dots, m, x \in \mathbb{R}^m$ , a basis of  $W$ -invariant polynomials, such that  $\deg(p_1(x)) = 1$ , and  $\widehat{P}(p)$  the corresponding  $\widehat{P}$ -matrix. Let  $H \subset \mathbb{R}^m$  be the hyperplane with equation  $p_1(x) = 0$ ,  $R \in SO(m)$  such that  $p_1(R^\top x') \propto x'_m$ ,  $W_H$  the  $R$ -induced reflection group,  $q_a(\bar{x}'), a = 1, \dots, m-1$ , the  $R$ -induced basis of  $W_H$ -invariant polynomials, and let  $\widehat{P}(q)$  be the corresponding  $\widehat{P}$ -matrix. Then,*

$$[\widehat{P}(p)]^q = \widehat{P}(q), \quad \text{if and only if} \quad \widehat{P}_{1a}(p) \Big|_{p_1=0} = 0, \quad \forall a = 2, \dots, m.$$

2. If  $[\widehat{P}(p)]^q = \widehat{P}(q)$ , then if  $a(p)$  is an active polynomial relative to  $\widehat{P}(p)$ , with  $\lambda$ -vector  $\lambda^{(a)}(p)$ , then  $a(q) = [a(p)]^q$  is an active polynomial relative to  $\widehat{P}(q)$ , with  $\lambda$ -vector  $\lambda^{(a)}(q) = [\lambda^{(a)}(p)]^q$ .

**Proof.** In this proof we shall use both the gradients  $\nabla p'_a(x')$  and  $\nabla p'_a(\bar{x}')$ . The first one is a vector of  $m$  components, because  $x' = Rx \in \mathbb{R}^m$ , while the second one is a vector of  $m-1$  components, because  $\bar{x}' \in \mathbb{R}^{m-1}$ . We shall also use the orbit maps  $\bar{p} : \mathbb{R}^m \rightarrow \mathbb{R}^m$ , and  $\bar{q} : \mathbb{R}^{m-1} \rightarrow \mathbb{R}^{m-1}$ , relative to bases of  $W$  and  $W_H$ , respectively. Given the basis:  $p'_a(x') = p_a(R^\top x'), \forall a = 1, \dots, m$ , Proposition 1 states that  $\widehat{P}(p') = \widehat{P}(p) \Big|_{p=p'}$ , and this implies  $[\widehat{P}(p)]^q = [\widehat{P}(p')]^q$ .

1. We first suppose  $[\widehat{P}(p)]^q = \widehat{P}(q)$ , that is,  $\widehat{P}_{a,b}(\bar{p}'(x')) \Big|_{x'_m=0} = \widehat{P}_{a-1,b-1}(\bar{q}(\bar{x}')), \forall a, b = 2, \dots, m, x' \in \mathbb{R}^m$ . For all  $a = 2, \dots, m$ , and  $x' \in \mathbb{R}^m$ , we have:

$$\begin{aligned} \widehat{P}_{a,a}(\bar{p}'(x')) \Big|_{x'_m=0} &= (\nabla p'_a(x') \cdot \nabla p'_a(x')) \Big|_{x'_m=0} \\ &= \nabla p'_a(\bar{x}') \cdot \nabla p'_a(\bar{x}') + \left( \frac{\partial p'_a(x')}{\partial x'_m} \right)^2 \Big|_{x'_m=0} = \widehat{P}_{a-1,a-1}(\bar{q}(\bar{x}')) + \left( \frac{\partial p'_a(x')}{\partial x'_m} \right)^2 \Big|_{x'_m=0}, \end{aligned}$$

but for hypothesis,  $\widehat{P}_{a,a}(\bar{p}'(x')) \Big|_{x'_m=0} = \widehat{P}_{a-1,a-1}(\bar{q}(\bar{x}'))$ , so that it must be:

$$\frac{\partial p'_a(x')}{\partial x'_m} \Big|_{x'_m=0} = 0, \quad \forall a = 2, \dots, m, \quad x' \in \mathbb{R}^m. \quad (17)$$

As  $p'_1(x') = kx'_m, k \in \mathbb{R}, k \neq 0$ , we have,  $\forall a = 2, \dots, m, x' \in \mathbb{R}^m$ :

$$\widehat{P}_{1,a}(\bar{p}'(x')) \Big|_{x'_m=0} = (\nabla p'_1(x') \cdot \nabla p'_a(x')) \Big|_{x'_m=0} = \left( k \frac{\partial p'_a(x')}{\partial x'_m} \right) \Big|_{x'_m=0}, \quad (18)$$

and using (17) in (18) we obtain  $\widehat{P}_{1,a}(\bar{p}'(x')) \Big|_{x'_m=0} = 0$ , that is,  $\widehat{P}_{1,a}(p) \Big|_{p_1=0} = 0, \forall a = 2, \dots, m$ .

Let us now suppose  $\widehat{P}_{1,a}(p) \Big|_{p_1=0} = 0, \forall a = 2, \dots, m$ , that is,  $\widehat{P}_{1,a}(\bar{p}'(x')) \Big|_{x'_m=0} = 0, \forall a = 2, \dots, m, x' \in \mathbb{R}^m$ . From the equalities in (18), we obtain Eq. (17), but then,  $\forall a, b = 2, \dots, m, x' \in \mathbb{R}^m$ , one has:

$$\begin{aligned} \widehat{P}_{a,b}(\bar{p}'(x')) \Big|_{x'_m=0} &= (\nabla p'_a(x') \cdot \nabla p'_b(x')) \Big|_{x'_m=0} \\ &= \nabla p'_a(\bar{x}') \cdot \nabla p'_b(\bar{x}') + \left( \frac{\partial p'_a(x')}{\partial x'_m} \frac{\partial p'_b(x')}{\partial x'_m} \right) \Big|_{x'_m=0} = \nabla q_{a-1}(\bar{x}') \cdot \nabla q_{b-1}(\bar{x}') + 0 = \widehat{P}_{a-1,b-1}(\bar{q}(\bar{x}')), \end{aligned}$$

that implies  $[\widehat{P}(p')]^q = \widehat{P}(q)$ , that is,  $[\widehat{P}(p)]^q = \widehat{P}(q)$ .

2. Suppose now  $[\widehat{P}(p)]^q = \widehat{P}(q)$ , so that  $\widehat{P}_{1,a}(p)|_{p_1=0} = 0, \forall a = 2, \dots, m$ . By hypothesis,  $a(p)$  satisfies the boundary equation (9), with  $\lambda$ -vector  $\lambda^{(a)}(p)$ . By taking  $p_1 = 0$  in Eq. (9), we obtain at the first member,  $\forall b = 2, \dots, m$ :

$$\left( \sum_{d=1}^m \widehat{P}_{bd}(p) \frac{\partial a(p)}{\partial p_d} \right) \Big|_{p_1=0} = \left( \sum_{d=2}^m \widehat{P}_{bd}(p) \frac{\partial a(p)}{\partial p_d} \right) \Big|_{p_1=0} = \sum_{d=2}^m \widehat{P}_{bd}(p) \Big|_{p_1=0} \frac{\partial a(p)|_{p_1=0}}{\partial p_d},$$

and at the second member:

$$\left( \lambda_b^{(a)}(p) a(p) \right) \Big|_{p_1=0} = \lambda_b^{(a)}(p) \Big|_{p_1=0} a(p) \Big|_{p_1=0}.$$

These two expressions imply the equality:

$$\sum_{d=2}^m [\widehat{P}_{bd}(p)]^q \frac{\partial [a(p)]^q}{\partial p_d} = [\lambda_b^{(a)}(p)]^q [a(p)]^q, \quad \forall b = 2, \dots, m,$$

where the operator  $[\cdot]^q$  has been applied only to scalars, that implies:

$$\sum_{d=1}^{m-1} \widehat{P}_{bd}(q) \frac{\partial a(q)}{\partial q_d} = [\lambda_{b+1}^{(a)}(p)]^q a(q), \quad \forall b = 1, \dots, m-1.$$

By comparison with Eq. (9), this equation proves that  $a(q) = [a(p)]^q$  is an active polynomial relative to  $\widehat{P}(q)$ , with  $\lambda$ -vector  $\lambda^{(a)}(q) = [\lambda^{(a)}(p)]^q$ , such that:

$$\lambda_b^{(a)}(q) = [\lambda_{b+1}^{(a)}(p)]^q, \quad \forall b = 1, \dots, m-1.$$

We also know that  $\widehat{P}_{11}(p) = \widehat{P}_{11}(p') = k^2 > 0$ , because  $p'_1(x') = k x'_m, k \neq 0$ , and then,  $[\det(\widehat{P}(p))]^q = k^2 \det(\widehat{P}(q))$ , proving that  $[\det(\widehat{P}(p))]^q$  and  $\det(\widehat{P}(q))$  have the same  $\lambda$ -vector  $\lambda^{(\det(\widehat{P}))}(q)$ .  $\diamond$

### III. GENERATING FORMULAS FOR THE SYMMETRIC GROUPS $S_n$

$S_n$  acts on  $\mathbb{R}^n$  by permuting in all possible ways the  $n$  coordinates  $x_1, \dots, x_n$  of  $x \in \mathbb{R}^n$ . It can be generated by the reflections about the  $\binom{n}{2}$  hyperplanes of equation  $x_i - x_j = 0, 1 \leq i < j \leq n$ , to which correspond the positive roots  $e_i - e_j, 1 \leq i < j \leq n$ , in which  $e_1, \dots, e_n$  are the unit vectors of a canonical basis of  $\mathbb{R}^n$ .

For that choice of the positive roots, a basis of invariant polynomials of  $S_n$ , is formed by the *elementary symmetric polynomials* in  $n$  variables, defined as follows:

$$\sigma_a(x) = \sum_{1 \leq i_1 < i_2 < \dots < i_a \leq n} x_{i_1} x_{i_2} \dots x_{i_a}. \quad (19)$$

If  $x \in \mathbb{R}^n$ ,  $\sigma_a(x)$  has  $\binom{n}{a}$  terms: all the monomials that are products of  $a$  different coordinates of  $x$ . In Section VIII we shall also consider  $\sigma_a(y), y \in \mathbb{R}^{n+1}$ , and in that case  $\sigma_a(y)$  will have  $\binom{n+1}{a}$  terms. We shall also use the notation  $\sigma_a(x^2)$  for the elementary symmetric polynomials in the squared coordinates, for example,  $\sigma_2(x^2) = x_1^2 x_2^2 + x_1^2 x_3^2 + \dots$

**Theorem 3** 1. The  $\widehat{P}$ -matrix  $\widehat{P}(\sigma)$ , corresponding to the basis of invariant polynomials of  $S_n$  written in Eq. (19), has matrix elements that can be obtained from the following generating formula:

$$\widehat{P}_{ab}(\sigma) = (n+1 - \min(a, b)) \sigma_{a-1} \sigma_{b-1} - \sum_{i=\max(1, a+b-n-1)}^{\min(a, b)} (a+b-2i) \sigma_{i-1} \sigma_{a+b-1-i}, \quad \forall a, b = 1, \dots, n, \quad (20)$$

in which one has to consider  $\sigma_0 = 1$ .

2. The determinant of the  $\widehat{P}$ -matrix  $\widehat{P}(\sigma)$ , obtained from Eq. (20), satisfies the boundary equation (9) with its  $\lambda$ -vector with the following elements:

$$\lambda_a^{(\det(\widehat{P}))}(\sigma) = -(n-a+2)(n-a+1)\sigma_{a-2}, \quad \forall a = 1, \dots, n, \quad (21)$$

in which one has to consider  $\sigma_{-1} = 0$  and  $\sigma_0 = 1$ .

