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Published
DOI:10.2307/2275470

THE JOURNAL OF SYMBOLIC LOGIC
Volume 56, Number 4, Dec. 1991

EXPRESSING INFINITY WITHOUT FOUNDATION

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Abstract. The axiom of infinity can be expressed by stating the existence of sets satisfying a formula which involves restricted universal quantifiers only, even if the axiom of foundation is not assumed.

The problem of expressing the existence of infinite sets in the first order set-theoretic language by means of formulae of low logical complexity has been addressed in [PP88] and [PP90b]. While the usual formulations of the infinity axiom (Inf) make use of formulae involving (at least) alternations of universal and existential restricted quantifiers, [PP88] provided the first example of a formula involving only restricted universal quantifiers, whose satisfiability entails the existence of infinite sets, provided the foundation axiom (FA) is assumed together with the usual axioms of Zermelo-Fraenkel except, of course, the infinity axiom. It was then observed in [PP90b] that an even shorter formula had the same property. As explained in [PP88], the above problem is related to the so-called decision problem for fragments of set theory (see [CFO90]).

Set theories not assuming FA but rather contradicting it in various forms have come to attract considerable interest (see [Acz88]), and the corresponding decision problem has begun to be investigated (see [PP90a]). It is therefore of particular interest to ask whether there are restricted purely universal formulae which are satisfiable but not finitely satisfiable, even when FA is dropped.

In this note we show that a positive answer can be obtained through an appropriate merging of the two formulae in [PP88] and [PP90b], although neither of them suffices alone.

Let $\mathscr{L}_\in$ be the first order set-theoretic language with identity, based on the membership relation $\in$. A formula of $\mathscr{L}_\in$ is restricted if it does not contain quantifiers except for the restricted quantifiers $(\forall x \in y)$ and $(\exists x \in y)$.

Let ZF^- denote $ZF - Inf$ and ZF^{--} denote $ZF^- - FA$. In ZF^{--} one can define the ordinals as transitive sets well-ordered by $\in$ and the nonzero natural numbers as successor ordinals with zero and successor ordinals only, as elements. Finiteness is taken to stand for equinumerousity with a natural number, and Inf can be stated as the existence of a set containing all the natural numbers.

Received May 30, 1990; revised November 14, 1990.
This work was supported by funds from the MPI, and by the AXL project of ENI and ENIDATA.

0022-4812/91/5604-0005/$01.60

In ZF$^-$, but not in ZF^{--}, Inf is equivalent to any of the other formulations of the infinity axiom in use. Note that Inf states the satisfiability of a formula that, besides restricted quantifiers of both types, involves also an unrestricted universal quantifier needed to express well-foundedness.

Let φ_1 and φ_2 be the following formulae from [PP88] and [PP90b]:

$$\begin{aligned} & a \neq b \wedge a \notin b \wedge b \notin a \\ & \quad \wedge (\forall x \in a)(\forall u \in x)(u \in b) \wedge (\forall x \in b)(\forall u \in x)(u \in a) \wedge (\forall x \in a)(x \notin b) \\ & \quad \wedge (\forall x, y \in a)(\forall z, w \in b)(z \in x \wedge x \in w \wedge w \in y \rightarrow z \in y) \\ & \quad \wedge (\forall x, y \in b)(\forall z, w \in a)(z \in x \wedge x \in w \wedge w \in y \rightarrow z \in y), \end{aligned}$$

and

$$\begin{aligned} & a \neq b \wedge a \notin b \wedge b \notin a \\ & \quad \wedge (\forall x \in a)(\forall u \in x)(u \in b) \wedge (\forall x \in b)(\forall u \in x)(u \in a) \\ & \quad \wedge (\forall x \in a)(\forall y \in b)(x \in y \vee y \in x). \end{aligned}$$

From [PP88] and [PP90b] we have the following property:

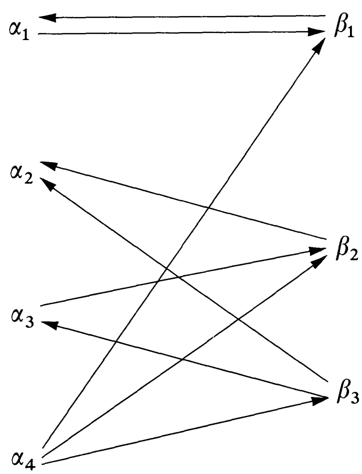
PROPOSITION 1. 1) *$ZF^- \vdash (\exists a, b)\varphi_1(a, b) \rightarrow Inf$.*

2) *$ZF^- \vdash (\exists a, b)\varphi_2(a, b) \rightarrow Inf$.*

Actually 2) follows immediately from 1) since ZF$^-$ $\vdash \varphi_2(a, b) \rightarrow \varphi_1(a, b)$. [PP90b] provides specific examples which show that ZF$^-$ $\nvdash \varphi_1(a, b) \rightarrow \varphi_2(a, b)$.

