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Abstract

Motivated by a recent theorem of Comfort and van Mill, we study when a pseudocompact abelian group admits proper dense minimal pseudocompact subgroups and give a complete answer in the case of compact abelian groups. Moreover we characterize abelian groups that admit proper dense minimal subgroups.

Key words: compact (abelian) group, minimal group, essential subgroup, pseudocompact group.
1991 MSC: Primary: 22A05, 54H11; Secondary: 22C05, 54D25.

1 Introduction

Throughout this paper all topological groups are Hausdorff. A topological group G is *pseudocompact* if every continuous real-valued function of G is bounded [22]. Moreover a pseudocompact group G is *s-extremal* if it has no proper dense pseudocompact subgroup and *r-extremal* if there exists no strictly finer pseudocompact group topology on G [2]. Recently in [8] Comfort and van Mill proved the following relevant result about pseudocompact abelian groups, which solves a problem raised in 1982 [3,7] and studied intensively since then.

Theorem 1.1 [8] *A pseudocompact abelian group G is either s- or r-extremal if and only if G is metrizable (hence compact).*

Note that according to Theorem 1.1 a pseudocompact abelian group G has a proper dense pseudocompact subgroup if and only if G is non-metrizable. The

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validity of this result for compact abelian groups was proved in [3] by Comfort and Robertson.

In this paper we study the pseudocompact abelian groups which admit proper dense pseudocompact subgroups with some other properties.

As an immediate corollary of Theorem 1.1 we obtain the following theorem in case the dense pseudocompact subgroup is required to be also either s- or r-extremal.

Corollary 1.2 *A proper dense pseudocompact subgroup of a topological abelian group cannot be either s- or r-extremal.*

PROOF. If H is a dense either s- or r-extremal pseudocompact subgroup of G, then H is compact by Theorem 1.1. Hence H is closed in G and so $H = G$ since H is also dense in G.

A topological group G is *minimal* if there exists no strictly coarser group topology on G. The concept of r-extremal pseudocompact group is "dual" to that of minimal pseudocompact group in the sense that a pseudocompact group is minimal if there exists no strictly coarser pseudocompact group topology on G. So Corollary 1.2 suggests the following problem which turns out to be less trivial:

Problem 1.3 *When does a pseudocompact abelian group admit a proper dense minimal pseudocompact subgroup?*

A description of dense minimal subgroups of compact groups was given in [23,25] in terms of essential subgroups; a subgroup H of a topological group G is *essential* if H non-trivially intersects every non-trivial closed normal subgroup of G [23,25]. The "minimality criterion" given in [23,25] asserts that a dense subgroup H of a compact group G is minimal if and only if H is essential in G. So in the case of compact abelian groups Problem 1.3 becomes:

Problem 1.4 *Determine when a compact abelian group admits a proper dense essential pseudocompact subgroup.*

In this paper we study and completely solve Problem 1.4.

A particular case of Problem 1.4 has already been considered by many authors, namely the problem of the existence of proper (strongly) totally dense pseudocompact subgroups of compact groups. A subgroup H of a topological group G is *strongly totally dense* if H densely intersects every closed subgroup of G, and it is *totally dense* if H densely intersects every closed normal subgroup of G [24]. Note that these two concepts coincide in the case of abelian groups. The totally dense subgroups of a compact group K are precisely those dense subgroups of K that satisfy the open mapping theorem [15,16,21]; the groups with this property were introduced in [15] under the name *totally minimal*. Note that totally dense $\Rightarrow$ dense essential (totally minimal $\Rightarrow$ minimal).

This question of when a compact abelian group K admits a proper totally dense pseudocompact subgroup was studied for the first time by Comfort and Soundararajan [7], and they solved it in case K is a connected compact abelian group (the answer is if and only if K is non-metrizable).

Later Comfort and Robertson studied the compact groups K that admit a strongly totally dense pseudocompact subgroup H which is *small* (i.e., $|H| < |K|$). They showed that ZFC cannot decide whether there exists a compact group with small strongly totally dense pseudocompact subgroups [4, Theorem 6.3]. We discuss the counterpart of this argument for small essential subgroups of compact groups elsewhere [11].

A topological group G is *countably compact* if every countable open cover of G has a finite subcover (compact $\Rightarrow$ countably compact $\Rightarrow$ pseudocompact). In [13, Theorem 1.4] Dikranjan and Shakmatov proved that a compact group K has no proper strongly totally dense countably compact subgroup.

On the other hand there exist compact abelian groups with proper dense essential countably compact subgroups [14,18]. Moreover the problem of the existence of connected compact abelian groups with proper dense essential countably compact subgroups is not decidable in ZFC, yet it is equivalent to that of the existence of measurable cardinals — see [17] and [9, Theorem 5.7].

A necessary condition for a topological group G to have a strongly totally dense pseudocompact subgroup was given in [13]: G does not admit any torsion closed G_δ-subgroup. In the case of compact abelian groups this condition was proved to be also sufficient under some set-theoretical assumption.