**Proof.** In Section VIII A. ◇

#### IV. GENERATING FORMULAS FOR THE GROUPS OF TYPE $A_n$

The action of the symmetric group  $S_{n+1}$  in  $\mathbb{R}^{n+1}$  is completely reducible: the hyperplane  $H$ , defined by the equation  $x_1 + x_2 + \dots + x_{n+1} = 0$ , is invariant under this action. The reflection group acting in the  $n$ -dimensional subspace  $H$  of  $\mathbb{R}^{n+1}$  that is generated by the same reflections as those of  $S_{n+1}$ , is an irreducible finite reflection group of type  $A_n$ . At the end of Section II, where we defined the  $R$ -induced groups and the  $R$ -induced bases of invariant polynomials, we described how to write the matrices of this  $n$ -dimensional representation of  $A_n$  in  $H$  as elements of  $O(n)$ , and its basic invariant polynomials as polynomials in  $n$  variables. We use here the method and notation introduced there, in particular, the definitions of  $x' = Rx$ , with  $R \in SO(n+1)$ , that transforms the unit vector  $u_H = (1, 1, \dots, 1)/\sqrt{n+1} \in \mathbb{R}^{n+1}$ , orthogonal to the invariant hyperplane  $H$ , into the canonical unit vector  $e_{n+1} \in \mathbb{R}^{n+1}$ , and of  $\bar{x}'$  for the projection of  $x' \in \mathbb{R}^{n+1}$  in  $\mathbb{R}^n$  obtained by dropping  $x_{n+1}$ . Let us first define a matrix  $R \in SO(n+1)$  that transforms  $u_H$  into  $e_{n+1}$ .

$$R : \begin{cases} R_{k,i} = 1/\sqrt{k(k+1)}, & 1 \leq k \leq n, & 1 \leq i \leq k, \\ R_{k,k+1} = -k/\sqrt{k(k+1)}, & 1 \leq k \leq n, \\ R_{k,i} = 0, & 1 \leq k \leq n, & k+1 < i \leq n+1, \\ R_{n+1,i} = 1/\sqrt{n+1}, & & 1 \leq i \leq n+1. \end{cases} \quad (22)$$

By construction, all row vectors of  $R$  are mutually orthogonal and normalized to 1, and this is sufficient to say that the matrix  $R$  is orthogonal. It is not difficult to verify that  $\det(R) = 1$ .

We can now define in 4 steps the basic invariant polynomials of  $A_n$  we shall use in Theorem 4.

1. Start from the  $n+1$  elementary symmetric polynomials  $\sigma_1(x), \dots, \sigma_{n+1}(x)$ , that form a basis of invariant polynomials for the action of the group  $S_{n+1}$  in  $\mathbb{R}^{n+1}$ . They are written in Eq. (19), with  $n$  replaced by  $n+1$ .

2. Rotate the system of coordinates with the matrix  $R$ , defined in Eq. (22), that is  $x' = Rx$ . For Proposition 1, item 2, the basic invariant polynomials of  $S_{n+1}$  in the rotated system of reference have the following expressions:

$$s_a(x') = \sigma_a(R^\top x'), \quad a = 1, \dots, n+1, \quad (23)$$

in which the polynomial  $s_1(x')$  is proportional to  $x'_{n+1}$ :  $s_1(x') = \sqrt{n+1}x'_{n+1}$ , and so  $H$  has equation  $x'_{n+1} = 0$ .

3. Take the restrictions of the basic polynomials  $s_a(x')$ ,  $\forall a = 1, \dots, n+1$ , to  $H$ . One finds  $s_1(x')|_{x'_{n+1}=0} = 0$ , and

$$q_a(\bar{x}') = s_{a+1}(x')|_{x'_{n+1}=0}, \quad a = 1, \dots, n, \quad (24)$$

that are expressed in terms of the  $n$  variables in  $\bar{x}'$ , only, form a basis of invariant polynomials for the action of  $A_n$  in  $H$ . This is an  $R$ -induced basis of invariant polynomials, according to the definition given in Section II.

4. Perform a scale transformation on the  $n$  basic invariant polynomials  $q_1(\bar{x}'), \dots, q_n(\bar{x}')$  in the following way:

$$p_a = \widehat{p}_a(q) = -2\sqrt{(n+1)^{a-1}}q_a, \quad a = 1, \dots, n, \quad (25)$$

where the arguments  $\bar{x}'$  are omitted. The scale transformation (25) is optional. It is necessary only if one wants the quadratic invariant polynomial with the standard form (5) and a  $\widehat{P}$ -matrix  $\widehat{P}(p)$  with only integer coefficients. The transformation in Eq. (25) is only one of the infinitely many possibilities to obtain that.

**Theorem 4** 1. The  $\widehat{P}$ -matrix  $\widehat{P}(p)$ , corresponding to the basis of invariant polynomials of  $A_n$  written in Eq. (25), has matrix elements that can be obtained from the following generating formula:

$$\widehat{P}_{ab}(p) = ((n+1) \min(a,b) - ab) p_{a-1} p_{b-1} + 2(a+b) p_{a+b-1}$$

$$-(n+1) \sum_{i=\max(2, a+b-n-1)}^{\min(a,b)-1} (a+b-2i) p_{i-1} p_{a+b-1-i}, \quad \forall a, b = 1, \dots, n, \quad (26)$$

in which one has to consider  $p_0 = 0$  and  $p_a = 0, \forall a > n$ .

2. The determinant of the  $\widehat{P}$ -matrix  $\widehat{P}(p)$ , obtained from Eq. (26), satisfies the boundary equation (9) with its  $\lambda$ -vector with the following elements:

$$\lambda_1^{(\det(\widehat{P}))}(p) = 2n(n+1), \quad \lambda_a^{(\det(\widehat{P}))}(p) = -(n+1)(n-a+2)(n-a+1)p_{a-2}, \quad \forall a = 2, \dots, n, \quad (27)$$

in which one has to consider  $p_0 = 0$ .

**Proof.** In Section VIII B. ◇

A generating formula similar to Eq. (26) was written in Remark 3.9 of [2], for a basis  $q'_a(x)$ ,  $a = 1, \dots, n$ ,  $x \in \mathbb{R}^n$ , almost equal to that one in Eq. (24), that is:  $q'_a = (-1)^{a-1} q_a$ ,  $a = 1, \dots, n$ . An equation that differs very little from Eq. (27), was written in Eq. (2.1.5.2) of [23].

In Example 1., in Section VII, it is shown how to determine the basic invariant polynomials of  $A_3$ , with Eqs. (23)–(25), and to use Eqs. (26) and (27) to write the corresponding  $\widehat{P}$ -matrix and  $\lambda$ -vector of its determinant.

## V. GENERATING FORMULAS FOR THE GROUPS OF TYPE $B_n$

Let  $e_1, \dots, e_n$  be the unit vectors of a canonical basis of  $\mathbb{R}^n$ . The roots of  $B_n$  form two root systems, each one containing all the roots of a same length. One may consider the  $n$  vectors  $e_i$ ,  $i = 1, \dots, n$ , as the positive short roots, and the  $2\binom{n}{2}$  vectors  $e_i \pm e_j$ ,  $1 \leq i < j \leq n$ , as the positive long roots. For this choice of positive roots, a possible choice of the basis of invariant polynomials of  $B_n$ , already proposed by Coxeter in [5], is obtained by using the  $n$  elementary symmetric polynomials in the squared variables, that is:

$$p_a(x) = \sigma_a(x^2), \quad \forall a = 1, \dots, n. \quad (28)$$

**Theorem 5** 1. The  $\widehat{P}$ -matrix  $\widehat{P}(p)$ , corresponding to the basis of invariant polynomials of  $B_n$  written in Eq. (28), has matrix elements that can be obtained from the following generating formula:

$$\widehat{P}_{ab}(p) = \sum_{i=\max(0, a+b-n-1)}^{\min(a,b)-1} 4(a+b-1-2i) p_i p_{a+b-1-i}, \quad \forall a, b = 1, \dots, n, \quad (29)$$

in which one has to consider  $p_0 = 1$ .

2. The determinant of the  $\widehat{P}$ -matrix  $\widehat{P}(p)$ , obtained from Eq. (29), has two irreducible active factors. One active factor is the polynomial  $\widehat{s}(p) \equiv p_n$ , such that  $\widehat{s}(\overline{p}(x)) = \prod_{i=1}^n x_i^2 = p_n(x)$ , and the other active factor is the polynomial  $\widehat{l}(p) = 4^{-n} \det(\widehat{P}(p))/p_n$ , such that  $\widehat{l}(\overline{p}(x)) = \prod_{i < j=1}^n (x_i^2 - x_j^2)^2$ .

The  $\lambda$ -vectors of the active polynomials  $\widehat{s}(p)$ ,  $\widehat{l}(p)$ , and  $\det(\widehat{P}(p))$ , have following elements:

$$\lambda_a^{(s)}(p) = 4(n-a+1)p_{a-1}, \quad \lambda_a^{(l)}(p) = 4(n-a+1)(n-a)p_{a-1}, \quad \forall a = 1, \dots, n, \quad (30)$$

$$\lambda_a^{(\det(\widehat{P}))}(p) = 4(n-a+1)^2 p_{a-1}, \quad \forall a = 1, \dots, n, \quad (31)$$

respectively, where one has to consider  $p_0 = 1$ .

**Proof.** In Section VIII C. ◇

Eq. (29) was first written in Eq. (\*\*) of [7]. An equation that differs very little from Eq. (31) was first written in Eq. (2.2.5.2) of [23].

## VI. GENERATING FORMULAS FOR THE GROUPS OF TYPE $D_n$

Let  $e_1, \dots, e_n$  be the unit vectors of a canonical basis of  $\mathbb{R}^n$ . The  $2\binom{n}{2}$  positive roots can be taken, for example, to be  $e_i \pm e_j$ ,  $1 \leq i < j \leq n$ . For this choice of positive roots, a possible choice of the basis of invariant polynomials of  $D_n$ , already proposed by Coxeter in [5], is obtained by using the the following polynomials:

$$p_a(x) = \sigma_a(x^2), \quad \forall a = 1, \dots, n-1, \quad p_n(x) = \sigma_n(x). \quad (32)$$

The labeling of the basic invariant polynomials given in Eq. (32) is not the standard one of Eq. (4), but it turns out to be more convenient here.

**Theorem 6** 1. The  $\widehat{P}$ -matrix  $\widehat{P}(p)$ , corresponding to the basis of invariant polynomials of  $D_n$ , written in Eq. (32), has matrix elements that can be obtained from the following generating formula:

$$\left\{ \begin{array}{l} \widehat{P}_{ab}(p) = \sum_{i=\max(0, a+b-n-1)}^{\min(a,b)-1} 4(a+b-1-2i) p_i p_{a+b-1-i} \Big|_{p_n \rightarrow p_n^2} \quad \forall a, b = 1, \dots, n-1, \\ \widehat{P}_{an}(p) = \widehat{P}_{na}(p) = 2(n-a+1) p_{a-1} p_n \quad \forall a = 1, \dots, n-1, \\ \widehat{P}_{nn}(p) = p_{n-1}, \end{array} \right. \quad (33)$$

in which one has to consider  $p_0 = 1$ .

2. The determinant of the  $\widehat{P}$ -matrix  $\widehat{P}(p)$ , obtained from Eq. (33), satisfies the boundary equation (9) with its  $\lambda$ -vector with the following elements:

$$\lambda_a^{(\det(\widehat{P}))}(p) = 4(n-a+1)(n-a) p_{a-1}, \quad \forall a = 1, \dots, n, \quad (34)$$

in which one has to consider  $p_0 = 1$ .

**Proof.** In Section VIII D. ◇

Eq. (33) was first written in [7], at the end of p. 88. An equation that differs very little from Eq. (34) was first written in Eq. (2.3.5.2) of [23].