PROPOSITION 2. *$ZF^{--} \nvdash (\exists a, b)\varphi_1(a, b) \rightarrow Inf$.*

PROOF. Consider the following graph G_1:



By making use of a suitable permutation of the universe it is easy to define a model $\mathcal{M}$ of ZF^{--} + $\neg$Inf in which there are elements a, a_1, a_2, a_3, a_4 and b, b_1, b_2, b_3 such that

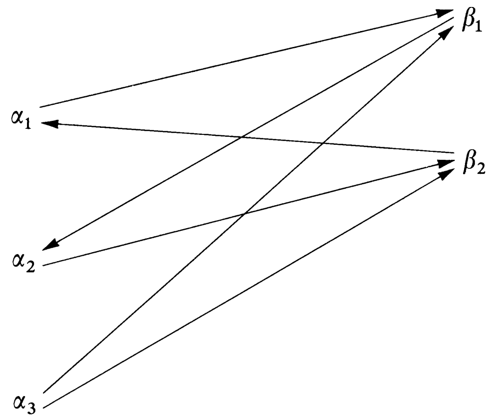
$$a = \{a_1, a_2, a_3, a_4\}^{\mathcal{M}},$$
$$b = \{b_1, b_2, b_3\}^{\mathcal{M}},$$

$$a_i \in^{\mathscr{M}} b_j \Leftrightarrow \alpha_i \to \beta_j \quad \text{for } 1 \leq i \leq 4,\ 1 \leq j \leq 3,$$
$$b_i \in^{\mathscr{M}} a_j \Leftrightarrow \beta_i \to \alpha_j \quad \text{for } 1 \leq i \leq 3,\ 1 \leq j \leq 4.$$

It is then straightforward to check that a and b satisfy φ_1 in $\mathscr{M}$. ■

PROPOSITION 3. *$ZF^{--} \nvdash (\exists a, b)\varphi_2(a,b) \to Inf$.*

PROOF. Use the same kind of argument as for the previous property, starting with the following graph G_2:



■

REMARK. Since G_1 and G_2 are extensional graphs, i.e. two different nodes have different sets of predecessors, the consistency results following from Proposition 2 and Proposition 3 can be improved to claim that the existence of finite sets satisfying φ_1 and φ_2 is actually a theorem of $\mathrm{ZF}^{--} + \mathrm{BA}_1$, where BA_1 stands for the weak form of Boffa's antifoundation axiom discussed in [Acz88].

The transitive closures of the finite sets a and b described by G_1 and G_2 contain a loop of the form $a_1 \in b_1 \in a_1$ and $a_1 \in b_1 \in a_2 \in b_2 \in a_1$ respectively. That is no exception. Let us begin by noticing that because the two conjuncts

$$(\forall x \in a)(\forall u \in x)(u \in b)$$

and

$$(\forall x \in b)(\forall u \in x)(u \in a)$$

are in both φ_1 and φ_2, if a and b satisfy either φ_1 or φ_2 then $a \cup b$ contains both the transitive closure of a and the transitive closure of b. Thus if a and b satisfy either φ_1 or φ_2 and are finite, then they are actually *hereditarily finite*, by which, in absence of FA, it is meant that they have finite pictures (like G_1 and G_2 in the above examples) (see [Acz88]).

PROPOSITION 4 (In ZF^{--}). *If a and b are finite and $\varphi_1(a,b)$ holds, then there is a_1 in a and b_1 in b such that $a_1 \in b_1$ and $b_1 \in a_1$.*

PROOF. From the proof in [PP88] it follows that if a and b are finite and $\varphi_1(a,b)$ holds, then $a \cup b$ cannot be well-founded. Hence $a \cup b$ must contain a cycle with respect to membership, say $c_1 \in c_2 \in \cdots \in c_n \in c_1$.