Topological groups G such that there exists a closed G_δ-subgroup N of G with $N \subseteq t(G)$ are called *singular*. Singular groups have been defined in a different (equivalent) way in [12, Definition 1.2]; we discuss this in Lemma 2.5. By means of a detailed study of the structure of singular compact abelian groups, it became possible to give a complete solution to the problem of when a compact abelian group has a proper totally dense pseudocompact subgroup:

Theorem 1.5 [10, Theorem 5.2] *A compact abelian group K has no proper totally dense pseudocompact subgroup if and only if K is singular.*

Example 3.1 shows that a compact abelian group can have a proper dense essential pseudocompact subgroup even if it has no proper totally dense subgroup.

Singular groups have a “big” torsion part and they were useful to prove Theorem 1.5; so we analogously define groups with a “big” socle:

Definition 1.6 *A topological abelian group G is* super-singular *if there exists a closed G_δ-subgroup N of G such that $N \subseteq Soc(G)$.*

Each super-singular topological abelian group G is singular since $Soc(G) \subseteq t(G)$. In this paper we describe the structure of super-singular compact abelian groups and solve Problem 1.4. In fact, in view of Theorem 1.5, we can consider the case when K is a singular non-metrizable compact abelian group, and Theorem 1.7 gives necessary and sufficient conditions in order that a singular non-metrizable compact abelian group K admits a proper dense essential pseudocompact subgroup.

Let p be a prime number, and let K be a compact abelian group. The *topological p-component* of K is $K_p = \{x \in K : p^n x \to 0 \text{ in } K \text{ where } n \in \mathbb{N}\}$. The *$p$-component* $T_p(K)$ of K is the closure of K_p (for more details see Remark 2.1).

Theorem 1.7 *Let K be a compact abelian group. Then the following conditions are equivalent*

(a) K admits no proper dense essential pseudocompact subgroup;
(b) K is singular and $pT_p(K)$ is metrizable for every prime p; and
(c) K is super-singular.

This theorem solves Problem 1.3 in the compact case, that is Problem 1.4, but it does not give an answer in the general case, that is for pseudocompact abelian groups. Moreover other related questions arise (note that (a) coincides with Problem 1.3):

Problem 1.8 *(a) Describe the pseudocompact abelian groups that admit proper dense minimal pseudocompact subgroups.*
(a^*) Describe the pseudocompact abelian groups that admit proper dense essential pseudocompact subgroups.
(b) Describe the pseudocompact abelian groups that admit proper dense totally minimal pseudocompact subgroups.
(b^*) Describe the pseudocompact abelian groups that admit proper totally dense pseudocompact subgroups.

The groups considered in (a) are necessarily minimal, and similarly the groups considered in (b) are necessarily totally minimal.

For compact abelian groups (a) coincides with (a*) and (b) coincides with (b*), and we have a complete description given respectively by Theorems 1.5 and 1.7.

So we would like to extend Theorem 1.5 to pseudocompact abelian groups to solve Problem 1.8(a*). Theorem 1.5 makes essential recourse to compactness. Indeed, one of the main results applied there is a consequence of [10, Lemma 3.2] (see Lemma 4.1 below), which constructs a totally dense pseudocompact subgroup of a compact abelian group starting from a dense pseudocompact subgroup with positive co-rank. So one has to construct a dense pseudocompact subgroup with positive co-rank and then apply this result. In the case of non-singular pseudocompact abelian groups the note added at the end of [12] yields the existence of a dense pseudocompact subgroup of co-rank $\geq \mathfrak{c}$. But we would need a version of Lemma 4.1 for pseudocompact abelian groups to get a totally dense pseudocompact subgroup.

In Proposition 2.4 we show that the groups of (b) cannot be super-singular.

In [8] Comfort and van Mill considered dense pseudocompact subgroups while Theorem 1.7 is about dense minimal pseudocompact subgroups. So it is natural to ask when a pseudocompact abelian group G admits proper dense minimal subgroups; note that such a group G is necessarily minimal. As a corollary of Theorem 1.7 we find a characterization of compact abelian groups that admit some proper dense minimal subgroup:

Theorem 1.9 *Let K be a compact abelian group. Then the following conditions are equivalent*

(a) K has no proper dense essential subgroup;
(b) K is torsion and $pT_p(K)$ is finite for every $p \in \mathbb{P}$; and
(c) $Soc(K)$ is open.

Theorem 1.9 suggests the following counterpart of Problem 1.8.

Problem 1.10 *(a) Describe the pseudocompact abelian groups that admit proper dense minimal subgroups.*
(b) Describe the pseudocompact abelian groups that admit proper dense essential subgroups.

The same questions can be posed about dense totally minimal and totally dense subgroups, as in Problem 1.8.