## VII. EXAMPLES

### Example 1.

This example shows how to write the explicit forms of the basic invariant polynomials of  $A_3$ , by means of Eqs. (23)–(25), and to use Eqs. (26) and (27) for  $A_3$ . When  $n = 3$ , the matrix  $R$ , defined in Eq. (22), is the following:

$$R = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} & 0 \\ 1/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} & -3/\sqrt{12} \\ 1/2 & 1/2 & 1/2 & 1/2 \end{pmatrix}.$$

One may easily verify that  $RR^\top = R^\top R = I_4$ , with  $I_4$  the unit matrix of order 4, that  $R$  transforms the column unit vector  $u_H = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$  into the column unit vector  $e_4 = (0, 0, 0, 1)$ , and that  $R^\top$  makes the opposite transformation. To obtain the explicit expressions of the basic invariant polynomials  $s_a(x')$ ,  $a = 1, \dots, 4$ , in Eq. (23), one has to substitute  $x$  with  $R_4^\top x'$  in the elementary symmetric polynomials  $\sigma_a(x)$ ,  $a = 1, \dots, 4$ , written in Eq. (19), for  $n = 4$ . One so finds the following basic invariant polynomials of  $S_4$ , in the coordinates  $x'$ :

$$\begin{aligned} s_1(x') &= \left( \frac{1}{\sqrt{2}}x'_1 + \frac{1}{\sqrt{6}}x'_2 + \frac{1}{\sqrt{12}}x'_3 + \frac{1}{2}x'_4 \right) + \left( \frac{-1}{\sqrt{2}}x'_1 + \frac{1}{\sqrt{6}}x'_2 + \frac{1}{\sqrt{12}}x'_3 + \frac{1}{2}x'_4 \right) + \left( \frac{-2}{\sqrt{6}}x'_2 + \frac{1}{\sqrt{12}}x'_3 + \frac{1}{2}x'_4 \right) + \left( \frac{-3}{\sqrt{12}}x'_3 + \frac{1}{2}x'_4 \right) \\ &= 2x'_4, \\ s_2(x') &= \left( \frac{1}{\sqrt{2}}x'_1 + \frac{1}{\sqrt{6}}x'_2 + \frac{1}{\sqrt{12}}x'_3 + \frac{1}{2}x'_4 \right) \left( \frac{-1}{\sqrt{2}}x'_1 + \frac{1}{\sqrt{6}}x'_2 + \frac{1}{\sqrt{12}}x'_3 + \frac{1}{2}x'_4 \right) + \dots + \left( \frac{-2}{\sqrt{6}}x'_2 + \frac{1}{\sqrt{12}}x'_3 + \frac{1}{2}x'_4 \right) \left( \frac{-3}{\sqrt{12}}x'_3 + \frac{1}{2}x'_4 \right) \\ &= -\frac{1}{2} (x_1'^2 + x_2'^2 + x_3'^2) + \frac{3}{2} x_4'^2, \\ s_3(x') &= \dots = \frac{1}{18} (3\sqrt{6}x_1'^2x_2' - \sqrt{6}x_2'^3 + 3\sqrt{3}x_1'^2x_3' + 3\sqrt{3}x_2'^2x_3' - 2\sqrt{3}x_3'^3 - 9x_1'^2x_4' - 9x_2'^2x_4' - 9x_3'^2x_4' + 9x_4'^3), \\ s_4(x') &= \dots = \frac{1}{144} (-36\sqrt{2}x_1'^2x_2'x_3' + 12\sqrt{2}x_2'^3x_3' + 18x_1'^2x_3'^2 + 18x_2'^2x_3'^2 - 3x_3'^4 + 12\sqrt{6}x_1'^2x_2'x_4' - 4\sqrt{6}x_2'^3x_4' + 12\sqrt{3}x_1'^2x_3'x_4' \\ &\quad + 12\sqrt{3}x_2'^2x_3'x_4' - 8\sqrt{3}x_3'^3x_4' - 18x_1'^2x_4'^2 - 18x_2'^2x_4'^2 - 18x_3'^2x_4'^2 + 9x_4'^4). \end{aligned}$$

The basic invariant polynomials of  $A_3$  are then obtained from the basic invariant polynomials  $s_a(x')$ ,  $a = 1, \dots, 4$ , in the way specified in Eqs. (24) and (25). These calculations give:

$$\begin{aligned} p_1(x) &= x_1^2 + x_2^2 + x_3^2, \\ p_2(x) &= -\frac{2\sqrt{3}}{9} \left( 3\sqrt{2}x_1^2x_2 - \sqrt{2}x_2^3 + 3x_1^2x_3 + 3x_2^2x_3 - 2x_3^3 \right), \\ p_3(x) &= \frac{1}{6} x_3 \left( 12\sqrt{2}x_1^2x_2 - 4\sqrt{2}x_2^3 - 6x_1^2x_3 - 6x_2^2x_3 + x_3^3 \right), \end{aligned} \quad (35)$$

where the primes in  $x'_i$ ,  $i = 1, 2, 3$ , have been dropped.

The corresponding  $\widehat{P}$ -matrix and  $\lambda$ -vector obtained from the generating formulas (26) and (27) and are the following:

$$\widehat{P}(p) = \begin{pmatrix} 4p_1 & 6p_2 & 8p_3 \\ 6p_2 & 4p_1^2 + 8p_3 & 2p_1p_2 \\ 8p_3 & 2p_1p_2 & 3p_2^2 - 8p_1p_3 \end{pmatrix}, \quad \lambda^{(\det(\widehat{P}))}(p) = (24, 0, -8p_1). \quad (36)$$

One may explicitly verify the results in Eqs. (35) and (36) using Eqs. (1), (2), and (9). One may further rescale  $p_2(x)$  and  $p_3(x)$  in order to eliminate the denominators in (35), by maintaining the  $\widehat{P}$ -matrix with integer coefficients.

### Example 2.

This example shows how to use Eqs. (15) and (16) for basis transformations. Let us consider the case of  $A_3$ , so we can use the basic invariant polynomials, the  $\widehat{P}$ -matrix and the  $\lambda$ -vector obtained in Example 1.

Avoiding scale transformations like  $p'_a = c_a p_a$ ,  $c_a \in \mathbb{R}$ ,  $c_a \neq 1$ ,  $a = 1, 2, 3$ , the most general basis transformation  $p' = \widehat{p}'(p)$  is the following:

$$p'_1 = \widehat{p}'_1(p) = p_1, \quad p'_2 = \widehat{p}'_2(p) = p_2, \quad p'_3 = \widehat{p}'_3(p) = p_3 + c p_1^2, \quad (37)$$

with  $c$  a real constant. This transformation implies the following jacobian matrix (14):

$$J(p) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2c p_1 & 0 & 1 \end{pmatrix},$$

that gives, using Eqs. (15), (16), and (36), the following expressions:

$$\widehat{P}(p') = \begin{pmatrix} 4p_1 & 6p_2 & 8(c p_1^2 + p_3) \\ 6p_2 & 4(p_1^2 + 2p_3) & 2(1 + 6c)p_1p_2 \\ 8(c p_1^2 + p_3) & 2(1 + 6c)p_1p_2 & 16c^2 p_1^3 + 3p_2^2 - 8(1 - 4c)p_1p_3 \end{pmatrix} \Big|_{p=\widehat{p}'(p')},$$

$$\lambda^{(\det(\widehat{P}))}(p') = (24, 0, 8(6c - 1)p_1) \Big|_{p=\widehat{p}'(p')}.$$

By substituting the variables  $p$  with those obtained from the inverse transformation of Eq. (37):

$$p_1 = \widehat{p}_1(p') = p'_1, \quad p_2 = \widehat{p}_2(p') = p_2, \quad p_3 = \widehat{p}_3(p') = p'_3 - c p_1'^2,$$

one obtains:

$$\widehat{P}(p') = \begin{pmatrix} 4p'_1 & 6p'_2 & 8p'_3 \\ 6p'_2 & 4(1 - 2c)p_1'^2 + 8p'_3 & 2(1 + 6c)p'_1p'_2 \\ 8p'_3 & 2(1 + 6c)p'_1p'_2 & 8c(1 - 2c)p_1'^3 + 3p_2'^2 - 8(1 - 4c)p'_1p'_3 \end{pmatrix},$$

$$\lambda^{(\det(\widehat{P}))}(p') = (24, 0, 8(6c - 1)p'_1).$$

The constant  $c$  is here not specified. For example, when  $c = \frac{1}{4}$  one obtains a flat basis, in which  $p'_3$  appears only along the antidiagonal of  $\widehat{P}(p')$  [15], when  $c = \frac{1}{6}$  one obtains a  $\det(\widehat{P})$ -basis, in which  $\lambda_1^{(\det(\widehat{P}))}(p')$  is the only non-vanishing element of  $\lambda^{(\det(\widehat{P}))}(p')$  [17], and when  $c = \frac{1}{10}$  one obtains a canonical basis [6, 10].

## VIII. PROOFS OF THE GENERATING FORMULAS

### A. Proof of the generating formulas for the groups of type $S_n$

#### Proof of the generating formula (20).

The  $\widehat{P}$ -matrix is symmetric, so it is sufficient to prove Eq. (20) for  $a \leq b$ . It is convenient to define  $\sigma_a = 0, \forall a < 0$  and  $\forall a > n$ , so Eq. (20), for  $a \leq b$ , can be simplified in the following way:

$$\widehat{P}_{ab}(\sigma) = (n+1-a) \sigma_{a-1} \sigma_{b-1} - \sum_{i=a+b-n-1}^a (a+b-2i) \sigma_{i-1} \sigma_{a+b-1-i}, \quad \forall a \leq b = 1, \dots, n, \quad (38)$$

in which one has to consider  $\sigma_0 = 1$  and  $\sigma_a = 0, \forall a < 0$  and  $\forall a > n$ .

We shall prove Eq. (38) by induction on the rank  $n$ . The lowest possible value for  $n$  is  $n = 1$ , and in that case Eq. (38) is trivially verified: when  $n = 1$ , Eq. (19) gives only one basic invariant polynomial:  $\sigma_1(x) = x_1$ , with which Eq. (1) gives  $P_{11}(x) = 1$ , and Eq. (38) gives  $\widehat{P}_{11}(\sigma) = 1$ . Suppose now, for a general  $n \geq 2$ , that Eq. (38) is true.

We use the symbols  $x$  and  $y$  for vectors of  $\mathbb{R}^n$  and  $\mathbb{R}^{n+1}$ , that is:  $x = (x_1, x_2, \dots, x_n)$ , and  $y = (x_1, x_2, \dots, x_n, x_{n+1})$ . The basic invariant polynomials of the action of  $S_n$  in  $\mathbb{R}^n$  are  $\sigma_a(x), a = 1, \dots, n$ , written in Eq. (19), and the basic invariant polynomials of the action of  $S_{n+1}$  in  $\mathbb{R}^{n+1}$  are also specified by Eq. (19), but with  $n$  substituted by  $n+1$ , and we write them as  $\sigma_a(y), a = 1, \dots, n+1$ .

The basic invariant polynomials of  $S_{n+1}$  can be expressed in terms of those of  $S_n$  in the following way:

$$\sigma_a(y) = \sigma_a(x) + \sigma_{a-1}(x) x_{n+1} \quad \forall a = 1, 2, \dots, n+1, \quad (39)$$

where the definitions  $\sigma_0(x) = 1$  and  $\sigma_{n+1}(x) = 0$  must be taken into account.

Let us define the following ‘‘nabla’’ operators that map a scalar function into a vector of  $\mathbb{R}^{n+1}$ :

$$\nabla_x = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}, 0 \right), \quad \nabla_y = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_{n+1}} \right). \quad (40)$$

One then has, for Eqs. (39) and (40),  $\forall a = 1, \dots, n+1$ :

$$\nabla_y \sigma_a(y) = \nabla_y (\sigma_a(x) + \sigma_{a-1}(x) x_{n+1}) = \nabla_x \sigma_a(x) + x_{n+1} \nabla_x \sigma_{a-1}(x) + \sigma_{a-1}(x) e_{n+1}, \quad (41)$$

where  $e_{n+1}$  is the  $(n+1)$ -th canonical unit vector of  $\mathbb{R}^{n+1}$ .

We can now calculate, from Eqs. (1) and (41), a general element of the  $\widehat{P}$ -matrix  $P(y)$ , corresponding to the basic invariant polynomials  $\sigma_a(y), a = 1, \dots, n+1$ , of  $S_{n+1}$ . For all  $a \leq b = 1, \dots, n+1$ , we have:

$$\begin{aligned} P_{ab}(y) &= \nabla_y \sigma_a(y) \cdot \nabla_y \sigma_b(y) \\ &= (\nabla_x \sigma_a(x) + x_{n+1} \nabla_x \sigma_{a-1}(x) + \sigma_{a-1}(x) e_{n+1}) \cdot (\nabla_x \sigma_b(x) + x_{n+1} \nabla_x \sigma_{b-1}(x) + \sigma_{b-1}(x) e_{n+1}) \\ &= P_{ab}(x) + x_{n+1} (P_{a,b-1}(x) + P_{a-1,b}(x)) + x_{n+1}^2 P_{a-1,b-1}(x) + \sigma_{a-1}(x) \sigma_{b-1}(x), \end{aligned} \quad (42)$$

where  $P_{ab}(x) = \nabla_x \sigma_a(x) \cdot \nabla_x \sigma_b(x)$ , and analogously with the other indices appearing in (42). Of course, in the preceding expression,  $P_{cd}(x) = 0$ , if one of the indices  $c, d$  (that in the above expression can be equal to  $a, b, a-1, b-1$ ) is equal to 0 or to  $n+1$ , because the gradients of  $\sigma_0(x)$  and  $\sigma_{n+1}(x)$  are null vectors. By the induction hypothesis, Eq. (38) is true for  $S_n$ , so we can use it in the preceding equation. Noting that Eq. (38) implies  $P_{ab}(\sigma) = 0$  when  $a = 0$  or  $b = n+1$ , we can ignore the above constraints. We can use Eq. (38) only for  $a \leq b$ , so it is not possible to use it for  $P_{a,b-1}(x)$ , unless we distinguish the cases  $a = b$  and  $a < b$ . Using Eq. (38) in the case  $a = b, \forall a = 1, \dots, n+1$ , the term in Eq. (42) inside the brackets that factorizes  $x_{n+1}$  becomes:

$$P_{a,a-1}(x) + P_{a-1,a}(x) = 2P_{a-1,a}(x) = 2 \left( (n+2-a) \sigma_{a-1}(x) \sigma_{a-2}(x) - \sum_{i=2a-n-2}^{a-1} (2a-1-2i) \sigma_{i-1}(x) \sigma_{2a-2-i}(x) \right). \quad (43)$$

