A unifying model of variables and names*

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Abstract. We investigate a category theoretic model where both "variables" and "names", usually viewed as separate notions, are particular cases of the more general notion of *distinction*. The key aspect of this model is to consider functors over the category of irreflexive, symmetric finite relations. The models previously proposed for the notions of "variables" and "names" embed faithfully in the new one, and initial algebra/final coalgebra constructions can be transferred from the formers to the latter. Moreover, the new model admits a definition of *distinction-aware* simultaneous substitutions. As a substantial application example, we give the first semantic interpretation of Miller-Tiu's $FO\lambda^{\nabla}$ logic.

1 Introduction

In recent years, many models for *dynamically allocable* entities, such as (bound) variables, (fresh) names, reference, etc., have been proposed. Most of (if not all) these models are based on some (sub)category of *(pre)sheaves*, i.e., functors from a suitable index category to *Set* [18, 6, 10, 8, 5, 17]. The basic idea is to stratify datatypes according to various "stages" representing different degrees of information, such as number of allocated variables. A simple example is that of set-valued functors over $\mathbb{F}$, which is the category of finite subsets $C \subset \mathbb{A}$ of a given enumerable set $\mathbb{A}$ of abstract symbols ("variable names") [6, 10]; here, the datatype of untyped λ-terms is the functor $\Lambda : \mathbb{F} \to Set$, $\Lambda_C = \{t \mid FV(t) \subseteq C\}$. Morphisms between objects of the index category describe how we can move from one stage to the others; in $\mathbb{F}$, morphisms are any function $\sigma : C \to D$, that is any variable renaming possibly with unifications. Correspondingly, $\Lambda_\sigma : \Lambda_C \to \Lambda_D$ is the usual (capture-avoiding) variable renaming $-\{\sigma\}$ on terms.

Different index categories lead to different notions of "allocable entities". The notion of *name*, particularly important for process calculi, can be modeled using the subcategory $\mathbb{I}$ of $\mathbb{F}$ of only injective functions. Thus, stages of $\mathbb{I}$ can be still "enlarged" by morphisms (which corresponds to allocation of new names), but they cannot be "contracted", which means that two different symbols can never coalesce to the same. Categories of set- and domain-valued functors over $\mathbb{I}$ have been used for modeling π-calculus, ν-calculus, etc. [18, 5].

According to this view, *variables* and *names* are quite different concepts, and as such they are rendered by different index categories. This separation is a drawback when we have to model calculi or logics where *both* aspects are present

* Work supported by EU projects IST-2001-33100 PROFUNDIS and IST-510996 TYPES.

and must be dealt with at once. Some examples are: the fusion calculus, where names can be unified under some conditions; the open bisimulation of π-calculus, which is defined by closure under all (also unifying) distinction-preserving name substitutions; even, a (still unknown) algebraic model for the Mobile Ambients is supposed to deal with both variables and names (which are declared as different entities in capabilities); and finally, the logic $FO\lambda^{\nabla}$ [14], featuring a peculiar interplay between "global variables" and "locally scoped constants".

Why are $\mathbb{F}$ and $\mathbb{I}$ not sufficient to model these situations? The problem is that these models force the behaviour of atoms *a priori*. Atoms will always act as variables in $\mathbb{F}$, as names in $\mathbb{I}$. This is to be contrasted with the situations above, where the behaviour of an atom is not known beforehand.

A way for circumventing this problem is to distinguish *allocation* of atoms, from *specifications* of their behaviour. Behaviour of atoms is given a *symmetric, irreflexive* relation, called *distinction*: two atoms are related if and only if they cannot be unified, in any reachable stage. These relations can change dynamically, *after* that atoms are introduced. Thus a stage is a finite set of atoms, together with a distinction over it. These stages form the objects of a new index category $\mathbb{D}$, which subsumes both the idea of variables and that of names.

The aim of this paper is to give a systematic presentation of the model of set-valued functors over $\mathbb{D}$, first introduced by Ghani, Yemane and Victor for characterizing open bisimulation of π-calculus [9]. Following similar previous work about [6, 5], we focus on algebraic, coalgebraic and logical properties of this category, relating these results with the corresponding ones in $Set^{\mathbb{F}}$ and $Set^{\mathbb{I}}$.

In Section 2, we present the category $\mathbb{D}$, its properties and relations with $\mathbb{F}$ and $\mathbb{I}$. In Section 3 we study the structure of $Set^{\mathbb{D}}$, and its relations with $Set^{\mathbb{F}}$, $Set^{\mathbb{I}}$. In particular, due to their importance for modeling process calculi, we will study initial algebras and final coalgebras of polynomial functors over $Set^{\mathbb{D}}$.

In Section 4, we give a general definition of the key notions of *support* and *apartness*, and then apply and compare their instances in the cases of $Set^{\mathbb{D}}$, $Set^{\mathbb{F}}$ and $Set^{\mathbb{I}}$. An application of apartness is in Section 5, where we present a monoidal definition of "apartness-preserving" simultaneous substitution.

In Section 6 we turn to the logical aspects of $Set^{\mathbb{D}}$: restricting to the subcategory of pullback-preserving functors, we define a self-dual quantifier similar to Gabbay-Pitts' И. This quantifier, and the structure of $Set^{\mathbb{D}}$, will be put at work in Section 7 in giving the first denotational semantics of Miller-Tiu's $FO\lambda^{\nabla}$.

Final remarks and directions for future work are in Section 8.

2 Distinctions

Let us fix an infinite, countable set of *atoms* $\mathbb{A}$. Atoms are abstract elements with no structure, intended to act both as *variables* and as *names* symbols.

We denote finite subsets of $\mathbb{A}$ as $n, m, \dots$. Functions among these finite sets are "atom substitutions". The category of all these finite sets, and any maps among them is $\mathbb{F}$. The subcategory of $\mathbb{F}$ with only *injective* maps is $\mathbb{I}$. Thus, while a morphism in $\mathbb{F}$ may map different atoms to the same target, this cannot happen in $\mathbb{I}$. This corresponds to the difference between variables and names, that is, the formers can be identified and replaced, while names cannot. In fact, we can see a name essentially as an atom which must be kept apart from the others. We can formalize this concept as follows:

Definition 1. (The category $\mathbb{D}$) *The category $\mathbb{D}$ of* distinctions relations *is the full subcategory of Rel of irreflexive, symmetric binary relations over $\mathbb{A}$ with a finite carrier set. (Here Rel is the category of relations and monotone functions.)*

A distinction relation (n, d) is thus a finite set n of atoms and a symmetric relation $d \subseteq n \times n$ such that for all $i \in n : (i, i) \notin d$. In the following we will write (n, d) as $d^{(n)}$, possibly dropping the superscript when clear from the context. A morphisms $f : d^{(n)} \to e^{(m)}$ is any *monotone* function $f : n \to m$, that is a substitution of atoms for atoms which preserves the distinction relation (if $(a, b) \in d$ then $(f(a), f(b)) \in e$). In other words, substitutions cannot map two related (i.e., definitely distinct) atoms to the same atom of a later stage, while unrelated atoms can coalesce to a single one.

Structure of $\mathbb{D}$. The category $\mathbb{D}$ inherits from *Rel* products and coproducts. More explicitly, products and coproducts can be defined on objects as follows:

$$\begin{aligned} d_1^{(m)} \times d_2^{(n)} &\triangleq (m \times n, \{((i_1, j_1), (i_2, j_2)) \mid (i_1, i_2) \in d_1 \text{ and } (j_1, j_2) \in d_2\}) \\ d_1^{(m)} + d_2^{(n)} &\triangleq (m + n, d_1 \cup \{(l + i, l + j) \mid (i, j) \in d_2\}) \qquad (l \triangleq \max(m) + 1) \end{aligned}$$

where $m+n \triangleq m \cup \{l+i \mid i \in n\}$. Note that $\mathbb{D}$ has no terminal object, but it has initial object $(\emptyset, \emptyset)$. In fact, $\mathbb{D}$ inherits meets, joins and partial order from $\wp(\mathbb{A})$:

- $d^{(n)} \wedge e^{(m)} = (d \cap e)^{(m \cap n)}$, and $d^{(n)} \vee e^{(m)} = (d \cup e)^{(m \cup n)}$
- $d^{(n)} \leq e^{(m)}$ iff $d \wedge e = d$, that is, iff $d \subseteq e$.

For each n, let us denote $\mathbb{D}_n$ the full subcategory of $\mathbb{D}$ whose objects are all relations over n. Then, $\mathbb{D}_n$ is a complete Boolean algebra. Let $\bot^{(n)} \triangleq (n, \emptyset)$ and $\top^{(n)} \triangleq (n, n^2 \setminus \Delta_n)$ be the *empty* and *complete* distinction on n, respectively, where $\Delta : \mathbb{F} \to \mathit{Rel}$ is the *diagonal* functor defined as $\Delta_n = (n, \{(i, i) \mid i \in n\})$. For $d_i^{(n)}$ $(i \in J)$ a set of distinctions of $\mathbb{D}_n$, we define $\bigvee_{i \in J} d_i^{(n)} \triangleq \bigcup_{i \in J} d_i$ as sets; similarly for meets. Finally, $\neg d^{(n)} = (n, n^2 \setminus (d \cup \Delta_n))$, and as usual, $d \Rightarrow e \triangleq (\neg d) \vee e$.

$\mathbb{D}$ can be given another monoidal structure. Let us define $\oplus : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$ as

$$d_1^{(m)} \oplus d_2^{(n)} = (m + n, d_1 \cup d_2 \cup \{(i, j), (j, i) \mid i \in m, j \in n\}).$$

Proposition 2. $(\mathbb{D}, \oplus, \bot^{(0)})$ *is a symmetric monoidal category.*

By applying coproduct and tensor to $\bot^{(1)}$ we get two distinguished *dynamic allocation* functors $\delta^-, \delta^+ : \mathbb{D} \to \mathbb{D}$, as $\delta^- \triangleq \bot^{(1)} + \_$ and $\delta^+ \triangleq \bot^{(1)} \oplus \_$. More explicitly, the action of δ^+ on objects is $\delta^+(d^{(n)}) = d_{+1}^{(n+1)}$ where $d_{+1} = d \cup \{(*, i), (i, *) \mid i \in n\}$. Thus both δ^- and δ^+ add an extra element to the carrier, but, as the superscript $^+$ is intended to suggest, δ^+ adds in *extra* distinctions.

The extra element can be used to represent a bound variable; δ^+ asks that, in addition, this new element is made distinct from the other elements. The functor δ^+ will be used for the binding associated with restriction to ensure that the extruded name cannot be renamed to other name as in open semantics of π-calculus, while the δ^- functor is used for bound input where no such restrictions are necessary.

Embedding $\mathbb{I}$ and $\mathbb{F}$ in $\mathbb{D}$. Let $\mathbb{D}_e$ denote the full subcategory of $\mathbb{D}$ of empty distinctions $\bot^{(n)} = (n, \emptyset)$, and $\mathbb{D}_c$ the full subcategory of complete distinctions $\top^{(n)} = (n, n^2 \setminus \Delta_n)$. Notice that all morphisms in $\mathbb{D}_c$ are *mono* morphisms of $\mathbb{D}$—that is, injective maps.