Notation and terminology

We denote by $\mathbb{Z}$, $\mathbb{P}$, $\mathbb{N}$ and $\mathbb{N}_+$ respectively the set of integers, the set of primes, the set of non-negative integers, and the set of natural numbers. For $m \in \mathbb{N}_+$, we use $\mathbb{Z}(m)$ for the finite cyclic group of order m. For $p \in \mathbb{P}$ the symbol $\mathbb{Z}_p$ is used for the group of p-adic integers. Let G be an abelian group. The subgroup of torsion elements of G is $t(G)$ and $G[m] = \{x \in G : mx = 0\}$. For a cardinal α we denote by $G^{(\alpha)}$ the direct sum of α many copies of G, that is $\bigoplus_\alpha G$. The socle of G is $Soc(G) = \bigoplus_{p\in\mathbb{P}} G[p]$. If $G = H^\alpha$, where H is a group and α a non-countable cardinal, ΣG is the Σ-product centered at 0 of G, that is the set of all elements of G with countable support; and moreover, $\Delta G = \{\mathbf{x} = (x_i) \in G : x_i = x_j \text{ for every } i, j < \alpha\}$ is the diagonal subgroup of G. The symbol $\mathfrak{c}$ stands for the cardinality of the continuum.

For a topological group G we denote by $c(G)$ the connected component of the identity e_G in G. If $c(G)$ is trivial, the group G is said to be *totally disconnected*. If M is a subset of G then $\langle M\rangle$ is the smallest subgroup of G containing M and $\overline{M}$ is the closure of M in G. The symbol $w(G)$ stands for the weight of G. The Pontryagin dual of a topological abelian group G is denoted by $\widehat{G}$.

For undefined terms see [19,20].

2 Super-singular groups

Let us start by recalling some properties of compact abelian groups.

Remark 2.1 *Let K be a compact abelian group, and let $p \in \mathbb{P}$. We defined the p-component $T_p(K)$ of K in the introduction. Following [16] it can be defined also as $T_p(K) = \bigcap\{nK : n \in \mathbb{N}_+, (n,p) = 1\}$. Thanks to this equivalent definition it is easy to see that $T_p(K) \supseteq c(K)$ for every $p \in \mathbb{P}$ and so $mT_p(K) \supseteq mc(K) = c(K)$ for every $m \in \mathbb{N}_+$ and for every $p \in \mathbb{P}$.*

(a) If N is a closed subgroup of K then $T_p(N) = \overline{N_p} \subseteq \overline{K_p} = T_p(K)$.
(b) If K is totally disconnected, then K_p is closed and so $T_p(K) = K_p$ for every $p \in \mathbb{P}$; each K_p is a compact $\mathbb{Z}_p$-module. Moreover $K = \prod_{p\in\mathbb{P}} K_p$ and every closed subgroup N of K is of the form $N = \prod_{p\in\mathbb{P}} N_p$, where each N_p is a closed subgroup of K_p [1],[16, Proposition 3.5.9].
(c) Let $L = K/c(K)$, and let π be the canonical projection of K onto L. Then $\pi \restriction_{T_p(K)}: T_p(K) \to T_p(L) = L_p$ is surjective and $T_p(K) = \pi^{-1}(L_p)$ [16, Proposition 4.1.5].
(d) If $c(K)$ is metrizable, then $T_p(K)$ is metrizable if and only if L_p is metrizable and $pT_p(K)$ is metrizable if and only if pL_p is metrizable.

Pseudocompact groups are G_δ-dense in their completion [6] and moreover a dense subgroup of a pseudocompact group is pseudocompact if and only if it is G_δ-dense. The family $\Lambda(G)$ of all closed G_δ-subgroups of G is a filter-base. As a consequence of [5, Lemma 6.1] and the results in [12, Section 2] we obtain the following lemma.

Lemma 2.2 *Let G be a pseudocompact abelian group. Then:*

(a) $N \in \Lambda(G)$ if and only if G/N is metrizable;
(b) if $f : G \to H$ is an open continuous surjective homomorphism and $N \in \Lambda(G)$, then $f(N) \in \Lambda(H)$;
(c) N is pseudocompact for every $N \in \Lambda(G)$;
(d) if $N \in \Lambda(G)$ and M is a closed subgroup G such that $N \subseteq M$, then $M \in \Lambda(G)$;
(e) if $N \in \Lambda(G)$ and $M \in \Lambda(N)$, then $M \in \Lambda(G)$; and
(f) a subgroup D of G is G_δ-dense in G if and only if $G = D + N$ for every $N \in \Lambda(G)$.

Lemma 2.3 [5] *Let G be a pseudocompact abelian group. If G is torsion then G is bounded-torsion.*

It is immediately possible to prove the following proposition, which gives a necessary condition for a pseudocompact abelian group to have proper dense essential pseudocompact subgroups. This result is related to Problem 1.8.

Proposition 2.4 *Let G be a pseudocompact abelian group. If G is super-singular, then G has no proper dense essential pseudocompact subgroup.*

PROOF. Suppose that G is super-singular. Let H be a dense essential pseudocompact subgroup of G. Since H is essential in G, it follows that $H \supseteq Soc(G)$ and so $H \supseteq N \in \Lambda(G)$. By the G_δ-density of H in G, Lemma 2.2(f) implies that $H + N = G$ and so $H = G$.