Using Eq. (38) in the case  $1 \leq a < b \leq n+1$ , the term in Eq. (42) inside the brackets that factorizes  $x_{n+1}$  becomes:

$$P_{a,b-1}(x) + P_{a-1,b}(x) = (n+1-a) \sigma_{a-1}(x) \sigma_{b-2}(x) - \sum_{i=a+b-n-2}^a (a+b-1-2i) \sigma_{i-1}(x) \sigma_{a+b-2-i}(x)$$

$$\begin{aligned}
& + (n+2-a) \sigma_{a-2}(x) \sigma_{b-1}(x) - \sum_{i=a+b-n-2}^{a-1} (a+b-1-2i) \sigma_{i-1}(x) \sigma_{a+b-2-i}(x) \\
& = (n+1-a) \sigma_{a-1}(x) \sigma_{b-2}(x) - (b-a-1) \sigma_{a-1}(x) \sigma_{b-2}(x) \\
& + (n+2-a) \sigma_{a-2}(x) \sigma_{b-1}(x) - \sum_{i=a+b-n-2}^{a-1} 2(a+b-1-2i) \sigma_{i-1}(x) \sigma_{a+b-2-i}(x) \\
& = (n+2-b) \sigma_{a-1}(x) \sigma_{b-2}(x) + (n+2-a) \sigma_{a-2}(x) \sigma_{b-1}(x) - \sum_{i=a+b-n-2}^{a-1} 2(a+b-1-2i) \sigma_{i-1}(x) \sigma_{a+b-2-i}(x).
\end{aligned}$$

We can see that if we take  $b = a$  in the last member of the preceding equation, we obtain the last member of Eq. (43), so we have no longer the necessity to distinguish the cases  $a = b$  and  $a < b$ , and we can consider the preceding expression valid  $\forall a \leq b = 1, \dots, n+1$ . By substituting this expression and using Eq. (38) in Eq. (42), we obtain then,  $\forall a \leq b = 1, \dots, n+1$ :

$$\begin{aligned}
P_{ab}(y) & = (n+2-a) \sigma_{a-1}(x) \sigma_{b-1}(x) - \sum_{i=a+b-n-1}^a (a+b-2i) \sigma_{i-1}(x) \sigma_{a+b-1-i}(x) \\
& + x_{n+1} \left( (n+2-b) \sigma_{a-1}(x) \sigma_{b-2}(x) + (n+2-a) \sigma_{a-2}(x) \sigma_{b-1}(x) - \sum_{i=a+b-n-2}^{a-1} 2(a+b-1-2i) \sigma_{i-1}(x) \sigma_{a+b-2-i}(x) \right) \\
& + x_{n+1}^2 \left( (n+2-a) \sigma_{a-2}(x) \sigma_{b-2}(x) - \sum_{i'=a+b-n-2}^a (a+b-2i') \sigma_{i'-2}(x) \sigma_{a+b-2-i'}(x) \right), \quad (44)
\end{aligned}$$

where in the last sum the index  $i$  has been changed to  $i' = i + 1$ .

Using Eq. (38) for  $S_{n+1}$ , that is with  $n$  substituted by  $n+1$ , and Eq. (39), one would instead find,  $\forall a \leq b = 1, \dots, n+1$ :

$$\begin{aligned}
P_{ab}(y) & = ((n+1)+1-a) \sigma_{a-1}(y) \sigma_{b-1}(y) - \sum_{i=a+b-(n+1)-1}^a (a+b-2i) \sigma_{i-1}(y) \sigma_{a+b-1-i}(y) \\
& = (n+2-a) (\sigma_{a-1}(x) + \sigma_{a-2}(x) x_{n+1}) (\sigma_{b-1}(x) + \sigma_{b-2}(x) x_{n+1}) \\
& - \sum_{i=a+b-n-2}^a (a+b-2i) (\sigma_{i-1}(x) + \sigma_{i-2}(x) x_{n+1}) (\sigma_{a+b-1-i}(x) + \sigma_{a+b-2-i}(x) x_{n+1}) \\
& = (n+2-a) \sigma_{a-1}(x) \sigma_{b-1}(x) - \sum_{i=a+b-n-1}^a (a+b-2i) \sigma_{i-1}(x) \sigma_{a+b-1-i}(x) \\
& + x_{n+1} \left( (n+2-a) (\sigma_{a-1}(x) \sigma_{b-2}(x) + \sigma_{a-2}(x) \sigma_{b-1}(x)) \right. \\
& \left. - \sum_{i=a+b-n-2}^a (a+b-2i) \sigma_{i-1}(x) \sigma_{a+b-2-i}(x) - \sum_{i=a+b-n-1}^a (a+b-2i) \sigma_{i-2}(x) \sigma_{a+b-1-i}(x) \right)
\end{aligned}$$



$$+ x_{n+1}^2 \left( (n+2-a) \sigma_{a-2}(x) \sigma_{b-2}(x) - \sum_{i=a+b-n-2}^a (a+b-2i) \sigma_{i-2}(x) \sigma_{a+b-2-i}(x) \right). \quad (45)$$

In the 1st and the 3rd of the four sums in the last member of Eq. (45), the lower limits  $i = a + b - n - 2$  have been increased by one unit, because the factor  $\sigma_{a+b-1-i}(x)$  inside the sum, would be zero if  $i = a + b - n - 2$ . The terms in this expression that do not factorize  $x_{n+1}$  and that one that factorizes  $x_{n+1}^2$  are the same as those in Eq. (44). The terms in Eq. (45) inside the brackets that factorizes  $x_{n+1}$  must be further transformed to see that they also coincide with those in Eq. (44). This can be done in the following way:

$$\begin{aligned} (\dots) &= \left( (n+2-a) (\sigma_{a-1}(x) \sigma_{b-2}(x) + \sigma_{a-2}(x) \sigma_{b-1}(x)) - \sum_{i=a+b-n-2}^{a-1} (a+b-2i) \sigma_{i-1}(x) \sigma_{a+b-2-i}(x) \right. \\ &\quad \left. - (b-a) \sigma_{a-1}(x) \sigma_{b-2}(x) - \sum_{i'=a+b-n-2}^{a-1} (a+b-2-2i') \sigma_{i'-1}(x) \sigma_{a+b-2-i'}(x) \right) \\ &= \left( (n+2-b) \sigma_{a-1}(x) \sigma_{b-2}(x) + (n+2-a) \sigma_{a-2}(x) \sigma_{b-1}(x) - \sum_{i=a+b-n-2}^{a-1} 2(a+b-1-2i) \sigma_{i-1}(x) \sigma_{a+b-2-i}(x) \right). \end{aligned}$$

This expression coincides with the expression inside the brackets multiplying  $x_{n+1}$  in Eq. (44). We conclude that Eq. (38) is true for  $a \leq b$  for all values of  $n$ . This also proves that the generating function in Eq. (20) is true for  $a \leq b$  for all values of  $n$ , because of the equivalence for  $a \leq b$  of Eqs. (20) and (38), given the definitions  $\sigma_a = 0, \forall a < 0$  and  $\forall a > n$ . For the symmetry of the  $\hat{P}$ -matrix  $\hat{P}(\sigma)$  we conclude that Eq. (20) is true  $\forall a, b = 1, \dots, n$ .  $\diamond$

### Proof of the generating formula (21).

We use Theorem 1, item 1.. The positive roots of  $S_n$  we used to define the basic invariant polynomials (19), and the  $\hat{P}$ -matrix  $\hat{P}(\sigma)$  in Eq. (20), are reported at the beginning of Section III. Eq. (10) gives then,  $\forall a = 1, \dots, n$ :

$$\lambda_a^{(\det(\hat{P}))}(\sigma(x)) = 2 \sum_{r \in \mathcal{R}_+} \frac{\nabla \sigma_a(x) \cdot r}{l_r(x)} = 2 \sum_{i < j=1}^n \frac{\nabla \sigma_a(x) \cdot (e_i - e_j)}{x_i - x_j} = 2 \sum_{i < j=1}^n \frac{1}{x_i - x_j} \left( \frac{\partial \sigma_a(x)}{\partial x_i} - \frac{\partial \sigma_a(x)}{\partial x_j} \right).$$

Using Eq. (19), we see that, for  $a = 1$  it is:  $\frac{\partial \sigma_1(x)}{\partial x_i} = 1, \forall i = 1, \dots, n$ , so that  $\lambda_1^{(\det(\hat{P}))}(\sigma(x)) = 0$ , while for  $a > 1$  it is:  $\frac{\partial \sigma_a(x)}{\partial x_i} = \sigma_{a-1}(x)|_{x_i=0}, \forall i = 1, \dots, n$ , that gives,  $\forall a = 2, \dots, n$ :

$$\begin{aligned} \lambda_a^{(\det(\hat{P}))}(\sigma(x)) &= 2 \sum_{i < j=1}^n \frac{1}{x_i - x_j} \left( \sigma_{a-1}(x)|_{x_i=0} - \sigma_{a-1}(x)|_{x_j=0} \right) \\ &= 2 \sum_{i < j=1}^n \frac{1}{x_i - x_j} \left( \sigma_{a-2}(x)|_{x_i=x_j=0} x_j - \sigma_{a-2}(x)|_{x_i=x_j=0} x_i \right) = -2 \sum_{i < j=1}^n \sigma_{a-2}(x)|_{x_i=x_j=0}. \end{aligned}$$

The sum in the last member is a  $w$ -homogeneous polynomial of weight  $a - 2$ , symmetric in the variables  $x_1, \dots, x_n$ , and is formed by terms contained in  $\sigma_{a-2}(x)$ , only, so it must be a real multiple of  $\sigma_{a-2}(x)$ . To find the multiplying factor it is sufficient to count the number of terms,  $c_1$ , in the sum in the last member of the preceding equation, and the number of terms,  $c_2$ , in  $\sigma_{a-2}(x)$ . The multiplying factor is then equal to the ratio  $\frac{c_1}{c_2}$ . One easily finds that  $c_1 = \binom{n}{2} \binom{n-2}{a-2}$  and  $c_2 = \binom{n}{a-2}$ , so that  $\frac{c_1}{c_2} = \frac{1}{2} (n - a + 2)(n - a + 1)$ . One then finds:

$$\lambda_a^{(\det(\hat{P}))}(\sigma(x)) = -2 \frac{1}{2} (n - a + 2)(n - a + 1) \sigma_{a-2}(x) = -(n - a + 2)(n - a + 1) \sigma_{a-2}(x), \quad \forall a = 2, \dots, n.$$

We see that we can allow  $a = 1$  in the last equation, if we define  $\sigma_{-1}(x) = 0$ . This completes the proof of Eq. (21).  $\diamond$

## B. Proof of the generating formulas for the groups of type $A_n$

### Proof of the generating formula (26).

We adopt here the notation already used in Section IV, and in particular, the definitions of  $R$  in Eq. (22), of  $x' = Rx$ , of  $\bar{x}'$ , for the projection of  $x'$  in  $\mathbb{R}^n$  obtained by dropping  $x'_{n+1}$ , and of the basic invariant polynomials  $s_a(x')$ ,  $a = 1, \dots, n+1$ ,  $q_a(\bar{x}')$ , and  $p_a(\bar{x}')$ ,  $a = 1, \dots, n$ , as defined in Eqs. (23)–(25). The  $\widehat{P}$ -matrix  $\widehat{P}(s)$ , corresponding to the basic invariant polynomials  $s_a(x') = \sigma_a(R^\top x')$ ,  $a = 1, \dots, n+1$ , by Proposition 1, item 4., has the same form as  $\widehat{P}(\sigma)$ , only the variables  $\sigma_a$  have to be substituted by the variables  $s_a$ ,  $\forall a = 1, \dots, n+1$ . It is here convenient to write explicitly the generating formula for  $\widehat{P}(s)$ , obtained from Eq. (20), by replacing  $n$  with  $n+1$ , and  $\sigma$  with  $s$ :

$$\widehat{P}_{ab}(s) = (n+2 - \min(a, b)) s_{a-1} s_{b-1} - \sum_{i=\max(1, a+b-n-2)}^{\min(a, b)} (a+b-2i) s_{i-1} s_{a+b-1-i}, \quad \forall a, b = 1, \dots, n+1, \quad (46)$$

where one has to consider, by definition,  $s_0 = 1$ .