Let us consider the forgetful functor $U : \mathbb{D} \to \mathbb{F}$, dropping the distinction relation. The functor $\mathsf{v} : \mathbb{F} \to \mathbb{D}_e$ mapping each n in $\mathbb{F}$ to $\bot^{(n)}$, and each $f : n \to m$ to itself, is inverse of the restriction of U to $\mathbb{D}_e$.

On the other hand, the restriction of U to $\mathbb{D}_c$ is a functor $U : \mathbb{D}_c \to \mathbb{I}$, because the only morphisms in $\mathbb{D}_c$ are the injective ones. The functor $\mathsf{t} : \mathbb{I} \to \mathbb{D}_c$ mapping each n in $\mathbb{I}$ to $\top^{(n)}$, and each $f : n \rightarrowtail m$ to itself, is inverse of U. Hence:

Proposition 3. *$\mathbb{D}_e \cong \mathbb{F}$, and $\mathbb{D}_c \cong \mathbb{I}$.*

Therefore, we can say that the category of $\mathbb{D}$ generalises both $\mathbb{I}$ and $\mathbb{F}$. In fact, it is easy to check that the forgetful functor $U : \mathbb{D} \to \mathbb{F}$ is the right adjoint of the inclusion functor $\mathsf{v} : \mathbb{F} \hookrightarrow \mathbb{D}$.

Remark 4. While we are on this subject, we define the functor $V : \mathbb{D} \to \mathbb{I}$ which singles out from each d the (atoms of the) largest complete distinction contained in d. More precisely, V is defined on objects as $V(d^{(n)}) = \max\{m \mid \top^{(m)} \leq d^{(n)}\}$ and on morphisms as the restriction. This defines a functor: if $f : d^{(n)} \to e^{(m)}$ is a morphism, then it preserves distinctions, and thus for $i \in V(d)$, since i is part of a complete subdistinction of d, it must be mapped in a complete subdistinction of e, and hence $f(i) \in V(e)$. However, V is not an adjoint of t. □

We recall finally that $\mathbb{F}$ has finite products (and hence also $\mathbb{D}_e$), while $\mathbb{I}$ has binary products only. Disjoint unions are finite coproducts in $\mathbb{F}$, but not in $\mathbb{I}$. Actually, disjoint union $\uplus : \mathbb{I} \times \mathbb{I} \to \mathbb{I}$ is only a monoidal structure over $\mathbb{I}$, which quite clearly corresponds to the restriction of $\oplus$ to $\mathbb{D}_c$:

Proposition 5. *$\oplus \circ \langle \mathsf{t}, \mathsf{t} \rangle = \mathsf{t} \circ \uplus$, that is, for $n, m \in \mathbb{I}$: $\top^{(n \uplus m)} = \top^{(n)} \oplus \top^{(m)}$.*

As a consequence, for Proposition 3, we have $\uplus = U \circ \oplus \circ \langle \mathsf{t}, \mathsf{t} \rangle$. On the other hand, $\oplus$ restricted to $\mathbb{D}_e$ is *not* equivalent to the coproduct $+$ in $\mathbb{F}$.

3 Presheaves over $\mathbb{D}$

$Set^{\mathbb{D}}$ is the category of functors from $\mathbb{D}$ to *Set* (often called *presheaves (over $\mathbb{D}^{op}$)*) and natural transformations. Note that the restriction to finite distinction relations means that there are no size problems when talking about the category of presheaves. The structure of $\mathbb{D}$ lifts to $Set^{\mathbb{D}}$, which has:[3]

[3] We shall use the same symbols for the lifted structure, but ensuring the reader has enough information to deduce which category we are working in.

1. *Products and coproducts*, which are computed pointwise (as with all limits and colimits in functor categories); e.g. $(P \times Q)_{d^{(n)}} = P_{d^{(n)}} \times Q_{d^{(n)}}$. The terminal object is the constant functor $\mathcal{K}_1 = \mathbf{y}(\bot^{(\emptyset)})$: $\mathcal{K}_1(d) = 1$.
2. A presheaf of *atoms* $Atom \in Set^{\mathbb{D}}$, $Atom = \mathbf{y}(\bot^{(1)}) = \mathbf{y}(\top^{(1)})$. The action on objects is $Atom(d^{(n)}) = n$.
3. Two *dynamic allocation* functors $\delta^-, \delta^+ : Set^{\mathbb{D}} \to Set^{\mathbb{D}}$, induced by each $\kappa \in \{\delta^+, \delta^-\}$ on $\mathbb{D}$ as $\_ \circ \kappa : Set^{\mathbb{D}} \to Set^{\mathbb{D}}$.
4. Let $\wp_f$ be the finite (covariant) powerset functor on Set; then $\wp_f \circ \_ : Set^{\mathbb{D}} \to Set^{\mathbb{D}}$ is the *finite powerset* operator on $\mathbb{D}$-presheaves.
5. *Exponentials* are defined as usual in functor categories:

$$(B^A)_d \triangleq Set^{\mathbb{D}}(A \times \mathbb{D}(d, \_), B)$$
$$(B^A)_f(m) \triangleq m \circ (id_A \times (\_ \circ f)) \quad \text{for } f : d \to e \text{ in } \mathbb{D}, m : A \times \mathbb{D}(d, \_) \longrightarrow B$$

In particular, exponentials of representable functors have a nice definition:

Proposition 6. *For all $d \in \mathbb{D}$, B in $Set^{\mathbb{D}}$: $B^{\mathbf{y}(d)} \cong B_{d+\_}$.*

Proof.
$$\begin{aligned} (B^{\mathbf{y}(d)})_e &= Set^{\mathbb{D}}(\mathbf{y}(d) \times \mathbf{y}(e), B) && \text{by definition of exponential} \\ &\cong Set^{\mathbb{D}}(\mathbf{y}(d + e), B) && \text{since } \mathbf{y} \text{ preserves coproducts} \\ &\cong B_{d+e} && \text{by Yoneda Lemma.} \end{aligned}$$
□

This allows us to point out a strict relation between $Atom$ and δ^-:

Proposition 7. $(\_)^{Atom} \cong \delta^-$*, and hence* $\_ \times Atom \dashv \delta^-$.

Proof. Since $Atom = \mathbf{y}(\bot^{(1)})$, by Proposition 6 we have that $F^{Atom} \cong F_{\bot^{(1)}+\_} = F_{\delta^-(\_)} = \delta^-(F)$. The second part is an obvious consequence, because in CCC's it is always $\_ \times B \dashv (\_)^B$. □

The categories $Set^{\mathbb{F}}$ and $Set^{\mathbb{I}}$ can be embedded into $Set^{\mathbb{D}}$. Let us consider first the functors $\mathsf{v} : \mathbb{F} \hookrightarrow \mathbb{D}$ and $U : \mathbb{D} \to \mathbb{F}$.

Proposition 8. *The functor $\mathsf{v} : \mathbb{F} \hookrightarrow \mathbb{D}$ induces an essential geometric morphism $\mathsf{v} : Set^{\mathbb{F}} \to Set^{\mathbb{D}}$, that is two adjunctions $\mathsf{v}_! \dashv \mathsf{v}^* \dashv \mathsf{v}_*$, where $\mathsf{v}_! \cong \_ \circ U$, $\mathsf{v}^* = \_ \circ \mathsf{v}$, and $\mathsf{v}_*(F)(d^{(n)}) = F_n$ if $d^{(n)} = \bot^{(n)}$, 1 otherwise.*

$$Set^{\mathbb{F}} \underset{\mathsf{v}_*}{\overset{\mathsf{v}_! \cong \_\circ U}{\rightleftarrows}} Set^{\mathbb{D}} \ (\mathsf{v}^* = \_\circ\mathsf{v} : Set^{\mathbb{D}} \to Set^{\mathbb{F}}) \qquad \textit{where} \qquad \mathsf{v}_*(F)(d^{(n)}) = \begin{cases} F_n & \textit{if } d^{(n)} = \bot^{(n)} \\ 1 & \textit{otherwise} \end{cases}$$

Proof. The existence of the essential geometric morphism, and that the inverse image is $\_ \circ \mathsf{v}$, is a direct application of [12, VII.2, Theorem 2].

Let us prove that the direct image v_* has the definition above. By [12, VII.2, Theorem 2], we know that

$$\mathsf{v}_* = \underline{\mathrm{Hom}}_{\mathbb{F}^{op}}({}^\bullet\mathbb{D}^{op}_{\mathsf{v}}, \_) \tag{1}$$

where ${}^\bullet\mathbb{D}^{op}_{\mathsf{v}} : \mathbb{D}^{op} \times \mathbb{F} \to \mathit{Set}$ is the bifunctor defined on objects as ${}^\bullet\mathbb{D}^{op}_{\mathsf{v}}(d, n) = \mathbb{D}^{op}(\mathsf{v}(n), d) = \mathbb{D}(d, \bot^{(n)})$. By expanding the equation (1), we have that for all $F : \mathbb{F} \to \mathit{Set}$ and $d^{(m)}$ in $\mathbb{D}$:

$$\mathsf{v}_*(F)(d^{(m)}) = \mathit{Set}^{\mathbb{F}}(\mathbb{D}(d, \mathsf{v}(\_)), F) : \mathit{Set} \tag{2}$$

Now, an element of the set $\mathit{Set}^{\mathbb{F}}(\mathbb{D}(d, \mathsf{v}(\_)), F)$ is a natural transformation $\phi : \mathbb{D}(d, \mathsf{v}(\_)) \longrightarrow F$, that is a family of functions $\phi_n : \mathbb{D}(d, \bot^{(n)}) \to F_n$.

If d is not an empty distinction, then the set $\mathbb{D}(d, \bot^{(n)})$ is always empty, because there is no monotone map from $d \neq \bot^{(m)}$ to $\bot^{(n)}$. Therefore ϕ_n can be only $? : \emptyset \to F_n$, and hence $\mathit{Set}^{\mathbb{F}}(\mathbb{D}(d, \mathsf{v}(\_)), F) = \{? : \mathcal{K}_\emptyset \longrightarrow F\} = 1$.

If d is the empty distinction $\bot^{(m)}$, then $\mathbb{D}(d, \bot^{(n)}) = \mathbb{D}(\bot^{(m)}, \bot^{(n)}) = \mathbb{F}(m, n)$ by Proposition 3. Hence we can write equation (2) as

$$\mathsf{v}_*(F)(\bot^{(m)}) = \mathit{Set}^{\mathbb{F}}(\mathbb{F}(m, \_), F) = F_m$$

by Yoneda lemma, hence the thesis.