In the sequel $\widetilde{G}$ denotes the completion of a topological abelian group G. In the next two lemmas we give conditions equivalent to singularity and super-singularity in the case of pseudocompact abelian groups. In particular we prove that a pseudocompact abelian group G is (super-)singular if and only if $\widetilde{G}$ is (super-)singular.

Lemma 2.5 *Let G be a pseudocompact abelian group. Then the following conditions are equivalent*

(a) G is singular;
(b) there exists an $m \in \mathbb{N}_+$ such that $G[m] \in \Lambda(G)$;

(c) *there exists an* $m \in \mathbb{N}_+$ *such that* mG *is metrizable; and*
(d) *there exists an* $m \in \mathbb{N}_+$ *such that* $m\widetilde{G}$ *is metrizable.*

PROOF. (b)$\Leftrightarrow$(c) is [12, Lemma 4.11] and (b)$\Rightarrow$(a) is obvious.

(a)$\Rightarrow$(b) Let $N \in \Lambda(G)$ be such that $N \subseteq t(G)$. Then N is pseudocompact by Lemma 2.2(c) and so N is bounded-torsion by Lemma 2.3. Therefore there exists an $m \in \mathbb{N}_+$ such that $mN = \{0\}$. Thus $N \subseteq G[m]$ and so $G[m] \in \Lambda(G)$ by Lemma 2.2(d).

(d)$\Rightarrow$(c) is obvious since $mG \subseteq m\widetilde{G}$.

(c)$\Rightarrow$(d) Being a continuous image of the pseudocompact group G, mG is pseudocompact. Since it is also metrizable, mG is compact and so closed in $m\widetilde{G}$. Hence $mG = m\widetilde{G}$.

We remind the reader that a pseudocompact abelian group G is super-singular if there exists an $N \in \Lambda(G)$ such that $N \subseteq Soc(G)$.

Lemma 2.6 *Let G be a pseudocompact abelian group. Then the following conditions are equivalent*

(a) *G is super-singular;*
(b) *there exist distinct primes $p_1, \dots, p_n$ ($n \in \mathbb{N}_+$) such that $G[p_1 \cdot \dots \cdot p_n] \in \Lambda(G)$;*
(c) *there exist distinct primes $p_1, \dots, p_n$ ($n \in \mathbb{N}_+$) such that $(p_1 \cdot \dots \cdot p_n)G$ is metrizable; and*
(d) *there exist distinct primes $p_1, \dots, p_n$ ($n \in \mathbb{N}_+$) such that $(p_1 \cdot \dots \cdot p_n)\widetilde{G}$ is metrizable.*

PROOF. (b)$\Leftrightarrow$(c) follows from [12, Lemma 4.11] and (b)$\Rightarrow$(a) is obvious.

(a)$\Rightarrow$(b) Let $N \in \Lambda(G)$ be such that $N \subseteq Soc(G)$. Then N is pseudocompact by Lemma 2.2(c) and so N is bounded-torsion by Lemma 2.3. Therefore there exists an $m \in \mathbb{N}_+$ such that $mN = \{0\}$. Since $N \subseteq Soc(G)$, $N = Soc(N)$. Consequently, if $p_1, \dots, p_n$ ($n \in \mathbb{N}_+$) are the primes that divide m, then $m' = p_1 \cdot \dots \cdot p_n$ is such that $m'N = \{0\}$. Thus $N \subseteq G[m']$ and hence $G[m'] \in \Lambda(G)$ by Lemma 2.2(d).

(d)$\Rightarrow$(c) is obvious since $(p_1 \cdot \dots \cdot p_n)G \subseteq (p_1 \cdot \dots \cdot p_n)\widetilde{G}$.

(c)$\Rightarrow$(d) Being a continuous image of the pseudocompact group G, $(p_1 \cdot \dots \cdot p_n)G$ is pseudocompact. By hypothesis $(p_1 \cdot \dots \cdot p_n)G$ is also metrizable; thus it is compact and so closed in $(p_1 \cdot \dots \cdot p_n)\widetilde{G}$. Hence $(p_1 \cdot \dots \cdot p_n)G = (p_1 \cdot \dots \cdot p_n)\widetilde{G}$.

Lemma 2.7 *Let G be a pseudocompact abelian group.*

(a) If G is super-singular, then every pseudocompact subgroup of G is singular.
(b) If $N \in \Lambda(G)$, then G is super-singular if and only if N is super-singular.

PROOF. (a) immediately follows from Lemma 2.6.

(b) Assume that N is super-singular. Then there exists $M \in \Lambda(N)$ such that $M \subseteq Soc(N)$. Therefore $M \subseteq Soc(G)$. By Lemma 2.2(e) $M \in \Lambda(G)$ and so G is super-singular.

Since super-singular groups are singular, they have also the following properties that are used in the proof of Theorem 1.7.