According to Eq. (24), the basis  $q_a(\bar{x}')$ ,  $a = 1, \dots, n$ , of  $R$ -induced  $A_n$ -invariant polynomials, is obtained by putting  $x'_{n+1} = 0$  in the basic invariant polynomials  $s_a(x')$ ,  $a = 1, \dots, n+1$ . The  $\widehat{P}$ -matrix  $\widehat{P}(q)$ , might be easily obtained from  $\widehat{P}(s)$  if we could use Theorem 2, but it is not possible to use at once Theorem 2, because Eq. (46) gives:

$$\widehat{P}_{1a}(s) = (n+2-a) s_{a-1}, \quad \forall a = 1, \dots, n+1, \quad (47)$$

in which  $s_0 = 1$ , and for all  $a > 2$  these elements are different from 0, even when  $s_1 = 0$ . To apply Theorem 2, we have first to change the basis of invariant polynomials. To this end, we prove the following Lemma.

**Lemma 1.** The basis of invariant polynomials  $t_a(x')$ ,  $a = 1, \dots, n+1$ , of  $S_{n+1}$ , defined as follows:

$$\begin{aligned} t_1 &= \widehat{t}_1(s) = s_1, & t_2 &= \widehat{t}_2(s) = s_2, \\ t_a &= \widehat{t}_a(s) = s_a + c_a s_1 s_{a-1}, & c_a &= -\frac{n+2-a}{n+1}, \quad \forall a = 3, \dots, n+1, \end{aligned} \quad (48)$$

in which the arguments  $x'$  are omitted, define a  $\widehat{P}$ -matrix  $\widehat{P}(t)$ , in which  $\widehat{P}_{1a}(t)|_{t_1=0} = 0$ ,  $\forall a = 2, \dots, n+1$ .

**Proof.** As  $t_1(x') = s_1(x') = \sqrt{n+1} x'_{n+1}$ , one has  $t_1(x')|_{x'_{n+1}=0} = 0$ . The definitions of  $t_1$  and  $t_2$ , and Eq. (46), imply  $\widehat{P}_{11}(t) = n+1$ , and  $\widehat{P}_{12}(t)|_{t_1=0} = 0$ . Using Eqs. (1), (48), and (47), one finds,  $\forall a = 3, \dots, n+1$ :

$$\begin{aligned} \widehat{P}_{1a}(t(x'))|_{x'_{n+1}=0} &= (\nabla t_1(x') \cdot \nabla t_a(x'))|_{x'_{n+1}=0} = \left( \nabla s_1(x') \cdot \nabla (s_a(x') + c_a s_1(x') s_{a-1}(x')) \right)|_{x'_{n+1}=0} \\ &= \left( \widehat{P}_{1a}(s(x')) + c_a \widehat{P}_{11}(s(x')) s_{a-1}(x') + c_a s_1(x') \widehat{P}_{1, a-1}(s(x')) \right)|_{x'_{n+1}=0} \\ &= (n+2-a) s_{a-1}(\bar{x}') + c_a (n+1) s_{a-1}(\bar{x}') + 0 = \left( (n+2-a) - \frac{n+2-a}{n+1} (n+1) \right) s_{a-1}(\bar{x}') = 0, \end{aligned}$$

and hence,  $\widehat{P}_{1a}(t)|_{t_1=0} = 0$ ,  $\forall a = 2, \dots, n+1$ . ◇

From Eqs. (24) and (48) it also follows:

$$q_a(\bar{x}') = t_{a+1}(x')|_{x'_{n+1}=0}, \quad a = 1, \dots, n, \quad (49)$$

and because  $\widehat{P}_{1a}(t)|_{t_1=0} = 0$ ,  $\forall a = 2, \dots, n+1$ , Theorem 2 assures that  $[\widehat{P}(t)]^q = \widehat{P}(q)$ . As we need  $\widehat{P}(t)$  only to calculate the matrix  $[\widehat{P}(t)]^q$ , it is easier if we calculate at once the matrix  $\widehat{P}(t)|_{t_1=0}$ , so we already know its first row

and column (that by Lemma 1, are vanishing, except the element  $\widehat{P}_{11}(t) = n + 1$ ), and simplify also the rest of the calculation. We shall use Eq. (15), that implies:

$$\widehat{P}_{ab}(t)\Big|_{t_1=0} = \left( J_{ac}(s) \widehat{P}_{cd}(s) J_{db}^\top(s) \right)\Big|_{s_1=0, s=\widehat{s}(t)}, \quad \forall a, b = 2, \dots, n+1. \quad (50)$$

The matrix elements  $J_{ab}(s) = \frac{\partial \widehat{t}_a(s)}{\partial s_b}$ ,  $a, b = 1, \dots, n+1$ , of the jacobian matrix of the transformation (48), are the following:

$$J_{1b}(s) = \delta_{1b}, \quad J_{2b}(s) = \delta_{2b}, \quad \forall b = 1, \dots, n+1,$$

$$J_{ab}(s) = \delta_{ab} + c_a \delta_{1b} s_{a-1} + c_a s_1 \delta_{a-1, b}, \quad \forall a = 3, \dots, n+1, \quad \forall b = 1, \dots, n+1,$$

and when  $s_1 = 0$ , corresponding to  $t_1 = 0$ , the last two equalities simplify to:

$$J_{ab}(s)\Big|_{s_1=0} = \delta_{ab} + c_a \delta_{1b} s_{a-1}\Big|_{s_1=0}, \quad \forall a = 2, \dots, n+1, \quad \forall b = 1, \dots, n+1. \quad (51)$$

Using Eqs. (51), (46), (47), and (48), in Eq. (50), we then find,  $\forall a, b = 2, \dots, n+1$ :

$$\begin{aligned} \widehat{P}_{ab}(t)\Big|_{t_1=0} &= \left( (\delta_{ac} + c_a \delta_{1c} s_{a-1}) \widehat{P}_{cd}(s) (\delta_{bd} + c_b \delta_{1d} s_{b-1}) \right)\Big|_{s_1=0, s=\widehat{s}(t)} \\ &= \left( \widehat{P}_{ab}(s) + c_a \widehat{P}_{1b}(s) s_{a-1} + \widehat{P}_{a1}(s) c_b s_{b-1} + c_a c_b \widehat{P}_{11}(s) s_{a-1} s_{b-1} \right)\Big|_{s_1=0, s=\widehat{s}(t)} \\ &= \left( (n+2 - \min(a, b)) s_{a-1} s_{b-1} - \sum_{i=\max(1, a+b-n-2)}^{\min(a, b)} (a+b-2i) s_{i-1} s_{a+b-1-i} \right. \\ &\quad \left. + c_a (n+2-b) s_{a-1} s_{b-1} + c_b (n+2-a) s_{a-1} s_{b-1} + c_a c_b (n+1) s_{a-1} s_{b-1} \right)\Big|_{s_1=0, s=\widehat{s}(t)} \\ &= \left( \frac{(n+2 - \min(a, b))(\max(a, b) - 1)}{n+1} s_{a-1} s_{b-1} - \sum_{i=\max(1, a+b-n-2)}^{\min(a, b)} (a+b-2i) s_{i-1} s_{a+b-1-i} \right)\Big|_{s_1=0, s=\widehat{s}(t)}. \end{aligned}$$

The first term in the last member is easily obtained by distinguishing the cases  $a \leq b$  and  $a > b$ . The inverse transformation  $s = \widehat{s}(t)$  of that one in Eq. (48) is the following one:

$$s_1 = \widehat{s}_1(t) = t_1, \quad s_2 = \widehat{s}_2(t) = t_2, \quad s_a = \widehat{s}_a(t) = t_a - c_a t_1 \widehat{s}_{a-1}(t), \quad \forall a = 3, \dots, n+1,$$

where the expressions for  $s_a$ ,  $\forall a > 2$ , must be calculated by recursion. However, when  $t_1 = 0$ , this inverse transformation simplifies to:  $s_a = \widehat{s}_a(t) = t_a$ ,  $\forall a = 1, \dots, n+1$ . We then find,  $\forall a, b = 2, \dots, n+1$ :

$$\widehat{P}_{ab}(t)\Big|_{t_1=0} = \left( \frac{(n+2 - \min(a, b))(\max(a, b) - 1)}{n+1} t_{a-1} t_{b-1} - \sum_{i=\max(1, a+b-n-2)}^{\min(a, b)} (a+b-2i) t_{i-1} t_{a+b-1-i} \right)\Big|_{t_1=0},$$

in which one has to consider  $t_0 = 1$ , corresponding to  $s_0 = 1$ . In the sum of the preceding formula, when  $a+b-n-2 \leq 1$ , that is, when  $a+b-2 \leq n+1$ , there is a linear term corresponding to the index  $i = 1$ :  $-(a+b-2) t_{a+b-2}$ . To take easily into account this condition, we define  $t_a = 0$  whenever  $a > n+1$ . Moreover, when  $a+b-n-2 \leq 2$ , there is a term corresponding to the index  $i = 2$  that is vanishing for the condition  $t_1 = 0$ . Taking both these terms out of the sum, we can rewrite the formula in the following way:

$$\widehat{P}_{ab}(t)\Big|_{t_1=0} = \frac{(n+2 - \min(a, b))(\max(a, b) - 1)}{n+1} t_{a-1} t_{b-1} - (a+b-2) t_{a+b-2}$$

$$- \sum_{i=\max(3, a+b-n-2)}^{\min(a, b)} (a+b-2i) t_{i-1} t_{a+b-1-i}, \quad \forall a, b = 2, \dots, n+1,$$

where the conditions  $t_1 = 0$  and  $t_a = 0, \forall a > n+1$ , must be taken into account ( $t_0$  no longer appears).

The formula just obtained also specifies the general formula for the element  $\widehat{P}_{a-1, b-1}(q)$  of the  $\widehat{P}$ -matrix  $\widehat{P}(q) = [\widehat{P}(t)]^q$ , corresponding to the basic invariant polynomials  $q_a(\bar{x}')$ ,  $a = 1, \dots, n$ , of  $A_n$ , written in Eq. (49), this because when calculating  $\widehat{P}(q)$  with the formula  $\widehat{P}(q) = [\widehat{P}(t)]^q$ , one has to drop the first line and column of the matrix  $\widehat{P}(t) \Big|_{t_1=0}$ .

It is then convenient to substitute  $a$  and  $b$  with  $a = a' + 1$  and  $b = b' + 1$ , drop the primes, and substitute the variables  $t_a$  with  $q_{a-1}, \forall a = 2, \dots, n+1$ . Doing this we obtain,  $\forall a, b = 1, \dots, n$ :

$$\widehat{P}_{ab}(q) = \frac{(n+1 - \min(a, b)) \max(a, b)}{n+1} q_{a-1} q_{b-1} - (a+b) q_{a+b-1} - \sum_{i=\max(3, a+b-n)}^{\min(a, b)+1} (a+b+2-2i) q_{i-2} q_{a+b-i},$$

where one has to consider  $q_0 = 0$ , and  $q_a = 0, \forall a > n$ .

We can take the term corresponding to the greatest index  $i = \min(a, b) + 1$  out of the sum. This term is:

$$-(a+b+2-2(\min(a, b)+1)) q_{\min(a, b)+1-2} q_{a+b-(\min(a, b)+1)} = -(a+b-2\min(a, b)) q_{a-1} q_{b-1},$$

and can be summed to the first term. We then find the following generating formula for the  $\widehat{P}$ -matrix  $\widehat{P}(q)$ , corresponding to the basic invariant polynomials of  $A_n$  in Eq. (24), valid  $\forall a, b = 1, \dots, n$ :

$$\widehat{P}_{ab}(q) = \frac{(n+1) \min(a, b) - ab}{n+1} q_{a-1} q_{b-1} - (a+b) q_{a+b-1} - \sum_{i=\max(3, a+b-n)}^{\min(a, b)} (a+b+2-2i) q_{i-2} q_{a+b-i}, \quad (52)$$

where one has to consider  $q_0 = 0$ , and  $q_a = 0, \forall a > n$ .