From $\varphi_1(a,b)$ it follows that an element of an element of a cannot be itself an element of a, since otherwise $a \cap b \neq \varnothing$. Similarly an element of an element of b

cannot be itself an element of b. From that it follows that $c_1, \dots, c_n$ contains alternatively an element of a and an element of b, and furthermore that n is even. Then by induction on n, using the condition

$$(\forall x_1, x_2 \in a)(\forall y_1, y_2 \in b)(x_1 \in y_1 \wedge y_1 \in x_2 \wedge x_2 \in y_2 \rightarrow x_1 \in y_2),$$

when $c_1 \in a$, or the condition

$$(\forall x_1, x_2 \in b)(\forall y_1, y_2 \in a)(x_1 \in y_1 \wedge y_1 \in x_2 \wedge x_2 \in y_2 \rightarrow x_1 \in y_2)$$

when $c_1 \in b$, it follows immediately that in any such cycle we must have $c_1 \in c_n$, and our claim is proved by taking c_1 for a_1 and c_n for b_1. ■

As a straightforward consequence of the previous proposition we have that if we add the conjunct

$$(\forall x \in a)(\forall y \in b)(x \in y \rightarrow y \notin x)$$

then we obtain a formula φ'_1 involving only restricted universal quantifiers whose satisfiability entails the existence of infinite sets even if FA is dropped.

COROLLARY 1. *$ZF^{--} \vdash (\exists a, b)(\varphi_1(a, b) \wedge (\forall x \in a)(\forall y \in b)(x \in y \rightarrow y \notin x) \rightarrow Inf$.*

PROOF. By Proposition 4, the existence of a and b satisfying φ'_1 entails the existence of nonfinite sets. By a well-known argument (see [Lev79]), using the power set axiom, the existence of a nonfinite set implies Inf in ZF^{--}. ■

A proof entirely similar to the one given for Proposition 4 shows that:

PROPOSITION 5 (In ZF^{--}). *If a and b are finite and $\varphi_2(a, b)$ holds and $a \cap b = \varnothing$, then there are a_1, a_2 in a and b_1, b_2 in b such that $a_1 \in b_1 \in a_2 \in b_2 \in a_1$ or $b_1 \in a_1 \in b_2 \in a_2 \in b_1$.*

Therefore the addition to φ_2 of $(\forall x \in a)(x \notin b)$ and the two further conditions

$$(\forall x_1, x_2 \in a)(\forall y_1, y_2 \in b)(x_1 \in y_1 \wedge y_1 \in x_2 \wedge x_2 \in y_2 \rightarrow y_2 \notin x_1)$$

and

$$(\forall x_1, x_2 \in a)(\forall y_1, y_2 \in b)(x_1 \in y_1 \wedge y_1 \in x_2 \wedge x_2 \in y_2 \rightarrow y_2 \notin x_1)$$

also yields a formula φ'_2 which satisfies Corollary 1.

It follows immediately from [PP88] that φ'_1 and φ'_2 are irredundant, in the sense that if one drops one of the conjuncts then this property fails. We can however provide an even simpler formula whose satisfiability entails Inf in ZF^{--}. Such a formula is of particular interest also in connection with the decision problem since, unlike φ'_1 and φ'_2, it does not introduce conjuncts which are implications whose consequent is a negative literal.

Let $\varphi(a, b)$ be the following formula:

$$\begin{aligned} & a \neq b \wedge a \notin b \wedge b \notin a \\ & \quad \wedge (\forall x \in a)(\forall y \in x)(y \in b) \wedge (\forall x \in a)(\forall y \in x)(y \in b) \\ & \quad \wedge (\forall x_1, x_2 \in a)(\forall y_1, y_2 \in b)(x_1 \in y_1 \wedge y_1 \in x_2 \wedge x_2 \in y_2 \rightarrow x_1 \in y_2) \\ & \quad \wedge (\forall x \in a)(\forall y \in b)(x \in y \vee y \in x). \end{aligned}$$

PROPOSITION 6. *$ZF^{--} \vdash (\exists a, b)\varphi(a, b) \rightarrow Inf$.*

PROOF. Working in ZF^{--}, we show that if a and b satisfy φ then $a \cup b$ cannot be finite. The conclusion follows as in Proposition 5.

Assuming that a and b satisfy φ, we show that if X is a finite nonempty subset of $a \cup b$ then there is an element $c_X \in X$ such that either $c_X \in a$ and $X \cap b \subseteq c_X$, or $c_X \in b$ and $X \cap a \subseteq c_X$. That is proved by induction on the cardinality of X.

