VITTORIO CHECCUCCI AND HIS CONTRIBUTIONS TO MATHEMATICS EDUCATION: A HISTORICAL OVERVIEW

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Abstract

This study deals with Vittorio Checcucci's ideas and proposals as to mathematics education. The scopes of this work are twofold: 1) the first scope is historical: my aim is to reconstruct Checcucci's thought. This is a novelty because almost no contribution dedicated to Checcucci exists. The few existing contributions are brief articles whose aim is not to provide a general picture of his ideas; 2) the second scope is connected to mathematics education in the 21st century. A series of argumentations will be proposed to prove that many Checcucci's ideas could be fruitfully exploited nowadays. For the first time, the thought of this mathematician is exposed to non-Italian readers because his ideas are worthy to be known, rethought and discussed in an international context.
Key words: *mathematics education, relations between theoretical and practical mathematics in the teaching, experimentations in mathematics education.*

Outline of the Problem and Scopes of the Research

The aim of this research is to present the work of Vittorio Checcucci (1918-1991) in mathematics education. Checcucci did not begin his academic and scientific career in this field of research: in fact, in 1939 he achieved his degree in mathematics at the Scuola Normale Superiore in Pisa; the following year he became assistant to the chair of geometry, held by Professor Salvatore Cherubino in Pisa. During the second world war, Checcucci participated to the El Alamein battle and, captured by the English, was transferred to India as a prisoner of war. He came back to Italy in 1946 and returned to his geometrical studies. Checcucci was the author of some significant papers (see, for example, Checcucci, 1950a, 1950b, 1955) and became professore aggregato (aggregate professor) for geometry at the university of Pisa. However, Checcucci's vocation was not geometry: after some initial results, he lost interest in active mathematical research and for a long period, between the half of the 50s and the 60s, he also crossed a personal crisis, partially connected to his work. But, starting from the second half of the 60s, Checcucci developed a strong interest in mathematical education. He became rapidly enthusiastic of this subject, because he was convinced that didactics had a fundamental educational and social role: with a solid cultural basis everyone could have the chance to improve his social status and this was particularly significant for young people belonging to the less prosperous classes. As to the educational questions and their connections with social problems, Checcucci was particularly impressed by the book *Lettera a una professoressa*, published by the priest Don Lorenzo Milani (1923-1967) in 1967, the year itself of his death. This book and his author became rapidly famous in Italy for the new pedagogical conceptions expressed in *Lettera a una professoressa*, in which a radically innovative paradigm of education was proposed. Starting from the end of the 60s, Checcucci began to write a series of contributions concerning mathematics teaching and to participate to experimental projects conceived for Italian schools (elementary, middle and high). As a consequence of this intense activity, in 1973, he became full Professor of Mathematics Education at Pisa University. This was the first chair in Italy with denomination *Didattica* *della matematica*. He was active until 1983, year in which he prematurely retired because of health problems. Nevertheless, he gave still significant contributions until his death, occurred in 1991 (for the news concerning Checcucci's life, an important source is Prodi, 1993. See also Guerraggio, 2011, Internet source).

The personal story of Checcucci is tied to another story: starting from the 60s, one of the poorest quarters of the city of Livorno (Toscana), the quarter Corea, saw the birth and the development of an innovative and particular didactical experimentation: a priest, Don Alfredo Nesi (1923-2003), was able to construct a whole *Villaggio scolastico* (scholastic Village), inside which a kindergartner, an elementary school and a middle school were built. Don Nesi was a friend of Don Milani and shared many ideas with him. The management of didactical activity was original. The aim was the *educazione permanente* (permanent education): a part from the ordinary lessons, the middle school – whose name is "Nicola Pistelli" - of the Villaggio Scolastico, had a *Comitato Scientifico* (Scientific Board), composed not only by the teachers working in the school itself, but by experts coming from the academic world. The scientific board had the aim to coordinate the whole didactical work and to provide and experiment new ideas. The young students remained in the school in the afternoon, too, and in this part of the day, complementary activities were developed. Some students of Pisa university slept gratuitously in the *Villaggio Scolastico*, offering a tutorial activity for the pupils who needed. Meetings and conferences with the most important Italian personalities were organized by the Villaggio Scolastico. Despite the main subject of this meetings was pedagogy, this was not the only one: often problems connected with the recent events were discussed. There were frequent projections of movies with consequent discussion. An important journal, the *Quaderni di Corea*, in which the whole activity of Villaggio Scolastico was summarised and in which the members of the Scientific Board gave their didactical contributions (often these contributions had the dimensions of a book) was published. The life of Villaggio Scolastico was hence well structured and it was open to the whole citizenship. Let us think that Livorno was a city in which the Italian Communist party was the biggest one (at the half of the 70s it reached the absolute majority of the votes) and that this party obtained about the 70% of the votes in the quarter Corea. Despite this, the inhabitancies of Corea were enthusiastic with Villaggio Scolastico and many teachers belonging to the Communist party worked actively in Villaggio Scolastico. This is one of those cases in which a priest of the Catholic Church was able to construct a positive synergy with a communist administration and the result was outstanding from a social point of view because a whole quarter was proud for more than twenty years (the Villaggio continued until the end of 80s) of his experimental scholastic institution (for the news concerning the Villaggio Scolastico, an important source is Scotto di Liquori, 2004). It is necessary to underline that Don Nesi and his group worked for a reformation of the public Italian school. They did not have the aim to propose an educative model based on the private Catholic school (See Pieri-Roncaglia, 1972 and in particular Nesi, 1972).

Vittorio Checcucci, who belonged to the left, was a member of the Scientific board of the school Pistelli at the beginning of the 70s. He became rapidly one of the inspirers of the whole scientific-didactical activity of the Villaggio Scolastico. He participated to the teaching organization with original ideas and published three important contributions (as a matter of fact three brief books) on the *Quaderni di Corea* (see, Checcucci, 1971a, 1971b, 1972). In the meantime Checcucci was also participating to an experimentation in mathematics education in a high school of Pisa. He brought some of these ideas inside Villaggio Scolastico. Checcucci changed the life of Villaggio Scolastico and Villaggio Scolastico gave Checcucci an enthusiasm he had never got before. The end of the 60s and the 70s were in fact the years in which Checcucci was more productive. In the 80s and 90s some ideas developed by Checcucci and by the authors who had inspired him were applied in the middle Italian school, but this was not a general situation and, however, from the middle of 90s the Italian textbooks of mathematics for the middle schools tendentially returned to a more traditional approach. As a matter of fact, Checcucci's point of view demands a constant application from part of teachers and students, much time to dedicate to the work and an engagement that goes far beyond the hours of lesson.

For all the exposed reasons in this introductory section, an exposition of Checcucci's ideas deserves to be known in an international context.