This formula implies some fractional coefficients due to the denominator  $n+1$ . In addition, the quadratic basic invariant  $q_1(\bar{x}')$  is not of the standard form (5). To see this, we use the identities:

$$\sigma_2(x) = \frac{1}{2} \left( (x_1 + x_2 + \dots + x_{n+1})^2 - \sum_{i=1}^{n+1} x_i^2 \right) = \frac{1}{2} ((\sigma_1(x))^2 - |x|^2), \quad \text{and} \quad \sigma_1(R^\top x') = \sqrt{n+1} x'_{n+1},$$

to obtain the following expression for  $t_2(x')$ :

$$t_2(x') = s_2(x') = \sigma_2(R^\top x') = \frac{1}{2} ((\sigma_1(R^\top x'))^2 - |R^\top x'|^2) = \frac{1}{2} \left( (n+1) x'_{n+1}^2 - \sum_{i=1}^{n+1} x_i'^2 \right) = \frac{1}{2} \left( n x'_{n+1}^2 - \sum_{i=1}^n x_i'^2 \right),$$

and hence:

$$q_1(\bar{x}') = t_2(x') \Big|_{x'_{n+1}=0} = -\frac{1}{2} \sum_{i=1}^n x_i'^2.$$

The basic invariant polynomials  $p_a(\bar{x}')$ ,  $a = 1, \dots, n$ , are obtained from the  $q_a(\bar{x}')$ ,  $a = 1, \dots, n$ , with the scale transformation (25). It is then easy to see that  $p_1(\bar{x}')$  has the standard form (5). Let us now determine the  $\widehat{P}$ -matrix  $\widehat{P}(p)$  corresponding to the basic invariant polynomials  $p_a(\bar{x}')$ ,  $a = 1, \dots, n$ . We shall use Eq. (15). The diagonal jacobian matrix of the scale transformation (25) has elements:

$$J_{ab}(q) = \frac{\partial \widehat{p}_a(q)}{\partial q_b} = -2(n+1)^{\frac{a-1}{2}} \delta_{ab}, \quad \forall a, b = 1, \dots, n, \quad (53)$$

and the inverse transformation  $q = \widehat{q}(p)$  of the scale transformation (25) is the following one:

$$q_a = \widehat{q}_a(p) = -\frac{1}{2} (n+1)^{\frac{1-a}{2}} p_a, \quad \forall a = 1, \dots, n. \quad (54)$$

Using Eqs. (15), (52), (53) and (54), we find,  $\forall a, b = 1, \dots, n$ :

$$\widehat{P}_{ab}(p) = J_{ac}(q) \widehat{P}_{cd}(q) J_{db}^\top(q) \Big|_{q=\widehat{q}(p)} = 4 \left( (n+1)^{\frac{a-1}{2}} \delta_{ac} \widehat{P}_{cd}(q) (n+1)^{\frac{b-1}{2}} \delta_{db} \right) \Big|_{q=\widehat{q}(p)} = 4(n+1)^{\frac{a+b-2}{2}} \widehat{P}_{ab}(q) \Big|_{q=\widehat{q}(p)}$$

$$\begin{aligned}
&= 4(n+1)^{\frac{a+b-2}{2}} \left( \frac{(n+1)\min(a,b) - ab}{4(n+1)} (n+1)^{\frac{1-(a-1)}{2}} p_{a-1} (n+1)^{\frac{1-(b-1)}{2}} p_{b-1} + \frac{1}{2} (a+b) (n+1)^{\frac{1-(a+b-1)}{2}} p_{a+b-1} \right. \\
&\quad \left. - \frac{1}{4} \sum_{i=\max(3,a+b-n)}^{\min(a,b)} (a+b+2-2i) (n+1)^{\frac{1-(i-2)}{2}} p_{i-2} (n+1)^{\frac{1-(a+b-i)}{2}} p_{a+b-i} \right) \\
&= ((n+1)\min(a,b) - ab) p_{a-1} p_{b-1} + 2(a+b) p_{a+b-1} - (n+1) \sum_{i=\max(3,a+b-n)}^{\min(a,b)} (a+b+2-2i) p_{i-2} p_{a+b-i},
\end{aligned}$$

where one has to consider  $p_0 = 0$ , and  $p_a = 0, \forall a > n$ .

In the preceding formula, all the coefficients of the monomials in the variables  $p_a, a = 1, \dots, n$ , are integer numbers. If we change the index of the sum from  $i$  to  $i' = i - 1$ , and then drop the primes, we obtain Eq. (26).  $\diamond$

### Proof of the generating formula (27).

As the invariant polynomials  $p_a(\bar{x}')$ ,  $a = 1, \dots, n$ , of  $A_n$ , are obtained from the elementary symmetric polynomials  $\sigma_a(x)$ ,  $a = 1, \dots, n+1$ , through Eqs. (23)–(25), we shall determine the  $\lambda$ -vector  $\lambda^{(\det(\hat{P}))}(p)$ , corresponding to the determinant of the  $\hat{P}$ -matrix  $\hat{P}(p)$ , relative to  $A_n$ , starting from the  $\lambda$ -vector  $\lambda^{(\det(\hat{P}))}(\sigma)$ , corresponding to the determinant of the  $\hat{P}$ -matrix  $\hat{P}(\sigma)$ , relative to  $S_{n+1}$ . From Eq. (21), by substituting  $n$  with  $n+1$ , we obtain:

$$\lambda_a^{(\det(\hat{P}))}(\sigma) = -(n-a+3)(n-a+2) \sigma_{a-2}, \quad a = 1, \dots, n+1, \quad (55)$$

in which one has to consider  $\sigma_{-1} = 0$  and  $\sigma_0 = 1$ .

For the transformation (23), Proposition 1, item 5., implies  $\lambda^{(\det(\hat{P}))}(s) = \lambda^{(\det(\hat{P}))}(\sigma) \Big|_{\sigma=s}$ , so that Eq. (55), with  $\sigma$  substituted by  $s$ , and the definitions  $s_{-1} = 0$ , and  $s_0 = 1$ , also gives the  $\lambda$ -vector  $\lambda^{(\det(\hat{P}))}(s)$ , corresponding to the determinant of the  $\hat{P}$ -matrix  $\hat{P}(s)$ , relative to  $S_{n+1}$ .

In the proof of the generating formula (26) we determined the  $\hat{P}$ -matrix  $\hat{P}(q)$  by determining first the matrix  $\hat{P}(t) \Big|_{t_1=0}$ , where the  $\hat{P}$ -matrix  $\hat{P}(t)$  corresponds to the basic invariant polynomials  $t_a(x')$ ,  $a = 1, \dots, n+1$ , of  $S_{n+1}$ , defined in Eq. (48), and then by applying Theorem 2. We shall then follow the same steps, and determine first the vector  $\lambda^{(\det(\hat{P}))}(t) \Big|_{t_1=0}$  from  $\lambda^{(\det(\hat{P}))}(s)$ , using Eq. (16), and then the  $\lambda$ -vector  $\lambda^{(\det(\hat{P}))}(q)$  from  $\lambda^{(\det(\hat{P}))}(t) \Big|_{t_1=0}$ , using Theorem 2. Actually, in view to use Theorem 2, we do not need to determine the first component of  $\lambda^{(\det(\hat{P}))}(t) \Big|_{t_1=0}$ . The jacobian matrix elements of the transformation (48), when  $s_1 = 0$ , corresponding to  $t_1 = 0$ , are written in Eq. (51), so we have, from Eq. (16),  $\forall a = 2, \dots, n+1$ :

$$\begin{aligned}
\lambda_a^{(\det(\hat{P}))}(t) \Big|_{t_1=0} &= J_{ab}(s) \lambda_b^{(\det(\hat{P}))}(s) \Big|_{s_1=0, s=\hat{s}(t)} = \left( (\delta_{ab} + c_a \delta_{1b} s_{a-1}) \lambda_b^{(\det(\hat{P}))}(s) \right) \Big|_{s_1=0, s=\hat{s}(t)} \\
&= \lambda_a^{(\det(\hat{P}))}(s) \Big|_{s_1=0, s=\hat{s}(t)} + \left( c_a s_{a-1} \lambda_1^{(\det(\hat{P}))}(s) \right) \Big|_{s_1=0, s=\hat{s}(t)} = -(n-a+3)(n-a+2) t_{a-2} + 0,
\end{aligned}$$

in which one has to consider  $t_1 = 0$  and  $t_0 = 1$ , corresponding to  $s_0 = 1$ . In the last member we used the inverse transformation  $s_a = \hat{s}_a(t) = t_a$ , valid  $\forall a > 1$ , when  $t_1 = 0$ . Next, using Theorem 2, item 2., one has,  $\forall a = 1, \dots, n$ :

$$\lambda_a^{(\det(\hat{P}))}(q) = [\lambda_{a+1}^{(\det(\hat{P}))}(t)]^q = [-(n-a+2)(n-a+1) t_{a-1}]^q = -(n-a+2)(n-a+1) q_{a-2}, \quad (56)$$

in which one has to consider  $q_{-1} = 1$  and  $q_0 = 0$ , corresponding to  $t_0 = 1$  and  $t_1 = 0$ .

The  $\lambda$ -vector in Eq. (56) corresponds to the determinant of the matrix  $\hat{P}(q)$  generated by Eq. (52).

To the basis transformation in Eq. (25), it corresponds the jacobian matrix in Eq. (53), and the inverse transformation in Eq. (54). By using Eqs. (16), (53), and (56), we find,  $\forall a = 1, \dots, n$ :

$$\lambda_a^{(\det(\hat{P}))}(p) = J_{ab}(q) \lambda_b^{(\det(\hat{P}))}(q) \Big|_{q=\hat{q}(p)} = -2(n+1)^{\frac{a-1}{2}} \delta_{ab} \lambda_b^{(\det(\hat{P}))}(q) \Big|_{q=\hat{q}(p)}$$

$$= -2(n+1)^{\frac{a-1}{2}} \lambda_a^{(\det(\widehat{P}))}(q) \Big|_{q=\widehat{q}(p)} = 2(n+1)^{\frac{a-1}{2}} (n-a+2)(n-a+1) q_{a-2} \Big|_{q=\widehat{q}(p)} .$$

To substitute the inverse transformation  $q = \widehat{q}(p)$ , we have to distinguish the cases  $a \leq 2$  and  $a > 2$ , because  $\lambda_1^{(\det(\widehat{P}))}(p) = 2n(n+1)$  and  $\lambda_2^{(\det(\widehat{P}))}(p) = 0$  are numbers and no substitution must be done in these components. By substituting  $q = \widehat{q}(p)$ , written in (54), in all the other components of  $\lambda^{(\det(\widehat{P}))}(p)$ , we obtain,  $\forall a = 3, \dots, n$ :

$$\lambda_a^{(\det(\widehat{P}))}(p) = 2(n+1)^{\frac{a-1}{2}} (n-a+2)(n-a+1) \left( -\frac{1}{2} (n+1)^{\frac{1-(a-2)}{2}} p_{a-2} \right) = -(n+1)(n-a+2)(n-a+1) p_{a-2} .$$

By defining  $p_0 = 0$ , the previous expression is valid also for  $a = 2$ . The proof of Eq. (27) is complete.  $\diamond$

### C. Proof of the generating formulas for the groups of type $B_n$

#### Proof of the generating formula (29).

The proof of Eq. (29) will be obtained by induction on the rank  $n$ . The lowest possible value for  $n$  is  $n = 2$  (because of the isomorphism  $B_1 \simeq A_1$ ), so we first prove that Eq. (29) is true for  $n = 2$  (it would be also true for  $n = 1$ ). The basic invariant polynomials of  $B_2$  are written in Eq. (28), with  $n = 2$ , and are:  $p_1(x) = x_1^2 + x_2^2$ , and  $p_2(x) = x_1^2 x_2^2$ . The elements of the matrix  $P(x)$ , calculated with Eqs. (1) and (2), are:  $P_{11}(x) = 4(x_1^2 + x_2^2) = 4p_1(x)$ ,  $P_{12}(x) = P_{21}(x) = 4(x_1^2 x_2^2 + x_1^2 x_2^2) = 8p_2(x)$ ,  $P_{22}(x) = 4(x_1^2 x_2^4 + x_1^4 x_2^2) = 4p_1(x)p_2(x)$ . Using Eq. (29) one finds the same expressions:  $\widehat{P}_{11}(p) = 4p_1$ ,  $\widehat{P}_{12}(p) = \widehat{P}_{21}(p) = 8p_2$ ,  $\widehat{P}_{22}(p) = 4p_1 p_2$ . Then, for  $n = 2$ , Eq. (29) is true.