Let us prove that $\mathsf{v}_! \cong \_ \circ U$. Again by [12, VII.2, Theorem 2], we have

$$\mathsf{v}_! = \_ \otimes_{\mathbb{F}^{op}} {}_{\mathsf{v}}\mathbb{D}^{op\bullet} \tag{3}$$

where ${}_{\mathsf{v}}\mathbb{D}^{op\bullet} : \mathbb{F}^{op} \times \mathbb{D} \to \mathit{Set}$ is the bifunctor defined on objects as

$${}_{\mathsf{v}}\mathbb{D}^{op\bullet}(n, d^{(m)}) = \mathbb{D}^{op}(d, \mathsf{v}(n)) = \mathbb{D}(\bot^{(n)}, d).$$

By expanding the equation (3), we can give the following more elementary definition of $\mathsf{v}_!$ on objects $F : \mathbb{F} \to \mathit{Set}$, $d^{(m)} : \mathbb{D}$:

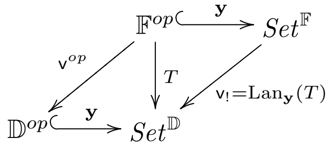
$$\begin{aligned} \mathsf{v}_!(F)(d) &= F \otimes_{\mathbb{F}^{op}} \mathbb{D}(\mathsf{v}(\_), d) = (\amalg_{n\in\mathbb{N}} F_n \times \mathbb{D}(\mathsf{v}(n), d))_{/\sim} \\ &= (\amalg_{n\in\mathbb{N}} F_n \times \mathbb{F}(n, m))_{/\sim} = (\amalg_{n\in\mathbb{N}} F_n \times m^n)_{/\sim} \end{aligned}$$

since $\mathbb{D}(\mathsf{v}(n), d) = \mathbb{D}(\bot^{(n)}, d) = \mathbb{F}(n, m)$, and where $\sim$ is the equivalence relation on pairs defined as follows: for $n, n' \in \mathbb{N}$, $f : n \to n'$, $g : n' \to m$, $a \in F_n$:

$$F_{n'} \times m^{n'} \ni (a[f], g) \sim (a, g \circ f) \in F_n \times m^n$$

By definition of $\sim$, any pair $(a, f) \in F_n \times m^n$ is equivalent to $(a[f], id) \in F_m \times m^m$. On the other hand, each $a \in F_m$ identifies uniquely an equivalence class $[(a, id)]_\sim$. Therefore, each equivalence class $\mathsf{v}_!(F)(d)$ can be given a unique canonic representantive $a \in F_m$. This means that there is a bijective equivalence between $\mathsf{v}_!(F)(d)$ and F_m, and hence $\mathsf{v}_!(F) \cong F \circ U$.

Alternative proof of $\mathsf{v}_! \cong \_ \circ U$: $\mathsf{v}_!$ can be defined as the left Kan extension along $\mathbf{y} : \mathbb{F}^{op} \hookrightarrow \mathit{Set}^{\mathbb{F}}$ of the functor $T : \mathbb{F}^{op} \to \mathit{Set}^{\mathbb{D}}$, $T(n) = \mathbb{D}(\bot^{(n)}, \_) = \mathbf{y} \circ \mathsf{v}^{op}$:



. Hence:

$$\mathsf{v}_!(F) = (\mathrm{Lan}_{\mathbf{y}}(T))(F) = \int^{m\in\mathbb{F}} Set^{\mathbb{F}}(\mathbf{y}(m), F) \cdot \mathbb{D}(\bot^{(m)}, \_)$$
$$= \int^{m\in\mathbb{F}} F_m \cdot \mathbb{F}(m, U(\_)) = \left(\int^{m\in\mathbb{F}} F_m \cdot \mathbb{F}(m, \_)\right) \circ U = F \circ U \quad \square$$

Proposition 9. $\mathsf{v} : Set^{\mathbb{F}} \to Set^{\mathbb{D}}$ *is an embedding, that is:* $\mathsf{v}^* \circ \mathsf{v}_* \cong Id$.

Proof. For $F : \mathbb{F} \to Set$, we have to prove that $\mathsf{v}_*(F) \circ \mathsf{v} \cong F$. For $n \in \mathbb{F}$, we have $\mathsf{v}_*(F)_{\mathsf{v}(n)} = \mathsf{v}_*(F)_{\bot^{(n)}} = F_n$ by definition. Analogously, it is easy to prove that for $f : n \to m$ in $\mathbb{F}$, it is $\mathsf{v}_*(F)(\mathsf{t}(f)) = \mathsf{v}_*(F)(f) = F_f$. $\square$

As a consequence, by [12, VII.4, Lemma 1] we have also $\mathsf{v}^* \circ \mathsf{v}_! \cong Id$, and hence both v_* and $\mathsf{v}_!$ are full and faithful.

A similar result holds also for $\mathsf{t} : \mathbb{I} \hookrightarrow \mathbb{D}$, although the adjoints have not a neat description as in the previous case.

Proposition 10. t *induces an essential geometric morphism* $\mathsf{t} : Set^{\mathbb{I}} \to Set^{\mathbb{D}}$, *that is two adjunctions* $\mathsf{t}_! \dashv \mathsf{t}^* \dashv \mathsf{t}_*$,

$$Set^{\mathbb{I}} \;\underset{\mathsf{t}_*}{\overset{\mathsf{t}_!}{\rightrightarrows}}\; Set^{\mathbb{D}}, \qquad Set^{\mathbb{I}} \xleftarrow{\;i^* = \_\circ\mathsf{t}\;} Set^{\mathbb{D}}$$

where for all $G : \mathbb{I} \to Set$, *and* $d \in \mathbb{D}$, *it is* $\mathsf{t}_*(G)(d) = Set^{\mathbb{I}}(\mathbb{D}(d, \mathsf{t}(\_)), G)$ *and* $\mathsf{t}_!(G)(d) = GV(d) \times \mathbb{D}(\top^{(V(d))}, d)$.

Proof. The definition of t_* is a direct application of [12, VII.2, Theorem 2]. Let us prove the definition of $\mathsf{t}_!$. We know that

$$\mathsf{t}_! = \_ \otimes_{\mathbb{I}^{op}} {}_{\mathsf{t}}\mathbb{D}^{op\bullet} \tag{4}$$

where ${}_{\mathsf{t}}\mathbb{D}^{op\bullet} : \mathbb{I}^{op} \times \mathbb{D} \to Set$ is the bifunctor defined on objects as

$${}_{\mathsf{t}}\mathbb{D}^{op\bullet}(n, d^{(m)}) = \mathbb{D}^{op}(d, \mathsf{t}(n)) = \mathbb{D}(\top^{(n)}, d).$$

By expanding the equation (4), we can give the following more elementary definition of $\mathsf{t}_!$ on objects $G : \mathbb{I} \to Set$, $d^{(m)} : \mathbb{D}$:

$$\mathsf{t}_!(G)(d) = G \otimes_{\mathbb{I}^{op}} \mathbb{D}(\mathsf{t}(\_), d) = \left(\amalg_{n\in\mathbb{N}} G_n \times \mathbb{D}(\top^{(n)}, d)\right)_{/\sim}$$

where $\sim$ is the equivalence relation on pairs defined as follows: for $n, n' \in \mathbb{N}$, $f : n \rightarrowtail n'$, $g : \top^{(n')} \to d^{(m)}$, $a \in G_n$:

$$G_{n'} \times \mathbb{D}(\top^{(n')}, d) \ni (a[f], g) \sim (a, g \circ f) \in G_n \times \mathbb{D}(\top^{(n)}, d)$$

Now, notice that for any $h \in \mathbb{D}(\top^{(n)}, d^{(m)})$ is a function $h : n \rightarrowtail m$ which can be factorized as $h = g \circ in$, where $in : n \hookrightarrow V(d)$ is the inclusion (and thus $in : \top^{(n)} \to \top^{(V(d))}$) and $g : \top^{(V(d))} \to d$ is a suitable monomorphism $g : V(d) \rightarrowtail m$. Therefore, for any pair $(a, h) \in G_n \times \mathbb{D}(\top^{(n)}, d)$ there is an equivalent one $(a[in], g) \in G_{V(d)} \times \mathbb{D}(\top^{(V(d))}, d)$, hence the thesis. $\square$

Proposition 11. $\mathtt{t} : Set^{\mathbb{I}} \to Set^{\mathbb{D}}$ *is an embedding, that is:* $\mathtt{t}^* \circ \mathtt{t}_* \cong Id$.

Proof. For $F : \mathbb{I} \to Set$, we have to prove that $\mathtt{t}_*(F) \circ \mathtt{t} \cong F$. For $n \in \mathbb{I}$, we have $\mathtt{t}_*(F)_{\mathtt{t}(n)} \cong F_n$, since $\mathtt{t}_*(F)_{\mathtt{t}(n)} = Set^{\mathbb{I}}(\mathbb{D}(\top^{(n)}, \mathtt{t}(\_)), F) \cong Set^{\mathbb{I}}(\mathbb{I}(n, \_), F) \cong F_n$.

It is similarly easy to prove that on morphisms, the action of $\mathtt{t}_*(F) \circ \mathtt{t}$ is isomorphic to that of F. Let $f : n \to m$ in $\mathbb{I}$; then, $\mathtt{t}_*(F)(\mathtt{t}(f))$ maps a natural transformation $\phi : \mathbb{I}(n, \_) \to F$ to the natural transformation $\phi : \mathbb{I}(m, \_) \to F$ whose components are $\psi_k = \phi_k(\_ \circ f) = F_f \circ \phi_k$, hence the thesis. □

This means that also $\mathtt{t}^* \circ \mathtt{t}_! \cong Id$, and hence both $\mathtt{t}_*$ and $\mathtt{t}_!$ are full and faithful.

Algebras and coalgebras of polynomial functors. It is well-known that any polynomial functor over *Set* (i.e., defined only by constant functors, finite products/coproducts and finite powersets) has initial algebra. This result has been generalized to $Set^{\mathbb{F}}$ [6, 10] in order to deal with signatures with *variable bindings*; in this case, polynomials can contain also Var, the functor of *variables*, and a dynamic allocation functor $\delta_{\mathbb{F}} : Set^{\mathbb{F}} \to Set^{\mathbb{F}}$. For instance, the datatype of λ-terms up-to α-conversion can be defined as the initial algebra of the functor

$$\Sigma_{\Lambda}(X) = Var + X \times X + \delta_{\mathbb{F}}(X) \tag{5}$$

that is, for all $n \in \mathbb{F}$: $\Sigma_{\Lambda}(X)_n = n + X_n \times X_n + X_{n+1}$. A parallel generalization for dealing with *name generation* use the category $Set^{\mathbb{I}}$ (and its variants) [10, 8, 5], which provides the functor of *names* N and a dynamic allocation functor $\delta_{\mathbb{I}} : Set^{\mathbb{I}} \to Set^{\mathbb{I}}$. The datatype of λ-terms where all bound variables are "fresh"[4] is defined as the initial algebra of the functor

$$\Sigma_{\Lambda}(X) = N + X \times X + \delta_{\mathbb{I}}(X) \tag{6}$$

that is, for all $n \in \mathbb{I}$: $\Sigma_{\Lambda}(X)_n = n + X_n \times X_n + X_{n+1}$.

The domain for late semantics of π-calculus [5] can be defined as the final coalgebra of the functor $B : Set^{\mathbb{I}} \to Set^{\mathbb{I}}$

$$\begin{aligned} BP &\triangleq \wp_f(\overbrace{N \times P^N}^{\text{input}} + \overbrace{N \times N \times P}^{\text{output}} + \overbrace{N \times \delta_{\mathbb{I}} P}^{\text{bound output}} + \overbrace{P}^{\tau}) \\ (BP)_X &= \wp_f(X \times (P_X)^X \times P_{X \uplus 1} + X \times X \times P_X + X \times P_{X \uplus 1} + P_X). \end{aligned} \tag{7}$$

In $Set^{\mathbb{D}}$, we can generalize a step further. We say that a functor $F : Set^{\mathbb{D}} \to Set^{\mathbb{D}}$ is *polynomial* if it be defined by using only $Atom$, constant functors, finite products/coproducts, dynamic allocations δ^+ and δ^- and finite powersets.