Let K be a totally dense compact abelian group. Then $K = \prod_{p\in\mathbb{P}} K_p$ by Remark 2.1(b). Following [10], we define

$$P_s = \{p \in \mathbb{P} : K_p \text{ is singular}\} \text{ and } P_m = \{p \in \mathbb{P} : K_p \text{ is metrizable}\}$$

and observe that $P_m \subseteq P_s \subseteq \mathbb{P}$. The group K is metrizable if and only if $P_m = \mathbb{P}$. Moreover these two sets describe completely the singularity of K:

Lemma 2.8 [10, Lemma 4.4] *Let K be a totally disconnected compact abelian group. Then K is singular if and only if $P_s = \mathbb{P}$ and $\mathbb{P} \setminus P_m$ is finite.*

Lemma 2.9 *Let K be a singular compact abelian group. Then:*

(a) [10, Lemma 3.4] *every quotient and every closed subgroup of K is singular; and*
(b) $c(K)$ is metrizable.

PROOF. (b) Since K is singular, there exists $n \in \mathbb{N}_+$ such that nK is metrizable. But $c(K)$ is divisible and so $c(K) = nc(K) \subseteq nK$ is metrizable.

3 Dense minimal pseudocompact subgroups

The next example shows that there exist compact abelian groups that have proper dense essential pseudocompact subgroups, although they have no proper totally dense subgroup.

Example 3.1 *Let $p \in \mathbb{P}$ and consider the compact p-group $K = \mathbb{Z}(p^m)^{\omega_1}$ with $m \in \mathbb{N}, m > 1$. Then $K[p] + \Sigma K$ is a proper dense essential pseudocompact subgroup of K. In fact $K[p]$ is essential in K and ΣK is G_δ-dense in K. Moreover $K[p] + \Sigma K$ is proper in K. Indeed fix an element $x \in \mathbb{Z}(p^m) \setminus \mathbb{Z}(p)$ and define $\boldsymbol{x} = (x_i) \in K$ by $x_i = x$ for every i. Since ΣK intersects trivially the diagonal subgroup ΔK of K and $x \notin \mathbb{Z}(p)$, it follows that $\boldsymbol{x} \in K \setminus (K[p] + \Sigma K)$.*

The property of compact abelian groups to admit a proper dense essential pseudocompact subgroup is preserved by inverse image under continuous surjective homomorphisms:

Lemma 3.2 *Let G and G_1 be topological abelian groups such that there exists a continuous surjective homomorphism $f : G \to G_1$.*

(a) If H is a proper subgroup of G_1 then $f^{-1}(H)$ is a proper subgroup of G.
(b) Suppose that f is open; then
(b1) if H is a dense subgroup of G_1, then $f^{-1}(H)$ is a dense subgroup of G; and
(b2) if G is pseudocompact and H is a G_δ-dense subgroup of G_1 then $f^{-1}(H)$ is a G_δ-dense subgroup of G.
(c) If G is compact and H is an essential subgroup of G_1 then $f^{-1}(H)$ is an essential subgroup of G.

PROOF. (a) The subgroup $f^{-1}(H)$ is proper in G because H is proper in G_1 and $f(f^{-1}(H)) = H$ since f is surjective.

(b1) We have to prove that $\overline{f^{-1}(H)} = G$. Since $f^{-1}(H)$ contains $\ker f$, this is equivalent to $f(\overline{f^{-1}(H)}) = G_1$. Observe that

$$H = f(f^{-1}(H)) \subseteq f(\overline{f^{-1}(H)}) \subseteq \overline{f(f^{-1}(H))} = \overline{H} = G_1.$$

Since $\overline{f^{-1}(H)}$ is closed and contains $\ker f$, its image $f(\overline{f^{-1}(H)})$ is closed. Hence $f(\overline{f^{-1}(H)}) = \overline{H} = G_1$.

(b2) Let us prove that $f^{-1}(H)$ is G_δ-dense in G using the criterion given by Lemma 2.2(f). Let $N \in \Lambda(K)$. By Lemma 2.2(b) $f(N) \in \Lambda(G_1)$ and so $G_1 = H + f(N)$ thanks to Lemma 2.2(f). Since $f(f^{-1}(H) + N) = H + f(N) = G_1 = f(G)$ and both $f^{-1}(H) + N$ and G contain $\ker f$, it follows that $f^{-1}(H) + N = G$.

(c) Let N be a non-trivial closed subgroup of G. Since N is compact and f is continuous, $f(N)$ is compact, and so $f(N)$ closed in G_1. If $f(N)$ is non-trivial, then $H \cap f(N) \neq \{0\}$ and so there exists an $h \neq 0$ in $H \cap f(N)$; hence $h = f(n)$ for some $n \neq 0$ in N and consequently $0 \neq n \in f^{-1}(H) \cap N$. If $f(N) = \{0\}$, then $N \subseteq \ker f \subseteq f^{-1}(H)$. Hence $f^{-1}(H)$ is essential in G.