Checcucci's General Pedagogical Ideas and Sources of Inspiration

Checcucci wrote three kinds of contributions as to mathematics teaching: 1) methodological papers; 2) papers and booklets in which he concretely developed his ideas, applying them to a series of different subjects (teaching of arithmetic, algebra, geometry, applied mathematics, and so on). In these contributions, a series of methodological ideas were recalled, too; 3) a textbook for the middle school (in Italian *scuola media*). Despite Checcucci also dealt with mathematics education in the elementary school and at the university, his main contributions concern middle and high school. I will basically focus on them.

Even if Checcucci's production on mathematics education concerns a plurality of themes and methods, it is possible to identify two basic ideas, on which it is founded:

1) mathematics can be learnt only if the pupil himself creates mathematics. A passive learning is neither stimulating nor useful. To develop the mathematical creativeness of the pupils every means can be suitable, if it is used in a correct and opportune way. Because of this Checcucci constructs stories with a mathematical content, uses riddles and games, utilizes instruments as the geoboard ideated by the pedagogue Caleb Gattegno (1911-1988), tries to connect the use and the functioning of the calculating machines with the elementary arithmetical operations, uses particular configurations as the *tangram*. Furthermore he also proposes the recourse to audio-visual media in order to clarify some situations and insists on the fact that the students must be able to use instruments as their own hands, papers and scissors to construct geometrical figures. This manual activity is considered by him significant for the pupils to become confident with the world of geometry. At the same time, drawing carefully is important because this is an active and creative situation that is a necessary propedeutic step to the comprehension of geometrical concepts. This way, applied and abstract mathematics has to be developed in unison and the manual activities become important to stimulate the intellectual ones. Coherently with this conception Checcucci writes of "[…] an exaggerated emphasis on the axiomatic and abstract aspect of mathematics. The consequence of this is a complete separation from every concrete problem" (Checcucci, 1968, 1-2, pp. 225-247. Quotation, p. 225. Original Italian text: "[…] un'enfasi sproporzionata dell'aspetto assiomatico e astratto della matematica, col risultato di una completa dissociazione da ogni problematica concreta."). Between the end of the 60s and the beginning of the 70s, insiemistic became in fashion in mathematical teaching, in Italy and abroad. Checcucci saw in the insiemistic a condensation of those abstract and, at the same time, imprecise, passive and superficial tendencies of mathematical education against which he was fighting. He was explicit at all: "One has just to wonder what is the utility for the child of this new mathematics [insiemistic], full of more and more complex operations; one has just to think that, eventually, our child will wonder […] what needs a binomial or a polynomial" (Checcucci, 1971a. Quotation, p. 6. Original Italian text: "Vien fatto di domandarsi che cosa potrà farsene il fanciullo di questa nuova matematica lastricata di operazioni sempre più complesse; vien fatto di pensare che, alla fine, il nostro fanciullo si ritroverà a domandarsi […] 'a cosa serve un binomio o un polinomio'").

In conclusion, mathematical teaching has to be: a) heuristic; b) to develop the creativity of the pupils; c) to start from situations.

2) Given these precise procedural indications concerning mathematics teaching, one can be surprised thinking that great part of the entire didactical production of Checcucci is centred, directly of indirectly, on the concepts of abstract algebra: a) when he speaks of geometry, he gives the prominence to the transformations and considers the relations between the structure of modern algebra and the transformations themselves; b) when he treats elementary algebra, with the structure of the real numbers, the polynomials, the negative numbers, and so on, Checcucci clarifies the nature of these structure in terms of groups, rings, bodies, fields, according to

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their characteristics; c) in the dense and interesting *Funzioni e grafici* (Ceccanti, Checcucci, Santoni, 1970), dedicated to mathematical teaching in the high schools, he connects the functions, the matrices, the derivatives and the integrals to abstract algebra; d) the same operation is carried out in the not less remarkable *Aspetti algebrici e metrici della geometria elementare. Numeri complessi e trigonometria* (Checcucci, 1969), where the fundamental trigonometric relations are deducted from the algebraic structure of the complex numbers z, such that $|z| = 1$.

One can ask how it is possible that Checcucci, aiming to a heuristic, concrete, creative, non formal and situational approach to mathematical teaching tries to frame numerous aspects of his pedagogical activity inside the highly formal and abstract structures of modern algebra. The answer is that there is no contradiction in these two apparently opposite tendencies: as Giovanni Prodi (Prodi, 1993, p. 429) underlines Checcucci was extremely well-informed on the whole debate concerning mathematical educations that was developing in Europe and in the world. In particular, he shared Piaget's (1896-1980) idea that the evolutive phases of the person, starting from her/his birth until 15/16 years are modelled by mental structures, acquired in the personal evolution, that are similar to the structures of abstract algebra from a formal and operational point of view. Checcucci, following explicitly Piaget, but also other scholars as Bruner (1915-living), writes: "a spontaneous and gradual construction of the elementary logical-mathematical structures exists; these structures are far more similar to those used in 'modern mathematics' than to those used in traditional mathematics" (Checcucci, 1973, pp. 19-23. Quotation, p. 21. Original Italian text: "Esiste una costruzione spontanea e graduale delle strutture elementari logico-matematiche; queste strutture sono assai più vicine a quelle usate dalla 'matematica moderna' di quelle usate dalla matematica tradizionale"). Therefore the explanation – of course at different levels of profoundness, according to the age of the pupils – of the mathematical concepts on the basis of their algebraic nature is coherent with the evolutive phases of the personality. In this sense, a fundamental reference point was, for Checcucci, the work of the "Commission internationale pour l'étude et l'amélioration de l'einseignement des mathématiques" (see references), founded in 1950. Starting from 1955 the Commission began to publish a series of books on mathematics education. The first of these books, whose original French title is *L'einsegnement des mathématiques* (1955), and was translated into Italian in 1960 (I will refer to this edition consulted by Checcucci himself, see, references, Piaget, 1960), contains five contributions, respectively by: Piaget, Dieudonné, Lichnerowicz, Choquet, Gattegno. The first four contributions represent a compact conceptual block because Piaget expresses the ideas already referred; Jean Dieudonné (1906-1992), a famous mathematician and a member of the Bourbaki group, in his contribution traces a brief history of algebra having the aim to show that a series of results obtained in the course of history become clear, from a conceptual and operational point of view, only if they are framed inside groups and rings theory. André Lichnerowicz (1915-1998) titles explicitly his contribution "L'introduzione dello spirito dell'agebra moderna nell'algebra e nella geometria elementare" ("The introduction of the spirit of modern algebra in elementary algebra and geometry"). He is convinced of the necessity to introduce the abstract algebraic structures starting from the beginning of mathematical teaching as follows: "in this teaching, it is necessary to avoid the introduction of abstract theories in a dogmatic way, but the fundamental concepts have to be drawn from the numerous elementary examples that occur in the teaching. Thus, the pupils will become confident with the main structures of abstract algebra, that recur from the beginning [of their mathematical education], even if this in not explicitly pointed out to them." (My translation from the Italian text, p. 65). Gustave Choquet (1915-2006), in his long and profound contribution, "L'insegnamento della geometria elementare" ("The teaching of elementary geometry"), proposes an axiomatization of geometry that starts from a practical point of view: he establishes a priori and in an explicit way certain instruments (rule and compass are typical of Euclid) whose operations determine the axioms of the theory. In this manner the pupils are in front of axioms deduced in a "practical" way from the instruments (while in Euclid and in the classical presentations of Euclidean geometry, the axioms are the first step and there is no direct reference to the instruments from which they are deduced or, anyway the instruments are introduced only after the axioms themselves). Once exposed the axioms like this, the treatment of Choquet become progressively more and more abstract until reaching the isometries and their properties as a group, the basic properties of the Euclidean non-Archimedean plane and the treatment of particular subsets of the Euclidean plane in which a particular metric is defined so that these subsets behave, as a matter of fact, as non-Euclidean planes. Checcucci was deeply influenced by this order of ideas: we have already seen Piaget's influence; but, let us think, that one of the most significant and dense Checcucci's works, *Alla conquista di un contenuto. La geometria delle trasformazioni, ambiente di base per l'apprendimento della matematica* (Checcucci, 1971b), a contribution of almost 100 pages, has Choquet's ideas as an explicit reference point (see Checcucci, 1971b, p. 7 and p. 47). Certainly the pedagogical situation was not easy: the problem was to transform the basic didactical ideas of the Bourbaki group, founded on the conviction that the structures of abstract algebra had to be introduced starting from the school, in a form different from that used by Bourbaki. It is, in fact, known that this group of mathematicians was favourable to an axiomatic and abstract approach in the didactics, too, not only in active mathematical research. Checcucci overcame this difficulty drawing inspiration from another fundamental didactical contribution appeared in the 60s, the "didactical encyclopedia" *The School Mathematics Project*, a very masterpiece in this field, published by the Cambridge University Press, in which all sections of mathematics teaching were dealt with, for elementary, middle and high schools. Checcucci was able to connect the "practical" approach given by *The School Mathematics Project* with the one, more formal and abstract, proposed by the Bourbaki group. This, together with a conspicuous knowledge of other sources, allowed Checcucci to construct a series of original didactic itineraries centred on the explained basic ideas. The best way to clarify the nature of these itineraries is to analyse one of the most significant of them. It concerns geometry and is ideally divided into three steps.