There is a precise relation among initial algebras of polynomial functors on $Set^{\mathbb{F}}$ and $Set^{\mathbb{D}}$. Let us recall a general result (see e.g. [10]):

Proposition 12. *Let* $\mathcal{C}, \mathcal{D}$ *be two categories and* $f : \mathcal{C} \longrightarrow \mathcal{D}$, $T : \mathcal{C} \longrightarrow \mathcal{C}$ *and* $T' : \mathcal{D} \longrightarrow \mathcal{D}$ *be three functors such that* $T' \circ f \cong f \circ T$ *for some natural isomorphism* $\phi : T' \circ f \longrightarrow f \circ T$.

[4] This is what Barendregt called "hygienic convention".

1. *If f has a right adjoint f^*, and $(A, \alpha : TA \to A)$ is an initial T-algebra in $\mathcal{C}$, then $(f(A), f(\alpha) \circ \phi_A : T'(f(A)) \to f(A))$ is an initial T'-algebra in $\mathcal{D}$.*
2. *If f has a left adjoint f^*, and $(A, \alpha : A \to TA)$ is a final T-coalgebra in $\mathcal{C}$, then $(f(A), \phi_A^{-1} \circ f(\alpha) : f(A) \to T'(f(A)))$ is a final T'-coalgebra in $\mathcal{D}$.*

Proof. 1. The adjoint pair $f \dashv f^*$ can be lifted to a pair of adjoint functors between the categories of T- and T'- algebras. Since any functor with a right adjoint preserves colimits and the initial object is a colimit, then the initial object of the former category is preserved in the latter.
2. Like in the previous case, the adjoint $f^* \dashv f$ can be lifted to the categories of coalgebras, and functors with a left adjoint preserve limits. □

For a polynomial functor $T : Set^{\mathbb{D}} \to Set^{\mathbb{D}}$, let us denote $\bar{T} : Set^{\mathbb{F}} \to Set^{\mathbb{F}}$ the functor obtained by replacing $Atom$ with Var and δ^+, δ^- with $\delta_{\mathbb{F}}$ in T.

Theorem 13. *The polynomial functor $T : Set^{\mathbb{D}} \to Set^{\mathbb{D}}$ has initial algebra, which is (isomorphic to) $F \circ U$, where (F, α) is the initial $\bar{T}$-algebra in $Set^{\mathbb{F}}$.*

Proof. The functor $\bar{T}$ has initial algebra (see e.g. [6, 10]); let us denote it by (F, α). In order to prove the result, we apply Proposition 12(1), where $f : \mathcal{C} \longrightarrow \mathcal{D}$ is the functor $\mathsf{v}_! = \_ \circ U : Set^{\mathbb{F}} \to Set^{\mathbb{D}}$ of Proposition 8, whose right adjoint is v^*. Then $\mathsf{v}_!(F) = F \circ U$. We have only to prove that $T \circ \mathsf{v}_! \cong \mathsf{v}_! \circ \bar{T}$. It is easy to see that this holds for products, coproducts, constant functors and finite powersets. It is also trivial to see that $Atom \cong Var \circ U$.

It remains to prove that $\kappa \circ \mathsf{v}_! \cong \mathsf{v}_! \circ \delta_{\mathbb{F}}$, for $\kappa = \delta^+, \delta^-$. For F a functor in $Set^{\mathbb{F}}$, we prove that there is a natural isomorphism $\phi : \kappa(\mathsf{v}_!(F)) = \kappa(F \circ U) \longrightarrow \mathsf{v}_!(\delta_{\mathbb{F}}(F)) = \delta_{\mathbb{F}}(F) \circ U$. This is trivial, because for $d^{(n)}$ a distinction in $\mathbb{D}$, it is $\kappa(F \circ U)_d = (F \circ U)_{\kappa d} = F_{U(\kappa d)} = F_{n+1} = \delta_{\mathbb{F}}(F)_n = (\delta_{\mathbb{F}}(F) \circ U)_d$. □

Therefore, initial algebras of polynomial functors in $Set^{\mathbb{D}}$ are exactly initial algebras of the corresponding functors in $Set^{\mathbb{F}}$. This means that $Set^{\mathbb{D}}$ can be used in place of $Set^{\mathbb{F}}$ for defining datatypes with variable binding, as in e.g. [9].

There is a similar connection between $Set^{\mathbb{I}}$ and $Set^{\mathbb{D}}$, about final coalgebras.

Lemma 14. $\delta^+ \circ \mathsf{t}_* \cong \mathsf{t}_* \circ \delta_{\mathbb{I}}$.

Proof. Let $F : \mathbb{I} \to Set$ be a functor, and $d^{(n)} \in \mathbb{D}$; we have to prove that

$$Set^{\mathbb{I}}(\mathbb{D}(\delta^+ d, \mathsf{t}(\_)), F) \cong Set^{\mathbb{I}}(\mathbb{D}(d, \mathsf{t}(\_)), \delta_{\mathbb{I}} F)$$

natural in d and F.

For $\phi : \mathbb{D}(\delta^+ d, \mathsf{t}(\_)) \longrightarrow F$, the corresponding natural transformation $\psi : \mathbb{D}(d, \mathsf{t}(\_)) \longrightarrow \delta_{\mathbb{I}} F$ has components $\psi_n \triangleq \phi_{n+1} \circ \delta^+$. More explicitly, for $f : d \to \top^{(n)}$, we have $\delta^+(f) : \delta^+ d \to \top^{(n+1)}$, thus $\phi_{n+1}(\delta^+(f)) \in F_{n+1} = (\delta_{\mathbb{I}}(F))_n$.

On the other hand, for $\psi : \mathbb{D}(d, \mathsf{t}(\_)) \longrightarrow \delta_{\mathbb{I}} F$, the components of the corresponding natural transformation $\phi : \mathbb{D}(\delta^+ d, \mathsf{t}(\_)) \longrightarrow F$ are defined as follows. Trivially, $\phi_0 = ? : \emptyset \to F_0$, because $\mathbb{D}(\delta^+ d, \top^{(0)}) = \emptyset$. Let us consider $n \neq 0$, and $f : \delta^+ d \to \top^{(n)}$, we have to define $\psi_n(f) \in F_n$. Now, let $n = m + 1$, where the element in 1 is the image along f of the element added by δ^+ to d. The restrict of f to d is a morphism $f_{|d} : d \to \top^{(m)}$. Thus, we define $\phi_n(f) \triangleq \psi_m(f_{|d})$.

It is easy to check that these two mappings are inverse of each other. □

Lemma 15. $\delta^- \circ \mathtt{t}_* \cong \mathtt{t}_* \circ (\_)^N$.

Proof.
$$\begin{aligned}
\mathtt{t}_*(F^N)_d &= \mathit{Set}^{\mathbb{I}}(\mathbb{D}(d, \mathtt{t}(_)), F^{\mathbf{y}(1)}) && \text{by definition}\\
&\cong \mathit{Set}^{\mathbb{I}}(\mathbb{D}(d, \mathtt{t}(_)) \times \mathbf{y}(1), F) && \text{since we are in a CCC}\\
&= \mathit{Set}^{\mathbb{I}}(\mathbb{D}(d, \mathtt{t}(_)) \times \mathbb{D}(\top^{(1)}, \mathtt{t}(_)), F) &&\\
&\cong \mathit{Set}^{\mathbb{I}}(\mathbb{D}(d + \top^{(1)}, \mathtt{t}(_)), F) && \mathbb{D}(_, e) \text{ preserves products}\\
&= \mathit{Set}^{\mathbb{I}}(\mathbb{D}(\delta^- d, \mathtt{t}(_)), F) && \text{by definition}\\
&= \mathtt{t}_*(F)_{\delta^- d} = \delta^-(\mathtt{t}_*(F))_d &&
\end{aligned}$$
□

Let $T : \mathit{Set}^{\mathbb{I}} \to \mathit{Set}^{\mathbb{I}}$ be a polynomial functor. Let us denote by $\tilde{T} : \mathit{Set}^{\mathbb{D}} \to \mathit{Set}^{\mathbb{D}}$ the functor obtained by replacing in (the polynomial of) T, every occurrence of N with $\mathtt{t}_*(N)$, δ with δ^+, $(\_)^N$ with δ^-. Then, we have the following:

Theorem 16. *The functor $\tilde{T} : \mathit{Set}^{\mathbb{D}} \to \mathit{Set}^{\mathbb{D}}$ has final coalgebra, which is (isomorphic to) $\mathtt{t}_*(F)$, where (F, β) is the final T-coalgebra in $\mathit{Set}^{\mathbb{I}}$.*

Proof. Follows from previous lemmas and Proposition 12(2). □

Therefore, in $\mathit{Set}^{\mathbb{D}}$ we can define coalgebrically all the objects definable by polynomial functors in $\mathit{Set}^{\mathbb{I}}$, like that for late bisimulation [5]. Moreover, $\mathit{Set}^{\mathbb{D}}$ provides other constructors, such as *Atom*, which do not have a natural counterpart in $\mathit{Set}^{\mathbb{I}}$. An example of application of these distinctive constructors, following [9], is the characterization of open semantics of π-calculus as the final coalgebra of the functor $B_o : \mathit{Set}^{\mathbb{D}} \to \mathit{Set}^{\mathbb{D}}$:

$$B_o P \triangleq \wp_f(\overbrace{\mathit{Atom} \times \delta^- P}^{\text{input}} + \overbrace{\mathit{Atom} \times \mathit{Atom} \times P}^{\text{output}} + \overbrace{\mathit{Atom} \times \delta^+ P}^{\text{bound output}} + \overbrace{P}^{\tau}) \qquad (8)$$

Notice that, although similar in shape, B_o is not the lifting of the functor B of strong late bisimulation in $\mathit{Set}^{\mathbb{I}}$ (Equation 7), nor can be defined on $\mathit{Set}^{\mathbb{I}}$. More precisely, open bisimulation is closed under all name substitutions keeping apart extruded names. Thus, names are actually atoms, which can be unified if the distinctions allow so. A bound output adds a new atom to the distinction, which must be kept apart from any other previously known atom—hence the usage of δ^+. On the other hand, an input action introduces an atom which can be unified with any other name—hence the usage of δ^-.

4 Support and apartness

A key feature of categories for modeling names, such as $\mathit{Set}^{\mathbb{I}}$ and similar functor categories, is to provide some notion of *support* of terms/elements, and of *non-interference*, or "apartness" [18, 8]. In this section, we first introduce a general definition of *support* and *apartness*, and then we examine these notions in the case of $\mathit{Set}^{\mathbb{D}}$, and related categories.

4.1 Support

Definition 17 (support). *Let $\mathcal{C}$ be a category, $F : \mathcal{C} \to Set$ be a functor. Let C be an object of $\mathcal{C}$, and $a \in F_C$. A subobject $i : D \rightarrowtail C$ of C* supports a (at C) *if there exists a (not necessarily unique) $b \in F_D$ such that $a = F_i(b)$.*

A support is called proper *iff it is a proper subobject.*

We denote by $\mathrm{Supp}_{F,C}(a)$ the set of subobjects of C supporting a. The intuition is that D supports $a \in F_C$ if D is "enough" for defining a. It is clear that the definition does not depend on the particular subobject representative. As a consequence, a is affected by what happens to elements in D only:

Proposition 18. *For all $D \in \mathrm{Supp}_{F,C}(a)$, and for all $h, k : C \to C'$: if $h_{|D} = k_{|D}$ then $F_h(a) = F_k(a)$.*

Notice that in general, the converse of Proposition 18 does not hold.