The following example shows that the condition “open” in Lemma 3.2(b) cannot be removed even if the subgroup H is totally dense.

Example 3.3 *Let $K = \mathbb{T}^{\mathfrak{c}}$, and let τ be the usual product topology on K. If $H = H_1 + H_2$, where $H_1 = \Sigma K$ and $H_2 = (\mathbb{Q}/\mathbb{Z})^{\mathfrak{c}}$, then H is G_δ-dense and totally dense in (K,τ). Moreover $r_0(K/H) \geq \mathfrak{c}$; here r_0 denotes the free rank. To see this note that $r_0(\Delta K) = \mathfrak{c}$ because $\Delta K \cong \mathbb{T}$. Let $\pi : K \to K/H$ be the canonical projection. Then $\pi(\Delta K) \cong \Delta K/(\Delta K \cap H)$. But $r_0(\Delta K \cap H) = 0$ since $\Delta K \cap H = \Delta(\mathbb{Q}/\mathbb{Z})^{\mathfrak{c}}$ and so $r_0(\pi(\Delta K)) = \mathfrak{c}$.*

Since $r_0(K/H) \geq \mathfrak{c}$, there exists a surjective homomorphism $\varphi : K/H \to \mathbb{T}$. Let $h = \varphi \circ \pi : K \to \mathbb{T}$, and let τ_h be the weakest topology on K such that $\tau_h \geq \tau$ and h is continuous (for more details see [12, Remark 2.12]). Let $G = \ker h$. Since $H \subseteq G$, it follows that G is G_δ-dense and totally dense in (K,τ). It is possible to prove that (K,τ_h) is pseudocompact, as it is homeomorphic to the graph of h endowed with the topology inherited from $(K,\tau) \times \mathbb{T}$ (the graph of h with this topology is pseudocompact as it is G_δ-dense in $(K,\tau) \times \mathbb{T}$ by [12, Lemma 3.7]). Moreover $\tau_h > \tau$, since h is not τ-continuous because the graph of h is not closed in $(K,\tau) \times \mathbb{T}$. Consider now the non-open continuous isomorphism $id_K : (K,\tau_h) \to (K,\tau)$. We have that G is a G_δ-dense and totally dense subgroup of (K,τ). But G is proper and closed in (K,τ_h) and so G cannot be dense in (K,τ_h).

The following lemma reveals important information on the structure of singular compact $\mathbb{Z}_p$-modules needed in the “local” step of the proof of Theorem 1.7.

Lemma 3.4 *Let $p \in \mathbb{P}$, and let K be a compact $\mathbb{Z}_p$-module. Suppose that there exists an $m \in \mathbb{N}_+$ such that $p^m K$ is metrizable, but $p^{m-1}K$ is non-metrizable. Then there exist a metrizable compact $\mathbb{Z}_p$-module K' and a bounded-torsion compact group $B \cong \prod_{n=1}^{m} \mathbb{Z}(p^n)^{\alpha_n}$, for some cardinals α_n, such that $K \cong K' \times B$ and $\alpha_m > \omega$.*

PROOF. Let $X = \widehat{K}$. Since K is compact abelian, we have $w(K) = |X|$ and so $|X| > \omega$. Moreover $|p^m X| \leq \omega$ and $|p^{m-1}X| > \omega$ because $w(p^n K) = |\widehat{p^n K}|$, by the compactness of $p^n K$, and $\widehat{p^n K} \cong p^n X$. By [20, Theorem 32.3] there exists a basic subgroup B_0 of X. In other words

$B_0 \cong \bigoplus_{n=1}^{\infty} \mathbb{Z}(p^n)^{(\alpha_n)}$,
B_0 is pure (i.e., $B_0 \cap p^n X = p^n B_0$ for every $n \in \mathbb{N}$) and
X/B_0 is divisible.

Let $B_1 = \bigoplus_{n=1}^{m} \mathbb{Z}(p^n)^{(\alpha_n)}$ and $B_2 = \bigoplus_{n=m+1}^{\infty} \mathbb{Z}(p^n)^{(\alpha_n)}$. Then $B_0 = B_1 \oplus B_2$. We set $X' = p^m X + B_2$ and observe that $|X'| \leq \omega$ because $|p^m X| \leq \omega$ and this fact implies also $|B_2| \leq \omega$.

Since X/B_0 is divisible, $X/B_0 = p(X/B_0)$. Note that $p(X/B_0) = (pX+B_0)/B_0$ and so $X = pX + B_0$. This equality applied m times gives $X = p^m X + B_0$. Consequently $X = X' + B_1$. Moreover this expression is a direct sum: $X = X' \oplus B_1$. Indeed, $X' \cap B_1 = \{0\}$, because if $z \in X' \cap B_1$, $z = b \in B_1$ and $z = x + b'$, where $x \in p^m X$ and $b' \in B_2$; then $x = b - b' \in B_0 \cap p^m X = p^m B_0 \subseteq B_2$, where $B_0 \cap p^m X = p^m B_0$ because B_0 is pure. This result yields $b \in B_2$ and consequently $b = 0$.