Suppose now Eq. (29) true for a general  $n \geq 2$ . We use the symbols  $x$  and  $y$  for general vectors of  $\mathbb{R}^n$  and  $\mathbb{R}^{n+1}$ , that is:  $x = (x_1, x_2, \dots, x_n)$ , and  $y = (x_1, x_2, \dots, x_n, x_{n+1})$ .  $B_n$  acts in  $\mathbb{R}^n$  and its basic invariant polynomials are  $p_a(x)$ ,  $a = 1, \dots, n$ , defined in Eq. (28), and  $B_{n+1}$  acts in  $\mathbb{R}^{n+1}$  and its basic invariant polynomials are  $p_a(y)$ ,  $a = 1, \dots, n+1$ , also defined in Eq. (28), but with  $n$  substituted by  $n+1$ . The basic invariant polynomials of  $B_{n+1}$  can be expressed in terms of those of  $B_n$  in the following way:

$$p_a(y) = p_a(x) + p_{a-1}(x) x_{n+1}^2 \quad \forall a = 1, 2, \dots, n+1, \quad (57)$$

where the definitions  $p_0(x) = 1$  and  $p_{n+1}(x) = 0$  must be taken into account.

Let us use the ‘‘nabla’’ operators  $\nabla_x$  and  $\nabla_y$ , defined in Eq. (40). One then has, for Eqs. (57) and (40),  $\forall a = 1, \dots, n+1$ :

$$\nabla_y p_a(y) = \nabla_y (p_a(x) + p_{a-1}(x) x_{n+1}^2) = \nabla_x p_a(x) + x_{n+1}^2 \nabla_x p_{a-1}(x) + p_{a-1}(x) 2 x_{n+1} e_{n+1}, \quad (58)$$

where  $e_{n+1}$  is the  $(n+1)$ -th canonical unit vector of  $\mathbb{R}^{n+1}$ .

We can now calculate a general element of the matrix  $P(y)$  of  $B_{n+1}$ , using Eqs. (1) and (58). For all  $a, b = 1, \dots, n+1$  we have:

$$\begin{aligned} P_{ab}(y) &= \nabla_y p_a(y) \cdot \nabla_y p_b(y) \\ &= (\nabla_x p_a(x) + x_{n+1}^2 \nabla_x p_{a-1}(x) + p_{a-1}(x) 2 x_{n+1} e_{n+1}) \cdot (\nabla_x p_b(x) + x_{n+1}^2 \nabla_x p_{b-1}(x) + p_{b-1}(x) 2 x_{n+1} e_{n+1}) \\ &= P_{ab}(x) + x_{n+1}^2 (P_{a-1,b}(x) + 4 p_{a-1}(x) p_{b-1}(x) + P_{a,b-1}(x)) + x_{n+1}^4 P_{a-1,b-1}(x). \end{aligned}$$

Of course, in the preceding expression,  $P_{cd}(x) = 0$ , if one of the indices  $c, d$  (that in the above expression can be equal to  $a, b, a-1, b-1$ ) is equal to 0 or to  $n+1$ , because the gradients of  $p_0(x)$  and  $p_{n+1}(x)$  are null vectors. By the induction hypothesis, Eq. (29) is true for  $B_n$ , so we can use it in the preceding equation. We obtain then,  $\forall a, b = 1, \dots, n+1$ :

$$\begin{aligned} P_{ab}(y) &= \sum_{i=\max(0, a+b-n-1)}^{\min(a,b)-1} 4(a+b-1-2i) p_i(x) p_{a+b-1-i}(x) \\ &+ 4 x_{n+1}^2 \left( \sum_{i=\max(0, a+b-n-2)}^{\min(a-1, b)-1} (a+b-2-2i) p_i(x) p_{a+b-2-i}(x) + p_{a-1}(x) p_{b-1}(x) \right) \end{aligned}$$

$$+ \sum_{i=\max(0, a+b-n-2)}^{\min(a, b-1)-1} (a+b-2-2i) p_i(x) p_{a+b-2-i}(x) + 4x_{n+1}^4 \sum_{i=\max(0, a+b-n-3)}^{\min(a, b)-2} (a+b-3-2i) p_i(x) p_{a+b-3-i}(x).$$

The terms inside the brackets multiplying  $x_{n+1}^2$  can be simplified. One notes that both sums inside the brackets contain equal terms up to the value  $i = \min(a-1, b-1) - 1 = \min(a, b) - 2$ . Moreover the first sum contains one more term if  $\min(a-1, b-1) \neq \min(a-1, b)$ , that is if  $a > b$ , that corresponds to  $i = b-1$  and the second sum contains one more term if  $\min(a-1, b-1) \neq \min(a, b-1)$ , that is if  $a < b$ , that corresponds to  $i = a-1$ . These two terms never appear simultaneously and one can write that this additional term appears for  $i = \min(a, b) - 1$ . One can then write the term inside the brackets factorizing  $x_{n+1}^2$  as follows:

$$\left( \dots \right) = \left( \sum_{i=\max(0, a+b-n-2)}^{\min(a, b)-2} 2(a+b-2-2i) p_i(x) p_{a+b-2-i}(x) \right. \\ \left. + p_{a-1}(x) p_{b-1}(x) + (a+b-2-2(\min(a, b)-1)) p_{\min(a, b)-1}(x) p_{a+b-2-(\min(a, b)-1)}(x) \right).$$

By distinguishing the cases  $a < b$ ,  $a = b$  and  $a > b$ , one may simplify the terms in the second line of the above expression in:  $(|a-b|+1) p_{a-1}(x) p_{b-1}(x)$ . In the sum multiplying  $4x_{n+1}^4$ , it is convenient to change the index  $i$  into  $i' = i + 1$ , and then rename  $i'$  in  $i$ . All this gives:

$$P_{ab}(y) = \sum_{i=\max(0, a+b-n-1)}^{\min(a, b)-1} 4(a+b-1-2i) p_i(x) p_{a+b-1-i}(x) \\ + 4x_{n+1}^2 \left( \sum_{i=\max(0, a+b-n-2)}^{\min(a, b)-2} 2(a+b-2-2i) p_i(x) p_{a+b-2-i}(x) + (|a-b|+1) p_{a-1}(x) p_{b-1}(x) \right) \\ + 4x_{n+1}^4 \sum_{i=\max(1, a+b-n-2)}^{\min(a, b)-1} (a+b-1-2i) p_{i-1}(x) p_{a+b-2-i}(x). \quad (59)$$

One has now to prove that this expression coincides with that one one finds from Eq. (29) in the case of  $B_{n+1}$ . Using Eq. (57) in Eq. (29), one finds,  $\forall a, b = 1, \dots, n+1$ :

$$\widehat{P}_{ab}(\bar{p}(y)) = \sum_{i=\max(0, a+b-(n+1)-1)}^{\min(a, b)-1} 4(a+b-1-2i) p_i(y) p_{a+b-1-i}(y) \\ = \sum_{i=\max(0, a+b-n-2)}^{\min(a, b)-1} 4(a+b-1-2i) (p_i(x) + p_{i-1}(x) x_{n+1}^2) (p_{a+b-1-i}(x) + p_{a+b-2-i}(x) x_{n+1}^2) \\ = \sum_{i=\max(0, a+b-n-1)}^{\min(a, b)-1} 4(a+b-1-2i) p_i(x) p_{a+b-1-i}(x) + 4x_{n+1}^2 \left( \sum_{i=\max(0, a+b-n-2)}^{\min(a, b)-1} (a+b-1-2i) p_i(x) p_{a+b-2-i}(x) \right. \\ \left. + \sum_{i=\max(1, a+b-n-1)}^{\min(a, b)-1} (a+b-1-2i) p_{i-1}(x) p_{a+b-1-i}(x) \right) + 4x_{n+1}^4 \sum_{i=\max(1, a+b-n-2)}^{\min(a, b)-1} (a+b-1-2i) p_{i-1}(x) p_{a+b-2-i}(x).$$

In the first, third and fourth sums in the last member, the lower limit of the sums has been changed from  $\max(0, a+b-n-2)$  to  $\max(0, a+b-n-1)$ ,  $\max(1, a+b-n-1)$ ,  $\max(1, a+b-n-2)$ , respectively, because  $p_{a+b-1-i}(x)$  vanishes when  $i = a+b-n-2$ , and because  $p_{i-1}(x)$  vanishes when  $i = 0$ .

The first and the last sums coincide now exactly with the first and the last sums in Eq. (59). The expression inside the brackets multiplying  $x_{n+1}^2$  needs to be transformed somewhat more to see that it is identical to the expression inside the brackets multiplying  $x_{n+1}^2$  in Eq. (59). One has:

$$\begin{aligned}
(\dots) &= \left( \sum_{i=\max(0, a+b-n-2)}^{\min(a,b)-1} (a+b-1-2i) p_i(x) p_{a+b-2-i}(x) + \sum_{i=\max(0, a+b-n-2)}^{\min(a,b)-2} (a+b-3-2i) p_i(x) p_{a+b-2-i}(x) \right) \\
&= \left( \sum_{i=\max(0, a+b-n-2)}^{\min(a,b)-2} (2a+2b-4-4i) p_i(x) p_{a+b-2-i}(x) \right. \\
&\quad \left. + \left( a+b-1-2(\min(a,b)-1) \right) p_{\min(a,b)-1}(x) p_{a+b-2-(\min(a,b)-1)}(x) \right) \\
&= \left( \sum_{i=\max(0, a+b-n-2)}^{\min(a,b)-2} 2(a+b-2-2i) p_i(x) p_{a+b-2-i}(x) + (|a-b|+1) p_{a-1}(x) p_{b-1}(x) \right).
\end{aligned}$$

In the second sum of the first line the index  $i$  has been changed into  $i' = i + 1$  and then  $i'$  has been renamed in  $i$ . The simplification in the last line is obtained by distinguishing the cases  $a > b$ ,  $a = b$  and  $a < b$ , as already done above. The last member of this expression is identical to the expression inside the brackets multiplying  $x_{n+1}^2$  in Eq. (59). We have so proved that the expression for the matrix element  $P_{ab}(y)$ , obtained from Eq. (29), for the group  $B_{n+1}$ , coincides with Eq. (59), that is obtained from Eq. (1), and the assumed validity of Eq. (29) for the group  $B_n$ . This proves that Eq. (29) is valid for  $B_n$  for all values of  $n$ .  $\diamond$

### Proof of the generating formulas (30) and (31).

The groups of type  $B_n$  have two different root systems, described at the beginning of Section V. Theorem 1, item 2. states then that there are two different active factors  $\widehat{s}(p)$  and  $\widehat{l}(p)$  of  $\det(\widehat{P}(p))$ , and describes how to construct both polynomials  $\widehat{s}(p)$  and  $\widehat{l}(p)$  and their  $\lambda$ -vectors. Let us start with the positive short roots, those in  $\mathcal{R}_{+,s}$ . The linear forms vanishing in the reflection hyperplanes of the short roots, obtained from Eq. (3), are the variables themselves:  $x_i$ ,  $i = 1, \dots, n$ , and the invariant polynomial  $s(x)$ , obtained as specified in the proof of Theorem 1, item 2., coincides with the highest degree basic invariant polynomial  $p_n(x) = \sigma_n(x^2)$ , that implies  $\widehat{s}(p) = p_n$ . The  $\lambda$ -vector of  $\widehat{s}(p)$  can be obtained from the first of Eq. (12), but the proof of the first of Eq. (30) is much quicker if one makes use of the explicit form of the  $\widehat{P}$ -matrix given in Eq. (29), and takes  $a(p) = p_n$  in the boundary equation (9). The first member of Eq. (9) then simplifies to the following one:

$$\sum_{b=1}^n \widehat{P}_{ab}(p) \frac{\partial p_n}{\partial p_b} = \sum_{b=1}^n \widehat{P}_{ab}(p) \delta_{bn} = \widehat{P}_{an}(p) = 4(n-a+1) p_{a-1} p_n, \quad a = 1, \dots, n,$$

and comparing with the second member of Eq. (9), given that  $a(p) = p_n$ , one finds the first of Eq. (30).