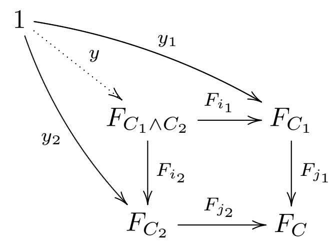
Remark 19. When $\mathcal{C} = \mathbb{F}, \mathbb{I}$, the supports of $a \in F_n$ can be seen as *approximations at stage* n of the free variables/names of a—that is, the free variables/names which are observable from n. For instance, let us consider $t \in \Lambda_n$, where Λ is the algebraic definition of untyped λ-calculus in equation 5. It is easy to prove by induction on t that for all $m \subseteq n$: $m \in \mathrm{Supp}_{\Lambda,n}(t) \iff FV(t) \subseteq m$.

Supports are viewed as "approximations" because elements may have not any proper support, at any stage. For example, consider the presheaf *Stream* : $\mathbb{F} \to Set$ constantly equal to the set of all *infinite* lists of variables. The stream $s = (x_1, x_2, x_3, \dots)$, which has infinite free variables, belongs to $Stream_n$ for all n, but also $\mathrm{Supp}_{Stream,n}(s) = \{n\}$. □

$\mathrm{Supp}_{F,C}(a)$ is a poset, inheriting its order from $\mathrm{Sub}(a)$, and C itself is always its top, but it may be that there are no proper supports, as shown in the remark above. Even in the case that an element has some finite (even proper) support, still it may be that it does not have a *least* support. (Consider, e.g., $G : \mathbb{F} \to Set$ such that $G_n = \emptyset$ for $|n| < 2$, and $= \{x\}$ otherwise; then $x \in G_{\{x,y,z\}}$ is supported by $\{x, y\}$ and $\{x, z\}$ but not by $\{x\}$ alone.) However, we can prove the following

Proposition 20. *Let $\mathcal{C}$ have pullbacks, $F : \mathcal{C} \to Set$ be pullback-preserving, C be in $\mathcal{C}$, and $x \in F_C$. If both C_1, C_2 support x at C, then $C_1 \wedge C_2$ supports x.*

Proof. $C_1 \wedge C_2$ is the pullback of the inclusions $j_1 : C_1 \rightarrowtail C$, $j_2 : C_2 \rightarrowtail C$; hence, by hypothesis the square in the diagram below is a pullback in *Set*:



Let $y_1 \in F_{C_1}$ and $y_2 \in F_{C_2}$ be the witnesses of x at stages C_1, C_2 by the definition of support. Due to the pullback there exists a (unique) $y \in F_{C_1 \wedge C_2}$ such that $F_{j_1 \circ i_1}(b) = F_{j_2 \circ i_2}(b) = a$, hence the thesis. □

Remark 21. In the case that $\mathcal{C} = \mathbb{I}$, pullback-preserving functors correspond to sheaves with respect to the atomic topology, that is the Schanuel topos [12]. This subcategory of $Set^{\mathbb{I}}$ has been extensively used in previous work for modeling names and nominal calculi; see [10, 4] among others, and ultimately also the FM techniques by Gabbay and Pitts [8, 16], since the category of nominal sets with finite support is equivalent to the Schanuel topos [8, Section 7].

We will use pullback-preserving functors over $\mathbb{D}$ in Section 6 below. □

Along the same line of Definition 17, we can introduce an abstract general notion of "closed element":

Definition 22. *Let $\mathcal{C}$ be a category with initial object 0. For $A : \mathcal{C} \to Set$, an element $a \in A_C$ is* closed *if $0 \in \mathrm{Supp}_{A,C}(a)$.*

Closed elements are not affected by any action on atoms whatsoever:

Proposition 23. *Let $\mathcal{C}$ be a category with initial object 0. For all $A : \mathcal{C} \to Set$, $C \in \mathcal{C}$, $a \in A_C$, if a is closed then for all $h, k : C \to D$ in $\mathcal{C}$: $A_h(a) = A_k(a)$.*

Proof. Follows from Proposition 18, noticing that $h_{|0} = k_{|0}$ always. □

In the rest of the paper, we focus on the case when $\mathcal{C}$ is one of $\mathbb{F}$, $\mathbb{I}$, $\mathbb{D}$, which do have pullbacks and initial object ($\emptyset$, $\emptyset$ and $\bot^{(\emptyset)}$ respectively). As one may expect, the support in $\mathbb{D}$ is a conservative generalization of those in $\mathbb{F}$ and $\mathbb{I}$:

Proposition 24.
1. *Let $n, m \in \mathbb{F}$, and $F : \mathbb{F} \to Set$. For all $a \in F_n$: $m \in \mathrm{Supp}_{F,n}(a) \iff \mathsf{v}(m) \in \mathrm{Supp}_{\mathsf{v}_!(F),\mathsf{v}(n)}(a)$.* [5]
2. *Let $n, m \in \mathbb{I}$, and $F : \mathbb{I} \to Set$. For all $a \in F_n$: $m \in \mathrm{Supp}_{F,n}(a) \iff \mathsf{t}(m) \in \mathrm{Supp}_{\mathsf{t}_*(F),\mathsf{t}(n)}(a)$.*

4.2 Apartness

We can now give the following general key definition, generalizing that used sometimes in $Set^{\mathbb{I}}$ (see e.g. [18]).

Definition 25 (Apartness). *Let $\mathcal{C}$ be a category with pullbacks and initial object. For $A, B : \mathcal{C} \to Set$, the functor $A \#_{\mathcal{C}} B : \mathcal{C} \to Set$ ("A apart from B") is defined on objects as follows:*

$$\begin{aligned}(A \#_{\mathcal{C}} B)_C = \{&(a, b) \in A_C \times B_C \mid \textit{for all } f : C \to D : \\ &\textit{there exist } s_1 \in \mathrm{Supp}_{A,D}(A_f(a)), s_2 \in \mathrm{Supp}_{B,D}(B_f(b)) \textit{ s.t. } s_1 \wedge s_2 = 0\} \end{aligned} \quad (9)$$

For $f : C \to D$, it is $(A \#_{\mathcal{C}} B)_f \triangleq A_f \times B_f$.

As a syntactic shorthand, we will write pairs $(a, b) \in (A \#_{\mathcal{C}} B)_c$ as $a \# b$. In the following, we will drop the index $_{\mathcal{C}}$ when clear from the context.

Let us now apply this definition to the three categories $Set^{\mathbb{I}}$, $Set^{\mathbb{F}}$, and $Set^{\mathbb{D}}$.

[5] Recall that $\mathsf{v}_!(F)_{\mathsf{v}(n)} \cong F_n$, and hence it is consistent to consider $a \in \mathsf{v}_!(F)_{\mathsf{v}(n)}$.

$\mathcal{C} = \mathbb{F}$ In this case we have that $a \# b$ iff at least one of a, b is closed, i.e., it is supported by the empty set: if both a and b have only non-empty supports, then some variable can be always unified by a suitable morphism. So the definition above simplifies as follows:

$$(A \#_{\mathbb{F}} B)_n = \{(a, b) \in A_n \times B_n \mid \emptyset \in \mathrm{Supp}_{A,n}(a) \text{ or } \emptyset \in \mathrm{Supp}_{B,n}(b)\} \quad (10)$$

$\mathcal{C} = \mathbb{I}$ In this case, names are subject only to injective renamings, and therefore can be never unified. So it is sufficient to look at the present stage, that is, the definition above simplifies as follows:

$$\begin{aligned}(A \#_{\mathbb{I}} B)_n = \{(a, b) \in A_n \times B_n \mid \\ \text{there exist } n_1 \in \mathrm{Supp}_{A,n}(a), n_2 \in \mathrm{Supp}_{B,n}(b) \text{ s.t. } n_1 \cap n_2 = \emptyset\}\end{aligned} \quad (11)$$

which corresponds to say that $a \# b$ iff a, b do not share any free name.

$\mathcal{C} = \mathbb{D}$ This case subsumes both previous cases: informally, $(a, b) \in (A \# B)_d$ means that if i is an atom appearing free in a, then any j occurring free in b can never be unified with i, that is $(i, j) \in d$:

$$\begin{aligned}(A \#_{\mathbb{D}} B)_{d^{(n)}} = \{(a, b) \in A_d \times B_d \mid \\ \text{there exist } s_1 \in \mathrm{Supp}_{A,d}(a), s_2 \in \mathrm{Supp}_{B,d}(b) \text{ s.t. } s_1 \oplus s_2 \leq d\}\end{aligned} \quad (12)$$

Actually, all these tensors arise from the monoidal structures $\oplus$ and $\uplus$ of the categories $\mathbb{I}$ and $\mathbb{D}$, via the following general construction due to Day [3]:

Proposition 26. *Let $(\mathcal{C}, \star, I)$ be a (symmetric) monoidal category. Then, $(Set^{\mathcal{C}}, \star_{\mathcal{C}}, \mathbf{y}(I))$ is a (symmetric) closed monoidal category, where*

$$(A \star_{\mathcal{C}} B)_C = \int^{C_1} A_{C_1} \times \int^{C_2} B_{C_2} \times \mathcal{C}(C_1 \star C_2, C) \quad (13)$$

Theorem 27. *The monoidal structure $(\mathbb{D}, \oplus, \bot^{(\emptyset)})$ induces, via equation 13, the monoidal structure $(Set^{\mathbb{D}}, \#_{\mathbb{D}}, \mathbf{y}(\bot^{(0)}) = \mathcal{K}_1 = 1)$ of equation 12.*

Proof. Let $A, B : \mathbb{D} \to Set$, and $d^{(n)} \in \mathbb{D}$; by applying Proposition 26 and since products preserves coends, we have

$$\begin{aligned}(A \star_{\mathbb{D}} B)_d &= \iint^{d_1, d_2} A_{d_1} \times B_{d_2} \times \mathbb{D}(d_1 \oplus d_2, d) \\ &= \left(\coprod_{d_1, d_2 \in \mathbb{D}} A_{d_1} \times B_{d_2} \times \mathbb{D}(d_1 \oplus d_2, d) \right)_{/\approx}\end{aligned} \quad (14)$$

where the equivalence $\approx$ is defined on triples as follows

$$\begin{aligned}(a, b, f : d_1 \oplus d_2 \to d) \approx (a', b', g : d_1' \oplus d_2' \to d) \\ \iff A_{f \circ inl}(a) = A_{g \circ inl}(a') \text{ and } B_{f \circ inr}(b) = B_{g \circ inr}(b')\end{aligned}$$

For each class $[(a,b,f:d_1\oplus d_2\to d)]\in(A\star_{\mathbb{D}}B)_d$ we can associate a unique pair $(A_{f\circ inl}(a),B_{f\circ inr}(b))\in(A\#_{\mathbb{D}}B)_d$; the definition does not depend on the particular representative we choose.

On the converse, let us consider a pair $(a,b)\in(A\#_{\mathbb{D}}B)_d$; this means that

- there exists $f_1:s_1\rightarrowtail d$, $a'\in A_{s_1}$ such that $a=A_{f_1}(a')$
- there exists $f_2:s_2\rightarrowtail d$, $b'\in B_{s_2}$ such that $b=B_{f_2}(b')$

and such that $[f_1,f_2]:s_1\oplus s_2\rightarrowtail d$. We can associate this pair (a,b) to the equivalence class of the triple $(a',b',[f_1,f_2])$ in the coend 14. The class defined in this way does not depend on the particular a' and b' we choose.