Put $B = \widehat{B_1}$ and $K' = \widehat{X'}$. Then $K \cong K' \times B$, where $B = \prod_{n=1}^{m} \mathbb{Z}(p^n)^{\alpha_n}$. By the hypothesis $p^{m-1}K$ is non-metrizable. Moreover $p^{m-1}K \cong p^{m-1}(K' \times B) = p^{m-1}K' \times p^{m-1}B$ and K' is metrizable because $w(K') = |X'| \leq \omega$. Hence $p^{m-1}B$ is non-metrizable, that is $\alpha_m > \omega$.

Lemma 3.5 *Let K be a compact abelian group. Then there exists a totally disconnected closed subgroup N of K such that $K = N + c(K)$.*

PROOF. Let $X = \widehat{K}$ and let M be a maximal independent subset of X. Put $Y = \langle M \rangle$ and $N = A(Y)$, where $A(Y) = \{x \in K : \chi(x) = \{0\} \ \forall \chi \in Y\}$ is the annihilator of Y in K. Thus N is a closed subgroup of K. Moreover N is totally disconnected since $N \cong \widehat{X/Y}$ and X/Y is torsion. Recall that $c(K) = A(t(X))$. Consequently $K = N + c(K)$, because $Y \cap t(X) = \{0\}$ and so $K = A(Y \cap t(X)) = A(Y) + A(t(X)) = N + c(K)$.

PROOF. OF THEOREM 1.7. (a)$\Rightarrow$(b) Suppose that K has no proper dense essential pseudocompact subgroup. In particular K has no proper totally dense pseudocompact subgroup; so by Theorem 1.5 K is singular. Lemma 2.9(b) yields that $c(K)$ is metrizable and Lemma 2.9(a) implies that $L = K/c(K)$ is singular. Since L is totally disconnected, $L = \prod_{p \in \mathbb{P}} L_p$ by Remark 2.1(b), where each L_p is singular by Lemma 2.9(a). If there exists a $p \in \mathbb{P}$ such that $pT_p(K)$ is non-metrizable, then pL_p is non-metrizable by Remark 2.1(d).

In particular L_p is a singular non-metrizable compact $\mathbb{Z}_p$-module. Thus there exists an $m \in \mathbb{N}_+$ such that $p^m L_p$ is metrizable and $p^{m-1}L_p$ is non-metrizable. By Lemma 3.4 there exist a metrizable compact $\mathbb{Z}_p$-module L'_p and a bounded-torsion compact abelian group $B = \prod_{n=1}^{m} \mathbb{Z}(p^n)^{\alpha_n}$ such that $L_p \cong L'_p \times B$ and $\alpha_m > \omega$. Since pL_p is non-metrizable, we have $m > 1$ and consequently there exists a proper dense essential pseudocompact subgroup of $\mathbb{Z}(p^m)^{\alpha_m}$ (see Example 3.1).

The composition of the projections $K \to L$, $L \to L_p$, $L_p \cong L'_p \times B \to B$ and $B = \prod_{n=1}^{m} \mathbb{Z}(p^n)^{\alpha_n} \to \mathbb{Z}(p^m)^{\alpha_m}$ is a continuous surjective homomorphism. So apply Lemma 3.2 to find a proper dense essential pseudocompact subgroup of K.

(b)⇒(c) By Lemma 3.5 there exists a totally disconnected closed subgroup N of K such that $K = N + c(K)$. Since K is singular, $c(K)$ is metrizable by Lemma 2.9(b) and so K/N is metrizable because $K/N \cong c(K)/(N \cap c(K))$. Hence $N \in \Lambda(K)$ by Lemma 2.2(a). According to Lemma 2.7 it suffices to prove that N is super-singular.

Since N is totally disconnected and compact, it follows that $N = \prod_{p \in \mathbb{P}} N_p$ by Remark 2.1(b). By Remark 2.1(a) pN_p is metrizable for every $p \in \mathbb{P}$. Moreover N is singular. According to Lemma 2.8 $\mathbb{P} = P_s$ and $\mathbb{P} \setminus P_m$ is finite, which means that $N = M \times \prod_{\mathbb{P} \setminus P_m} N_p$, where $M = \prod_{p \in P_m} N_p$ is metrizable and $\prod_{\mathbb{P} \setminus P_m} N_p$ is a finite product; each N_p with $p \in \mathbb{P} \setminus P_m$ is a singular non-metrizable compact $\mathbb{Z}_p$-module. Since pN_p is metrizable for every $p \in \mathbb{P}$, for $p \in \mathbb{P} \setminus P_m$ Lemma 3.4 (with $m = 1$) implies $N_p \cong N'_p \times B_p$, where N'_p is a metrizable compact $\mathbb{Z}_p$-module and $B_p = \mathbb{Z}(p)^{\alpha_p}$ with $\alpha_p > \omega$. Therefore $N \cong M \times \prod_{p \in \mathbb{P} \setminus P_m} N'_p \times \prod_{p \in \mathbb{P} \setminus P_m} B_p$. Let B be a subgroup of N isomorphic to $\prod_{p \in P_m} \{0_{N_p}\} \times \prod_{p \in \mathbb{P} \setminus P_m} \{0_{N'_p}\} \times \prod_{p \in \mathbb{P} \setminus P_m} B_p$. Then $Soc(N) \supseteq B$ as $\mathbb{P} \setminus P_m$ is finite. Since N/B is metrizable, $B \in \Lambda(N)$ by Lemma 2.2(a), which means that N is super-singular.