A Didactical Itinerary Drawn from Checcucci's Works: Geometry. 1) The tangram play and geometry at the first year of middle school

At the beginning of the middle school, therefore when the pupils are about 11 years old, Checcucci proposes to introduce the concept of isometry and the link between isometric transformations and the area of a flat region trough the play of *tangram*. The *tangram* is an old Chinese play: it is constituted by a square divided into seven parts in this manner:

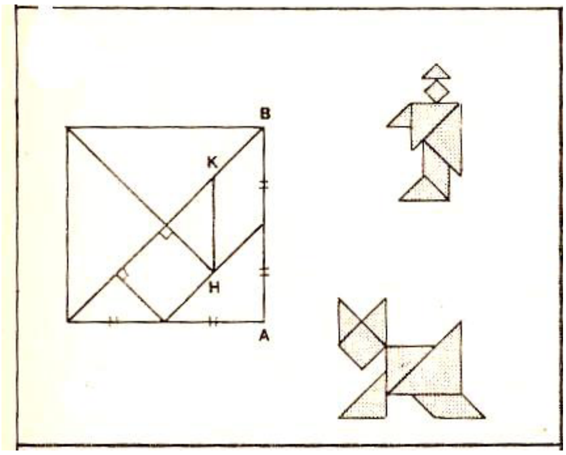

Figure 1: Basic elements of the *tangram* and figures constructed by them (drawn from Checcucci, 1971a, p. 58).

Checcucci was basically inspired by Read to the use of the *tangram* (Checcucci, 1971a,

p. 53 quotes the Italian translation of Read, 1965. In the References I will quote the edition consulted by Checcucci) but part of the way in which the *tangram* is used in mathematical education is original of his.

At the beginning, a first situation can be proposed to the students: let us suppose that the *tangram* square is given with its seven wedges. They can be by paper or wood.

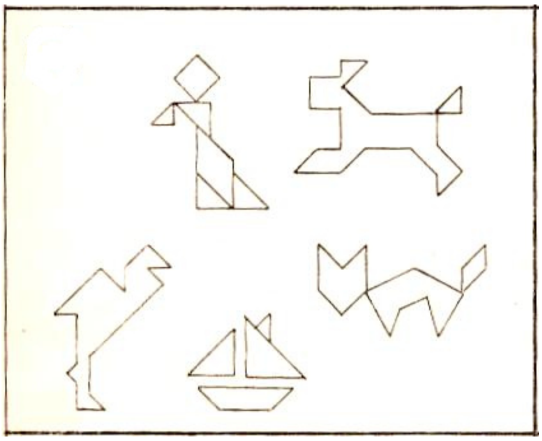

Figure 2: Other figures constructed with *tangram* elements (drawn from Checcucci, 1971a, p. 58).

The play consists in constructing figures like the ones in figure 2: birds, cats, dogs, and every other possible figure, using all the seven wedges. Checcucci claims that this can be enjoyable for the pupils. After this play-phase a question can be posed to the class: what do all configurations have in common?

Here a discussion can begin and this discussion has to be guided by the teacher in a free way, Checcucci gives no indication in this sense, the scene is open. The conclusion of the discussion should be that all configurations have in common a quantity we call the *area* of a surface.

Second situation: the *tangram* square is not constructed; the single wedges are given and the pupils have to construct the square. Here, Checcucci suggests, it is possible to show that, if a white paper is available and the teacher allows the students to use a graduate rule and a graduate set-square, the construction of the *tangram* square is easy, but, if only a compass and a non-graduate rule are allowed, the construction needs a certain ability. Starting from this practical situation, it is possible to touch, in a class-discussion, the intuitive aspect of problems connected to the possibility to carry out certain constructions, given certain instruments. This way of reasoning, empirically justified by the work in the class, can help the students to enter, in a heuristic manner, into questions that, once formalized, are the basis of many of the most profound aspects of mathematics, in particular those connected to the axioms posed in a theory and, consequently, with the operations and constructions allowed in the geometrical environment created by the axioms. At this stage, Checcucci does not speak of axioms, but the whole proposed work has – among other aims – the one to facilitate the eventual full comprehension of the concept of axiomatic method through an intuitive, creative and practical approach. Third situation: the wedges of the *tangram* can be used to introduce the concept of rotation and specular symmetry. Once posed that two configurations has to be considered identical if they derive from a rotation or a specular symmetry of the whole *tangram* square, the pupils can be asked which of the two following configurations has to be considered identical to the initial one (Figure 3).