It is easy to check that these two mappings are inverse of each other. □

A similar constructions applies also to $Set^{\mathbb{I}}$, as observed e.g. in [18]:

Proposition 28. *The monoidal structure* $(\mathbb{I},\uplus,0)$ *induces, via equation 13, the monoidal structure* $(Set^{\mathbb{I}},\#_{\mathbb{I}},\mathbf{y}(0)=1)$ *of equation 11.*

Using Theorem 27, we can show that $\#_{\mathbb{F}}$ is a particular case of $\#_{\mathbb{D}}$:

Proposition 29. $\#_{\mathbb{F}}=\mathsf{v}^*\circ\#_{\mathbb{D}}\circ\langle\mathsf{v}_*,\mathsf{v}_*\rangle$.

Proof. Let us prove that for $F,G:\mathbb{F}\to Set$, it is $(\mathsf{v}_*(F)\#_{\mathbb{D}}\mathsf{v}_*(G))_{\perp^{(n)}}\cong(F\#_{\mathbb{F}}G)_n$. By applying Theorem 27, we have

$$(\mathsf{v}_*(F)\#_{\mathbb{D}}\mathsf{v}_*(G))_{\perp^{(n)}}=\left(\coprod_{d_1^{(n_1)},d_2^{(n_2)}\in\mathbb{D}}\mathsf{v}_*(F)_{d_1}\times\mathsf{v}_*(G)_{d_2}\times\mathbb{D}(d_1\oplus d_2,\perp^{(n)})\right)_{/\approx}$$
$$=\left(\coprod_{d_1^{(n_1)},d_2^{(n_2)}\in\mathbb{D}}F_{n_1}\times G_{n_2}\times\mathbb{D}(d_1\oplus d_2,\perp^{(n)})\right)_{/\approx}$$

Let us consider the set $\mathbb{D}(d_1\oplus d_2,\perp^{(n)})$. If $d_1\oplus d_2=\perp^{(m)}$ for some m, then $\mathbb{D}(d_1\oplus d_2,\perp^{(n)})=\mathbb{F}(m,n)$. Otherwise, $\mathbb{D}(d_1\oplus d_2,\perp^{(n)})=\emptyset$.

Now, the only way for having $d_1\oplus d_2=\perp^{(m)}$ is that both d_1 and d_2 are empty relations $\perp^{(n_1)},\perp^{(n_2)}$, and at least one of them has no atoms at all (otherwise the $\oplus$ would add a distinction in any case). Therefore, the equivalence above can be continued as follows:

$$\ldots=\left(\left(\coprod_{n_1\in\mathbb{F}}F_{n_1}\times G_{\emptyset}\times\mathbb{F}(n_1,n)\right)+\left(\coprod_{n_2\in\mathbb{F}}F_{\emptyset}\times G_{n_2}\times\mathbb{F}(n_2,n)\right)\right)_{/\approx}$$

This means that the triples are either of the form $(a\in F_{\emptyset},b\in G_{n_2},f:n_2\to n)$, or of the form $(a\in F_{n_1},b\in G_{\emptyset},f:n_1\to n)$. The first is equivalent to the pair $(F_?(a),G_f(b))$, the second to the pair $(F_f(a),G_?(b))$, both in $(F\#_{\mathbb{F}}G)_n$. □

The next corollary is a consequence of Theorem 27 and Proposition 26:

Corollary 30. *The functor $A\#\_ : Set^{\mathbb{D}} \to Set^{\mathbb{D}}$ has a right adjoint $[A]\_$, defined on objects by $([A]B)_d = Set^{\mathbb{D}}(A, B_{d\oplus\_})$.*

Proof. By the general construction in [3], the right adjoint of $A\star\_$ is $[A]\_$, defined as $([A]B)_d = \int_{e^{(m)}} Set(A_e, B_{d\oplus e})$, which yields the thesis. ☐

Remark 31. Let us consider the counit $ev_{A,B} : A \# [A]B \longrightarrow B$ of this adjunction. For $d \in \mathbb{D}$, the component $ev_d : (A \# [A]B)_d \longrightarrow B_d$ maps an element $a \in A_d$ and a natural transformation $\phi : A \to B_{d\oplus\_}$, *apart from each other*, to an element in B_d, which can be described as follows. Let $s_1, s_2 \in \mathrm{Sub}(d)$ supporting ϕ and a, respectively, and such that $s_1 \oplus s_2 \leq d$. By the definition of support, let $\phi' : A \to B_{s_1\oplus\_}$ and $a' \in A_{s_2}$ be the witnesses of ϕ and a at s_1 and s_2, respectively. Then, $\phi'_{s_2}(a') \in B_{s_1\oplus s_2}$, which can be mapped to an element in B_d by the inclusion $s_1 \oplus s_2 \leq d$. ☐

Finally, for $A = Atom$ we have the counterpart of Proposition 7:

Proposition 32. *$[Atom]\_ \cong \delta^+$, and hence $\_ \# Atom \dashv \delta^+$.*

Proof. Since $Atom = \mathbf{y}(\bot^{(1)})$, we have $([Atom]B)_d = Set^{\mathbb{D}}(\mathbf{y}(\bot^{(1)}), B_{d\oplus\_}) = B_{d\oplus\bot^{(1)}} = (\delta^+(B))_d$ by Yoneda. ☐

5 Substitution monoidal structure of $Set^{\mathbb{D}}$

Let us define a tensor product $\bullet : Set^{\mathbb{D}} \times Set^{\mathbb{D}} \to Set^{\mathbb{D}}$ as follows:

$$\text{for } A, B \in Set^{\mathbb{D}} : \qquad A \bullet B \triangleq \int^{e\in\mathbb{D}} A_e \cdot B^e$$

$$\text{that is, for } d \in \mathbb{D} : \qquad (A \bullet B)_d = \int^{e\in\mathbb{D}} A_e \times (B^e)_d$$

where, for $e^{(n)}$ in $\mathbb{D}$, $B^e : \mathbb{D} \to Set$ is the functor defined by

$$(B^e)_d = \{(b_1, \ldots, b_n) \in (B_d)^n \mid \text{if } (i,j) \in e \text{ then } (b_i, b_j) \in (B \# B)_d\}$$
$$(B^e)_f = (B_f)^n \qquad \text{for } f : d^{(m)} \to d'^{(m')}$$

Unfolding the coend, we obtain the following explicit description of $A \bullet B$:

$$(A \bullet B)_d = \left(\coprod_{e\in\mathbb{D}} A_e \times (B^e)_d \right)_{/\approx}$$

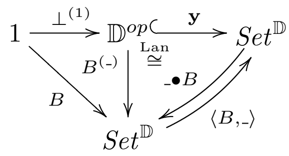
where $\approx$ is the equivalence relation defined by

$$(a; b_{\rho(1)}, \ldots, b_{\rho(n)}) \approx (A_\rho(a); b_1, \ldots, b_{n'}) \quad \text{for } \rho : e^{(n)} \to e'^{(n')}.$$

Actually, $B^{(\text{-})}$ can seen as a functor $B^{(\text{-})} : \mathbb{D}^{op} \to Set^{\mathbb{D}}$, adding the "reindexing" action on morphisms: for $\rho : e^{(n)} \to e'^{(n')}$, define $B^f : B^{e'} \longrightarrow B^e$ as the natural transformation with components $B^f_d : (B^{e'})_d \longrightarrow (B^e)_d$, $B^f_d(b_1, \ldots, b_{n'}) =$

$(b_{f(1)}, \ldots, b_{f(n)})$. It is easy to check that B^f is well defined: if $(i,j) \in e'^{(n')}$, then $(f(i), f(j)) \in e^{(n)}$ and hence $(b_{f(i)}, b_{f(j)}) \in (B \# B)_d$. The functor $B^{(\_)}$ is a generalization of Cartesian extension; for instance, $B^{\perp^{(2)}} = B \times B$, $B^{\top^{(2)}} = B \# B$.

We can give now a more abstract definition of $\_ \bullet B : \mathit{Set}^{\mathbb{D}} \to \mathit{Set}^{\mathbb{D}}$, for all $B \in \mathit{Set}^{\mathbb{D}}$. In fact, $\_ \bullet B$ arises as the left Kan extension of the functor $B^{(\_)}$:



(15)

where $\langle B, \_ \rangle$ is the right adjoint of $\_ \bullet B$, defined as $\langle B, A\rangle_d = \mathit{Set}^{\mathbb{D}}(B^d, A)$.

Proposition 33. *$(\mathit{Set}^{\mathbb{D}}, \bullet, \mathit{Atom})$ is a (non-symmetric) monoidal category.*

Proof. Since $\mathit{Atom} = \mathbf{y}(\perp^{(1)})$, the equivalence $A \bullet \mathit{Atom} \cong A$ follows from Diagram 15. The equivalence $\mathit{Atom} \bullet A \cong A$ is a simple calculation:

$$(\mathit{Atom} \bullet A)_d = \int^{e^{(m)}} \mathit{Atom}_{e^{(m)}} \times (A^{e^{(m)}})_d = \left(\coprod_{e^{(m)}} m \times (A^{e^{(m)}})_d \right)_{/\approx} \cong A_d$$

where the last equivalence holds because the class of a tuple $(i; a_1, \ldots, a_m) \in m \times (A^{e^{(m)}})_d$ corresponds uniquely to $a_i \in A_d$.

We prove now associativity of $\bullet$:

$$\begin{aligned}
((A \bullet B) \bullet C)_d &= \int^{e} (A \bullet B)_e \times (C^e)_d && \text{by definition,} \\
&= \int^{e} \left(\int^{e'} A_{e'} \times (B^{e'})_e \right) \times (C^e)_d && \text{by definition,} \\
&= \int^{e} \int^{e'} A_{e'} \times (B^{e'})_e \times (C^e)_d && \text{product preserves coends,} \\
&= \int^{e'} A_{e'} \times \int^{e} (B^{e'})_e \times (C^e)_d && \text{by Fubini result,} \\
&= \int^{e'} A_{e'} \times ((B^{e'}) \bullet C)_d && \text{by definition,} \\
&\cong \int^{e'} A_{e'} \times ((B \bullet C)^{e'})_d \\
&= (A \bullet (B \bullet C))_d
\end{aligned}$$

□

Monoids in $\mathit{Set}^{\mathbb{D}}$ satisfy the usual properties of clones. In particular, the multiplication $\sigma : A \bullet A \to A$ of a monoid (A, σ, v) can be seen as a *distinction-preserving* simultaneous substitution: for every $d^{(n)} \in \mathbb{D}$, σ_d maps (the class of)

$(a; a_1, \ldots, a_m) \in A_e \times (A^e)_d$ to an element in A_d, making sure that distinct atoms are "replaced by" elements which are apart (if $(i, j) \in e$, then $(a_i, a_j) \in (A\#A)_d$).

As in [6, 17], the monoidal structure of $Set^{\mathbb{D}}$ can be used for characterizing presheaves coherent with apartness-preserving substitution; in particular, presheaves generated by binding signatures with constructors for distinctions, such as the signature of D-Fusion [2]. Details will appear elsewhere.