(c)⇒(a) follows from Proposition 2.4.

4 Dense minimal subgroups

As a corollary of the next lemma, we obtain a very simple characterization of compact abelian groups having proper totally dense subgroups.

Lemma 4.1 [10, Lemma 3.2] *Let K be a compact abelian group that admits a subgroup B such that $r_0(K/B) \geq 1$. Then K has a proper totally dense subgroup H that contains B.*

Corollary 4.2 *Let K be a compact abelian group. Then K has no proper totally dense subgroup if and only if K is torsion.*

PROOF. Assume that K is not torsion, that is $K \neq t(K)$. Apply Lemma 4.1 with $B = t(K)$ to find a proper totally dense subgroup of K. If K is torsion, then K cannot have any proper totally dense subgroup because every totally dense subgroup contains $t(K) = K$.

The group in the following example is a (metrizable) compact abelian group that has a proper dense essential subgroup but no proper dense pseudocompact subgroup.

Example 4.3 *Let $p \in \mathbb{P}$ and $K = \mathbb{Z}(p^m)^\omega$ with $m \in \mathbb{N}$. Then $K[p]$ is a proper dense essential subgroup of K if and only if $m > 1$.*

PROOF. OF THEOREM 1.9. (a)$\Rightarrow$(b) Suppose that K has no proper dense essential subgroup, in particular K has no proper totally dense subgroup. Then Corollary 4.2 implies that K is torsion. By Lemma 2.3 K is bounded-torsion and so $K = K_{p_1} \times \ldots \times K_{p_n}$ for some $n \in \mathbb{N}_+$ and $p_1, \ldots, p_n \in \mathbb{P}$ and, in particular, K is totally disconnected and Remark 2.1(b) can be applied, which means that $K_p = \{0\}$ for all $p \in \mathbb{P} \setminus \{p_1, \ldots, p_n\}$.

The group K has no proper dense essential subgroup and, in particular, K has no proper dense essential pseudocompact subgroup, and so pK_p is metrizable for every $p \in \mathbb{P}$ by Theorem 1.7. According to Lemma 3.2 K_p cannot have any proper dense essential subgroup for every $p \in \mathbb{P}$. Let us prove that pK_p is finite for every $p \in \mathbb{P}$. Suppose that pK_p is infinite for some $p \in \mathbb{P}$; then $p \in \{p_1, \ldots, p_n\}$. Since K is bounded-torsion, then K_p is a bounded p-torsion compact $\mathbb{Z}_p$-module. Consequently $K_p \cong \mathbb{Z}(p)^{\alpha_1} \times \ldots \times \mathbb{Z}(p^n)^{\alpha_n}$ for some cardinals $\alpha_1, \ldots, \alpha_n$, where $n \in \mathbb{N}_+$. Since pK_p is infinite and metrizable, there exists an $m \in \mathbb{N}, 1 < m \leq n$ such that $\alpha_m = \omega$. By Example 4.3 $\mathbb{Z}(p^m)^{\alpha_m}$ has a proper dense essential subgroup. Consider the composition of the projections $K \to K_p$ and $K_p \to \mathbb{Z}(p^m)^{\alpha_m}$ and apply Lemma 3.2 to find a proper dense essential subgroup of K.

(b)$\Rightarrow$(c) Since K is torsion, it follows that K is bounded-torsion by Lemma 2.3. In particular it is totally disconnected, and we can apply Remark 2.1(b). We have $K \cong K_{p_1} \times \ldots \times K_{p_n}$, where $n \in \mathbb{N}_+, p_1, \ldots, p_n \in \mathbb{P}$. Since $K_p/K_p[p] \cong pK_p$ is finite for every $p \in \mathbb{P}$, the subgroup $Soc(K) = K_{p_1}[p_1] \oplus \ldots \oplus K_{p_n}[p_n]$ has finite index in K. Since $Soc(K)$ is closed, it is also open.

(c)$\Rightarrow$(a) Let H be a dense essential subgroup of K. Then $H \supseteq Soc(K)$; since $Soc(K)$ is open in K so is H, which implies that H is closed in K. Since H is also dense in K, we conclude that $H = K$.

Corollary 4.4 *Let K be a compact abelian group, and let H be an open essential subgroup of K. Then K has no proper dense essential subgroup if and only if H has no proper dense essential subgroup.*

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