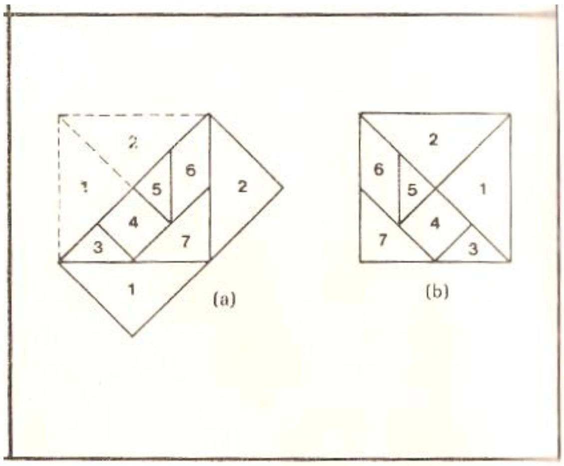

Figure 3: A rectangle and a square constructed by the *tangram* (drawn from Checcucci, 1971a, p. 58).

Once the pupils answered that (a) must not be considered identical, while (b) must, a further question arises: what does have the rectangle (a) in common with square (b)? Once again, the area. But here the teacher can underline that these two geometrical figures have many other shared properties: the angles, the fact that both of them are convex (important concept) quadrilaterals, and so on.

The fourth situation concerns the three fundamental isometries: rotation around a point, specular axial symmetry and translation. Their nature can be clarified by resorting to the wedges of the *tangram*, as, without entering all details dealt with in the text, Checcucci's original figure (see Figure 4) makes it clear.

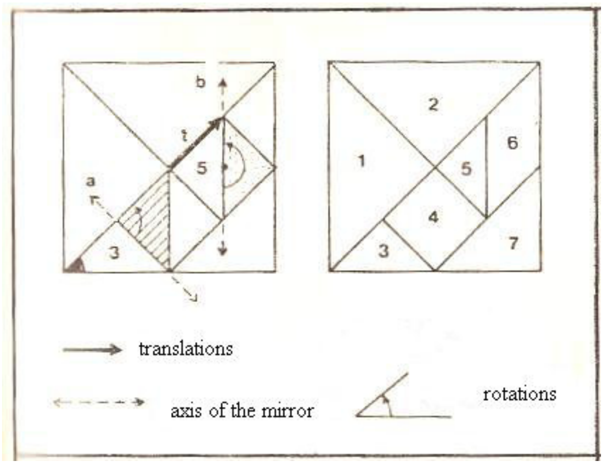

Figure 4: *Tangram* and isometries (Drawn with some modifications from Checcucci, 1971a, p. 59).

Now a series of interesting questions can be asked the pupils: which parts of the *tangram* can be transformed one into the other through the three previous transformations? What means a composition of transformations? Which properties of the figures are invariant if we apply an isometry or a series of isometric transformations? From the answer to these questions, it is possible to clarify the nature of the quantity already intuitively defined as *area* (let us indicate it by

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A): 1) if *R* and *R'* are two flat regions, such that one is the transformed of the other through an isometry, then *A*(*R*)=*A*(*R'*); 2) if *R* and *S* are two not superimposed regions, then $A(R \cup S) = A(R) + A(S)$.

In this manner, starting from an ancient Chinese play, a series of geometrical concepts can be introduced: according to Checcucci, the fact that these concepts have been proposed through a practical activity of the pupils, namely to construct figures with the *tangram* wedges, helps the pupils themselves to be creative and active. Thus, Checcucci thinks that, if the pupils can observe some geometrical facts by a direct and manual construction, they become confident with these facts. Therefore, in the successive educative phases, the frame of these facts inside axiomatic, abstract and rigorous theories will be seen as a natural development of this initial heuristic phase of their learning.

A Didactical Itinerary Drawn from Checcucci's Works: Geometry. 2) *Geometry and axioms at the third year of middle school*

An important contribution written by Checcucci is the already mentioned *Alla conquista di un contenuto. La geometria delle trasformazioni, ambiente di base per l'apprendimento della matematica* (Checcucci, 1971b). Here the interconnection between abstract algebra, geometry and theory of the real numbers is explicit. The booklet is divided into three parts: 1) "Aspetti intuitivi della geometria del piano" ("Intuitive aspects of plane geometry"); 2) "Aspetti assiomatici della geometria del piano" ("Axiomatic aspects of plane geometry"); 3) "I numeri reali" ("The real numbers"). This contribution is dedicated to the teachers of middle schools and, basically, concerns the pupils of the last year (13-14 years old). The part 2) "Axiomatic aspects of plane geometry" will be analysed because here Checcucci, inspired by the conceptions explicitly expressed for the first time by Felix Klein (1849-1925) in his "Erlangen program", introduces the axiomatic properties of plane geometry basing on the concept of transformation, a concept that, as well known, is strictly connected to abstract algebra and, in particular to the group of transformations. As to the axiomatic aspects, Checcucci follows Choquet's ideas. The result is an interesting, albeit complex, educational itinerary that reaches the proof of Tales theorem and the properties of the homotheties and of the similarities as most significant results. Everything is carried out without resorting to the Euclidean theory of proportions – deemed too complicated and limited -; however, to avoid the use of proportions a series of further concepts has to be introduced, in particular those of function, real number and numerical straight line. Checcucci thought that these concepts were more useful because the pupils face with them in many fields of mathematics and applied sciences, while the use of proportions is limited. In the axiomatic presented by Checcucci, the concept of *grandezza* (size) is not defined and assumed as a primitive one. The set **G** of all the sizes forms a commutative group, considering the product between sizes as the group operation. Furthermore the real positive numbers $\Re^+$ are a subgroup of the sizes, hence the number 1 (the multiplicative neutral element of $\Re^+$) is the neutral element of the whole group **G**. Given two sizes of the same type, it is possible to define their sum and to establish a comparison between them according the relation ">", as it is the case for the real numbers. After this introductory section, Checcucci briefly analyses the properties of the plane that can be obtained without introducing the concept of distance (pp. 50-54). Here the axiom A_1: "given two point *P* and *Q* only a straight line can be traced that contains *P* and *Q*" and A_2: "given a straight line *a* and a point *P* that not belong to *a*, a sole parallel to *a* can be traced trough *P*" are introduce. Two further axioms are: A_3, according to which two opposite verses are given on a straight line and A_4 establishing that the projection of a straight line *a* on the straight line *b*, in parallel to a third straight line *d*, transforms segments into segments. By these axioms some elementary properties of the parallel lines are proved (as the transitivity of the parallelism). The following step consists in introducing the metrical properties of the plane trough the concept of length of a segment. This concept is assumed as a primitive one. The two fundamental axioms concerning the length of a segment are: A_5: "The lengths of the segments have two properties; a) if *A,B,C* are three points of the plane, then $\overline{AB} \leq \overline{AC} + \overline{BC}$; b) it is $\overline{AB} = \overline{AC} + \overline{BC}$ if and only if *C* belongs to the segment *AB*" and A_6: "let *a* be a half line with origin in *O*; given a length *p*, a sole point *P* exists such that. With $\overline{OP} = p$ these axioms that establish the properties of the length of the segments, Checcucci comes back to the concept of isometry defining an isometry as a transformation that maintains the lengths of the segments. Beyond this property the isometries have the properties formulated in the following theorem **T**: "every isometry transforms: straight lines into straight lines; segments into segments; half lines into half lines; half planes into half planes, parallel lines into parallel lines, opposite half lines into opposite half lines; opposite half planes into opposite half planes". Checcucci suggests the teachers to ask the pupils a question: do the properties expressed by the theorem **T** connote only the isometries, or are there other more general transformations that fulfil them? The answer is affirmative. These transformations are the similarities with their similarities ratio *k*. Checcucci underlines that the similarities are a group and suggests to ask the pupils what this means and what the similarities with *k*=1 are.