6 Self-dual quantifier

As for any topos, $Set^{\mathbb{D}}$ can be used for modeling (higher-order) intuitionistic logic. However, like in $Set^{\mathbb{I}}$, the Schanuel topos, and FM-set theory, the extra structure given by apartness product brings in other, peculiar logical constructors. In this section we define a *self-dual* quantifier, in a suitable subcategory of $Set^{\mathbb{D}}$. We begin with a standard construction of categorical logic. For $A, B \in Set^{\mathbb{D}}$, let us consider the morphism $\theta : A \# B \hookrightarrow A \times B \xrightarrow{\pi} B$, given by inclusion in the cartesian product. We can define the *inverse image* of θ, $\theta^* : \mathrm{Sub}(B) \to \mathrm{Sub}(A \# B)$: for $U \in \mathrm{Sub}(A)$, the subobject $\theta^*(U) \in \mathrm{Sub}(A \# B)$ is the pullback of $U \rightarrowtail B$ along θ: $\quad \theta^*(U)_d = \{(x, y) \in (A \# B)_d \mid y \in U_d\}$.

By general and well-known results [15, 12], θ^* has both left and right adjoints, denoted by $\exists_\theta, \forall_\theta : \mathrm{Sub}(A \# B) \to \mathrm{Sub}(B)$, respectively. (If $\#$ is replaced by $\times$, these are the usual existential and universal quantifiers $\exists, \forall : \mathrm{Sub}(A \times B) \to \mathrm{Sub}(B)$.) Our aim is to prove that, under some conditions, it is $\exists_\theta = \forall_\theta$.

The condition is suggested by the following result, stating that if a property of a "well-behaved" type holds for a fresh atom, then it holds for *all* fresh atoms:

Proposition 34. *Let $B : \mathbb{D} \to Set$ be a pullback preserving functor, and let U a subobject of $Atom \# B$. Let $d \in \mathbb{D}$, and $(a, x) \in U_d$. Then for all $b \in Atom_d$ such that $b \# x$: $(b, x) \in U_d$.*

Proof. It suffices to define $f : d \to d$ in $\mathbb{D}$ such that $(Atom \# B)_f(a, x) = (b, x)$; that is, we have to find an $f : d \to d$ such that $f(a) = b$ and $B_f(x) = x$. By functoriality of U, this means that $U_f(a, x) = (b, B_f(x)) \in U_d$.

Since $(a, x) \in U$, it is $a \# x$; hence, let s_1, r_1 be the two subdistinctions supporting a and x at d, such that $s_1 \oplus r_1 \leq d$ (equation 12). Similarly, for $b \# x$, let s_2, r_2 be the two subdistinctions supporting b and x at d, such that $s_2 \oplus r_2 \leq d$. Both r_1 and r_2 support x; hence, by Proposition 20, also $r_1 \wedge r_2$ supports x. Thus we can define the map $f : d \to d$ as $f(a) = b$, $f(b) = a$, and $f(i) = i$ otherwise. f is well defined, and moreover $f_{|r_1 \wedge r_2} = id_{|r_1 \wedge r_2}$ because both $a, b \notin \mathrm{Im}(r_1 \wedge r_2)$ ($a \notin \mathrm{Im}(r_1)$ and $b \notin \mathrm{Im}(r_2)$). By Proposition 18, this means that $B_f(x) = x$, hence the thesis. □

Then, we have to restrict our attention to a particular class of subobjects:

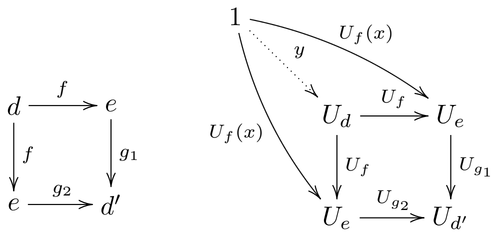
Definition 35. *Let $A : \mathbb{D} \to Set$ be an object of $Set^{\mathbb{D}}$. A subobject $U \leq A$ is* closed *if for all $d \in \mathbb{D}$, $f : d \to e$, $x \in A_d$: if $A_f(x) \in U_e$ then $x \in U_d$.*

The lattice of closed subobjects of A is denoted by $\mathrm{ClSub}(A)$.

However, pullback-preserving subobjects of pullback-preserving functors are automatically closed, so this requirement is implied by the first one:

Proposition 36. *Let $A : \mathbb{D} \to Set$ be a pullback preserving functor, and $U \leq A$ be a subobject of A. If also U is pullback preserving, then it is closed.*

Proof. Let $f : d \to e$ be a morphism in $\mathbb{D}$, and $x \in A_d$ such that $A_f(x) \in U_e$. Take any object d' and $g_1, g_2 : d \to d'$ such that d is the pullback of g_1 along g_2:



Then, the square of the diagram on the right is a pullback, and hence there exists a unique $y \in U_d$ such that $U_f(y) = A_f(x)$. It must be $y = x$, because there must be exactly one x satisfying a similar pullback diagram for A. □

Let us denote by $\mathcal{D}$ the full subcategory of $Set^{\mathbb{D}}$ of pullback preserving functors. By above, for all $A \in \mathcal{D}$, the lattice $\mathrm{Sub}(A)$ of pullback-preserving subobjects is $\mathrm{ClSub}(A)$, but we will keep writing $\mathrm{ClSub}(A)$ for avoiding confusions.

For "well-behaved" types, θ^* restricts to closed subobjects:

Proposition 37. *For all $A, B \in \mathcal{D}$ and $U \in \mathrm{ClSub}(A) : \theta^*(U) \in \mathrm{ClSub}(A \# B)$.*

Its left and right adjoints $\exists_\theta, \forall_\theta : \mathrm{ClSub}(A \# B) \to \mathrm{ClSub}(A)$ have the following explicit descriptions: for $U \leq A \# B$:

$$\exists_\theta(U)_d = \{y \in B_d \mid \text{there exist } f : d \to e, x \in A_e, \text{ such that } x \# B_f(y) \text{ and } (x, B_f(y)) \in U_e\}$$

$$\forall_\theta(U)_d = \{y \in B_d \mid \text{for all } f : d \to e, x \in A_e, \text{ if } x \# B_f(y) \text{ then } (x, B_f(y)) \in U_e\}$$

Proposition 38. *For all B in $\mathcal{D}$: $\theta^* \circ \exists_\theta = id_{\mathrm{ClSub}(Atom \# B)}$*

Proof. For $U \in \mathrm{ClSub}(Atom \# B)$, we have to prove that $\theta^*(\exists_\theta(U)) = U$. Inclusion $\supseteq$ is trivial. Let us prove $\subseteq$. If $(a, y) \in \theta^*(\exists_\theta(U))_d$, then $a \# y$, and by definition of $\exists_\theta$ there exist $f : d \to e$, $b \in Atom_e$ such that $(b, B_f(y)) \in U_e$ (and hence $b \# B_f(y)$). But also $f(a) \# B_f(y)$, and therefore by Proposition 34, this means that also $(f(a), B_f(y)) \in U_e$. By closure of U, it must be $(a, y) \in U_d$. □

Proposition 39. *Let $B \in \mathcal{D}$, and $U \in \mathrm{ClSub}(B)$; then, for all $x \in U_d$, there exist $f : d \to e$ and $a \in Atom_e$ such that $a \# B_f(x)$.*

Proof. We can "lift" this result from the subcategory of pullback preserving functors of $Set^{\mathbb{I}}$, i.e. the Schanuel topos, where this property is known to hold [8]. It is easy to check that if $F : \mathbb{D} \to Set$ is pullback-preserving, then also $F \circ t : \mathbb{I} \to Set$ is pullback preserving. As a consequence, if $x \in U_{d(n)}$, then $x \in U_{\top(n)} = (U \circ \mathtt{t})_n$, and hence there exist $f : n \rightarrowtail m$, $a \in N_m = Atom_{\top(m)}$ such that $a \#_{\mathbb{I}} (B \circ \mathtt{t})_f(x)$, and thus $a \# B_f(x)$. □

Proposition 40. *For all B in $\mathcal{D}$: $\exists_\theta \circ \theta^* = id_{\mathrm{ClSub}(B)}$.*

Proof. Let $U \in \mathrm{ClSub}(B)$ be a closed subobject. For any $d \in \mathbb{D}$, we have

$$\begin{aligned}\exists_\theta(\theta^*(U))_d &= \{x \in B_d \mid \text{there exist } f : d \to e, a \in Atom_e,\\ &\qquad \text{s.t. } a \# B_f(x) \text{ and } (a, B_f(x)) \in \theta^*(U)_e\}\\ &= \{x \in B_d \mid \text{there exist } f : d \to e, a \in Atom_e, \text{ s.t. } a \# B_f(x) \text{ and } B_f(x) \in U_e\}\\ &= \{x \in U_d \mid \text{there exist } f : d \to e, a \in Atom_e, \text{ s.t. } a \# B_f(x)\}\end{aligned}$$

For Proposition 39 above, this is exactly equal to U_d, hence the thesis. □

Corollary 41. *For $A \in \mathcal{D}$, the inverse image $\theta^* : \mathrm{ClSub}(A) \to \mathrm{ClSub}(Atom \# A)$ is an isomorphism, and hence $\theta^* \dashv \exists_\theta = \forall_\theta \dashv \theta^*$*

Let us denote by $\text{И} : \mathrm{ClSub}(Atom \# A) \to \mathrm{ClSub}(A)$ any of $\exists_\theta$ and $\forall_\theta$. There is a close connection between this quantifier and Gabbay-Pitts' (hence the notation); in fact, both quantifiers enjoy the following inclusions:

Proposition 42. *Let $i : A \# B \hookrightarrow A \times B$ be the inclusion map, and $i^* : \mathrm{ClSub}(A \times B) \to \mathrm{ClSub}(A \# B)$ its inverse image. Then: $\forall \leq \text{И} \circ i^* \leq \exists$, that is, for all $U \in \mathrm{ClSub}(A \times B)$: $\forall U \leq \text{И}(i^*(U)) \leq \exists U$.*

Proof. A direct check. □

7 A model for $FO\lambda^\nabla$

In this section we apply the structure of $\mathcal{D}$ for giving a semantic interpretation of the logic $FO\lambda^\nabla$ [14]. $FO\lambda^\nabla$ is a proof theory of *generic judgments*. Terms and typing judgments $\Sigma \vdash t : \tau$ of $FO\lambda^\nabla$ are as usual for simply typed λ-calculus, signatures Σ are sets $x_1{:}\tau_1, \ldots, x_m{:}\tau_m$. Sequents have the form

$$\Sigma : \sigma_1 \triangleright B_1, \ldots, \sigma_n \triangleright B_n \longrightarrow \sigma_0 \triangleright B_0$$

where Σ is the *global* signature, and each σ_i is a *local* signature. A judgment $\sigma_i \triangleright B_i$ is called *generic*; each B_i can use variables of the global signature Σ or in the local signature σ_i (formally: $\Sigma, \sigma_i \vdash B_i : o$). See [14] for further details.

Variable symbols in $FO\lambda^\nabla$ play two different roles. Those declared in global signatures act as variables of λ-calculus; instead, variables of local signatures act as "locally scoped constants", much like restricted names of π-calculus. A model of $FO\lambda^\nabla$ must account for both aspects *at once*, and this is the reason for neither $Set^{\mathbb{F}}$ nor $Set^{\mathbb{I}}$ (and their subcategories) can suffice. We can give an interpretation of both aspects in $\mathcal{D}$, taking advantage of its structure which subsumes those of $Set^{\mathbb{F}}$ and $Set^{\mathbb{I}}$: as we will see, the dynamic allocation functor δ^-, the apartness tensor (right adjoint to δ^+) and the И quantifier will come into play.