If X is a singleton the claim is clear, since $a \cap b = \varnothing$. So assume X has more than one element. If $X \cap a = \varnothing$ then every element in X can be taken as c_X; otherwise pick $a_i \in X \cap a$ and let $X' = X \setminus \{a_i\}$.

By the induction hypothesis there is $c_{X'}$ in X' satisfying our claim. If $c_{X'} \in a$ and $X' \cap b \subseteq c_{X'}$ then, since $a \cap b = \varnothing$, $X \cap b = X' \cap b$; hence $X \cap b \subseteq c_{X'}$ and we can take c_X to be $c_{X'}$ itself. On the other hand, if $c_{X'} \in b$ and $X' \cap a \subseteq c_{X'}$, we have two cases:

Case 1. $a_i \in c_{X'}$. Then it still suffices to let $c_X = c_{X'}$.

Case 2. $a_i \notin c_{X'}$. Then, since $(\forall x \in a)(\forall y \in b)(x \in y \vee y \in x)$, $c_{X'} \in a_i$. If $b \cap X \subseteq a_i$, then of course it suffices to let $c_X = a_i$. Otherwise there must be a $b_i \in b \cap X$ such that $b_i \notin a_i$. Then, as above, $a_i \in b_i$. In this case it suffices to let $c_X = b_i$. In fact since

$$(\forall x_1, x_2 \in a)(\forall y_1, y_2 \in b)(x_1 \in y_1 \wedge y_1 \in x_2 \wedge x_2 \in y_2 \rightarrow x_1 \in y_2),$$

from $X' \cap a \subseteq c_{X'}$, $c_{X'} \in a_i$ and $a_i \in b_i$ it follows that $X' \cap a \subseteq b_i$ and therefore that $X \cap a \subseteq b_i$.

We can now prove that $a \cup b$ cannot be finite.

First of all, $a \cup b \neq \varnothing$, since otherwise $a = b = \varnothing$ and $\varphi(a, b)$ would fail.

If $a \cup b$ were finite and nonempty, then we could take $a \cup b$ for X in the above claim and conclude that there is $c \in a \cup b$ such that either $c \in a$ and $(a \cup b) \cap b = b \subseteq c$, or $c \in b$ and $(a \cup b) \cap a = a \subseteq c$. In the former case it would follow that $c = b$, since from $c \in a$ it follows that $c \subseteq b$, because of the condition $(\forall x \in a)(\forall y \in x)(y \in b)$. But then $b \in a$, contradicting $b \notin a$ as required by φ. Similarly in the latter case we would get $a \in b$, contrary to $a \notin b$. ■

Obviously Proposition 6 still holds if we exchange a with b.

As for φ_1, in [PP88] it is easy to see that if any of the conjunct in φ is dropped then the resulting formula turns out to be satisfiable by finite (although not necessarily well-founded) sets.

REMARK. It would be interesting to prove Proposition 6 without using the power set axiom.

Note that the implication in Proposition 1, Corollary 1, and Proposition 6 can be reversed since φ, φ_1, φ_2, φ'_1 as well as φ'_2 are all satisfied by $\omega' = \{f_0, f_1, \ldots\}$ and $\omega'' = \{g_0, g_1, \ldots\}$, where $f_0 = \varnothing$, $g_n = \{f_0, \ldots, f_n\}$, $f_{n+1} = \{g_0, \ldots, g_n\}$, and the existence of ω' and ω'' is ensured in ZF^{--} provided Inf is assumed. Thus the existential closures of φ, φ_1, φ_2, φ'_1 and φ'_2 can all be used to express the axiom of infinity in ZF^{--}. To that end the presence of nesting of quantified variables, as in $(\forall x \in a)(\forall y \in x)(y \in b)$ is in general unavoidable; in fact it is unavoidable in any restricted universal formula which is satisfiable but not finitely satisfiable (see [PP88]). Furthermore the presence of at least two free variables is necessary—in fact, if a restricted universal formula with just one free variable is satisfied by a set a and there is a finite descending chain of memberships starting with a and ending with the empty set, its Mostowski's collapse provides an hereditarily finite set satisfying the same formula, whereas on the other hand if no such chain exists, then, under Aczel's antifoundation axiom AFA, a is just the finite set Ω.

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