To proceed towards Tales theorem without resorting to the properties of the proportions, it is necessary to introduce the numerical line. Therefore Checcucci poses a system of abscissas, assuming the origin of this system in the point *O* of the axiom A_6. In this way a positive and a negative part of the numerical line is univocally determined. The relation between the length of the segment $\overline{PQ}$ and the abscissas *x* and *y* of the two point *P* and *Q* is given by the numerical relation $\overline{PQ} = |x - y|$, namely $\overline{PQ}^{\,2} = (y - x)^2$. By the association of every point of a straight line with the real numbers, it is possible to study the similarities and, in particular the isometries, through an analytical approach. Now Checcucci introduces the concept of perpendicularity by the definition of the specular symmetry, through a last axiom A_7: "For every half plane D' and the opposite half plane D'', one and only one function f exists, $f : D' \to D''$, such that 1) *f* is biunivocal, 2) *f* is an isometry, 3) every point *P* of the line *d*, that separates D' from D'', is reflected on itself" (p. 58). This function is called specular symmetry with re- spect to the line *d*. The following step that Checcucci suggests to the teachers is the definition of the distance point-straight line. If the point *P* is on the given line *d*, there is nothing to say, if *P* is out of the line, then one considers the specular point P' of *P* in respect to *d*, calling P_o the point in which the segment $P\ '$ touches *d*. P_o is called the orthogonal projec- tion of *P* on *d*. The half of the segment $P\ '$ is the required distance. The perpendicularity is introduced by the following theorem: "let *a* be a straight line different from *b* and let *P* be a point of *a* out of *b*. Let $P' = s_a(P)$, the symmetric point of *P* in respect to *b*, P_0 the orthogonal projection of *P* an *b*, then the following facts are equivalent: 1) $s_b(a) = a$; b) *a* is the line $P\ '$; c) *a* is the line $P\ _0$" (p. 60). Once posed this theorem, the straight line *a* is called the perpendicular to *b* in P_0. Hence, coherently with his general vision, Checcucci deals with the perpendicularity, starting from a transformation: the specular symmetry. Now, many of the classical theorems on the perpendiculars are proved (i.e. the uniqueness of the perpendicular drawn from a point to a straight line). The following fundamental step (and the last one we will analyse) concerns the Tales theorem. However, before dealing with this theorem, Checcucci introduces another transformation: the symmetry with respect to a point, with the fundamental theorem that the symmetry with respect to a point *O*, is the product $s_a \circ s_b$ of two symmetries with respect to two perpendiculars passing through *O*.

The enunciation of Tales theorem is like this (see figure 5) : "let *a* and *b* are two half lines with origin in *O* and both of them are intersected by a line *d*, and let us project *a* on *b* in parallel to *d*. If P' is the projection of *P*, the relation $OP'/OP = k$ exists between the lengths of the segments *OP* and OP', where *k* is a constant" (p. 63).

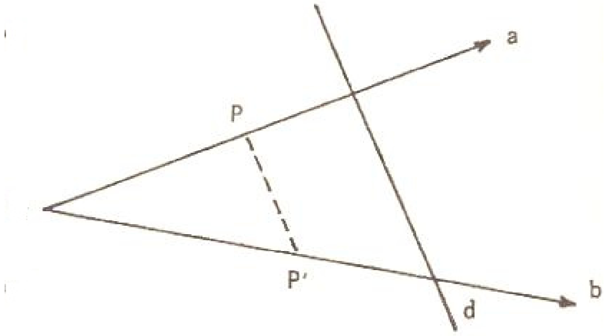

Figure 5: Tales theorem 1 (drawn from Checcucci, 1971b, p. 63).

The proof is articulated in the following steps: 1) the half lines *a* and *b* are made numerical (see figure 6). The projection *f* of *a* on *b* (in parallel to *d*) associates the number $x' = f(x)$, that is the abscissa of the projection P', to every positive number *x* (abscissa of *P*).

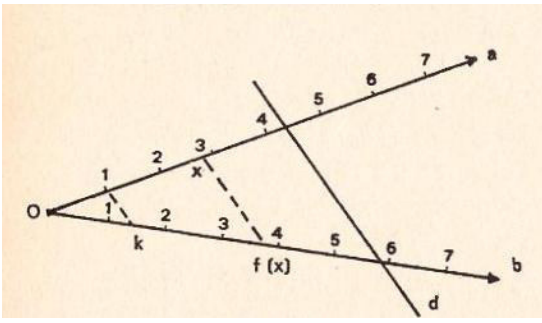

Figure 6: Tales theorem 2 (Drawn from Checcucci, 1971b, p. 68).

2) On the base of previous axioms and theorems it is easy to prove that *f* is an increasing function and that *f* is additive, namely $f(x+y) = f(x) + f(y)$; 3) let be $k = f(1)$, the theorem will be proved if, for every real positive number *x*, it is $f(x) = kx$. Since *f* is additive, it is $f(2x) = f(x+x) = f(x) + f(x) = 2f(x)$ and, through the same reasoning, it will be $f(mx) = mf(x)$, for every natural number *m*. Therefore: a) if $x=1$, $f(m) = fm(1) = km$, because $k = f(1)$; b) if $x = \frac{1}{n}$ and $m=n$, then $f\left(n \cdot \frac{1}{n}\right) = n \cdot f\left(\frac{1}{n}\right)$, therefore $f\left(\frac{1}{n}\right) = k \cdot \frac{1}{n}$; 3) finally if $x = \frac{1}{n}$ without further limitations, it is $f\left(\frac{m}{n}\right) = m \cdot f\left(\frac{1}{n}\right) = k \cdot \frac{m}{n}$ and this proves

the theorem for every fraction. A further refined *ad absurdum* reasoning is necessary to demonstrate this propositions for the irrational numbers. I do not face this proof (see, pp. 68-69).