Let us consider now the long roots. The linear forms vanishing in the reflection hyperplanes of the long roots, obtained from Eq. (3), are:  $x_i \pm x_j$ ,  $1 \leq i < j \leq n$ , and the invariant polynomial  $l(x)$ , obtained as specified in the proof of Theorem 1, item 2., is the following one:

$$l(x) = \prod_{i < j=1}^n (x_i + x_j)(x_i - x_j) = \prod_{i < j=1}^n x_i^2 - x_j^2.$$

As  $l(x)$  is invariant, we can write  $l(x) = \widehat{l}(\overline{p}(x))$ . The  $\lambda$ -vector of  $\widehat{l}(p)$  can be obtained from the second of Eq. (12):

$$\begin{aligned}
\lambda_a^{(l)}(\overline{p}(x)) &= 2 \sum_{r \in \mathcal{R}_{+,l}} \frac{\nabla p_a(x) \cdot r}{l_r(x)} = 2 \nabla p_a(x) \cdot \sum_{i < j=1}^n \frac{e_i + e_j}{x_i + x_j} + \frac{e_i - e_j}{x_i - x_j} = 2 \nabla p_a(x) \cdot \sum_{i < j=1}^n \frac{2x_i e_i - 2x_j e_j}{(x_i + x_j)(x_i - x_j)} \\
&= 4 \sum_{i < j=1}^n \frac{1}{x_i^2 - x_j^2} \left( x_i \frac{\partial p_a(x)}{\partial x_i} - x_j \frac{\partial p_a(x)}{\partial x_j} \right), \quad \forall a = 1, \dots, n.
\end{aligned} \tag{60}$$



The basic polynomials  $p_a(x)$ ,  $a = 1, \dots, n$ , are written in Eq. (28). By taking the derivatives, one easily finds:

$$x_i \frac{\partial p_a(x)}{\partial x_i} = 2 x_i^2 \left( \sum_{(i_1, i_2, \dots, i_{a-1}) \subset (1, 2, \dots, \widehat{i}, \dots, n)} x_{i_1}^2 x_{i_2}^2 \cdots x_{i_{a-1}}^2 \right),$$

where the sum is over all combinations of  $a - 1$  integers  $(i_1, i_2, \dots, i_{a-1})$  among the first  $n$  integers, excluding the integer  $i$  (the hat in  $\widehat{i}$  means exclusion). There are  $\binom{n-1}{a-1}$  such combinations. It is convenient to distinguish the terms containing the factor  $x_j^2$  from those that do not. We can then write:

$$x_i \frac{\partial p_a(x)}{\partial x_i} = 2 x_i^2 \left( \sum_{(i_1, i_2, \dots, i_{a-2}) \subset (1, 2, \dots, \widehat{i}, \dots, \widehat{j}, \dots, n)} x_j^2 \cdot x_{i_1}^2 x_{i_2}^2 \cdots x_{i_{a-2}}^2 + \sum_{(i_1, i_2, \dots, i_{a-1}) \subset (1, 2, \dots, \widehat{i}, \dots, \widehat{j}, \dots, n)} x_{i_1}^2 x_{i_2}^2 \cdots x_{i_{a-1}}^2 \right).$$

The expression for  $x_j \frac{\partial p_a(x)}{\partial x_j}$  is similar, with  $i$  and  $j$  interchanged. When making the difference of the two expressions so found, as required by Eq. (60), the first sums cancel, because of the symmetry in  $i$  and  $j$ , and one finds:

$$x_i \frac{\partial p_a(x)}{\partial x_i} - x_j \frac{\partial p_a(x)}{\partial x_j} = 2(x_i^2 - x_j^2) \left( \sum_{(i_1, i_2, \dots, i_{a-1}) \subset (1, 2, \dots, \widehat{i}, \dots, \widehat{j}, \dots, n)} x_{i_1}^2 x_{i_2}^2 \cdots x_{i_{a-1}}^2 \right).$$

Inserting this result in Eq. (60), one obtains,  $\forall a = 1, \dots, n$ :

$$\lambda_a^{(l)}(\overline{p}(x)) = 8 \sum_{i < j = 1}^n \left( \sum_{(i_1, i_2, \dots, i_{a-1}) \subset (1, 2, \dots, \widehat{i}, \dots, \widehat{j}, \dots, n)} x_{i_1}^2 x_{i_2}^2 \cdots x_{i_{a-1}}^2 \right).$$

The sum in the second member is a symmetric polynomial in  $x_1, \dots, x_n$ , with terms that are products of  $a - 1$  different squared variables. It must then be a multiple of the basic invariant polynomial  $p_{a-1}(x)$ . To find its multiplying factor, one can count the number of terms,  $c_1$ , in the expression above, and the number of terms,  $c_2$ , in  $p_{a-1}(x)$ . The multiplying factor is then equal to the ratio  $\frac{c_1}{c_2}$ . One easily finds:  $c_1 = \binom{n}{2} \binom{n-2}{a-1}$ , and  $c_2 = \binom{n}{a-1}$ , so that:

$$\frac{c_1}{c_2} = \frac{n(n-1)}{2} \frac{(n-2)!}{(a-1)!(n-a-1)!} \frac{(a-1)!(n-a+1)!}{n!} = \frac{(n-a+1)(n-a)}{2},$$

and one obtains,  $\forall a = 1, \dots, n$ :

$$\lambda_a^{(l)}(p(x)) = 8 \frac{(n-a+1)(n-a)}{2} p_{a-1}(x) = 4(n-a+1)(n-a) p_{a-1}(x).$$

The second of Eq. (30) is so proved. By substituting Eq. (30) in Eq. (13), one finds,  $\forall a = 1, \dots, n$ :

$$\lambda_a^{(\det(\widehat{P}))}(p) = \lambda_a^{(s)}(p) + \lambda_a^{(l)}(p) = 4(n-a+1) p_{a-1} + 4(n-a+1)(n-a) p_{a-1} = 4(n-a+1)^2 p_{a-1},$$

and Eq. (31) is proved.  $\diamond$

#### D. Proof of the generating formulas for the groups of type $D_n$

##### Proof of the generating formula (33).

The basic invariant polynomials of  $D_n$ , written in Eq. (32), differ from those of  $B_n$ , written in Eq. (28), only for the basic invariant polynomial  $p_n(x)$ : The square of the basic polynomial  $p_n(x)$  of  $D_n$  is the basic polynomial  $p_n(x)$  of  $B_n$ . It is convenient in this proof to write  $p_n^B(x)$  and  $p_n^D(x)$  to distinguish the  $n$ -th basic invariant polynomials of  $B_n$  and  $D_n$ , respectively. One has thus  $(p_n^D(x))^2 = p_n^B(x)$ . We also use the notations  $\widehat{P}^B(p)$  and  $\widehat{P}^D(p)$  for the  $\widehat{P}$ -matrices of  $B_n$  and  $D_n$ , respectively. For  $a, b < n$ , the matrix elements  $P_{ab}(x)$ , calculated from Eq. (1), have the same expressions either for  $B_n$  or for  $D_n$ , and when one has to use Eq. (2) to express these matrix elements in terms of the basic invariant polynomials, the expressions obtained for  $D_n$  differ from those of  $B_n$  only for the use of  $(p_n^D)^2$

in place of  $p_n^B$ . Using Eq. (29), the first line of Eq. (33) follows immediately.

To prove the second and third lines of Eq. (33) we shall use Eq. (29) for the elements of the  $\widehat{P}$ -matrix  $\widehat{P}^B(p)$ . As  $p_n^B(x) = (p_n^D(x))^2$ , one has  $\nabla p_n^B(x) = \nabla(p_n^D(x))^2 = 2p_n^D(x) \nabla p_n^D(x)$ . Then,  $\forall a = 1, \dots, n-1$ :

$$\begin{aligned} P_{an}^D(x) &= \nabla p_a(x) \cdot \nabla p_n^D(x) = \frac{1}{2p_n^D(x)} \nabla p_a(x) \cdot \nabla p_n^B(x) = \frac{1}{2p_n^D(x)} \widehat{P}_{an}^B(\bar{p}(x)) = \frac{1}{2p_n^D(x)} 4(n-a+1)p_{a-1}(x)p_n^B(x) \\ &= \frac{1}{p_n^D(x)} 2(n-a+1)p_{a-1}(x)(p_n^D(x))^2 = 2(n-a+1)p_{a-1}(x)p_n^D(x). \end{aligned}$$

The second line of Eq. (33) is thus proved. In a similar way, we have:

$$P_{nn}^D(x) = \nabla p_n^D(x) \cdot \nabla p_n^D(x) = \frac{1}{4(p_n^D(x))^2} \nabla p_n^B(x) \cdot \nabla p_n^B(x) = \frac{1}{4p_n^B(x)} \widehat{P}_{nn}^B(\bar{p}(x)) = \frac{1}{4p_n^B(x)} 4p_{n-1}(x)p_n^B(x) = p_{n-1}(x).$$

The third line of Eq. (33) is thus proved.  $\diamond$

### Proof of the generating formula (34).

We shall use Theorem 1, item 1.. Our choice of the positive roots of  $D_n$  is described at the beginning of Section VI. These roots are the same as the positive long roots we used for  $B_n$ . This means that using Eq. (10) with the positive roots of  $D_n$ , and  $a < n$ , one has to do the same calculations as those done in the proof of the second of Eq. (30) with the positive long roots of  $B_n$ , so the results are already readable in the second of Eq. (30),  $\forall a = 1, \dots, n-1$ . For  $a = n$ , one has  $p_n(x) = \sigma_n(x) = x_1 x_2 \cdots x_n$ , and Eq. (10) gives:

$$\begin{aligned} \lambda_n^{(\det(\widehat{P}))}(p(x)) &= 2 \sum_{r \in \mathcal{R}_+} \frac{\nabla p_n(x) \cdot r}{l_r(x)} = 2 \nabla p_n(x) \cdot \sum_{i < j=1}^n \frac{e_i + e_j}{x_i + x_j} + \frac{e_i - e_j}{x_i - x_j} = 2 \nabla p_n(x) \cdot \sum_{i < j=1}^n \frac{2x_i e_i - 2x_j e_j}{(x_i + x_j)(x_i - x_j)} \\ &= 4 \sum_{i < j=1}^n \frac{1}{x_i^2 - x_j^2} \left( x_i \frac{\partial p_n(x)}{\partial x_i} - x_j \frac{\partial p_n(x)}{\partial x_j} \right) = 4 \sum_{i < j=1}^n \frac{1}{x_i^2 - x_j^2} (p_n(x) - p_n(x)) = 0. \end{aligned}$$

As the second of Eq. (30), for  $a = n$ , gives  $\lambda_n^{(\det(\widehat{P}))}(p) = 0$ , we can use it also for  $a = n$ . Eq. (34) is so proved.  $\diamond$

## IX. APPENDIX

This Appendix lists alternative expressions of the generating formulas (20), (26), (29) and (33) in which the symmetry of the resulting matrices is hidden, but that can be of some interest, because they are somewhat simpler.

A formula equivalent to Eq. (20), for the groups of type  $S_n$  is:

$$(n+1-b)p_{a-1}p_{b-1} + \sum_{i=1}^{a-1} (a-b-2i)p_{a-1-i}p_{b-1+i}, \quad \forall a, b = 1, \dots, n,$$

in which one has to consider  $p_0 = 1$  and  $p_a = 0, \forall a > n$ .

A formula equivalent to Eq. (26), for the groups of type  $A_n$ :

$$\widehat{P}_{ab}(p) = a(n+1-b)p_{a-1}p_{b-1} + 2(a+b)p_{a+b-1} + (n+1) \sum_{i=1}^{a-2} (a-b-2i)p_{a-1-i}p_{b-1+i}, \quad \forall a, b = 1, \dots, n,$$

in which one has to consider  $p_0 = 0$  and  $p_a = 0, \forall a > n$ .

A formula equivalent to Eq. (29), for the groups of type  $B_n$  is:

$$\widehat{P}_{ab}(p) = 4 \sum_{i=1}^a (b-a-1+2i)p_{a-i}p_{b-1+i}, \quad \forall a, b = 1, \dots, n,$$

in which one has to consider  $p_0 = 1$  and  $p_a = 0, \forall a > n$ .

A formula equivalent to Eq. (33), for the groups of type  $D_n$  is:

$$\left\{ \begin{array}{l} \widehat{P}_{ab}(p) = 4 \sum_{i=1}^a (b-a-1+2i) p_{a-i} p_{b-1+i} \Big|_{p_n \rightarrow p_n^2} \quad \forall a, b = 1, \dots, n-1, \\ \widehat{P}_{an}(p) = \widehat{P}_{na}(p) = 2(n-a+1) p_{a-1} p_n \quad \forall a = 1, \dots, n-1, \\ \widehat{P}_{nn}(p) = p_{n-1}, \end{array} \right.$$

where one has to consider  $p_0 = 1$  and  $p_a = 0, \forall a > n$ .

To obtain the formulas written in this appendix from those in Eqs. (20), (26), (29) and (33), one starts to consider  $a \leq b$ , define  $p_a = 0 \forall a < 0$  and  $\forall a > n$ , in such a way to drop all min and max in Eqs. (20), (26), (29) and (33), and then change the summation index from  $i$  to  $i' = a - i$ .

The formulas written above in this appendix generate symmetric matrices, because they are equivalent to those in Eqs. (20), (26), (29) and (33), that clearly generate symmetric matrices. When one calculates the matrix elements  $\widehat{P}_{ab}(p)$  with the formulas above, it is convenient, although not necessary, to make the calculations only for  $a \leq b$ , because when  $a > b$  one has a larger number of terms originating from the sums than when  $a < b$ , and many of them cancel each others.

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