The interpretation of types and terms is standard: each type τ is interpreted as a functor $[\![\tau]\!]$ in $\mathcal{D}$; the interpretation is extended to global signatures using the cartesian product. A well-typed term $\Sigma \vdash t : \gamma$ is interpreted as a morphism (i.e., a natural transformation) $[\![t]\!] : [\![\Sigma]\!] \longrightarrow [\![\gamma]\!]$ in $\mathcal{D}$. Notice that here, "local" signatures do not have any special rôle, so that terms are simply typed λ-terms without any peculiar "freshness" or "scoping" constructor.[6]

On the other hand, in the interpretation of generic judgments we consider variables in local signatures as *distinguished* atoms. A declaration y appearing in a local signature σ, is intended as a "fresh, local" atom.

Remark 43. A correct model for $FO\lambda^{\nabla}$ would require a distinguished functor of atoms for each type (which can occur in local signatures) of the term language. Although it is technically possible to develop a typed version of the theory of $Set^{\mathbb{D}}$ (along the lines of [13] for $Set^{\mathbb{F}}$), it does not add anything substantial to our presentation; so in the following we assume variables of local signatures, or bound by ∇, can be only of one type (denoted by α). Hence, local signatures σ are of the form $(y_1{:}\alpha, \ldots, y_n{:}\alpha)$, or better $(y_1, \ldots, y_n)$ leaving α's implicit. □

The distinguished type of propositions, o, is interpreted as the classifier of *(closed) subobjects*: $[\![o]\!]_d = \mathrm{ClSub}(\mathbf{y}(d)) = \mathrm{ClSub}(\mathbb{D}(d, \_))$. A generic judgment $(y_1, \ldots, y_n) \rhd B$ in Σ (i.e., $\Sigma, y_1 : \alpha, \ldots y_n : \alpha \vdash B : o$) is interpreted as a closed subobject $[\![(y_1, \ldots, y_n) \rhd B]\!]_{[\![\Sigma]\!]} \leq [\![\Sigma]\!]$. More precisely, $[\![\sigma \rhd B]\!]_A \in \mathrm{ClSub}(A)$ is defined first by induction on the length of the local context σ, and then by structural induction on B. Local declarations and the ∇ quantifier are rendered by the functor $\text{И} : \mathrm{ClSub}(A \# Atom) \to \mathrm{ClSub}(A)$ above. Some interesting cases:

$$[\![(y, \sigma) \rhd B]\!]_A \triangleq \text{И}([\![\sigma \rhd B]\!]_{A\#Atom}) \qquad [\![\rhd B_1 \wedge B_2]\!]_A \triangleq [\![\rhd B_1]\!]_A \wedge [\![\rhd B_2]\!]_A$$
$$[\![\rhd \nabla y.B]\!]_A \triangleq \text{И}([\![\rhd B]\!]_{A\#Atom}) \qquad [\![\rhd \forall_\gamma x.B]\!]_A \triangleq \forall([\![\rhd B]\!]_{A\times[\![\gamma]\!]})$$

It is easy to prove by induction on σ that

Proposition 44. $[\![(\sigma, y) \rhd B]\!]_A = [\![\sigma \rhd \nabla y.B]\!]_A$.

Finally, a sequent $\Sigma : \mathcal{B}_1, \ldots, \mathcal{B}_n \longrightarrow \mathcal{B}_0$ is *valid* if $\bigwedge_{i=1}^n [\![\mathcal{B}_i]\!]_{[\![\Sigma]\!]} \leq [\![\mathcal{B}_0]\!]_{[\![\Sigma]\!]}$. A rule $\frac{\mathcal{S}_1 \ldots \mathcal{S}_n}{\mathcal{S}}$ is *sound* if, whenever all $\mathcal{S}_1, \ldots, \mathcal{S}_n$ are valid, also $\mathcal{S}$ is valid.

Using this interpretation, one can check that the rules of $FO\lambda^{\nabla}$ are sound. In particular, the rules $\nabla\mathcal{L}$ and $\nabla\mathcal{R}$ are trivial consequence of above. The verification of $\forall\mathcal{R}$, and $\exists\mathcal{L}$ requires some work. Here, we have to give a categorical account of a particular encoding technique, called *raising*, used to "gain access" to local constants from "outside" their scope. E.g.:

$$\frac{\Sigma, h{:}\sigma \to \gamma : \Gamma \longrightarrow \sigma \rhd B[(h\ \sigma)/x]}{\Sigma : \Gamma \longrightarrow \sigma \rhd \forall_\gamma x.B} \forall\mathcal{R}$$

A simpler (i.e., monadic) application of raising occurs, in the following equivalence, which is provable in $FO\lambda^{\nabla}$:

$$\nabla x \forall_\gamma y.B \equiv \forall_{\alpha\to\gamma} h \nabla x.B[(h\ x)/y] \qquad \text{where } \Sigma, x : \alpha, y : \gamma \vdash B : o \qquad (16)$$

We show first how to represent (monadic) raising as in the equation 16; interestingly, it is here where the δ^- comes into play. Referring to equation 16, let us

[6] As Miller and Tiu say, this is a precise choice in the design of $FO\lambda^{\nabla}$, motivated by the fact that standard unification algorithms still work unchanged.

denote $A = [\![\Sigma]\!]$ and $C = [\![\gamma]\!]$. By the definition above, the interpretation of B is a subobject of $(A \# Atom) \times C$, while $B[(h\ x)/y]$ corresponds to a subobject of $(A \times C^{Atom}) \# Atom$. Now, notice that $C^{Atom} = \delta^- C$ (Proposition 7); thus, $h : \alpha \to \gamma$ is actually a term $[\![h]\!] \in \delta^- C$, that is a term which can make use of a locally declared *variable*. We can define the *raising* morphism

$$\begin{aligned} \mathsf{r} : (A \times \delta^- C) \# Atom &\to (A \# Atom) \times C \\ (x, h, a) &\mapsto (x, a, h(a)) \end{aligned}$$

The inverse image of r is $\mathsf{r}^* : \mathrm{ClSub}((A \# Atom) \times C) \to \mathrm{ClSub}((A \times \delta^- C) \# Atom)$, defined by the following pullback:

$$\begin{array}{ccc} \mathsf{r}^*(U) & \longrightarrow & U \\ \downarrow & & \downarrow \\ (A \times \delta^- C) \# Atom & \xrightarrow{\ \mathsf{r}\ } & (A \# Atom) \times C \end{array}$$

This morphism r^* is the categorical counterpart of the syntactic raising:

Proposition 45. *Let $\Sigma, x{:}\alpha, y{:}\gamma \vdash B : o$. Let us denote $A = [\![\Sigma]\!]$, $C = [\![\gamma]\!]$. Then, $\mathsf{r}^*([\![y \rhd B]\!]_C) = [\![y \rhd B[(h\ y)/x]]\!]_{A \times \delta^- C}$.*

Then, quite obviously, the equation 16 states that $\text{И} \circ \forall_\gamma = \forall_{\alpha\to\gamma} \circ \text{И} \circ \mathsf{r}^*$, that is, the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{ClSub}((A \# Atom) \times C) & \xrightarrow{\ \mathsf{r}^*\ } & \mathrm{ClSub}((A \times \delta^- C) \# Atom) & \xrightarrow{\ \text{И}\ } & \mathrm{ClSub}(A \times \delta^- C) \\ \downarrow{\scriptstyle \forall_\gamma} & & & & \downarrow{\scriptstyle \forall_{\alpha\to\gamma}} \\ \mathrm{ClSub}(A \# Atom) & & \xrightarrow{\qquad\qquad \text{И} \qquad\qquad} & & \mathrm{ClSub}(A) \end{array}$$

which can be checked by calculation. The raising morphism can be easily generalized to the polyadic case (recall that $B^{\top(n)} = B \# \cdots \# B$, n times):

$$\begin{aligned} \mathsf{r} : (A \times \delta^{-n} C) \# Atom^{\top(n)} &\to (A \# Atom^{\top(n)}) \times C \\ (x, h, a_1, \ldots, a_n) &\mapsto (x, a_1, \ldots, a_n, h(a_1, \ldots, a_n)) \end{aligned}$$

Then, the soundness of the rule $\forall\mathcal{R}$ is equivalent to the following:

Proposition 46. *Let $A, C \in \mathcal{D}$ be functors, and $n \in \mathbb{N}$. Let $\pi : A \times \delta^{-n} C \to A$ be the projection, and $\mathsf{r} : (A \times \delta^{-n} C) \# Atom^{\top(n)} \longrightarrow (A \# Atom^{\top(n)}) \times C$ the raising morphism. For all $G \in \mathrm{ClSub}(A)$, and $U \in \mathrm{ClSub}((A \# Atom^{\top(n)}) \times C)$, if $\pi^*(G) \le \text{И}^n(\mathsf{r}^*(U))$ then $G \le \text{И}^n(\forall_\gamma(U))$.*

8 Conclusions

In this paper, we have studied a new model for dynamically allocable entities, based on the notion of *distinction*. Previous models for variables and for names can be embedded faithfully in this model, and also results about initial algebras/final coalgebras and simultaneous substitutions are extended to the more general setting. In a suitable subcategory of the model, it is possible to define also a self-dual quantifier, similar to Gabbay-Pitts' "И". This rich structure has allowed us to define the first denotational model for the logic $FO\lambda^{\nabla}$.

Future work. The rich structure of $Set^{\mathbb{D}}$ can be useful also for modeling process calculi featuring both variables and names at once, like e.g. ambients. Actually, the intuition behind distinctions is also at the base of the *D-Fusion* calculus [2]; in fact, we think that the two binders λ, ν of D-Fusion can be modeled precisely by δ^- and δ^+ in $Set^{\mathbb{D}}$, respectively. Details will appear elsewhere.

$FO\lambda^{\nabla}$ is not complete with respect to the model presented in this paper: the И quantifier enjoys properties which are not derivable in $FO\lambda^{\nabla}$ (e.g., $\forall x.B \supset \nabla x.B$ and $\nabla x.B \supset \exists x.B$). One main reason is that $FO\lambda^{\nabla}$ does not admit *weakening on local signature*; for instance, the sequent $\Sigma : \sigma \rhd B \longrightarrow (\sigma, y) \rhd B$ is not derivable. This has been already noticed by Gabbay and Cheney, in their interpretation of $FO\lambda^{\nabla}$ into *Fresh Logic* [7], another first-order logic with a self-dual quantifier. Actually, we think that the И quantifier of $\mathcal{D}$ is closer to the И quantifier of Fresh Logic, than to the ∇ of $FO\lambda^{\nabla}$. For this reason, it should be possible to model Fresh Logic in $\mathcal{D}$ quite easily—another future work.

On the other hand, the construction of a complete model for $FO\lambda^{\nabla}$ is still an open problem. Ulrich Schöpp has suggested to use a tripos over $Set^{\mathbb{B}}$, where $\mathbb{B}$ is the subcategory of $\mathbb{F}$ of bijective maps.

Acknowledgments. The authors wish to thank Dale Miller and Alwen Tiu for useful discussions about $FO\lambda^{\nabla}$, and Neil Ghani for hints about Kan extensions.

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