Many details have been exposed to show the coherence of Checcucci's way of thinking. In this case, the treatment of geometry has been based on the concept of transformations and on the way in which particular transformations operate; furthermore many specifications on the nature of the real numbers and of the functions are needed. In this section, the problem whether the educational itinerary proposed by Checcucci can be realistically carried out is not dealt with. This question will be addressed in the final section. The proposed pedagogical didactical itinerary will be concluded by its third step.

A Didactical Itinerary Drawn from Checcucci's Works: Geometry. 3) *Trigonometry and complex numbers at the final years of the high school*

Checcucci thought that the concept of complex number had to be introduced in the high schools because the complex numbers allow to see many parts of mathematics under a unitary point of view. He was aware of the difficulties that a treatment of the complex numbers implies and therefore he proposed to introduce only those properties that are more strictly connected with geometry and trigonometry, continuing to take into account the algebraic properties of particular subsets of complex numbers. The work in which these ideas are developed is *Aspetti algebrici e metrici della geometria elementare: numeri complessi e trigonometria* (Checcucci, 1969). Many interesting questions are dealt with. Here the explanation how Checcucci proposed to deduce the fundamental trigonometric relations from the complex number whose modulus is 1 will be provided. A necessary presupposition is that the student has been already introduced into the concepts of abstract algebra (structures: group, ring, body, field, quotient set. Relation between structures: homomorphism, isomorphism, automorphism) and that he is familiar with specific structures as the ordered body of the real numbers, the ring of the matrices 2x2, the ring of the polynomials with integral, rational and real coefficients. The elementary properties of the complex numbers can be introduced in more than one manner of which Checcucci speaks briefly (pp. 63-65). Given the set of the rotations R_o around the point *O*, let us establish a Cartesian system of axes with its origin in *O*. As it is classical, the *x*-axis represents the real part of a complex number and the *y*-axis the imaginary part. The analysis is limited to the complex number whose modulus is 1. Given a complex number $z = x + iy$ and its conjugate $z = x - iy$, only the following properties (whose proof is easy) are exploited: a) if *z* and z' are two complex numbers, then $\overline{z + z'} = \bar{z} + \bar{z}', \overline{z \cdot z'} = \bar{z} \cdot \bar{z}'$; b) if $z = x + iy$, then its modulus $|z|$ is defined as $\sqrt{x^2 + y^2}$ and it is $|z| = \sqrt{z \cdot \bar{z}} = \sqrt{x^2 + y^2}$, with the properties 1) $|z + z'| \leq |z| + |z'|$; 2) $|z \cdot z'| = |z| \cdot |z'|$; 3) $|\bar{z}| = |z|$.

Checcucci considers a circumference whose radius is 1 (Figure 7).

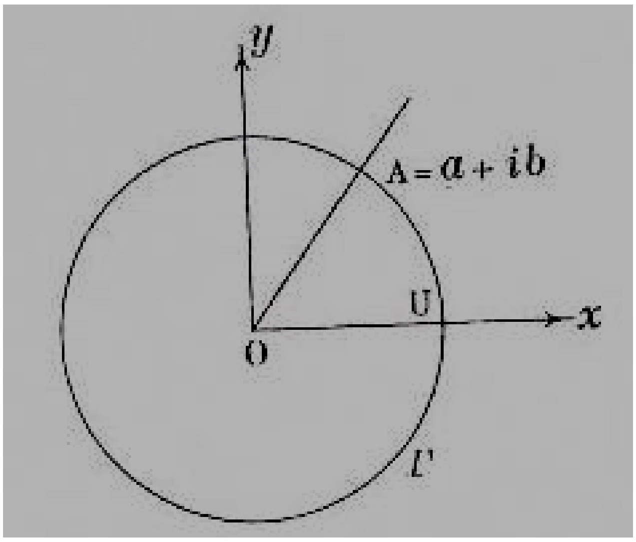

Figure 7: The complex numbers with modulus 1 and the rotations (Drawn from Checcucci, 1969, p. 66).

He poses the origin of the Cartesian system in the centre of the circumference, so that every point of the circumference can be interpreted as a complex number *z=a+ib* (*A* in the figure), whose modulus is 1. Since, given two half-lines *p* and *q*, a unique rotation exists that brings *p* on *q*, every rotation can be represented by the complex number *a+ib*, if we identify, as it is legitimate given the uniqueness of a rotation, one of the two half lines *p* or *q* with the positive half-line of the *x*-axis. The truth of this consideration is extremely plausibly with our heuristic reasoning and its proof is very easy, as Checcucci shows (p. 66). In this way it is possible to associate to every rotation R_o a complex number whose modulus is 1 (let us indicate this subset by *U*) through the correspondence $\gamma : R_o \to U$, being $\gamma(\theta) = \alpha$, , where θ is a rotation and $\alpha = a + ib$ is the complex number associated to θ, being $|\alpha| = 1$. Checcucci can now easily prove the following fundamental theorem: "the set R_o of the rotations around a point *O* is a commutative group of isometries. It is isomorphic, through γ, to the multiplicative group *U*." In particular this means that, the operation of group connoting the rotations, that is the addition, is biunivocally associated to the group operation of the complex numbers whose modulus is 1, that is the multiplication. In this manner, if $\theta_0, \theta_1, \ldots, \theta_n$ are rotations and $\alpha_0, \alpha_1, \ldots, \alpha_n$ are the complex numbers of *U* associated to them, it is $\gamma(\theta_0 + \theta_1 +, \ldots, +\theta_n) = \gamma(\theta_0) \cdot \gamma(\theta_1) \cdot, \ldots, \cdot \gamma(\theta_n) = \alpha_0 \cdot \alpha_1 \cdot, \ldots, \cdot \alpha_n$. Now let us see how the addition laws for cosine and sine can be obtained (p. 74): let θ and θ' be two angles, interpreted as rotations whose origin is the real positive axis of *U*. The writing $Cos(\theta + \theta') + iSin(\theta + \theta')$ indicates both the rotation $\theta + \theta'$ and the associated complex number whose modulus in 1. Because of the isomorphism $\gamma : R_o \to U$, the following identities are satisfied:

$$Cos(\theta + \theta') + iSin(\theta + \theta') = \gamma(\theta + \theta') = \gamma(\theta) \cdot \gamma(\theta') =$$
$$= (Cos\,\theta + iSin\theta)(Cos\,\theta' + iSin\theta') =$$
$$= Cos\,\theta \cdot Cos\,\theta' - Sin\theta \cdot Sin\theta' + i(Sin\theta \cdot Cos\,\theta' + Cos\,\theta \cdot Sin\theta')$$

From a didactical point of view, the result obtained in this manner is significant because the trigonometric formulas for the sum of cosines and sins have been achieved by means of the general structures and relations of abstract algebra showing to the students – starting from high school – that these structures do not have only a value as far as they allow to see different mathematical facts under a new unitary point of view, but they also allow to obtain "concrete" mathematical results, as the trigonometric formulas. Checcucci proceeds further on: in the theory of complex numbers, De Moivre formula is fundamental. It claims that $(Cos\,\theta + iSin\theta)^n = Con(n\theta) + iSin(n\theta)$. Because of the isomorphism $\gamma : R_o \rightarrow U$, one has:

$$(Cos\,\theta + iSin\theta)^n = \underbrace{\gamma(\theta)\cdot\gamma(\theta)\cdot\ldots\cdot\gamma(\theta)}_{n-times} =$$
$$= \gamma(\underbrace{\theta+\theta+\ldots+\theta}_{n-times}) = \gamma(n\theta) =$$
$$= Con(n\theta) + iSin(n\theta)$$

In this case the isomorphism has been exploit in the opposite sense than in the proof of the previous formula, that is as $\gamma :\ U \rightarrow R_o$, showing in the clearest manner the biunivocal nature of the isomorphic relations and the deep link between trigonometry and complex numbers. Checcucci adds many other interesting considerations, but what exposed is enough to understand the nature of his ideas with regard to the use of abstract algebraic structures in fields of mathematics different from abstract algebra itself.

In this manner an itinerary connecting geometry (starting from the most elementary plays with the *tangram* until reaching the geometrical properties of the complex numbers) to practical and empirical constructions – wedges of the *tangram* – and to the abstract algebraic structures has been traced. This itinerary has a formative purpose. It aims to give a a unitary vision of mathematics and of mathematical processes. Other itineraries could be traced basing on Checcucci's works, even if it is necessary to underline that Checcucci did not care much to propose explicitly these itineraries, rather he presented didactical proposals oriented in a certain direction, and the teachers had to construct the itineraries; this is their specific job. A question arises: is the way in which Checcucci proposed to frame mathematical teaching realistic? The next section will be dedicated to this important question.

In conclusion of this section, it is possible to summarize and expose Checcucci's fundamental ideas on mathematical education: it makes no sense to present mathematics as a set of isolated results and branches. According to Checcucci, mathematics has to be proposed as a unitary construction. This is why he is favourable to anticipate as soon as possible the properties of algebraic structures and to frame inside these structures the results and the concepts of different fields of mathematics as elementary algebra, geometry, trigonometry and analysis. Checcucci looks to propose and educative process like this: in the initial phase of the learning of a certain field of mathematics, the concepts have to be introduced with creative, intuitive, constructive and "practical" methods: we have seen this for geometry with the *tangram*, for pupils about 11 years old, but Checcucci thinks that such procedure has to be followed, i.e., also for the introduction of mathematical analysis in the last years of the high school (see Ceccanti, Checcucci, Santoni, 1970). The second step is the presentation of the axioms connoting a theory, basing on practical instruments and experiments. At the same time, the properties of the objects of the theory are interpreted with the notions of abstract algebra. Such notions have been progressively introduced in an intuitive manner, looking at the operations possible in a certain environment (i.e., the isometries learnt by the *tangram*). Finally, one shows how the relations and operations of abstract algebra can help to deduce a series of specific properties (we have seen the example of the trigonometric formulas). Checcucci is against the approach that could be called of the "local axiomatization and formalization", that is to an approach in which the properties of the single theories are introduced axiomatically without taking into account either the empirical and genetic bases of the theories or their frame inside the structures of abstract algebra. The man has hands and brain, while the "local axiomatization and formalization" does not allow to use either the hands because it despises the empirical aspects of mathematics, or the brain in a correct way because the human mental structures are similar to those of modern algebra and not to those imposed by the "local axiomatization".

The Fortune and Misfortune behind Checcucci's Ideas

To understand the fortune and misfortune behind Checcucci's ideas and to realize which of these ideas can be useful for mathematical teaching in the 21st century middle and high school, it is necessary: 1) to frame these ideas inside the historical period in which Checcucci worked; 2) to take into account his specific personal character and didactical convictions.

As to 1), in 1962 in Italy the law n. 1859 approved the new unified middle school. This was an important moment because starting from 1861, the year in which Italy was unified, there was never a sole middle school. The instruction for young between 11 and 14 was different for those who would have followed an eventual study itinerary with the *licei* (humanistic and, successively scientific, too, high schools) and for those who would have frequented technical high schools. In 1928 a middle professional school ("Scuola di avviamento al lavoro") was created for the pupils who after the middle school, would have entered into the work world. The law n. 1859 can be interpreted as a consequence of the so called economic-boom, a particularly favourable economic conjuncture that characterized Italy in the 60s of the 20th century. In this context, the industrialists themselves needed a more refined instruction for workers. From here the decision that the workers classes should have had an instruction similar to that of the higher classes, at least until middle school. Since the middle unified school was not a tradition in Italy, the law n.1859 posed a series of great problems on how was opportune to organize the education for a school that, at least in the intentions, should not have been classist. The argument is too broad to be dealt with in this context. Let us only remember that positions like those of Don Milani or Don Nesi have to be considered inside this context. A little – even if not secondary – part of the outlined picture concerns mathematical education. Among the proposals produced in Italy between the 60s and the 70s of the 20th century, Checcucci's was one of the most original. His didactical proposals are based on two inspirations: 1) social convictions: Checcucci was against an elitary school and favourable to a school that offered everyone the chances for a social ascent and that offered people, who after middle school would have stopped their studies, the chance to have a fruitful and useful basic education; 2) mathematical convictions: we have analysed them in the previous section. Taking into account 1), it is possible to understand a theme that was particularly dear to Checcucci: the comprehension of the "practical" value of mathematics in order to solve daily life problems. When it is possible to put in a quantitative form these problems, they can be dealt with far easier and more perspicuously than if they remain at a mere qualitative level. This capability must be assimilated from the beginning of mathematical instruction because it is an useful *forma mentis* both for young who will continue their studies and for those who will stop them at 14 (or nowadays at 16). To this problem is dedicated the whole contribution Checcucci, 1972. The book in three volumes Checcucci wrote for each of the three years of the middle schools (see, Checcucci 1974a) is based on this conviction, too. The title itself of the textbook is indicative: *Matematica e realtà per la scuola media* (*Mathematics and reality for middle school*). A famous mathematician as Bruno De Finetti, wrote a review of Checcucci's book writing: "the spread of this approach in Italian school would represent an improvement of enormous value" (De Finetti, 1974, p. 31. Original Italian text: "La diffusione di un'impostazione di questo genere nella scuola italiana significherebbe un 'salto di qualità' di portata inestimabile"). De Finetti clearly explains the basic approach of Checcucci's textbook. A conspicuous inspirations derived him from the "School Mathematics Project". Let us think that, when UMI (Unione Matematica Italiana, Italian Mathematical Union) decided to translate the first three books of SMP from English into Italian, Checcucci was one of the editors (see, references, UMI, 1972-1973). Beyond the general ideas, the way in which the content is proposed is important, as well. De Finetti stressed in particular: 1) the textbook by Checcucci teaches how to transcribe the problems of daily life in a mathematical form, in part basing on Polyas's ideas; 2) the Cartesian coordinates are introduced starting from a play, called "naval battle", in which a net of mutually perpendicular lines is necessary. In general the aspects of mathematics connected to plays are put in evidence; 3) many exercises are based on the practical capability of the pupils to fold and cut sheets in order to construct geometrical figures. In this way, De Finetti concludes, every new acquisition appears as a conquest and the pupil is never passive. This picture is coherent at all with what has been described in the former sections of this paper. This approach looks extremely reasonable from a pedagogical point of view. Nevertheless, Checcucci did not succeed (a part some exceptions) in making his ideas accepted and put in practice in Italian middle and high school. It is to wonder why.

In this sense two interviews with two former university students of Checcucci have been very useful: one interview has been carried out by me with Claudio Santoni, former Professor for Mathematical Analysis at Naval Academy, Livorno, now retired, and one with Maria Alessandra Mariotti, Professor for Mathematics Education ("Didattica della matematica") at Siena University. Both of them were Checcucci's students at the beginning of the 70s in Pisa University and both of them agreed on the merits and the defects of Checcucci's teaching. Merits: a) serious preparation both in mathematics and in mathematics education; b) continuous update as to the contributions in didactics of mathematics published in Europe and in the world; c) original ideas; d) enthusiasm in communicating with students, colleagues and high schools teachers; e) full involvement in many educational projects. The defect was one, but, according to them, remarkable: a) Checcucci had many original ideas and many ideas drawn from literature, but he had difficulty to give a systematic and clear form to these ideas. Both Santoni and Mariotti remember that often they left Checcucci in the evening, having established an appointment few days later to develop a certain train of thought and at the appointment Checcucci had changed his mind and the work had to restart from the beginning. This non systematic tendency can be deduced also reading some of his publications (for example Checcucci, 1971b) in which a series of ideas on didactics of geometry are present, but: 1) the proofs are almost completely lacking because Checcucci thought that every teacher had to research in his mind or in the literature such proofs; 2) sometimes it is difficult to follow the train of thought on which Checcucci's proposals are based. This tendency can also be interpreted as a superior point of view on education, that is: the scope of the studies in mathematics education is not to provide a completely refined "product", but some basic educational lines on which the university students in mathematics education and the teachers have to work giving their original contribution. This approach needs a lot of work and a complete dedication. Mariotti and Santoni claim that the dedication needed was, in their opinion, excessive, for students and teachers. The critics is hence that the line of mathematical education proposed by Checcucci was not always clear and it was often non realistic because of the conspicuous – and perhaps excessive - engagement required both from pupils's and teachers's part. According to Checcucci, mathematical instruction had to take into account, in every its step of: 1) a creative approach, based upon single problems, games, figures and every device that can develop the interest and the intuition of the students, this means an approach à la "School Mathematics Project", also influenced by Polya's profound works. In particular, with regard to the importance of presenting mathematical education in terms of problems solving, Checcucci refers to two Polya's works (see, Checcucci 1971b, p. 97. The references are to Polya, 1967, Italian translation of Polya, 1945; Polya, 1970, Italian translation of Polya, 1967; Polya, 1971, Italian translation of Polya, 1962); 2) a structuralist approach, based on the idea to introduce the structures of abstract algebra in an early phase of mathematical education. In the previous sections we have seen how Checcucci made these apparently incompatible approaches coherent. Nevertheless, it can be understood that many teachers and scholars, too, had difficulties to enter into Checcucci's way of thinking and in applying his proposals. In the 80s of the 20th century there were some textbooks that in part were inspired by Checcucci's work. The most important of these texts for middle schools was probably Cambini-Capecchi, 1980. In 1970 Checcucci also wrote an experimental textbook in mathematics for the first two

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years of the technical nautical high schools and the book was published by the technical nautical school of Livorno (see references, Checcucci, 1970a). Between the second half of the 70s and the 80s of the 20th century there were in Italy various experimental schools, but the experimentation in mathematics followed in general approaches different from those proposed by Checcucci. Nowadays in Italy, most part of textbooks in mathematics are conceived in a way that made it difficult to discover the didactical line (if it exists) followed by the author/s because they appear as a series of subjects and exercises conceived to satisfy ministerial programs rather than to indicate a precise didactical approach. Instead, most part of persons who have seriously studied the problems of mathematical educations think that a precise didactical line is necessary and certainly Checcucci thought this way. Thus, what part of Checcucci's ideas con be proposed for mathematical education in the 21th century?

A) The idea that specific problems must be the basis to introduce a certain mathematical theory. In the research, too, the different fields of mathematics originate from single problems. The theories are constructed in the course of the time. Starting from single problems, the pupils can understand why a theory become necessary. If we introduce a theory in an abstract-axiomatic manner, the pupils hardly can understand its utility and limit themselves to learn it in a passive manner. This is valid from the most elementary acquisitions at the beginning of the middle schools until the last years of high school where pupils learn the theory of functions.

B) In connection to this: a methodology that stimulate the creativity is necessary. Checcucci proposes methods based on practical activity, as those analysed in the previous sections, through which the pupils can construct the figures the teachers speak about. The mathematical riddles are another good means proposed by Checcucci to stimulate creativity. Sometimes audio-visual means, too, can be used. Nowadays development of informatics offers many means that, in Checcucci's time were not yet available, but that surely Checcucci would have used.

C) Application of mathematics: this is an important aspect because if applications of mathematics to daily life and to other sciences, as physics, chemistry, biology, ecc. is showed, the pupils immediately understand how broad and interesting is the range of human activities in which mathematics plays a fundamental role and avoid to consider mathematics only an abstract discipline for specialists.

D) Rigour in mathematics: the educational phase in which the formal structure of a theory and the role and nature of axioms is clearly explained is necessary. Checcucci insists on this fact, for example as to the axioms of geometry. The studies on the best way how to propose the axiomatization of geometry in the middle and high schools date back to the second half of the 19th century and there is not a general agreement among scholars. Checcucci, as we have seen, proposes his ideas following in part Choquet. However, nowadays in many textbooks, the concept of axiom is not clearly explained. Due to the almost complete replacement of Euclidean geometry with analytical geometry, the problems are solved through equations. In this manner: 1) the concept itself of hypothetical-deductive system risks to get lost; 2) the concept of proof risks to be identified with a mere preconceived procedural process with the consequence that; 3) the creative aspect of mathematics is missed

The story of Checcucci is the story of a man with a great passion and with a fertile mind rich of innovative ideas. To this passion he devoted more than 20 years of his life with original contributions. Part of his ideas could be fruitful revised nowadays. Therefore, this work wishes also to be a stimulus for non Italian experts in educational sciences to directly discover and analyse Checcucci's works.

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